1-2 Model, Dimers, and Clusters

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Abstract

A 1-2 model is a probability measure on subgraphs of a hexagonal lattice, satisfying the condition that the degree of present edges at each vertex is either 1 or 2. We prove that for any translation-invariant Gibbs measure of the 1-2 model on the plane, almost surely there are no infinite paths. We discover a measure-preserving correspondence between the 1-2 model and the dimer model on a decorated graph, and construct an explicit translation-invariant measure P for 1-2 model on the infinite periodic hexagonal lattice. We prove that the behavior of infinite clusters is different for small and large local weights under the measure P, which is an evidence of the existence of a phase transition.

1 1-2 Model

Computer Scientists M. Schwartz and J. Bruck ([12]) proposed the uniform 1-2 model (not-all-equal-relation), as a graphical model whose partition function (total number of configurations) can be computed by computing determinants via the holographic algorithm ([13]). A general version of the 1-2 model is explored in ([10]), as an application of a generalized holographic algorithm, and local statistics are computed. Holographic algorithm, although very general and beautiful, turns out to be not an efficient method to solve the 1-2 model. In this paper, we introduce a new approach to solve the 1-2 model, from which we prove the existence of a phase transition (Section 5), which has not been derived using the holographic algorithm.

Let $\mathbb{H} = (V, E)$ be a hexagonal lattice. A **1-2 model configuration** $\omega = (V_\omega, E_\omega)$ of $\mathbb{H}$ is a subgraph of $\mathbb{H}$, such that $V_\omega = V$, and the degree (number of incident edges) for each vertex of V_ω in ω is either 1 or 2. An example of a 1-2 model configuration is illustrated in Figure 1. From Figure 1, it is clear that for any configuration ω, the only possible connected components are either self-avoiding path or loops.

We look at a configuration ω locally at a vertex $v \in V$, and use $\omega|_v$ to denote the **local configuration** of ω at the vertex v, i.e, the intersection of E_ω with the set of incident edges of v. Each one of the 3 incident edges e_1, e_2, e_3 of v is either present in ω or not; all the configurations at v can be labeled by a 3-digit binary number. That is, each edge corresponds to a digit, present edges correspond to a number "1" on the digit while other edges correspond to "0". Examples of such correspondences are illustrated in Figure 1.

A positive number (**weight**) is associated to each local configuration at a vertex, i.e, a choice of subsets of incident edges of v, or a specific 3-digit binary number. This way we can write a dimension-8 vector labeled by the 8 different local configurations, such that the entry at a configuration, or a specific 3-digit binary number, is the weight of

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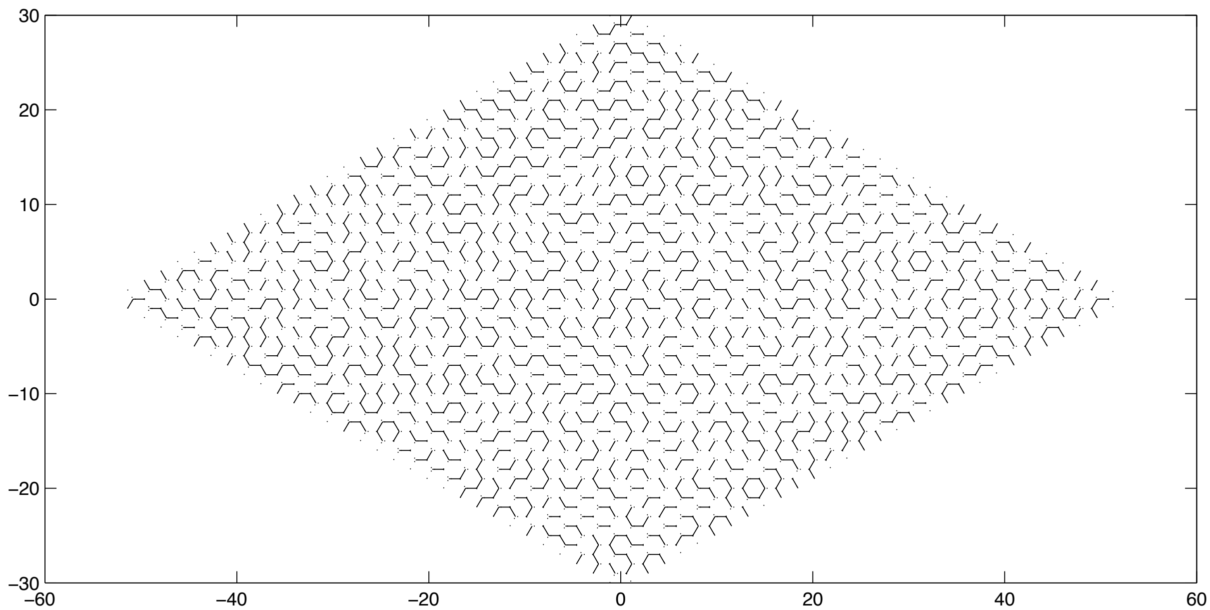

