3.1.

Take $N \in \mathbb{N}$. All $n \in \mathbb{N}$ can be written $n = kN + r$ with $r \leq N$. Thus

$$\frac{x_n}{n} \leq \frac{k x_N}{kN+r} + \frac{x_r}{kN+r} \xrightarrow[n \to \infty]{} \frac{x_N}{N}$$

So $\limsup \frac{x_n}{n} \leq \frac{x_N}{N}$ for all N

But $\limsup \frac{x_n}{n} \geq \inf_N \frac{x_N}{N}$, so $\lim \frac{x_n}{n}$ exists and is $\inf_N \frac{x_N}{N}$.

3.2. We will show that $\alpha_n = o(n)$ is a sufficient condition:

With the same notation:

$$\frac{x_n}{n} \leq \frac{k(x_N + \alpha_N)}{kN+r} + \frac{x_r + \alpha_k}{kN+r} \xrightarrow[n \to \infty]{} \frac{x_N + \alpha_N}{N}$$

So $\limsup \frac{x_n}{n} \leq \liminf \frac{x_N + \alpha_N}{N} \overset{\alpha_n = o(n)}{=\!=} \liminf \frac{x_n}{n}$ $\square$.

3.3/

(i) For a graph $G=(V,E)$; $p_c^{bond} \leq p_c^{site}$.

Take $p \in [0,1]$ and take $\omega \in \{0,1\}^E$ according to $\mathbb{P}_p^{bond}$ (i.e. product measure with intensity p).

We will create a configuration $\sigma \in \{0,1\}^V$ based on ω as follows:

Write $V = \{v_0, v_1, \dots\}$.

Let $\sigma(v_0) = \begin{cases} 0 & \text{with prob. } 1-p \\ 1 & \text{— " — } p \end{cases}$ indep of ω.

Let $C_0 = \begin{cases} v_0 & \text{if } \sigma(v_0)=1 \\ \emptyset & \text{else.} \end{cases}$

(*) Take the first vertex v_i st $\sigma(v_i)$ is not defined and v_i is connected to C_0 via an edge e. Let $\sigma(v_i) = \omega(e)$; $C_0 = C_0 \cup \{v_i\}$ if $\sigma(v_i)=1$ and $C_0 = C_0$ else. Go back to (*) until no vertex v_i can be found.

For the vertices v where σ is not define take $\sigma(v)$ iid, indep of ω, 1 with proba p, 0 else.

With this construction σ follows $\mathbb{P}_p^{site}$ and V_0 is in an finite cluster in ω $\Rightarrow$ V_0 is in a finite cluster in σ.

Hence $\mathbb{P}_p^{site}[V_0 \to \infty] \leq \mathbb{P}_p^{bond}[V_0 \to \infty]$ which implies the result.

ii) Suppose the maximal degree of the vertices of G is Δ. Then $p_c^{site} \leq 1 - (1 - p_c^{bond})^{\Delta - 1}$.

Take $p \in [0,1]$ and define $q := 1 - (1-p)^{\Delta-1}$.

Take $\omega \sim \mathbb{P}_p^{bond}$; we will construct $\sigma \in \{0,1\}^V$ as follows:

First take:

- $\sigma(V_0) = 1$
- The edges adjacent to V_0 are explored.
- $C_0 = \{V_0\}$

⊛ Take the first vertex v, adjacent to C_0 and with $\sigma(v)$ not defined.

Let E_v be the set of unexplored edges adjacent to v. Then $|E_v| \leq \Delta - 1$.

Let - $\sigma(v) = \begin{cases} 0 & \text{wp. } (1-p)^{\Delta - 1 - |E_v|} \text{ if all the edges in } E_v \text{ are closed in } \omega. \\ 1 & \text{if not.} \end{cases}$

- ~~Add~~ v to C_0 if v is $\sigma(v) = 1$.

Declare the edges of E_v explored.

Go back to ⊛ until there are no more such vertices left.

For vertices that have not received a value via the above procedure, take σ iid, indep. of ω, with $\sigma(v) = 1$ with proba q.

Since every edge is used for at most vertex, the $(\sigma(v))_{v \in V}$ form a indep. family of variables with $\mathbb{P}[\sigma(v) = 1] = q$. (except for v_0).

Also, if $v_0 \xleftrightarrow{\omega} \infty$ then the same path is open in σ. Hence

$$\mathbb{P}_p^{\text{bond}}[|C_{v_0}| = \infty] \leq q\, \mathbb{P}_q^{\text{site}}[|C_{v_0}| = \infty]$$

Where q comes from the initial conditioning on v_0.

