UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 7 June, 2006 1.30 to 4.30

PAPER 37

INTERACTING PARTICLE SYSTEMS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $G = (V, E)$ be a finite connected graph and let T be a spanning tree of G chosen uniformly at random. Let s, t be distinct vertices of G, and think of G as an electrical network with unit edge-resistances, and source s and sink t. Show that

$$i_{xy} = P(T \text{ has } \Pi(s,x,y,t)) - P(T \text{ has } \Pi(s,y,x,t)), \quad x, y \in V,$$

is a solution to the two Kirchhoff laws when a unit flow arrives at s and departs from t. Here, $\Pi(s,u,v,t)$ is the property of trees that the unique path from s to t passes along the edge $\langle u, v\rangle$ in the direction from u to v.

(b) Let $G = (V, E)$ be a subgraph of G', and let T (respectively, T') be a uniform spanning tree of G (respectively, G'). Show that $P(e \in T) \geqslant P(e \in T')$ for $e \in E$. [A clear statement should be given of any general principle used.]

2 (a) Let $(x_n : n \geqslant 1)$ and $(\alpha_n : n \geqslant 1)$ be real sequences satisfying $x_{m+n} \leqslant x_m + x_n + \alpha_m$ for $m, n \geqslant 1$. Show that the limit $\lambda = \lim_{n\to\infty} \{n^{-1}x_n\}$ exists and satisfies $x_n \geqslant n\lambda - \alpha_n$ for $n \geqslant 1$, under the assumption that $n^{-1}\alpha_n \to 0$ as $n \to \infty$.

(b) Consider bond percolation on $\mathbb{Z}^d$ with $d \geqslant 2$ and parameter $p \in (0, 1)$. Let $\Lambda_n = [-n, n]^d$ and $\partial\Lambda_n = \Lambda_n \backslash \Lambda_{n-1}$. Show that $\beta_n = P_p(0 \leftrightarrow \partial\Lambda_n)$ satisfies

$$\beta_{m+n} \leqslant |\partial\Lambda_m|\beta_m\beta_n, \qquad m, n \geqslant 1,$$

and deduce the existence of the limit $\gamma = \lim_{n\to\infty}\{n^{-1} \log \beta_n\}$. Show that $\beta_n \geqslant |\partial\Lambda_n|^{-1}e^{-n\gamma}$.

[A clear statement should be given of any general result to which you appeal.]

3 Write an essay on the uniqueness of the infinite open cluster for bond percolation on $\mathbb{Z}^d$ for $d \geqslant 2$. Your essay should include the main steps in the proof of uniqueness, with emphasis on clear communication of the arguments used.

4 (a) Explain the existence of the lower invariant measure $\underline{\nu}$ and the upper invariant measure $\overline{\nu}$ for the contact model with parameter λ on $\mathbb{Z}^d$. Prove that $\underline{\nu} = \overline{\nu}$ if and only if $\theta(\lambda) = 0$, where $\theta(\lambda)$ is the probability that infection persists for all time, having begun at the origin only.

(b) Let $p_c(\text{site})$ and $p_c(\text{bond})$ be the critical probabilities of oriented site percolation and oriented bond percolation on $\mathbb{Z}^2$. Show that

$$p_c(\text{site}) \leqslant 1 - (1 - p_c(\text{bond}))^2.$$

(c) Show that the critical value λ_c of the contact model on $\mathbb{Z}$ satisfies $\lambda_c < \infty$. [You may assume that $p_c(\text{bond}) < 1$.]

5 (a) Let E be a finite set and $\Omega = \{0,1\}^E$. Explain what is meant by saying that two probability measures μ_1 and μ_2 on Ω are stochastically ordered in that $\mu_1 \geqslant_{\text{st}} \mu_2$. State the Holley condition for $\mu_1 \geqslant_{\text{st}} \mu_2$ when μ_1 and μ_2 are strictly positive.

Let μ be a strictly positive probability measure on Ω. State the FKG lattice condition for μ. Show that μ is positively associated whenever it satisfies the FKG lattice condition. [You may appeal to the stochastic-ordering statement of the first part of the question.]

(b) Let $G = (V, E)$ be a finite graph, and $\Sigma = \{-1, +1\}^V$. Let π be the probability measure on Σ given by

$$\pi(\sigma) \propto \exp\left(\sum_{e \in E} \sigma_x \sigma_y\right), \quad \sigma = (\sigma_x : x \in V) \in \Sigma,$$

where the summation is over all edges $e = \langle x, y\rangle \in E$. With Σ viewed as a partially ordered set, show that π is positively associated. [You may assume that the FKG lattice condition holds for all pairs $\sigma_1, \sigma_2 \in \Sigma$ if it holds for all pairs that agree at all vertices except two.]

6 (a) Explain what is meant by the *exclusion process* on the integers $\mathbb{Z}$.

(b) Let η_t denote the exclusion process on $\mathbb{Z}$ with initial configuration η_0, and let A_t denote the set of occupied vertices of an exclusion process on $\mathbb{Z}$ with a finite number $|A_0|$ of particles. Show that

$$P^\eta(\eta_t \equiv 1 \text{ on } A) = P^A(\eta \equiv 1 \text{ on } A_t), \quad \eta \in \{0,1\}^{\mathbb{Z}},\ A \subseteq \mathbb{Z},\ |A| < \infty$$

where P^ξ denotes the probability measure governing the appropriate process with initial configuration ξ.

(c) A probability measure μ on $\{0,1\}^{\mathbb{Z}}$ is called *exchangeable* if the quantity $\mu(\{\eta : \eta \equiv 1 \text{ on } A\})$, as A ranges over the finite subsets of $\mathbb{Z}$, depends only on the cardinality of A. Show that every exchangeable probability measure is invariant for the exclusion process.

END OF PAPER