1. Ergodic theory[1]

1.1. **Bernoulli shifts.** Let m be a probability measure on $\mathbb{R}$. We may construct a (product) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there exists a sequence of independent random variables $(Y_n : n \in \mathbb{N})$, each having distribution m. Consider now the infinite product space

$$E = \mathbb{R}^{\mathbb{N}} = \big\{x = (x_n : n \in \mathbb{N}) : x_n \in \mathbb{R} \text{ for all } n\big\}$$

and the σ-algebra $\mathcal{E}$ on E generated by the coordinate maps $X_n(x) = x_n$,

$$\mathcal{E} = \sigma(X_n : n \in \mathbb{N}).$$

Note that $\mathcal{E}$ is also generated by the π-system

$$\mathcal{R} = \left\{ \prod_{n\in\mathbb{N}} A_n : A_n \in \mathcal{B} \text{ for all } n,\ A_n = \mathbb{R} \text{ for all large } n \right\}.$$

Define the function $Y : \Omega \to E$ by $Y(\omega) = (Y_n(\omega) : n \in \mathbb{N})$. It is easily checked that Y is measurable, and the image measure $\mu = \mathbb{P} \circ Y^{-1}$ satisfies,

$$\mu(A) = \prod_{n\in\mathbb{N}} m(A_n) \quad \text{for} \quad A = \prod_{n\in\mathbb{N}} A_n \in \mathcal{R}.$$

By the uniqueness-of-extension theorem, μ is the unique measure on $(E, \mathcal{E})$ having this property. Under the probability measure μ, the coordinate maps $(X_n : n \in \mathbb{N})$ are themselves a sequence of independent random variables with law m. The probability space $(E, \mathcal{E}, \mu)$ is called the *canonical model* for such sequences.

Define the *shift map* $\theta : E \to E$ by

$$\theta(x) = (x_2, x_3, \dots) \quad \text{where } x = (x_1, x_2, \dots).$$

Theorem 1.1.1. *The shift map θ is an ergodic measure-preserving transformation.*

Proof. The details of showing that θ is measurable and measure-preserving are left as an exercise. To see that θ is ergodic, we recall the definition of the tail σ-fields,

$$\mathcal{T}_n = \sigma(X_m : m \geq n+1), \quad \mathcal{T} = \bigcap_n \mathcal{T}_n.$$

For $A = \prod_{k\in\mathbb{N}} A_k \in \mathcal{R}$ we have

$$\theta^{-n}(A) = \{X_{n+k} \in A_k \text{ for all } k \geq 1\} \in \mathcal{T}_n.$$

Since $\mathcal{T}_n$ is a σ-field, we have that $\theta^{-n}(A) \in \mathcal{T}_n$ for all $A \in \mathcal{E}$. If A lies in the invariant σ-field $\mathcal{E}_\theta$, then $A = \theta^{-n}(A) \in \mathcal{T}_n$ for all n, whence $A \in \bigcap_n \mathcal{T}_n = \mathcal{T}$ and $\mathcal{E}_\theta \subseteq \mathcal{T}$. By the Kolmogorov zero-one law, $\mathcal{T}$ is trivial in the sense that every member has probability either 0 or 1, and it follows that $\mathcal{E}_\theta$ is trivial also. $\square$

[1]Notes in part by courtesy of James Norris

1.2. **Ergodic theorems.** Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space, on which is given a measure-preserving transformation θ. Let $f : \Omega \to \mathbb{R}$ be integrable, and set $S_0 = 0$ and

$$S_n = S_n(f) = f + f \circ \theta + \cdots + f \circ \theta^{n-1}, \qquad n \geq 1.$$

Lemma 1.2.1 (Garsia's maximal ergodic lemma)**.** *Let* $S^* = \sup_{n \geq 0} S_n$. *Then*

$$\int_{\{S^*>0\}} f \, d\mu \geq 0.$$

Proof. Set $S_n^* = \max_{0 \leq m \leq n} S_m$ and $A_n = \{S_n^* > 0\}$. Then

$$S_m = f + S_{m-1} \circ \theta \leq f + S_n^* \circ \theta, \qquad 1 \leq m \leq n.$$