This proves the result

4.5. a) If μ satisfies

$$\circledast \quad \mu(\omega_1 \wedge \omega_2)\mu(\omega_1 \vee \omega_2) \geq \mu(\omega_1)\mu(\omega_2)$$

for all pairs ω_1, ω_2, then it certainly does so for pairs that differ by exactly 2 elements.

Let us show the converse. We will prove by induction on n that $\circledast$ holds for pairs ω_1, ω_2 that differ by at most n elements.

$n = 1$ is trivial

$n = 2$ follows from the assumption.

For $n \geq 2$ prove $n \Rightarrow n+1$.

Take ω_1, ω_2 two configs with $\|\omega_1 - \omega_2\| = n+1$. (Not $\omega_1 \leq \omega_2$ or $\omega_2 \geq \omega_1$)

Take e s.t. $\omega_1(e) = 0$ $\omega_2(e) = 1$. Then $|\omega_1^e - \omega_2| = n$ so:

$$\mu(\omega_1^e \wedge \omega_2)\mu(\omega_1^e \vee \omega_2) \geq \mu(\omega_1^e)\mu(\omega_2)$$

$$\mu(\omega_1^e \wedge \omega_2) = \mu((\omega_1 \wedge \omega_2)^e) \qquad \mu(\omega_1^e \vee \omega_2) = \mu(\omega_1 \vee \omega_2)$$

But $|(\omega_1 \wedge \omega_2)^e - \omega_1| \leq n$

$$\mu(\omega_1)\mu((\omega_1 \wedge \omega_2)^e) \leq \mu(\omega_1^e)\mu(\omega_1 \wedge \omega_2)$$

So $\mu(\omega_1^e)\mu(\omega_1 \wedge \omega_2)\mu(\omega_1 \vee \omega_2) \geq \mu(\omega_1^e)\mu(\omega_1)\mu(\omega_2)$

We can simplify by $\mu(\omega_1^e)$ since μ <u>is positive</u>

b) if μ is monotone, then, for $\omega \in \Omega$ and e, f,

$$f(e, \omega_f) = \frac{\mu(\omega^e_f)}{\mu(\omega^e_f) + \mu(\omega_{fe})} \leq f(e, \omega^f) = \frac{\mu(\omega^{ef})}{\mu(\omega^{ef}) + \mu(\omega^f_e)}$$

Which implies $\circledast$ for pairs that differs on 2 elements.
By b) this implies FKG.

if FKG holds we only need to apply $*$ to ξ_1^e; ξ_{2e} to prove that for $\xi_1 \leq \xi_2$

$$f(e; \xi_1) \leq f(e, \xi_2) \quad . \square.$$

Ex 9 & 10. For $\mathbb{L} = \mathbb{H}$ or $\mathbb{A}$

Let $W_{(n)}(\mathbb{L})$ = SAW (of length n) starting at 0 on $\mathbb{L}$

Let $\tilde{W}_{(n)}(\mathbb{A})$ = SAW (of length n) starting with two edges of T_0 and ending with two edges of some triangle.

$\sigma_n(.) = \# W_n(.)$; $\tilde{\sigma}_n(\mathbb{A}) = \# \tilde{W}_n(\mathbb{A})$.

Claim 1. $\tilde{\sigma}_n(\mathbb{A}) \leq \sigma_n(\mathbb{A}) \leq 16\, \tilde{\sigma}_n(\mathbb{A})$.

Define: - $Z^{\mathbb{H}}(x) = \sum_{n \in \mathbb{N}} \sigma_n(\mathbb{H}) x^n = \sum_{\gamma \in W(\mathbb{H})} x^{|\gamma|}$

- $\tilde{Z}^{\mathbb{A}}(x) = \sum_{n \in \mathbb{N}} \tilde{\sigma}_n(\mathbb{A}) x^n = \sum_{\gamma \in \tilde{W}(\mathbb{A})} x^{|\gamma|}$

Claim 2: - The radius of cv of $Z^{\mathbb{H}}$ is $K(\mathbb{H})^{-1}$

— " — $\tilde{Z}^{\mathbb{A}}$ is $K(\mathbb{A})^{-1}$.

Claim 3: $Z^{\mathbb{A}}(x) = \sum_{x \in W(\mathbb{H})} x^{|\gamma|}(x+x^2)^{|\gamma|+1} = (x+x^2) Z^{\mathbb{H}}(x^2+x^3)$

Claims 1, 2 and 3 imply $K(\mathbb{A})^{-2} + K(\mathbb{A})^{-3} = K(\mathbb{H})^{-1}$. $\square$.

