

### Interacting particle systems, Handout I

Consider the  $d$ -dimensional lattice  $\mathbb{Z}^d$ , with  $(\Omega, \mathcal{F})$  given as usual, and let  $P$  be a product measure on this space with some constant density  $p$ . Let  $\tau$  be a translation of  $\mathbb{Z}^d$ , so that  $\tau$  acts on  $\Omega$  in the usual way by  $\tau\omega(e) = \omega(\tau^{-1}e)$ .

Let  $N$  be a random variable on  $(\Omega, \mathcal{F})$  which is  $\tau$ -invariant in that  $N(\omega) = N(\tau\omega)$  for  $\omega \in \Omega$ . We show below that  $N$  is necessarily  $P$ -almost-surely constant, in that there exists  $n \in \mathbb{R}$  such that  $P(N = n) = 1$ .

It suffices to show that, for any  $\tau$ -invariant event  $A$  (in that  $A = \tau A$ ), either  $P(A) = 1$  or  $P(A) = 0$ . The claim above will follow by applying this to the events  $\{N \leq x\}$  for  $x \in \mathbb{R}$ .

Let  $A$  be a  $\tau$ -invariant event, and let  $\epsilon > 0$ . Remembering that  $\mathcal{F}$  is the  $\sigma$ -field generated by the finite-dimensional cylinders, the following is a basic fact of measure theory. There exists a finite set  $F$  of edges, and an event  $A_F$  defined in terms of the states of edges in  $F$  only, such that  $P(A \Delta A_F) \leq \epsilon$ , where  $\Delta$  denotes symmetric difference.

By translation by  $\tau^k$ ,  $P(\tau^k A \Delta \tau^k A_F) \leq \epsilon$ . Now  $\tau^k A = A$ , so that  $P(A \Delta \tau^k A) \leq \epsilon$ , and therefore

$$P(A_F \Delta \tau^k A_F) \leq 2\epsilon.$$

We choose  $k$  such that  $F \cap \tau^k F = \emptyset$ . Since  $P$  is a product measure,

$$P(A_F \cap \tau^k A_F) = P(A_F)P(\tau^k A_F) = P(A_F)^2.$$

Finally,

$$\begin{aligned} 2\epsilon &\geq P(A_F \Delta \tau^k A_F) \\ &= P(A_F) + P(\tau^k A_F) - 2P(A_F \cap \tau^k A_F) \\ &= 2[P(A_F) - P(A_F)^2] \\ &\geq 2[P(A) - \epsilon - (P(A) + \epsilon)^2]. \end{aligned}$$

Let  $\epsilon \downarrow 0$  to find that  $P(A) \leq P(A)^2$ , so that  $P(A)$  equals either 0 or 1.

**Note:** This is a standard argument, and is stated often as ‘the product measure  $P$  is ergodic’. ‘Recall’ that a measure is called ergodic (or in this case, ergodic with respect to  $\tau$ ) if every  $\tau$ -invariant random variable is  $\mu$ -almost-surely constant. The conclusion is of course related to the Kolmogorov zero–one law, which states that every tail event is trivial with respect to the product measure  $P$ .