On the event A_n we have $S_n^* = \max_{1 \leq m \leq n} S_m$, so

$$S_n^* \leq f + S_n^* \circ \theta.$$

On the complement A_n^{c} we have $S_n^* = 0$, whence

$$S_n^* \leq S_n^* \circ \theta.$$

So, integrating and adding, we obtain

$$\int_\Omega S_n^* \, d\mu \leq \int_{A_n} f \, d\mu + \int_\Omega S_n^* \circ \theta \, d\mu.$$

But S_n^* is integrable and θ is measure-preserving, so

$$\int_\Omega S_n^* \circ \theta \, d\mu = \int_\Omega S_n^* \, d\mu < \infty$$

which implies that

$$\int_{A_n} f \, d\mu \geq 0.$$

The claim follows by taking the limit as $n \to \infty$ and appealing to monotone convergence. $\square$

Theorem 1.2.2 (Birkhoff's ergodic theorem)**.** *There exists an invariant function* $\bar{f}$, *with* $\mu(|\bar{f}|) \leq \mu(|f|)$, *such that* $S_n/n \to \bar{f}$ *a.e. as* $n \to \infty$.

Proof. We claim first that the functions $\liminf_n(S_n/n)$ and $\limsup_n(S_n/n)$ are invariant. To see this in the first case, note that

$$\begin{aligned}\left(\liminf \frac{S_n}{n}\right) \circ \theta &= \liminf \left(\frac{S_n \circ \theta}{n}\right) = \liminf \left(\frac{S_{n+1} - f}{n}\right) \\ &= \liminf \left(\frac{S_{n+1}}{n}\right) = \liminf \left(\frac{S_{n+1}}{n+1}\right).\end{aligned}$$

Let $a < b$. It follows from the above that

$$D = D(a,b) = \{\liminf_n (S_n/n) < a < b < \limsup_n (S_n/n)\}$$

is an invariant event. *We shall show that* $\mu(D) = 0$. First, by invariance, if $\omega \in D$ then $\theta^n \omega \in D$ for all n, and we may therefore restrict ourselves to the universe D; thus we may assume that $\Omega = D$. Note that either $b > 0$ or $a < 0$. We can interchange the two cases by replacing f by $-f$. Let us assume then that $b > 0$. Let $B \in \mathfrak{F}$ with $\mu(B) < \infty$, then $g = f - b1_B$ is integrable and, for each $\omega \in D$, for some n,

$$S_n(g)(\omega) \geq S_n(f)(\omega) - nb > 0.$$

Hence $S^*(g) > 0$ everywhere and, by the maximal ergodic lemma,

$$0 \leq \int_D (f - b1_B)\, d\mu = \int_D f\, d\mu - b\mu(B).$$

Since μ is σ-finite, we can let $B \uparrow D$ to obtain

$$b\mu(D) \leq \int_D f\, d\mu.$$

In particular, we see that $\mu(D) < \infty$. A similar argument applied to $-f$ and $-a$, this time with $B = D$, shows that

$$(-a)\mu(D) \leq \int_D (-f)\, d\mu.$$

Hence

$$b\mu(D) \leq \int_D f\, d\mu \leq a\mu(D).$$

Since $a < b$ and the integral is finite, this forces $\mu(D) = 0$.

Back to general Ω. Set

$$\Delta = \{\liminf_n (S_n/n) < \limsup_n (S_n/n)\}$$

and note that Δ is invariant. Also, $\Delta = \cup_{a,b\in\mathbb{Q},a<b} D(a,b)$, so $\mu(\Delta) = 0$. On the complement of Δ, S_n/n converges in $[-\infty, \infty]$, so we can define an invariant function $\bar{f}$ by

$$\bar{f} = \begin{cases} \lim_n (S_n/n) & \text{on } \Delta^{\mathrm{c}}, \\ 0 & \text{on } \Delta. \end{cases}$$

