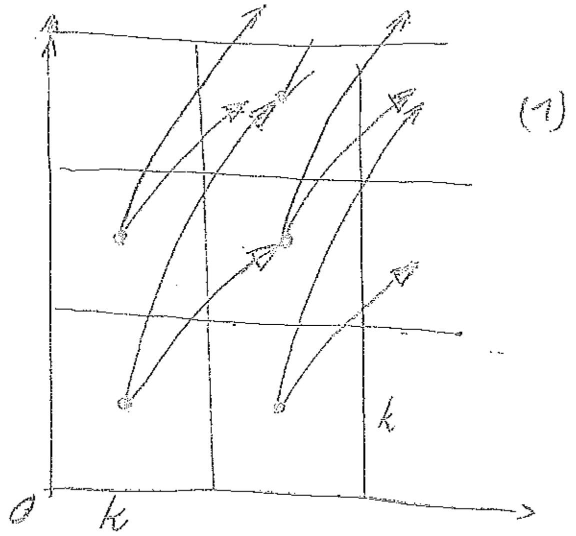
Percolation ; Sheet 2 - 2011

3.11

$P[\,[0,k-1]^2$ contains both a 0 and a $1] = 1 - p^{k^2} - (1-p)^{k^2} \xrightarrow[k\to\infty]{} 1$

Say a $k \times k$ square is open if it contains both a 0 and a 1.



Consider the system of squares of fig 1 with the directed connections drawn.

if 0 is in an infinite (directed) path of open squares in (1) then any word is $6k$ seen.

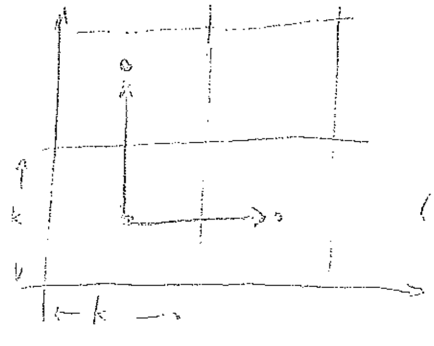
Notice that the graph of 1 is just $\vec{\mathbb{Z}}^2$.

We know that p_c(site perco on $\vec{\mathbb{Z}}^2$) < 1 so for k large enough $P[$a $k \times k$ square is good$] > p_c(\vec{\mathbb{Z}}^2)$

Thus for k large, the probability that any word is $6k$-seen is positive.

- Note that the squares in (1) do <u>not</u> overlap, thus their states are independent. This is essential for relating (1) to site percolation on $\vec{\mathbb{Z}}^2$.

- it might look interesting to consider percolation of open squares in (2). The existence of an infinite open path in (2) does <u>not</u> guarantee that any word is $4k$ seen.



4.9 $$\frac{d E_p[X]}{dp} = \frac{d}{dp} \sum_\omega p^{|\omega|}(1-p)^{N-|\omega|} X(\omega) =$$

$$= \sum_\omega \left(\frac{|\omega|}{p} - \frac{N-|\omega|}{1-p} \right) P_p(\omega) X(\omega) =$$

$$= \sum_{e,\omega} \left(\frac{1}{p} \mathbb{1}_{e\in\omega} - \frac{1}{1-p} \mathbb{1}_{e\notin\omega} \right) P_p(\omega) X(\omega)$$

$$= \sum_e \frac{1}{p} \mathbb{E}_p[\mathbb{1}_{e\in\omega} X] - \frac{1}{1-p} \mathbb{E}_p[\mathbb{1}_{e\notin\omega} X] =$$

$$= \sum_e \mathbb{E}_p[X^e] - \mathbb{E}[X_e] = \sum_e \mathbb{E}[\delta_e X].$$

For A increasing, $\delta_e \mathbb{1}_A \in \{0,1\}$, with

$\delta_e \mathbb{1}_A = 1 \Leftrightarrow \omega^e \in A$ and $\omega_e \notin A \Leftrightarrow e$ is pivotal for A (in ω).