Proof of Claim 3.

For $\gamma \in \tilde{W}(\mathbb{A})$ we define $\varphi(\gamma) \in W(\mathbb{H})$ as follows:

Let $T_0, \dots, T_n$ denote the triangles crossed by γ (in order). Then $T_0, \dots, T_n$ correspond to vertices $t_0, \dots, t_n$ in $\mathbb{H}$, with t_i adjacent to t_{i+1} and with no repetitions.

Let $\varphi(\gamma)$ be the SAW on $\mathbb{H}$ that visits $t_0, \dots, t_n$.

For $h \in W_n(\mathbb{H})$, let us study $\varphi^{-1}(h) := \{\gamma : \varphi(\gamma) = h\}$

→ Each of the $(n+1)$ points visited by h corresponds to a triangle of $\mathbb{A}$, which γ can cross in two ways.

→ Each of the n edges of h corresponds to a non triangular edge of $\mathbb{A}$, which γ uses.

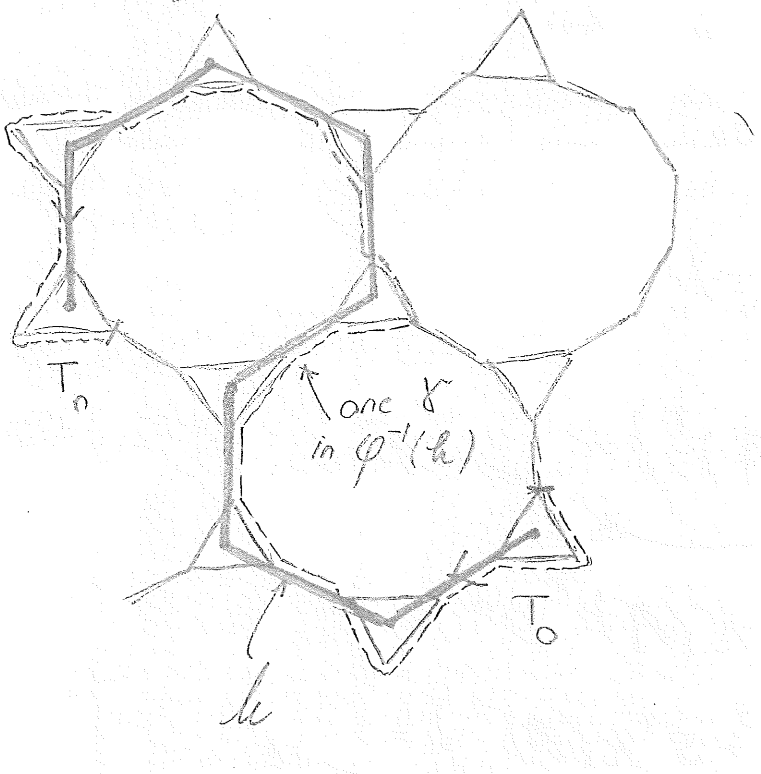
Hence: $\tilde{Z}^{\mathbb{A}}(x) = \sum_{h \in W(\mathbb{H})} \sum_{\gamma \in \varphi^{-1}(h)} x^{|\gamma|} = \sum_{h \in W(\mathbb{H})} x^{|h|} (x + x^2)^{|h|+1}$ $\square$

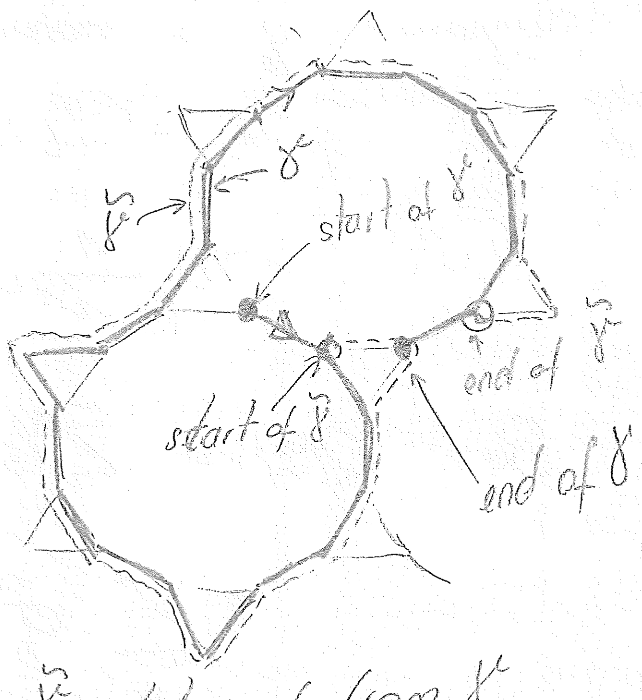
Proof of Claim 1: $\sigma_n(A) \geq \tilde{\sigma}_n(A)$ is obvious.

