ADVANCED PROBABILITY

Example Sheet 2

1. **Log-normal distribution**. Let X be $N(0,1)$, and let $Y = e^X$. Consider the function

$$g_a(y) = g(y)\{1 + a\sin(2\pi \log y)\}, \quad y \in \mathbb{R},$$

where g is the density function of Y. Show that, if $|a| \le 1$, then g_a is a density function all of whose moments agree with those of Y.

2. Let $\{F_n\}$ be a sequence of distributions such that: there exists a function $g : \mathbb{R} \to [0,\infty)$ such that $g(x) \to \infty$ as $|x| \to \infty$ and

$$\sup_n \int g(x) dF_n < \infty.$$

Show that $\{F_n\}$ generates a tight sequence of measures.

3. Suppose that $Y_n \Rightarrow Y$ and $\{c_n\}, \{d_n\}$ are real sequences satisfying $c_n \to c, d_n \to d$. Show that $c_n Y_n + d_n \Rightarrow cY + d$.

4. Let $X_n \Rightarrow X$ and g be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Show that $g(X_n) \Rightarrow g(X)$.

5. Show that $X_n \Rightarrow X$ if and only if $E(g(X_n)) \to E(g(X))$ for all bounded continuous $g : \mathbb{R} \to \mathbb{R}$. (Take as your definition of weak convergence the statement $P(X_n \le x) \to P(X \le x)$ at all continuity points x of $P(X \le \cdot)$.)

6. Let $Y_1, Y_2, \ldots$ be independent random variables, each taking values in $\{0,1,2,3\}$ with equal probability $\frac{1}{4}$. Show that

$$X_n = \sum_{i=1}^{n} Y_i 4^{-i}$$

converges weakly, and identify the limit distribution.

7. Show that $X_n \xrightarrow{P} 0$ if and only if

$$E\left(\frac{|X_n|}{1+|X_n|}\right) \to 0 \quad \text{as } n \to \infty.$$

8. Let $g : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Show that

$$\sum_{k=0}^{\infty} g(k/n) \frac{(n\lambda)^k}{k!} e^{-n\lambda} \to g(\lambda) \quad \text{as } n \to \infty.$$

9. Which of the following functions are characteristic functions? Give reasons. (a) $\max\{1-|t|, 0\}$, (b) $\cos t$, (c) $(1+t^4)^{-1}$, (d) $2(1-\cos t)/t^2$.

10. Prove that

$$e^{-n}\left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) \to \frac{1}{2} \quad \text{as } n \to \infty.$$

11. Find a sequence ϕ_n of characteristic functions with the property that the limit $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists for all t, but ϕ is not a characteristic function.

12. Let S_n be the number of heads in n tosses of coin, each toss of which shows heads with probability x. Let $f : [0,1] \to \mathbb{R}$ be continuous. By finding an upper bound for $|f(x) - Ef(S_n/n)|$, prove Weierstrass's theorem, that f may be approximated by a polynomial uniformly on $[0,1]$, in the following form:

$$\lim_{n\to\infty} \sup_{0\le x\le 1} \left| f(x) - \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k} \right| = 0.$$

13. Show that if X_n converges weakly to a constant c, then $X_n \xrightarrow{P} c$.