Finally, we have $\mu(|f \circ \theta^n|) = \mu(|f|)$, so $\mu(|S_n|) \leq n\mu(|f|)$ for all n. Hence, by Fatou's lemma,

$$\mu(|\bar{f}|) = \mu(\liminf_n |S_n/n|) \leq \liminf_n \mu(|S_n/n|) \leq \mu(|f|).$$

$\square$

Theorem 1.2.3 (von Neumann's L^p ergodic theorem)**.** *Assume that $\mu(\Omega) < \infty$. Let $p \in [1, \infty)$. Then, for $f \in L^p$, $S_n/n \to \bar{f}$ in L^p.*

Proof. We have

$$\|f \circ \theta^n\|_p = \left(\int_\Omega |f|^p \circ \theta^n\, d\mu\right)^{1/p} = \|f\|_p.$$

So, by Minkowski's inequality,

$$\|S_n(f)/n\|_p \leq \|f\|_p.$$

Given $\varepsilon > 0$, choose $K < \infty$ such that $\|f - g\|_p < \varepsilon/3$, where

$$g = \begin{cases} f & \text{if } |f| \leq K, \\ K & \text{if } f \geq K, \\ -K & \text{if } f \leq -K. \end{cases}$$

By Birkhoff's theorem, $S_n(g)/n \to \bar{g}$ a.e. We have $|S_n(g)/n| \leq K$ for all n so, by bounded convergence, there exists N such that, for $n \geq N$,

$$\|S_n(g)/n - \bar{g}\|_p < \varepsilon/3.$$

By Fatou's lemma,

$$\begin{aligned}\|\bar{f} - \bar{g}\|_p^p &= \int_\Omega \liminf_n |S_n(f-g)/n|^p \, d\mu \\ &\leq \liminf_n \int_\Omega |S_n(f-g)/n|^p \, d\mu \\ &\leq \|f-g\|_p^p.\end{aligned}$$

Hence, for $n \geq N$,

$$\begin{aligned}\|S_n(f)/n - \bar{f}\|_p &\leq \|S_n(f-g)/n\|_p + \|S_n(g)/n - \bar{g}\|_p + \|\bar{g} - \bar{f}\|_p \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

□

1.3. **Strong law of large numbers.**

Theorem 1.3.1. *Let m be a probability measure on $\mathbb{R}$, with*

$$\int_{\mathbb{R}} |x| \, m(dx) < \infty, \qquad \int_{\mathbb{R}} x \, m(dx) = \nu.$$

Let $(E, \mathcal{E}, \mu)$ be the canonical model for a sequence of independent random variables with law m. Then

$$\mu\big(\{x : (x_1 + \cdots + x_n)/n \to \nu \text{ as } n \to \infty\}\big) = 1.$$

Proof. The shift map θ on E is measure-preserving and ergodic. The coordinate function $f = X_1$ is integrable and

$$S_n(f) = f + f \circ \theta + \cdots + f \circ \theta^{n-1} = X_1 + \cdots + X_n.$$

So $(X_1 + \cdots + X_n)/n \to \bar{f}$ a.e. and in L^1, for some invariant function $\bar{f}$, by Birkhoff's ergodic theorem. Since θ is ergodic, $\bar{f} = c$ a.e., for some constant c and then

$$c = \mu(\bar{f}) = \lim_n \mu(S_n/n) = \nu.$$

□

Theorem 1.3.2 (Strong law of large numbers)**.** *Let $(Y_n : n \in \mathbb{N})$ be a sequence of independent, identically distributed, integrable random variables with mean ν. Set $S_n = Y_1 + Y_2 + \cdots + Y_n$. Then*

$$S_n/n \to \nu \quad \textit{a.s., as } n \to \infty.$$

Proof. In the notation of Theorem 1.3.1, take m to be the law of the random variables Y_n. Then $\mu = \mathbb{P} \circ Y^{-1}$, where $Y : \Omega \to E$ is given by $Y(\omega) = (Y_n(\omega) : n \in \mathbb{N})$. Hence

$$\mathbb{P}\big(S_n/n \to \nu \text{ as } n \to \infty\big) = \mu\big(\{x : (x_1 + \cdots + x_n)/n \to \nu \text{ as } n \to \infty\}\big) = 1.$$

□