Figure 1: 1-2 model configuration

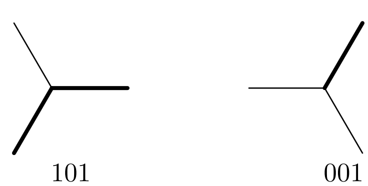

Figure 2: local configurations and binary numbers

the configuration. For a 1-2 model, the local configuration $\{000\}$ (no incident edges are included) or $\{111\}$ (all incident edges are included) are not allowed, and we give them a weight 0. Such a vector, is called the **signature** at a vertex v. The 1-2 model considered in this paper has the property that each local configuration and its complement have the same weight. Namely, the signature r_v at a vertex v has the following form

$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 0 & a & b & c & c & b & a & 0 \end{array}, \tag{1}$$

where a, b, c are positive numbers independent of v. The 1-2 model is called **uniform** if $a = b = c$.

We can construct a probability measure for 1-2 model configurations on a finite graph as follows. For simplicity, we consider only graphs embedded onto a finite torus, such that every vertex is a translation of another and there is no need to consider boundary vertices. The probability of a configuration ω, is defined to be proportional to the product of weights of its local configurations at each vertex, namely,

$$P(\omega) = \frac{\prod_{v \in V} w(\omega|_v)}{Z} \tag{2}$$

where $w(\omega|_v)$ is the weight of the local configuration $\omega|_v$, and Z is a normalizing constant called the **partition function**.

Let us call a connected set of vertices, each of which has a configuration with weight a, an a**-cluster**. We will construct a measure-preserving correspondence between the 1-2 model on $\mathbb{H} = (V, E)$ and the dimer model on a decorated graph $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$, such that for vertex $v \in V$, the bisectors of the three angles of $\mathbb{H}$ at v are edges of V_Δ, see Section 3 for details. The main result of this paper can be summarized as follows:

Theorem 1. *Consider the 1-2 model on a hexagonal lattice $\mathbb{H}$ with signature given by (1). The measure P_n on an $n \times n$ torus converges weakly to a translation-invariant measure P on the plane. For the limit measure P,*

1. *Let $\mathcal{L}$ be a path of length $\ell + 1$. Assume $E_{\mathcal{L}} = \{u_1 v_1, \cdots, u_\ell v_\ell\}$ are the set of all the bisector edges of angles with two sides in $\mathcal{L}$, then*

   $$P(\text{Path } \mathcal{L} \text{ appears in the } 1-2 \text{ model}) = \frac{1}{2} \prod_{k=1}^{\ell} w_{u_k v_k} (-1)^\ell \mathrm{Pf} K_{E_{\mathcal{L}}}^{-1}$$

   where $w_{u_k v_k}$ is the weight of the edge $u_k v_k$ in $\mathbb{H}_\Delta$, and $K_{E_{\mathcal{L}}}^{-1}$ is the submatrix the inverse of the weighted adjacency matrix of $\mathbb{H}_\Delta$ with rows and columns labeled by $u_1, v_1, \cdots, u_\ell, v_\ell$.

