Probability and Measure, Hand-out 3
Borel measure on $\mathbb{R}$

Theorem. There exists a unique measure μ on $\mathbb{R}$ such that $\mu((a,b]) = b - a$ for all $a < b$. [This μ is called *Borel measure.*]

Proof. Let $\mathcal{R}$ be the ring of all sets of the form $A = \bigcup_{i=1}^n (a_i, b_i]$, and define $\mu(A) = \sum_{i=1}^n (b_i - a_i)$. The presentation of such A is not unique, since $(a,b] \cup (b,c] = (a,c]$ whenever $a < b < c$. Nevertheless, it is easy to check that μ is well-defined and additive. We shall show that μ is countably additive on $\mathcal{R}$, and existence will follow by the extension theorem.

It suffices to show that: if $B \in \mathcal{R}$ and if $(B_n : n \geq 1)$ is an increasing sequence in $\mathcal{R}$ with $B_n \uparrow B$, then $\mu(B_n) \to \mu(B)$.

Set $\widetilde{B}_n = B \setminus B_n$, so that $\widetilde{B}_n \in \mathcal{R}$ and $\widetilde{B}_n \downarrow \varnothing$. It suffices to show that $\mu(\widetilde{B}_n) \to 0$. Suppose that there exists $\epsilon > 0$ such that $\mu(\widetilde{B}_n) \geq 2\epsilon$ for all n. For each n we can find $C_n \in \mathcal{R}$ with $\overline{C}_n \subseteq \widetilde{B}_n$ and $\mu(\widetilde{B}_n \setminus C_n) \leq \epsilon 2^{-n}$. Let $x \in \widetilde{B}_n \setminus (C_1 \cap \cdots \cap C_n)$. There exists $j \in \{1, 2, \ldots, n\}$ such that $x \notin C_j$, and evidently $x \in \widetilde{B}_j$ since $\widetilde{B}_j \supseteq \widetilde{B}_n$. Therefore $x \in \widetilde{B}_j \setminus C_j$, and hence

$$\mu(\widetilde{B}_n \setminus (C_1 \cap \cdots \cap C_n)) \leq \mu((\widetilde{B}_1 \setminus C_1) \cup \cdots \cup (\widetilde{B}_n \setminus C_n)) \leq \sum_n \epsilon 2^{-n} = \epsilon.$$

Since $\mu(\widetilde{B}_n) \geq 2\epsilon$, we must have $\mu(C_1 \cap \ldots \cap C_n) \geq \epsilon$, so $C_1 \cap \ldots \cap C_n \neq \varnothing$, and so $K_n = \overline{C}_1 \cap \ldots \cap \overline{C}_n \neq \varnothing$. Now $(K_n : n \geq 1)$ is a decreasing sequence of bounded non-empty closed sets in $\mathbb{R}$, whence

$$\varnothing \neq \bigcap_n K_n \subseteq \bigcap_n \widetilde{B}_n,$$

which is a contradiction.

Turning to the question of uniqueness, let μ' be a measure on $\mathcal{B}$ with $\mu'((a,b]) = b - a$ for all $a < b$. Fix n and consider

$$\mu'_n(A) = \mu'((n, n+1] \cap A).$$

Then μ'_n is a probability measure on $\mathcal{B}$ so, by the uniqueness theorem, μ'_n is uniquely determined by its values on the π-system $\mathcal{I} = \{(a,b] : a < b\}$ generating $\mathcal{B}$. Since

$$\mu'(A) = \sum_n \mu'_n(A),$$

it follows that μ' is also uniquely determined.