III — Michaelmas 2007 **GRG**

ADVANCED PROBABILITY

Example Sheet 4

Throughout this sheet, B_t denotes a standard Brownian Motion.

1. If X is distributed as $\mathrm{N}(0,1)$, and show that

$$P(X > x) \le \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \qquad x > 0.$$

2. Let $X_1, X_2, \dots$ be independent and equally likely to take the values ± 1. Let $S_n = X_1 + X_2 + \cdots + X_n$, and show that $M_n = S_n^2 - n$ is a martingale. Let a, b be strictly positive integers, and let T be the earliest passage time of S to the set $\{-a, b\}$. Show that $E(T) = ab$.

3. Suppose, instead, the X_i in Example 2 are independent and identically distributed with finite mean μ and variance σ^2. For any stopping time T with finite mean, show that $\mathrm{var}\,(Y_T) = \sigma^2 E(T)$ where $Y_n = \sum_{i=1}^n (X_i - \mu)$.

4. **Likelihood-ratio test.** Let $X_1, X_2, \dots$ be independent and identically distributed with common density function f satisfying either $f = p$ or $f = q$, and set

$$Y_n = \frac{p(X_1)p(X_2)\dots p(X_n)}{q(X_1)q(X_2)\dots q(X_n)}.$$

Show that (Y_n) is a martingale under the measure P_q corresponding to $f = q$. Show that $Y_n \to 0$ almost surely (under P_q). [Note that Y_n does not converge to 0 in L^1, since that would imply $Y_n = E(0 \mid \mathcal{F}_n) = 0$.]

Show that

$$P_q(Y_n \ge a \text{ for any } n \ge 1) \le \frac{1}{a},$$

and explain the relevance of this to the likelihood-ratio test.

5. Let $\alpha > 0$. Show that $W_t^1 = \alpha B_{t/\alpha^2}$, $W_t^2 = B_{t+\alpha} - B_\alpha$, $W_t^3 = tB_{1/t}$ and $W_0^3 = 0$, constitute three standard Brownian Motions.

6. Show that

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} = \infty, \qquad \liminf_{t\to\infty} \frac{B_t}{\sqrt{t}} = -\infty.$$

7. Let $a, b > 0$. Let T be the earliest time at which B visits either of the two points $-a$, b. Show that

$$P(B_T = b) = \frac{a}{a+b}, \qquad E(T) = ab.$$

In the case $a = b$, find $E(e^{-sT})$ for $s > 0$.

8. Let

$$M(t) = \int_0^t B_u\,du - \tfrac{1}{3}B_t^3.$$

Show that $M(t)$ is a martingale, and deduce that the expected area under the path of B until it first reaches one of the levels $-a$ (< 0) or b (> 0) is $\frac{1}{3}ab(b-a)$.

9. Compute the joint density function of the pair (B_t, M_t) where $M_t = \sup_{0\le s\le t} B_s$.

10. A *binary tree* is a tree in which each node has exactly two descendants. Suppose that each node of the tree is coloured black with probability p, and white otherwise, independently of all other nodes. For any path π containing n nodes beginning at the root of the tree, let $B(\pi)$ be the number of black nodes in π, and let $X_n(k)$ be the number of such paths π for which $B(\pi) \ge k$. Show that there exists β_c such that

$$E\{X_n(\beta n)\} \to \begin{cases} 0 & \text{if } \beta > \beta_c, \\ \infty & \text{if } \beta < \beta_c, \end{cases}$$

and show how to determine the value β_c.

Prove that

$$P\big(X_n(\beta n) \ge 1\big) \to 0 \qquad \text{if } \beta > \beta_c.$$

Can you show the converse?