Percolation and Related Topics
Example Sheet 1

adapted from Temple He's solutions

October 27, 2011

Problem 4(a) (Exercise 3.7): First note that $\mathbb{P}_p(r(k) \geq u) = p^u$ for any $k \in \mathbb{N}^+$. Thus

$$\mathbb{P}_p(L_n \geq u) = \mathbb{P}_p(r(k) \geq u \text{ for some } 1 \leq k \leq n)$$
$$\leq \sum_{k=1}^{n} \mathbb{P}_p(r(k) \geq u) = np^u.$$

Set $u_n = \frac{(1+\epsilon)\log n}{\log(1/p)} = \log_p n^{-1-\epsilon}$. Then

$$\mathbb{P}_p(L_n > u_n) \leq np^{u_n} = n^{-\epsilon},$$

which goes to zero as $n \to \infty$.

Problem 4(b) (Exercise 3.8): Set $v_n = \frac{(1+\epsilon)\log n}{\log(1/p)}$. We wish to study $\mathbb{P}_p(L_n < v_n)$.

$$\mathbb{P}_p(L_n < v_n) \leq P_p\left(r_{kv_n} < v_n,\ \forall k \in \{0, \dots, \tfrac{n}{v_n}\}\right)$$
$$\leq (1 - p_n^v)^{\frac{n}{v_n}} = e^{-n^\epsilon + o(n^\epsilon)} \to 0.$$

We now wish to apply Borel Cantelli to show that:

$$\mathbb{P}_p(v_n \leq L_n \leq u_n \text{ for } n \text{ large enough}) = 1.$$

First notice that $\mathbb{P}_p(L_n < v_n)$ is summable, so $\mathbb{P}_p(L_n < v_n$ infinitely often$) = 0$. The same can not be said about u_n, since $\mathbb{P}_p(L_n > u_n)$ is not summable. But $\mathbb{P}_p(r_n > u_n)$ is, so

$$\mathbb{P}_p(r_n \leq u_n \text{ for } n \text{ large enough}) = 1.$$

But if $r_n \leq u_n$ for $n \geq N_0$, then we can find $N_1 \geq N_0$ such that $u_{N_1} \geq \max\{r_n : n < N_0\}$. For $n \geq N_1$ we have $L_n \leq u_n$.

Problem 6 (Exercise 3.10): We approach this problem using a similar method to the one used in Section 3.4 of Grimmett to show the critical probability of an oriented lattice is strictly between 0 and 1.

Take Δ a cycle of the dual lattice surrounding the origin. Call Δ a blocking cycle if all the edges crossing Δ and oriented from the interior to the exterior are closed. Upon inspection of Δ we notice that every other intersected edge is oriented towards the exterior. Hence the probability for a given cycle, Δ, to be blocking is $(1-p)^{|\Delta|/2}$.

Let $\vec{C}$ be the cluster of the origin, i.e. the set of sites that can be reached from the origin via a directed open path. If $|\vec{C}| < \infty$, then the cycle surrounding $\vec{C}$ is a blocking cycle.

We will now use a classical counting argument. The number of cycles of length n around the origin is bounded by 4^n, hence:

$$\begin{aligned}\mathbb{P}_p(|\vec{C}| < \infty) &\leq \sum_{\Delta} \mathbb{P}_p(\Delta \text{ is a blocking cycle}) \\ &\leq \sum_{n \geq 4} 4^n (1-p)^{n/2}.\end{aligned}$$

The RHS can be made strictly less than 1 by taking p sufficiently close to 1.