ADVANCED PROBABILITY

Example Sheet 1

1. Show that a field of subsets of a set S has the property of being closed under taking finite intersections.

2. Let $S = (0, 1]$, and let $\mathcal{G}$ be the class of all subsets A of S of the form $A = (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n]$ for some a_i, b_j, and $n \geq 0$. Show that $\mathcal{G}$ is a field. Let $\mu : \mathcal{G} \to (0, 1]$ be given by $\mu(A) = \sum_i (b_i - a_i)$. Show that μ is σ-additive on $\mathcal{G}$, and explain the value of this property.

3. Prove, by using indicator functions or otherwise, the formula

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i \mu(A_i) - \sum_{i<j} \mu(A_i \cap A_j) + \cdots + (-1)^{n+1} \mu\left(\bigcap_i A_i\right).$$

4. Let X and Y be random variables on the probability space $(\Omega, \mathcal{F}, P)$. Show that the following are random variables: $-X$, $X + Y$, $\max\{X, Y\}$, $\min\{X, Y\}$.

5. Show that the limsup and liminf of a sequence of random variables are themselves random variables.

6. Let $A_1, A_2, \ldots, A_n$ be independent events. Show that $\overline{A}_1, A_2, \ldots, A_n$ are independent.

7. (a) Let X be a non-negative random variable, and show that

$$\sum_{i=1}^{\infty} P(X \geq i) \leq E(X) \leq 1 + \sum_{i=1}^{\infty} P(X \geq i).$$

(b) Let $X_1, X_2, \ldots$ be non-negative iid random variables, and let $Y_n = \min\{X_n, n\}$. Show that

$$P\big(X_n \neq Y_n \text{ i.o.}\big) = \begin{cases} 1 & \text{if } E(X_1) = \infty \\ 0 & \text{if } E(X_1) < \infty. \end{cases}$$

8. For two random variables X, Y, define their covariance by

$$\text{cov}\,(X, Y) = E\big((X - EX)(Y - EY)\big).$$

Show that $\text{cov}\,(X, Y) = 0$ if X and Y are independent, and also the Cauchy–Schwarz inequality:

$$|\text{cov}\,(X, Y)| \leq \sqrt{\text{var}\,(X)\text{var}\,(Y)}.$$

9. Prove that $P(Z_n \to Z \text{ as } n \to \infty) = 1$ if

$$\forall \epsilon > 0, \quad \sum_n P(|Z_n - Z| > \epsilon) < \infty.$$

10. Let $X_1, X_2, \ldots$ be independent random variables with common mean 0 and satisfying $E(X_i^4) \leq M$ for some M and all i. Show that $S_n = X_1 + X_2 + \cdots + X_n$ is such that

$$P\left(\frac{1}{n} S_n \to 0 \text{ as } n \to \infty\right) = 1.$$

11. (Lyapounov's inequality) Let Z be a random variable, and show that the *moments* of Z satisfy

$$\{E|Z^r|\}^{1/r} \geq \{E|Z^s|\}^{1/s} \quad \text{if } 0 < s \leq r.$$

In particular, if Z has finite rth moment, then it has finite sth moment for all $0 < s \leq r$.

12. Show that a distribution function F has the properties that: F is non-decreasing, continuous from the right, and satisfies

$$F(x) \to \begin{cases} 0 & \text{as } x \to -\infty \\ 1 & \text{as } x \to \infty. \end{cases}$$

Conversely, let F have these properties, and construct a probability space $(\Omega, \mathcal{F}, P)$ and a random variable X such that X has distribution function F.

13. A fair coin is tossed infinitely often. Let L_n be the length of the head-run beginning with the nth toss (so that, for example, $L_n = 0$ if the nth toss shows tails). Show that, for $\epsilon > 0$,

$$P\big(L_n \geq (1+\epsilon)\log_2 n \text{ i.o.}\big) = 0, \quad P\big(L_n \geq (1-\epsilon)\log_2 n \text{ i.o.}\big) = 1.$$

Here, i.o. means 'for infinitely many n'.