Thus $$\frac{d P_p(A)}{dp} = \mathbb{E}[\text{nb of pivotal edges}].$$

Using the computation twice we get:

$$\frac{d^2 E_p(X)}{dp^2} = \sum_{e,f} \mathbb{E}[\delta_e \delta_f X].$$

For A increasing event $\frac{d^2 P_p(A)}{dp^2} = \mathbb{E}_p[N_A^{ser} - N_A^{par}]$

N_A^{ser} = nb of pairs (e,f) st $\omega^{ef} \in A$ but $\omega_f^e, \omega_e^f \notin A$

N_A^{par} = nb of pairs (e,f) st $\omega_f^e, \omega_e^f \in A$ but $\omega_{ef} \notin A$.

4.11

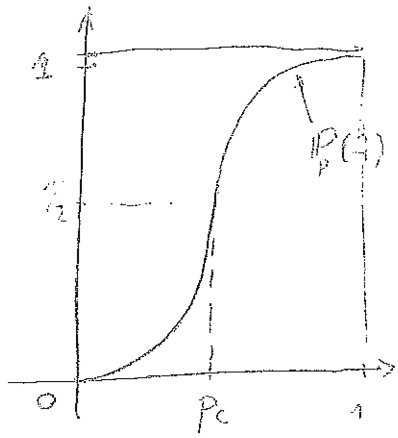
Since A is $\mathcal{A}$ invariant and $\mathcal{A}$ acts transitively on $\{1,\dots,n\}$; all influences are the same:

$$I_A(i) = I_A(j) \quad \forall\, i,j \in \{1,\dots,n\}.$$

Thus: $I_A(i) = \max_j I_A(j) \geq c\, \mathbb{P}_p[A](1-\mathbb{P}_p[A]) \frac{\log n}{n}$

And $$\frac{d\,\mathbb{P}_p(A)}{dp} = \sum_i I_A(i) \geq c\,\mathbb{P}_p(A)(1-\mathbb{P}_p(A)) \log n$$

For $p > p_c$; $\mathbb{P}_p(A) \geq 1/2$ (since A is increasing)



So $$\frac{d\,\mathbb{P}_p(A)}{dp} \geq \frac{c}{2}(1-\mathbb{P}_p(A)) \log n$$

thus $d \log_e (1-\mathbb{P}_p(A)) \leq -\frac{c}{2} \log n$. ⊛

Using ⊛ we get that, for $p > p_c$

$$\mathbb{P}_p(A) \geq 1 - n^{-\frac{c}{2}(p-p_c)}$$

For $p < p_c$ we can deduce, by the same method:

$$\mathbb{P}_p(A) \leq n^{\frac{c}{2}(p-p_c)}$$

5.1. Let $\mathcal{B} = [-n, n] \times [0, 2n-1]$.

$\mathcal{B}$ has width $2n+1$ and height $2n$, hence, by self-duality:

$$\mathbb{P}_{1/2}[\exists \text{ a horizontal crossing of } \mathcal{B}] = 1/2.$$

This implies that there exists $x \in \{0\} \times [0, 2n-1]$ s.t.

$$\mathbb{P}[x \text{ is on a horizontal crossing of } \mathcal{B}] \geq \frac{1}{4n}$$

But if the above occurs, then there are two disjoint open paths linking x to $\partial[-n;n]^2 + x$ so:

$$\frac{1}{4n} \leq \mathbb{P}[x \text{ is on a horizontal crossing of } \mathcal{B}] \overbrace{\leq}^{BK}$$

$$\leq \mathbb{P}[\mathrm{rad}(C_x) \geq n]^2.$$

This proves the result, since C_x and C_0 have same law (modulo translation)

7.3. (a) Let σ and σ' be two configurations in Σ.

