Percolation: Some questions 1

1. **Oriented percolation**. Each edge of the quadrant $\mathbb{Z}_+^2 = \{x = (x_1, x_2) : x_1, x_2 \geq 0\}$ is oriented upwards (if vertical) and rightwards (if horizontal). Assume each edge is open with probability p, as usual, and let $\theta(p)$ be the probability that 0 is the starting point of an infinite oriented open path (all of whose edges are directed away from the origin, i.e., north-east). Show that $p_c = \sup\{p : \theta(p) = 0\}$ satisfies $0 < p_c < 1$. [You will have to think a little for the upper bound.]

2. **Bond/site comparison**. Let G be an infinite connected graph with maximal vertex degree Δ. Show that the critical probabilities f r bond and site percolation on G satisfy nd and site percolation on G satisfy
$$p_c(\text{bond}) \leq p_c(\text{site}) \leq 1 - \big(1 - p_c(\text{bond})\big)^{\Delta}.$$
[The first inequality is a fair target. The second is a little harder.]

3. (continuation) Under what conditions on G is it the case that $p_c(\text{bond}) < p_c(\text{site})$? [Research problem. See me if you have ideas. One of last year's audience, Alan Stacey, and I are now writing a paper on this general subject.]

4. **Inhomogeneous percolation**. Each horizontal (resp. vertical) edge of $\mathbb{L}^2$ is declared *open* with probability p_1 (resp. p_2). Show that there is a.s. no infinite open cluster when $p_1 + p_2 \leq 1$. [Use the argument that $\theta(\frac{1}{2}) = 0$ for ordinary bond percolation on $\mathbb{L}^2$; see Section 9 of the notes.]

5. Let a_n be the number of self-avoiding walks on $\mathbb{L}^d$, i.e., the number of paths $x_0, e_0, x_1, e_1, \ldots, x_n$ of distinct vertices x_j and edges $e_j = \langle x_j, x_{j+1}\rangle$. Show that $a_{m+n} \leq a_m a_n$, and deduce the existence of the connective constant $\mu = \lim_{n\to\infty}\{a_n^{1/n}\}$. Show that $p_c \geq \mu^{-1}$.

6. Let A be an increasing event on the finite sample space $\Omega = \{0,1\}^E$. Find a formula for the second derivative $(d^2/dp^2)P_p(A)$, akin to Russo's formula.

7. **Subadditive inequality**. Let (x_n) be a real sequence satisfying $x_{m+n} \leq x_m + x_n$ for all m, n. Show that the limit $\lambda = \lim\{n^{-1}x_n\}$ exists and satisfies $\lambda = \inf_k\{k^{-1}x_k\}$.

8. (continuation) Can you find reasonable conditions on the sequence (α_n) such that: the generalised inequality
$$x_{m+n} \leq x_m + x_n + \alpha_m \qquad \text{for all } m, n$$
implies the existence of the limit $\lambda = \lim\{n^{-1}x_n\}$?

9. **One-dimensional percolation**. Each edge of the one-dimensional lattice $\mathbb{L}$ is *open* with probability p, as usual. For $k \in \mathbb{Z}$, let $r(k) = \max\{u : k \leftrightarrow k + u\}$, and $R_n = \max\{r(k) : 1 \leq k \leq n\}$. Show that $P(R_n > u) \leq np^u$, and hence, for $\epsilon > 0$,
$$\left(R_n > \frac{(1+\epsilon)\log n}{\log(1/p)}\right) \to 0 \qquad \text{as } n \to \infty.$$

10. (continuation) Show that, for $\epsilon > 0$,
$$P\left(R_n < \frac{(1-\epsilon)\log n}{\log(1/p)}\right) \to 0 \qquad \text{as } n \to \infty.$$
[This is the famous problem of the longest run of heads in n tosses of a coin.]

11. Read the proof in the notes of the general version of the FKG inequality for measures on the sample space $\Omega = \{0,1\}^E$.