2. *Almost surely there are no infinite paths.*

3. *Fix $b, c > 0$, when a is sufficiently small, almost surely there are no infinite a-clusters. When a is sufficiently large, almost surely there exists a unique infinite a-cluster.*

4. *Let $\mathcal{N}_{a,n}$ be the random variable denoting the number of configurations with weight a in a $n \times n$ torus, as $n \to \infty$, $\frac{\mathcal{N}_{a,n}}{2n^2}$ converges to P_a in probability, where P_a is the probability that an a-configuration appears at a vertex. Moreover, P_a is increasing in a.*

Here is an outlline of the paper. In Section 2, we prove results on the expected number of self-avoiding path of the 1-2 model, as well as the monotonicity of the expected number of specific local configurations with respect to local weights, which follow directly from the definition of the measure. In Section 3, we prove the non-existence of infinite path for any translation-invariant measure, with the help of mass-transport principle introduced in [1]. In Section 4, we introduce a measure-preserving correspondence between the 1-2 model on a hexagonal lattice $\mathbb{H}$ and the dimer model on a decorated lattice $\mathbb{H}_\Delta$, and prove the weak convergence of the measure using a large torus to approximate the infinite periodic graph. Then we prove an explicit formula to compute the probability of a self-avoiding path of the 1-2 model under the limit measure. In Section 5, we prove Part 2 of Theorem 1. In Section 6, we prove Part 3 of Theorem 1. The different behaviors of infinite clusters imply the existence of phase transition.

2 Non-loop Components and Monotonicity

In this section, we prove two propositions resulting from the definition of the measure of the 1-2 model. First of all, we notice that in each configuration (subgraphs of $\mathbb{H}$), there are two kinds of connected components: either self-avoiding paths or loops, see Figure 1. Proposition 2 is about the expected number of non-loop connected components, and Proposition 3 is about the monotonicity of the expected number of a specific local configuration with respect to local weights.

Proposition 2. *Consider the 1-2 model defined on a honeycomb lattice embedded into an $n\times n$ torus. That is, the total number of white vertices is n^2, so is the total number of black vertices. Let the random variable $\mathcal{C}_n$ denote the number of non-loop connected components of the 1-2 model, then*

$$\mathbb{E}\mathcal{C}_n = \frac{1}{2}n^2$$

where the expectation is taken for the natural probability measure of the 1-2 model defined in (2).

Proof. Since each non-loop connected component has two degree-1 vertices as endpoints, and all the other vertices are of degree 2, the number of non-loop connected components is one half of the number of degree-1 vertices ($\mathcal{N}_1$). By symmetry, each configuration and its complement have the same probability, namely

$$P(\mathcal{N}_1 = k) = P(\mathcal{N}_1 = 2n^2 - k)$$

therefore

$$\begin{aligned}
\mathbb{E}\mathcal{N}_1 &= \sum_{k=0}^{2n^2} kP(\mathcal{N}_1 = k) \\
&= \sum_{k=0}^{n^2-1} (k + (2n^2 - k))P(\mathcal{N}_1 = k) + n^2 P(\mathcal{N}_1 = n^2) \\
&= 2n^2[\sum_{k=0}^{n^2-1} P(\mathcal{N}_1 = k) + \frac{1}{2}P(\mathcal{N}_1 = n^2)] \\
&= n^2
\end{aligned}$$

As a result

$$\mathbb{E}\mathcal{C}_n = \frac{1}{2}\mathbb{E}\mathcal{N}_1 = \frac{1}{2}n^2$$

□

Let $\mathcal{N}_a$ be the number of vertices which have a configuration of weight a. In other words, $\mathcal{N}_a$ is the number of vertices which have configurations 001 or 110. We call all such vertices a-vertices. Whether or not a vertex is an a-vertex depends on the specific configuration taken.

Proposition 3. *Fix the size of the $n \times n$ torus. For any $\gamma > 0$, $\mathbb{E}\mathcal{N}_a^\gamma$ is increasing with respect to a.*

Proof. First let us prove that for any positive integer k, $P(\mathcal{N}_a \geq k)$ is increasing with respect to a. Let $Z_{k,a}^+$ be the partition function of the 1-2 model on an $n \times n$ torus, satisfying the condition that the number of a-vertices is no less than k. Let $Z_{k,a}^-$ be the partition function of the 1-2 model on an $n \times n$ torus, satisfying the condition that the number of a-vertices is less than k. Assume $0 < a_1 < a_2$. Then

$$\frac{Z_{k,a_2}^+}{Z_{k,a_1}^+} \geq \left(\frac{a_2}{a_1}\right)^k \geq \frac{Z_{k,a_2}^-}{Z_{k,a_1}^-}.$$

