Interacting particle systems II

1. Let $X_n, Y_n \in L^2(\Omega, \mathcal{F}, P)$ be such that $X_n \to X$, $Y_n \to Y$ in L^2. Show that $X_n Y_n \to XY$ in L^1. [Reminder: L^p is the set of random variables Z with $E(|Z|^p) < \infty$, and $Z_n \to Z$ in L^p if $E(|Z_n - Z|^p) \to 0$. You may use any standard fact such as the Cauchy–Schwarz inequality.]
2. **Coupling**. (a) Take $\Omega = [0,1]$, with the Borel σ-field and Lebesgue measure. For any distribution function F, define a random variable Z_F on Ω by
$$Z_F(\omega) = \inf\{z : \omega \le F(z)\}, \qquad \omega \in \Omega.$$
Prove that
$$P(Z_F \le z) = P\big([0, F(z)]\big) = F(z),$$
whence Z_F has distribution function F.
(b) For real-valued random variables X, Y, we write $X \le_{\mathrm{st}} Y$ if $P(X \le u) \ge P(Y \le u)$ for all u. Show that $X \le_{\mathrm{st}} Y$ if and only if there exist random variables X', Y' on Ω, with the same respective distributions as X and Y, such that $P(X' \le Y') = 1$.
3. Let P_p be a product measure on the space $\{0,1\}^n$ with density p; that is, P_p is the measure which governs the behaviour of n independent identically distributed random variables with distribution given by $P_p(Z = 1) = 1 - P_p(Z = 0) = p$. Show by induction on n that P_p satisfies the Harris–FKG inequality, which is to say that $P_p(A \cap B) \ge P_p(A)P_p(B)$ for any pair A, B of increasing events.
4. **Subadditive inequality**. Let (x_n) be a real sequence satisfying $x_{m+n} \le x_m + x_n$ for all m, n. Show that the limit $\lambda = \lim\{n^{-1}x_n\}$ exists and satisfies $\lambda = \inf_k\{k^{-1}x_k\}$.
5. (continuation) Can you find reasonable conditions on the sequence (α_n) such that: the generalised inequality
$$x_{m+n} \le x_m + x_n + \alpha_m \qquad \text{for all } m, n$$
implies the existence of the limit $\lambda = \lim\{n^{-1}x_n\}$?
6. **Bond/site comparison**. Let G be an infinite connected graph with maximal vertex degree Δ. Show that the critical probabilities for bond and site percolation on G satisfy
$$p_{\mathrm{c}}(\text{bond}) \le p_{\mathrm{c}}(\text{site}) \le 1 - \big(1 - p_{\mathrm{c}}(\text{bond})\big)^{\Delta}.$$
[The first inequality is a fair target. The second is a little harder.]
7. Let a_n be the number of self-avoiding walks on $\mathbb{L}^d$, i.e., the number of paths $x_0, e_0, x_1, e_1, \dots, x_n$ of distinct vertices x_j and edges $e_j = \langle x_j, x_{j+1}\rangle$. Show that $a_{m+n} \le a_m a_n$, and deduce the existence of the connective constant $\mu = \lim_{n\to\infty}\{a_n^{1/n}\}$. Show that $p_{\mathrm{c}}(\mathbb{L}^d) \ge \mu^{-1}$. Use duality and the FKG inequality to obtain $p_{\mathrm{c}}(\mathbb{L}^2) \le 1 - \mu^{-1}$ when $d = 2$.
8. **Russo's formula**. Let X be a random variable on the finite sample space $\Omega = \{0,1\}^E$. Show that
$$\frac{d}{dp}E_p(X) = \sum_{e\in E} E_p(\delta_e X)$$

where $\delta_e X(\omega) = X(\omega^e) - X(\omega_e)$, and ω^e (respectively, ω_e) is the configuration obtained from ω by replacing $\omega(e)$ by 1 (respectively, 0).
Let A be an increasing event, with indicator function I_A. An edge e is called *pivotal* for the event A in the configuration ω if $\delta_e I_A(\omega) = 1$. Show that the derivative of $P_p(A)$ equals the mean number of pivotal edges for A, and find a related formula for the second derivative of $P_p(A)$.

9. Let $\chi(p)$ be the mean number of vertices in the open cluster at the origin of bond percolation at density p, and suppose that p is such that $\chi(p) < \infty$. Show the existence of $\gamma > 0$ such that $P_p(0 \leftrightarrow \partial\Lambda_k) \le e^{-\gamma k}$ for all k. [You are not required to re-prove the inequality
$$g_n \le \sum_{x \in \partial\Lambda_m} P_p(0 \leftrightarrow x) g_{n-m}$$
proved already in lectures.]
10. **One-dimensional percolation**. Each edge of the one-dimensional lattice $\mathbb{L}$ is *open* with probability p, as usual. For $k \in \mathbb{Z}$, let $r(k) = \max\{u : k \leftrightarrow k+u\}$, and $R_n = \max\{r(k) : 1 \le k \le n\}$. Show that $P(R_n > u) \le np^u$, and hence, for $\epsilon > 0$,
$$P_p\left(R_n > \frac{(1+\epsilon)\log n}{\log(1/p)}\right) \to 0 \qquad \text{as } n \to \infty.$$
11. (continuation) Show that, for $\epsilon > 0$,
$$P_p\left(R_n < \frac{(1-\epsilon)\log n}{\log(1/p)}\right) \to 0 \qquad \text{as } n \to \infty.$$
[This is the famous problem of the longest run of heads in n tosses of a coin.]
12. Here is way to prove that $p_c(\mathbb{Z}^2) \ge \frac{1}{2}$ and more. Suppose on the contrary that $\theta(\frac{1}{2}) > 0$. Let $B_n = [0,n]^2$. Show that, with probability approaching 1 as $n \to \infty$, there exists some vertex on the top of B_n which lies in an infinite path which does not otherwise intersect B_n. By using this fact applied to both primal and dual lattices, show that there exists a.s. strictly greater than one infinite cluster in either primal or dual. This contradicts the result that such a cluster is a.s. unique, and therefore $\theta(\frac{1}{2}) = 0$ as required. More details can be found in Section 9.1 of paper 13 of

`http://www.statslab.cam.ac.uk/~grg/preprints.html`

Deduce in particular that, when $p = \frac{1}{2}$, there exists a.s. an open circuit with the origin in its interior.
13. The following random 'dynamical system' has acquired a certain notoriety. At each vertex of $\mathbb{L}^2$ we place a mirror. These mirrors are plane and double-backed, and are placed such that light incident along an axis is reflected along a perpendicular axis. Each mirror has two possible configurations, labelled in the natural way NW and NE, and we stipulate that, for each vertex x, the mirror at x is equally likely to be NW as NE (and that mirrors at different vertices have independent orientations). We light a candle at the origin, and let C be the set of illuminated vertices. Show that $P(|C| < \infty) = 1$. (*Hint*. Use the result of Question 12, on a suitable copy of $\mathbb{L}^2$ inclined at 45° to the horizontal.)