

ADVANCED PROBABILITY

Example Sheet 3

1. Suppose that the sequence X_n satisfies

$$\forall \epsilon > 0, \quad P(|X_m - X_n| > \epsilon) \rightarrow 0 \quad \text{as } m, n \rightarrow \infty$$

(Cauchy convergence in probability). Show that X_n converges in probability to some limit X .

2. Show that, for $p \geq 1$, $L^p(\Omega, \mathcal{F}, P)$ is complete in the sense that: for any Cauchy sequence (X_n) in L^p , there exists $X \in L^p$ such that $X_n \rightarrow X$ in L^p .

3. Let the X_i be independent random variables, and write $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ and $\mathcal{H}_n = \sigma(X_n, X_{n+1}, \dots)$. Let $A \in \mathcal{T} = \bigcap_n \mathcal{H}_n$. By considering the martingale $Y_n = E(I_A | \mathcal{F}_n)$, show that A has probability either 0 or 1.

4. (a) Let $Y = (Y_n)$ be a martingale with respect to the filtration $\mathcal{F} = (\mathcal{F}_n)$, and let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Show that $(u(Y_n))$ is a submartingale so long as $E(u(Y_n)^+) < \infty$ for all n . Deduce that $|Y_n|$, Y_n^2 , and Y_n^+ are submartingales, so long as the appropriate moment condition holds.

(b) Let Y be a submartingale, and let u be convex and non-decreasing. Show that $u(Y_n)$ defines a submartingale provided $E(u(Y_n)^+) < \infty$ for all n . Deduce that Y_n^+ defines a submartingale subject to the moment condition.

5. *Branching Process.* At time n , a population contains Z_n members. Each of these has a family of descendants in the $(n+1)$ th generation, different families having independent sizes drawn from the same distribution. Suppose $Z_0 = 1$. Write μ ($\neq 0$) for the mean family-size, and $\eta = P(Z_n = 0 \text{ for some } n)$ for the extinction probability. Show that $A_n = Z_n/E(Z_n)$ and $B_n = \eta^{Z_n}$ constitute martingales with respect to the sequence Z_1, Z_2, \dots .

Show that, if $\mu > 1$, there exists A such that $A_n \rightarrow A$ a.s., and also that the characteristic function ϕ of A satisfies the functional equation $\phi(\mu t) = G(\phi(t))$ where $G(x) = E(x^{Z_1})$.

6. Let Z_1, Z_2, \dots be independent random variables such that

$$Z_n = \begin{cases} a_n & \text{with probability } \frac{1}{2}n^{-2} \\ 0 & \text{with probability } 1 - n^{-2} \\ -a_n & \text{with probability } \frac{1}{2}n^{-2} \end{cases}$$

where $a_1 = 2$ and $a_n = 4 \sum_{j=1}^{n-1} a_j$. Show that $Y_n = \sum_{j=1}^n Z_j$ defines a martingale. Show that $Y = \lim Y_n$ exists a.s., but that there exists no M such that $E|Y_n| \leq M$ for all n .

7. Let X_1, X_2, \dots be independent random variables with

$$X_n = \begin{cases} 1 & \text{with probability } (2n)^{-1} \\ 0 & \text{with probability } 1 - n^{-1} \\ -1 & \text{with probability } (2n)^{-1}. \end{cases}$$

Let $Y_1 = X_1$ and for $n \geq 2$

$$Y_n = \begin{cases} X_n & \text{if } Y_{n-1} = 0 \\ nY_{n-1}|X_n| & \text{if } Y_{n-1} \neq 0. \end{cases}$$

Show that Y_n is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, Y_2, \dots, Y_n)$. Show that Y_n does not converge almost surely. Does Y_n converge in any way? Why does the martingale convergence theorem not apply?

8. A bag contains red and blue balls, with initially r red and b blue where $rb > 0$. A ball is drawn from the bag, its colour noted, and then it is returned to the bag together with a new ball of the same colour. Let R_n be the number of red balls after n such operations.

(a) Show that $Y_n = R_n/(n + r + b)$ is a martingale which converges a.s. and in mean.

(b) Let T be the number of balls drawn until the first blue ball appears, and suppose that $r = b = 1$. Show that $E\{(T + 2)^{-1}\} = \frac{1}{4}$.

(c) Suppose $r = b = 1$, and show that $P(Y_n \geq \frac{3}{4} \text{ for some } n) \leq \frac{2}{3}$.

9. Here is a martingale approach to the question of determining the mean number of tosses of a coin before the first appearance of the sequence HHH . A large casino contains infinitely many gamblers G_1, G_2, \dots , each with an initial fortune of \$1. A croupier tosses a coin repeatedly. For each n , gambler G_n bets as follows. Just before the n th toss he stakes his \$1 on the event that the n th toss shows heads. The game is assumed fair, so that he receives a total of $\$p^{-1}$ if he wins, where p is the probability of heads. If he wins this gamble, then he *repeatedly* stakes his entire current fortune on heads, at the same odds as his first gamble. At the first subsequent tail he loses his fortune and leaves the casino, penniless. Let S_n be the casino's profit (losses count negative) after the n th toss. Show that S_n is a martingale. Let N be the number of tosses before the first appearance of HHH ; show that N is a stopping time and hence find $E(N)$.

Now adapt this scheme to calculate the mean time to the first appearance of the sequence HTH .

10. The game 'Red Now' may be played by a single player with a well shuffled conventional pack of 52 playing cards. At times $n = 1, 2, \dots, 52$ the player turns over a new card and observes its colour. Just once in the game he must say, just before exposing a card, "Red Now". He wins the game if the next exposed card is red. Let R_n be the number of red cards remaining face down after the n th card has been turned over. Show that $X_n = R_n/(52 - n)$, $0 \leq n < 52$, defines a martingale. Show that there is no strategy for the player which results in a probability of winning different from $\frac{1}{2}$.

11. Let X_1, X_2, \dots be independent identically distributed random variables and suppose that $M(t) = E(e^{tX_1})$ satisfies $M(t) = 1$ for some $t > 0$. Show that $P(S_k \geq x \text{ for some } k) \leq e^{-tx}$ for $x > 0$ and such a value of t , where $S_k = X_1 + X_2 + \dots + X_k$.

12. If T_1 and T_2 are stopping times with respect to a filtration \mathcal{F} , show that $T_1 + T_2$, $\max\{T_1, T_2\}$, and $\min\{T_1, T_2\}$ are stopping times also.

13. On failure, a light bulb is replaced by a new bulb. The lifetimes of the bulbs are iid random variables with common mean μ satisfying $0 < \mu < \infty$. Let $N(t)$ be the number of failures by time t . Show that $N(t)$ is *not* a stopping time but that $N(t) + 1$ is a stopping time, with respect to the sequence X_1, X_2, \dots of lifetimes.

Show that $\liminf_{t \rightarrow \infty} E(N(t))/t \geq \mu^{-1}$. Can you explain how to prove the complementary fact that $\limsup_{t \rightarrow \infty} E(N(t))/t \leq \mu^{-1}$.