Therefore

$$\frac{Z_{a_2}}{Z_{a_1}} = \frac{Z_{k,a_2}^+ + Z_{k,a_2}^-}{Z_{k,a_1}^+ + Z_{k,a_1}^-} \leq \frac{Z_{k,a_2}^+}{Z_{k,a_1}^+}.$$

Hence we have

$$P(\mathcal{N}_{a_2} \geq k) = \frac{Z_{k,a_2}^+}{Z_{a_2}} \geq \frac{Z_{k,a_1}^+}{Z_{a_1}} = P(\mathcal{N}_{a_1} \geq k).$$

Then lemma follows from the fact that

$$\mathbb{E}\mathcal{N}_a^\gamma = \sum_{1 \leq k \leq 2n^2} k^\gamma P(\mathcal{N}_a = k) = \sum_{1 \leq k \leq 2n^2} (k^\gamma - (k-1)^\gamma) P(\mathcal{N}_a \geq k)$$

□

3 Nonexistence of Infinite Paths

Define a **uni-infinite path** to be an infinite self-avoiding path of the 1-2 model configuration, which has a finite endpoint and is infinite in just one direction. In other words, this infinite path should consist a single vertex of degree 1, while all the other (infinitely many) vertices on the path should have degree 2. In this section, we will apply the mass-transport principle ([1]) to prove the almost-sure nonexistence of uni-infinite path in the 1-2 model for all positive parameters a, b, c. First of all, we introduce the definition of unimodality.

Definition 4. *Let $\mathcal{G} = (V, E)$ be a quasi-transitive graph with automorphism group $Aut(\mathcal{G})$. For $v \in V(\mathcal{G})$, the* ***stabilizer*** *of v is defined as*

$$Stab(v) = \{\gamma \in Aut(\mathcal{G}) : \gamma v = v\}$$

*The graph $\mathcal{G}$ is said to be **unimodular** if for all $u, v \in V$ in the same orbit of $Aut(\mathcal{G})$, we have the symmetry*

$$|Stab(u)v| = |Stab(v)u|$$

In our case, the automorphism group of the hexagonal lattice $\mathbb{H}$, $Aut(\mathbb{H})$, is the group of translations, isomorphic to $\mathbb{Z}^2$. Each member in $Aut(\mathbb{H})$ corresponds to a color-preserving, weight-preserving isomorphism of $\mathbb{H}$. That is, it maps each black vertex to another black vertex with the same signature, and each white vertex to another white vertex with the same signature. The orbit of each black (resp. white) vertex is the set of all black (resp. white) vertices. Obviously $\mathbb{H}$ is uni-modular since $|Stab(u)v| = |Stab(v)u| = 1$, for all $u, v \in V(\mathbb{H})$.

Let $m(u, v, \omega)$ be a non-negative function of three variables: two vertices $u, v \in V(\mathbb{H})$, and the 1-2 model configuration ω. We assume $m(u, v, \omega) = 0$ unless u and v are in the same orbit of $Aut(\mathbb{H})$. More precisely, $m(u, v, \omega)$ is defined as follows. Each vertex u sitting in a uni-infinite path sends unit mass to the unit endpoint of this path, if the endpoint has the same color as u. Vertices not sitting in a uni-infinite path, or having a different color with the endpoint, send no mass at all. In other words, $m(u, v, \omega) = 1$ if and only if in the configuration ω there is a uni-infinite path starting at v, passing u and both u and v are black, or both u and v are white. Otherwise $m(u, v, \omega) = 0$. Obviously, $m(u, v, \omega)$ is invariant under the diagonal action of $Aut(\mathbb{H})$, meaning that $m(u, v, \omega) = m(\gamma u, \gamma v, \gamma\omega)$, for all u, v, ω, and $\gamma \in Aut(\mathbb{H})$.