For the converse: take $\gamma \in W(A)$,

we can transform γ into a $\tilde{\gamma} \in \tilde{W}(A)$ by starting at the first triangle crossed by γ and finishing at the last triangle crossed by γ.

One may check there are at most 16 γ for one $\tilde{\gamma}$. □





$\tilde{\gamma}$ obtained from γ.

Claim 2. Follows from subadditivity and claim 1. □

Ex 10. Showing that γ is the same in $\mathbb{H}$ and $\mathbb{A}$.

Suppose γ exists in $\mathbb{H}$. Then:

$$\sigma_n(\mathbb{H}) = n^{\gamma-1} K(n)^n L(n)$$

for some slowly varying function.

Then $$Z^{\mathbb{H}}(x) = Q(x K(\mathbb{H})) = \sum_n \sigma_n x^n = \sum_n \underbrace{\sigma_n K(\mathbb{H})^{-n}}_{\substack{\shortparallel \\ q_n}} \cdot (x K(\mathbb{H}))^n$$

Q is a power series with radius of cv 1

so we can apply the Tauberian thm to Q.

Supposing q_n is monotone, we have

$$q_n = n^{\gamma-1} L(n) \quad \text{so}$$

$$Q(s) \sim \Gamma(\gamma) \frac{1}{(1-s)^{\gamma}} L\left(\frac{1}{1-s}\right)$$

But $$\tilde{Q}(s) := \tilde{Z}^{\mathbb{A}}(s K(\mathbb{A})^{-1}) = \sum_n \overbrace{\tilde{\sigma}_n(\mathbb{A}) K(\mathbb{A})^{-n}}^{\tilde{q}_n} s^n =$$

$$= Z^{\mathbb{H}}(s^2 K(\mathbb{A})^{-2} + s^3 K(\mathbb{A})^{-3}) =$$

$$= Q(\overbrace{(s^2 K(\mathbb{A})^{-2} + s^3 K(\mathbb{A})^{-3}) K(\mathbb{H})}^{t}) \sim$$

$$\underset{\substack{t \to 1 \\ (\text{i.e. } s \to 1)}}{\sim} \Gamma(\gamma) \frac{1}{(1-t)^{\gamma}} L\left(\frac{1}{1-t}\right) \overset{\text{1 is a simple root of } K(\mathbb{H})(s^2 K(\mathbb{A})^{-2} + s^3 K(\mathbb{A})^{-3}) =}{=}$$

$$= \frac{1}{(1-s)^{\gamma}} \tilde{L}\left(\frac{1}{1-s}\right)$$

By a new application of the Tauberian thm we get the result.

The same type of argument shows that if γ exists in $\mathbb{A}$, then it exists and has the same value in $\mathbb{H}$.

(ii) Show that η is the same in $\mathbb{A}$ and in $\mathbb{H}$ (provided it exists for one of the lattices)

For a vertex v set $W^v_{(\cdot)}(\cdot) = \{ \gamma \in W_{(\cdot)}(\cdot)$ going from 0 to $v \}$.

Define $\tilde{W}^v$, σ^v_n and $\tilde{\sigma}^v_n$ in a similar way.

As in 9, we may show:

$$\tilde{Z}^{v,\mathbb{A}}(x) = (x + x^2)\, Z^{v,\mathbb{H}}(x^2 + x^3). \qquad (1)$$

$\exists\, c_1, c_2 > 0$ st $\quad c_1 \sigma^v_n(\mathbb{A}) \leq \tilde{\sigma}^v_n(\mathbb{A}) \leq c_2 \sigma^v_n(\mathbb{A}) \qquad (2)$

(c_1, c_2 not depending on v).

Then, if $\dfrac{Z^{\mathbb{H},v}(K(\mathbb{H})^{-1})}{|v|^{-\eta}}$ is bounded as $|v| \to \infty$

then, by (1), so is $\dfrac{\tilde{Z}^{\mathbb{A},v}(K(\mathbb{A})^{-1})}{|v|^{-\eta}}$ and by (2) so

is $\dfrac{Z^{\mathbb{A},v}(K(\mathbb{A}^{-1}))}{|v|^{-\eta}}$. $\square$