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MARKOV CHAINS

Example Sheet 1

1.1 Let $X = (X_n)$ be a Markov chain. Suppose that we are given that $X_m = i$. Show that $Z_k = X_{m+k}$, $k \geq 0$, is a Markov chain with starting state i.

1.2 Show that any sequence of independent random variables taking values in the countable set S is a Markov chain. Under what condition is this chain homogeneous?

1.3 Let X_n be the maximum reading obtained in the first n throws of a fair die. Show that X is a Markov chain, and find the transition probabilities $p_{ij}(n)$.

1.4 (Harder) Let $\{S_n : n \geq 0\}$ be a simple (potentially asymmetric) random walk with $S_0 = 0$, and show that $X_n = |S_n|$ defines a Markov chain; find the transition probabilities of this chain. Let $M_n = \max\{S_k : 0 \leq k \leq n\}$, and show that $Y_n = M_n - S_n$ defines a Markov chain.

1.5 Let X be a Markov chain and let $\{n_r : r \geq 0\}$ be an unbounded increasing sequence of positive integers. Show that $Y_r = X_{n_r}$ constitutes a (possibly inhomogeneous) Markov chain. Find the transition matrix of Y when $n_r = 2r$ and X is a simple random walk.

1.6 Let X and Y be Markov chains on the set $\mathbb{Z}$ of integers. Is the sequence $Z_n = X_n + Y_n$ necessarily a Markov chain? Explain.

1.7 A flea hops about at random on the vertices of a triangle (i.e., each hop is from the currently occupied vertex to one of the other two vertices each with probability $\frac{1}{2}$). Find the probability that after n hops the flea is back where it started.

A second flea also hops about on the vertices of a triangle, but this flea is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops this second flea is back where it started?
[*Hint*: $\frac{1}{2} \pm \frac{i}{2\sqrt{3}} = \frac{1}{\sqrt{3}} e^{\pm i\pi/6}$.]

1.8 A die is 'fixed' so that when it is rolled the score cannot be the same as the previous score, all other scores having probability $\frac{1}{5}$. If the first score is 6, what is the probability p that the nth score is 6? What is the probability that the nth score is j, where $j \neq 6$?

Suppose instead that the die cannot score one greater (mod 6) than the previous score, all other five scores having equal probability. What is the new value of p?
[*Hint*: Think about the relationship between the two dice.]

1.9 Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{1,2,3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ p & 1-p & 0 \end{pmatrix}.$$

Calculate $\mathbb{P}(X_n = 1 \mid X_0 = 1)$ in each of the following cases (a) $p = \frac{1}{16}$, (b) $p = \frac{1}{6}$, (c)* $p = \frac{1}{12}$.

1.10 Identify the communicating classes of the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Which of the classes are closed?

1.11 Show that every transition matrix on a finite state-space has at least one closed communicating class. Find an example of a transition matrix with no closed communicating class.

1.12 A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the side which actually turns up, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10.

Let $X_0 = 2$ and let X_n $(n \geq 1)$ be his capital after n throws. Prove that the gambler will achieve his aim with probability $\frac{1}{5}$.

What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

1.13 (Optional) Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0,1,\dots\}$ with transition probabilities given by

$$p_{0,1} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad i \geq 1.$$

Show that if $X_0 = 0$ the probability that $X_n \geq 1$ for all $n \geq 1$ is $6/\pi^2$.

1.14 (Optional) Let $Y_1, Y_2, \dots$ be independent identically distributed random variables with $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$ and set $X_0 = 1$, $X_n = X_0 + Y_1 + \cdots + Y_n$ for $n \geq 1$. Define

$$H_0 = \inf\{n \geq 0 : X_n = 0\}.$$

Find the probability generating function $\phi(s) = \mathbb{E}(s^{H_0})$.

Suppose the distribution of $Y_1, Y_2, \dots$ is changed to $\mathbb{P}(Y_1 = 2) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$. Show that ϕ now satisfies

$$s\phi^3 - 2\phi + s = 0.$$