We need to show:

$$\lambda(\sigma\wedge\sigma')\cdot\lambda(\sigma\vee\sigma')\geq\lambda(\sigma)\lambda(\sigma') \overset{\text{simplifies by norm. cons}}{\Longleftrightarrow}$$

$$\exp\Big(h\sum_{v\in V}\big((\sigma_v\wedge\sigma_v')+(\sigma_v\vee\sigma_v')\big)+$$

$$+\beta\sum_{e=\langle u,v\rangle\in E}\big((\sigma_u\wedge\sigma_u')(\sigma_v\wedge\sigma_v')+(\sigma_u\vee\sigma_u')(\sigma_v\vee\sigma_v')\big)\Big)\geq$$

$$\geq\exp\Big(h\sum_{v\in V}(\sigma_v+\sigma_v')+\beta\sum_{e=(u,v)\in E}(\sigma_u\cdot\sigma_v+\sigma_u'\cdot\sigma_v')\Big) \underset{\beta>0}{\overset{\sigma_v\wedge\sigma_v'+\sigma_v\vee\sigma_v'=\sigma_v+\sigma_v'}{\Longleftarrow\!\Longrightarrow}}$$

$$\Rightarrow\sum_{e=\langle u,v\rangle}(\sigma_u\wedge\sigma_u')(\sigma_v\wedge\sigma_v')+(\sigma_u\vee\sigma_u')(\sigma_v\vee\sigma_v')\geq\sum_{e=\langle u,v\rangle}\sigma_u\sigma_v+\sigma_u'\sigma_v' \quad (*)$$

The last inequality is true term by term.

The only case in which the terms are not equal is of the type $\sigma_u=+1$; $\sigma_v=-1$; $\sigma_u'=-1$; $\sigma_v'=+1$.

In this case:

$$(\sigma_u\wedge\sigma_u')(\sigma_v\wedge\sigma_v')+(\sigma_u\vee\sigma_u')(\sigma_v\vee\sigma_v')=2$$

$$\sigma_u\sigma_v+\sigma_u'\sigma_v'=-2.$$

so $(*)$ holds.

(b) Let A be an increasing event. Since λ is positively associated,

$$\circledast \quad \lambda(A \cap \{\sigma_v = 1\}) \geq \lambda(A)\lambda(\sigma = 1), \text{ so}$$

$$\lambda(A \mid \sigma_v = 1) = \frac{\lambda(A \cap \{\sigma_v = 1\})}{\lambda(\sigma_v = 1)} \geq \lambda(A).$$

Hence $\lambda(\cdot \mid \sigma_v = 1) \geq_{st} \lambda(A)$.

Moreover $\circledast$ also implies:

$$\lambda(A \cap \{\sigma_v = -1\}) \leq \lambda(A)\lambda(\sigma = -1)$$

which in turn implies:

$$\lambda(\cdot \mid \sigma_v = -1) \leq_{st} \lambda(A).$$

$\boxed{8.1}$

$$\phi_{pq}(\omega) = \frac{1}{Z_{pq}} \underbrace{p^{M}(1-p)^{|E|-M} q^{k(\omega)}}_{\varphi_{pq}(\omega)}$$

For an event A:

$$\frac{d\,\varphi(A)}{dp} = \frac{d}{dp} \sum_{\omega} \varphi(\omega)\, \mathbb{1}_{\omega\in A} =$$

$$= \frac{1}{p(1-p)} \sum_{\omega} \big(M(1-p) - (|E|-M)p \big) \varphi_{pq}(\omega)\, \mathbb{1}_{\omega \in A} =$$

$$= \frac{1}{p(1-p)} \big(\varphi_{pq}(M \mathbb{1}_A) - p|E|\,\varphi(A) \big)$$

Setting $A = \Omega$, we have:

$$0 = \frac{d}{dp} \phi_{pq}(\Omega) = \varphi_{pq}(\Omega) \frac{d}{dp} \frac{1}{Z_{pq}} + \frac{1}{Z_{pq}} \frac{d}{dp} \varphi_{pq}(\Omega)$$

So $$\frac{d}{dp} \frac{1}{Z_{pq}} = -\frac{1}{\varphi_{pq}(\Omega)} \cdot \frac{1}{Z_{pq}} \frac{d}{dp} \varphi_{pq}(\Omega) =$$

$$= -\frac{1}{Z_{pq}^{2}} \frac{1}{p(1-p)} \big(\varphi_{pq}(M) - p|E|\,\varphi(\Omega) \big) =$$

$$= -\frac{1}{p(1-p)} \left(\frac{\phi_{pq}(M)}{Z_{pq}} - \frac{p|E|}{Z_{pq}} \right)$$

