Interacting particle systems I

1. Let $G = (V, E)$ be a finite connected graph with positive edge weights $(w_e : e \in E)$. Show, in the notation of the lectures that

$$i_{ab} = \frac{N^*(s,a,b,t) - N^*(s,b,a,t)}{N^*}$$

constitutes a unit flow through G from s to t satisfying Kirchhoff's laws.

2. (continuation) Let $G = (V, E)$ be finite and connected with given conductances $(w_e : e \in E)$, and let $(x_v : v \in V)$ be reals satisfying $\sum_v x_v = 0$. Show that there exists a solution i to Kirchhoff's laws, viewed as two laws concerning current flow, such that the current leaving the network at each v satisfies $i_{v\infty} = x_v$.

3. Prove the series and parallel laws for electrical resistances.

4. Let $R(r)$ be the effective resistance between two given vertices of a finite network with edge-resistances $r = (r(e) : e \in E)$. Show that R is concave in that

$$\tfrac{1}{2}\big(R(r_1) + R(r_2)\big) \le R\big(\tfrac{1}{2}(r_1 + r_2)\big).$$

5. Let G be an infinite connected graph, and let $\partial\Lambda_n$ be the set of vertices distance n from the vertex labelled O. With E_n the number of edges joining $\partial\Lambda_n$ to $\partial\Lambda_{n+1}$, show that random walk on G is recurrent if $\sum_n E_n^{-1} = \infty$.

6. (continuation) Assume that G is 'spherically symmetric' in that: for all n, for all $x, y \in \partial\Lambda_n$, there exists a graph automorphism which fixes O and maps x to y. Show that random walk on G is transient if $\sum_n E_n^{-1} < \infty$.

7. Let G be a finite connected subgraph of the finite connected graph G'. Let T and T' be uniform spanning trees on G and G' respectively. Show that, for any edge e of G, $P(e \in T) \ge P(e \in T')$.

   Let T_n be a UST of the lattice box $[-n, n]^2$ of $\mathbb{Z}^2$. Show that the limit $\lambda(e) = \lim_{n\to\infty} P(e \in T_n)$ exists.

8. **Subadditive inequality**. Let (x_n) be a real sequence satisfying $x_{m+n} \le x_m + x_n$ for all m, n. Show that the limit $\lambda = \lim\{n^{-1}x_n\}$ exists and satisfies $\lambda = \inf_k\{k^{-1}x_k\}$.

9. (continuation) Can you find reasonable conditions on the sequence (α_n) such that: the generalised inequality

$$x_{m+n} \le x_m + x_n + \alpha_m \qquad \text{for all } m, n$$

implies the existence of the limit $\lambda = \lim\{n^{-1}x_n\}$?