ADVANCED PROBABILITY

Example Sheet 3

1. If Y and Y' are versions of the conditional expectation $E(X \mid \mathcal{G})$, show that $P(Y = Y') = 1$.

2.* (a) Suppose that the sequence X_n satisfies

$$\forall \epsilon > 0, \quad P(|X_m - X_n| > \epsilon) \to 0 \quad \text{as } m, n \to \infty$$

(Cauchy convergence in probability). Show that X_n converges in probability to some limit X.

(b) Suppose that X_n is 'Cauchy convergent is mean square', i.e., $E(|X_m - X_n|^2) \to 0$ as $m, n \to \infty$ (and also that $E|X_k^2| < \infty$ for all k). Show that $X_n \to X$ is mean square (i.e., L^2) for some X with $E(X^2) < \infty$.

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, and let $\mathcal{R}$ be the space of all $\mathcal{G}$-measurable random variables X satisfying $E(X^2) < \infty$. Suppose $Y \notin \mathcal{R}$, and $E(Y^2) < \infty$. Show that there exists $X \in \mathcal{R}$ such that

$$\|X - Y\|_2 \le \|Z - Y\|_2 \quad \text{for all } Z \in \mathcal{R}$$

and

$$E\big((X - Y)Z\big) = 0 \quad \text{for all } Z \in \mathcal{R}.$$

4. Adapt the given proof of the Strong Law of Large Numbers to prove the L^2 martingale convergence theorem, namely: If S_n is a martingale with respect to $(\mathcal{F}_n)$ and $E(S_n^2) \le M < \infty$ for all n, then there exists a random variable S such that $S_n \to S$ almost surely and in mean square (i.e., L^2).

5. *Branching Process*. At time n, a population contains Z_n members. Each of these has a family of descendants in the $(n+1)$th generation, different families having independent sizes drawn from the same distribution. Suppose $Z_0 = 1$. Write μ ($\neq 0$) for the mean family-size, and $\eta = P(Z_n = 0 \text{ for some } n)$ for the extinction probability. Show that $A_n = Z_n / E(Z_n)$ and $B_n = \eta^{Z_n}$ constitute martingales with respect to the sequence $Z_1, Z_2, \ldots$.

Show that, if $\mu \neq 1$, there exists A such that $A_n \to A$ a.s., and also that the characteristic function ϕ of A satisfies the functional equation $\phi(\mu t) = G(\phi(t))$ where $G(x) = E(x^{Z_1})$.

6. Let $Z_1, Z_2, \ldots$ be independent random variables such that

$$Z_n = \begin{cases} a_n & \text{with probability } \frac{1}{2}n^{-2} \\ 0 & \text{with probability } 1 - n^{-2} \\ -a_n & \text{with probability } \frac{1}{2}n^{-2} \end{cases}$$

where $a_1 = 2$ and $a_n = 4\sum_{j=1}^{n-1} a_j$. Show that $Y_n = \sum_{j=1}^{n} Z_j$ defines a martingale. Show that $Y = \lim Y_n$ exists a.s., but that there exists no M such that $E|Y_n| \le M$ for all n.

7. Let $X_1, X_2, \ldots$ be independent random variables with

$$X_n = \begin{cases} 1 & \text{with probability } (2n)^{-1} \\ 0 & \text{with probability } 1 - n^{-1} \\ -1 & \text{with probability } (2n)^{-1}. \end{cases}$$

Let $Y_1 = X_1$ and for $n \ge 2$

$$Y_n = \begin{cases} X_n & \text{if } Y_{n-1} = 0 \\ nY_{n-1}|X_n| & \text{if } Y_{n-1} \neq 0. \end{cases}$$

Show that Y_n is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$. Show that Y_n does not converge almost surely. Does Y_n converge in any way? Why does the martingale convergence theorem not apply?

8. A bag contains red and blue balls, with initially r red and b blue where $rb > 0$. A ball is drawn from the bag, its colour noted, and then it is returned to the bag together with a new ball of the same colour. Let R_n be the number of red balls after n such operations.

(a) Show that $Y_n = R_n/(n+r+b)$ is a martingale which converges a.s. and in mean.

(b) Let T be the number of balls drawn until the first blue ball appears, and suppose that $r = b = 1$. Show that $E\{(T+2)^{-1}\} = \frac{1}{4}$.

(c) Suppose $r = b = 1$, and show that $P\big(Y_n \geq \frac{3}{4} \text{ for some } n\big) \leq \frac{2}{3}$.

9. Here is a martingale approach to the question of determining the mean number of tosses of a coin before the first appearance of the sequence HHH. A large casino contains infinitely many gamblers $G_1, G_2, \dots$, each with an initial fortune of \$1. A croupier tosses a coin repeatedly. For each n, gambler G_n bets as follows. Just before the nth toss he stakes his \$1 on the event that the nth toss shows heads. The game is assumed fair, so that he receives a total of $\$p^{-1}$ if he wins, where p is the probability of heads. If he wins this gamble, then he *repeatedly* stakes his entire current fortune on heads, at the same odds as his first gamble. At the first subsequent tail he loses his fortune and leaves the casino, penniless. Let S_n be the casino's profit (losses count negative) after the nth toss. Show that S_n is a martingale. Let N be the number of tosses before the first appearance of HHH; show that N is a stopping time and hence find $E(N)$.

Now adapt this scheme to calculate the mean time to the first appearance of the sequence HTH.

10. The game 'Red Now' may be played by a single player with a well shuffled conventional pack of 52 playing cards. At times $n = 1, 2, \dots, 52$ the player turns over a new card and observes its colour. Just once in the game he must say, just before exposing a card, "Red Now". He wins the game if the next exposed card is red. Let R_n be the number of red cards remaining face down after the nth card has been turned over. Show that $X_n = R_n/(52-n)$, $0 \leq n < 52$, defines a martingale. Show that there is no strategy for the player which results in a probability of winning different from $\frac{1}{2}$.

11. Let $X_1, X_2, \dots$ be independent identically distributed random variables and suppose that $M(t) = E(e^{tX_1})$ satisfies $M(t) = 1$ for some $t > 0$. Show that $P\big(S_k \geq x \text{ for some } k\big) \leq e^{-tx}$ for $x > 0$ and such a value of t, where $S_k = X_1 + X_2 + \cdots + X_k$.

12. If T_1 and T_2 are stopping times with respect to a filtration $\mathcal{F}$, show that $T_1 + T_2$, $\max\{T_1, T_2\}$, and $\min\{T_1, T_2\}$ are stopping times also.

13. On failure, a light bulb is replaced by a new bulb. The lifetimes of the bulbs are iid random variables with common mean μ satisfying $0 < \mu < \infty$. Let $N(t)$ be the number of failures by time t. Show that $N(t)$ is *not* a stopping time but that $N(t)+1$ *is* a stopping time, with respect to the sequence $X_1, X_2, \dots$ of lifetimes.

Show that $\liminf_{t\to\infty} E(N(t))/t \geq \mu^{-1}$. Can you explain how to prove the complementary fact that $\limsup_{t\to\infty} E(N(t))/t \leq \mu^{-1}$.