Finally

$$\frac{d}{dp}\phi_{pq}(A) = \frac{d}{dp}\frac{1}{Z_{pq}}\cdot\varphi(A) + \frac{1}{Z_{pq}}\frac{d}{dp}\varphi(A) =$$

$$= \frac{1}{p(1-p)}\left(-\frac{\varphi(A)}{Z_{pq}}\left(\phi_{pq}(M) - p|E|\right) + \frac{1}{Z_{pq}}\left(\varphi_{pq}(M 1_A) - p|E|\varphi_{pq}(A)\right)\right) =$$

$$= \frac{1}{p(1-p)}\left(\phi_{pq}(M 1_A) - \phi_{pq}(M)\phi_{pq}(A)\right).$$

Throughout the computation we have used the definition of the normalising constant:

$$Z_{pq} = \varphi_{pq}(\Omega)$$

8.4 It suffices to prove the FKG lattice condition for ω_1, ω_2 differing on only two edges.

Take $\omega \in \Omega$ and $e, f \in E$, we need to show:

$$\phi_{pq}(\omega^{ef})\,\phi_{pq}(\omega_{ef}) \geq \phi_{pq}(\omega_e^f)\,\phi_{pq}(\omega_f^e) \iff$$

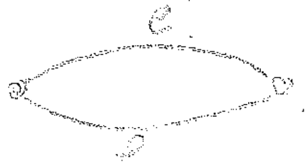
$$\iff q^{K(\omega^{ef}) + K(\omega_{ef})} \geq q^{K(\omega_e^f) + K(\omega_f^e)} \overset{q \geq 1}{\iff}$$

$$\iff K(\omega^{ef}) + K(\omega_{ef}) \geq K(\omega_e^f) + K(\omega_f^e).$$

$$\iff K(\omega_f^e) - K(\omega^{ef}) \leq K(\omega_{ef}) - K(\omega_e^f)$$

The last inequality may be read as:
if removing f from ω^{e} breaks a cluster,
then removing it from ω_e^f also breaks a cluster.

For $q < 1$ take G.



and $A = \{\omega_e = 1\}$; $B = \{\omega_f = 1\}$.

$$\phi_{pq}(A)\,\phi_{pq}(B) = \frac{1}{Z^2}\left(p^2 q + p(1-p)q\right)^2 = \frac{q^2 p^2}{(p^2 q + 2p(1-p)q + (1-p)^2 q^2)^2}$$

$$\phi_{pq}(A \cap B) = \frac{p^2 q}{p^2 q + 2p(1-p)q + (1-p)^2 q^2} < \phi(A)\,\phi(B).$$

8.6| Case 1 $p, q \to 0$; $\frac{q}{p} \to 0$

For ω a spanning tree on $G = (V, E)$

$$\phi_{pq}(\omega) = Z_{pq}^{-1} \, p^{|V|-1} (1-p)^{|E|-|V|+1} \, q =: \mu_{pq}$$

For ω containing more than one cluster:

$$\phi_{pq}(\omega) = Z_{pq}^{-1} \, p^{|\omega|} (1-p)^{|E|-|\omega|} \, q^{k(\omega)} \overset{\text{since } \frac{q}{p} \to 0, \ \& \ \underline{k(\omega)+|\omega| \geq |V|}}{=} o(\mu_{pq})$$

For ω containing more than $|V|-1$ edges and $k(\omega) = 1$:

$$\phi_{pq}(\omega) = Z_{pq}^{-1} \, p^{|\omega|} (1-p)^{|E|-|\omega|} \, q \overset{\text{since } p \to 0}{=} o(\mu_{pq}).$$

Thus: $Z_{pq} \sim (\#\text{ spanning trees}) \cdot \mu_{pq}$

and $\phi_{pq}(\omega) \longrightarrow \mathbb{1}_{\omega \text{ is a spanning tree}}$.

Case 2 $p = q \to 0$.

Using the same computation we obtain

$$\phi_{p,q}(\omega) \longrightarrow \mathbb{1}_{|\omega| + k(\omega) \text{ is minimal}}.$$

For a config ω, $|\omega| + k(\omega)$ is minimal if ω contains no cycles. Thus the limiting measure is the uniform measure on forests on G.

8.7 Take $q' \geq q$; $q' \geq 1$; $p' \leq p$. and ω, ω' two configurations. We want to show:

$$\phi_{pq}(\omega \vee \omega')\, \phi_{p'q'}(\omega \wedge \omega) \geq \phi_{pq}(\omega)\, \phi_{p'q'}(\omega') \iff$$

$$p^{|\omega\vee\omega'|}(1-p)^{N-|\omega\vee\omega'|}\, p'^{|\omega\wedge\omega'|}(1-p')^{N-|\omega\wedge\omega'|}\, q^{k(\omega\vee\omega')}\, q'^{k(\omega\wedge\omega')} \geq$$

$$\geq p^{|\omega|}(1-p)^{N-|\omega|}\, p'^{|\omega'|}(1-p')^{N-|\omega\wedge\omega'|} \cdot q^{k(\omega)}\, q'^{k(\omega')}$$

The last inequality holds since:

- $|\omega \wedge \omega'| + |\omega \vee \omega'| = |\omega| + |\omega'|$ and $|\omega \vee \omega'| \geq |\omega|$;

- $k(\omega \wedge \omega') - k(\omega') \geq k(\omega) - k(\omega \vee \omega')$ $\circledast$

(so: $q'^{\,k(\omega\wedge\omega')-k(\omega')} \geq q^{k(\omega)-k(\omega\vee\omega')}$)

$\circledast$ is a simple graph theory exercise;
see 8.2 for an idea.

Take $q' \geq q$, $q' \geq 1$, $\frac{p'}{q'(1-p')} \geq \frac{p}{q(1-p)}$

We need to show

$$\phi_{p'q'}(\omega \vee \omega')\,\phi_{pq}(\omega \wedge \omega') \geq \phi_{pq}(\omega)\,\phi_{p'q'}(\omega') \iff$$

$$\left(\frac{p}{1-p}\right)^{|\omega \wedge \omega'| - |\omega|} \left(\frac{p'}{1-p'}\right)^{|\omega \vee \omega'| - |\omega'|} q^{k(\omega \wedge \omega') - k(\omega)}\; q'^{\,k(\omega \vee \omega') - k(\omega')} \geq 1. \quad \circledast$$

Notice that $|\omega \vee \omega'| - |\omega'| = |\omega| - |\omega \wedge \omega'| \geq$

$$\geq k(\omega \wedge \omega') - k(\omega) \geq k(\omega') - k(\omega \vee \omega') \geq 0$$

Thus

$$\circledast \geq \frac{q'^{\,|\omega \vee \omega'| - |\omega'| + k(\omega \vee \omega') - k(\omega')}}{q^{|\omega \vee \omega'| - |\omega'| + k(\omega) - k(\omega \wedge \omega')}} \geq 1.$$

Since $q' \geq 1$; $q' \geq q$ and

$|\omega \vee \omega'| - |\omega'| + k(\omega \vee \omega') - k(\omega') \geq |\omega \vee \omega'| - |\omega'| + k(\omega) - k(\omega \wedge \omega')$

$\boxed{8.12}$ For $q \geq 1$ and $p \in [0;1]$

$$\phi_{p,q} \leq_{st} \phi_{p,1} \leq \phi_{\frac{qp}{1+(q-1)p},q}$$

$p < p_c(1) \Rightarrow \phi_{p,1}$ subcrit. $\Rightarrow \phi_{p,q}$ subcrit. $\Rightarrow p < p_c(q)$. So $p_c(1) \leq p_c(q)$

$p > p_c(1) \Rightarrow \phi_{p,1}$ supercrit. $\Rightarrow \phi_{\frac{qp}{1+(q-1)p},q}$ supercrit. So $\frac{q\,p_c(1)}{1+(q-1)p_c(1)} \geq p_c(q)$

$\square$