Lemma 5. *(The Mass-Transport Principle, Section 3 in [1]) Assume μ is a probability measure invariant under actions of $Aut(\mathcal{G})$. Given $\mathcal{G}$ and $m(\cdot, \cdot, \cdot)$ as above, let*

$$M(u, v) = \int_\Omega m(u, v, \omega) d\mu(\omega),$$

for any $u, v \in V(\mathcal{G})$. If $\mathcal{G}$ is unimodular, then the expected total mass transported out of any vertex v equals the expected total mass transported into v, that is

$$\sum_{u \in V(\mathcal{G})} M(v, u) = \sum_{u \in V(\mathcal{G})} M(u, v) \tag{3}$$

Proof. (in the case of hexagonal lattice $\mathbb{H}$ and the translation action $Aut(\mathbb{H})$) Without loss of generality, assume v is black. The terms contribute to the sum are $M(u, v)$'s for which u is also black. For any $u, v \in V(\mathbb{H})$, both u and v are black, there is a unique $h \in Aut(\mathbb{H}) \simeq \mathbb{Z}^2$, such that $u = hv$. In other words, h is a translation action mapping u to v. This gives

$$\sum_{u\ black} M(v, u) = \sum_{h \in Aut(\mathbb{H})} M(v, hv) = \sum_{h \in Aut(\mathbb{H})} M(h^{-1}v, v) = \sum_{u\ black} M(u, v),$$

where the second equality follows from translation-invariance of $m(\cdot, \cdot, \cdot)$ and μ. □

Lemma 6. *For all positive parameters a, b, c, let μ be a translation-invariant measure. The μ-a.s. there is no uni-infinite path in 1-2 model configurations.*

Proof. Since μ is a translation-invariant probability measure, for which Lemma 5 holds. The expected mass sent from a vertex is at most 1, while if uni-infinite paths exist with positive probability, then some vertex will receive infinite mass with positive probability, so that the expected mass received is infinite, contradicting (3). □

Theorem 7. *For all translation-invariant measure of the 1-2 model, a.s. there are no infinite paths.*

Proof. Without loss of generality, assume $0 \leq c \leq b \leq a$. Assume infinite paths exist with positive probability, then there exists a positive probability that an infinite path passes the origin, denoted by $p_0 (p_0 > 0)$. Note that the infinite path cannot end at any finite vertex, by lemma 6. We remove one incident edges of the origin occupied by the infinite path. This way we get a configuration including uni-infinite paths, and any configuration with an infinite path passing the origin have two corresponding configurations including uni-infinite paths under the rule described above. On the other hand, any configuration including uni-infinite paths which could be obtained from a configuration with an infinite path passing the origin by the previous process, it has a unique corresponding configuration with an infinite path passing the origin. Hence the probability that uni-infinite paths exist is at least $\left(\frac{c}{a}\right)^2 p_0 > 0$, which is a contradiction to Lemma 6. □

4 Correspondence with Dimer Model

A **dimer configuration**, or a **perfect matching**, of a graph is a collection of edges with the property that each vertex is incident to exactly one of these edges. Each edge e is associated with a positive weight w_e, and the probability of a dimer configuration on a finite graph is defined to be proportional to the product of weights of included edges. Namely, let $\mathcal{D}$ be a dimer configuration, then

$$P(\mathcal{D}) \propto \prod_{e \in \mathcal{D}} w_e$$

We construct a measure-preserving correspondence between the 1-2 model on a honeycomb lattice $\mathbb{H} = (V, E)$, and the dimer configurations of a decorated graph $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$, as illustrated in Figure 4, where the dashed lines are edges of the honeycomb lattice $\mathbb{H}$.

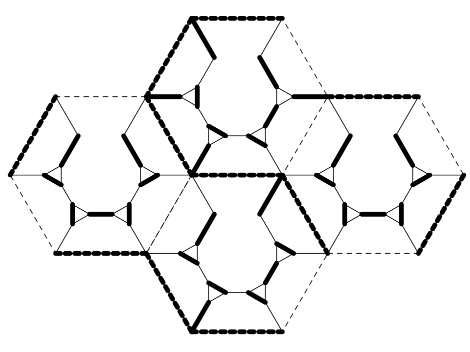

Figure 3: 1-2 Model and Dimers

Each vertex of $\mathbb{H}$ is also a vertex of $\mathbb{H}_\Delta$, namely $V \subset V_\Delta$. For each $v \in V$, The three incident edges $e_{1,\Delta}, e_{2,\Delta}, e_{3,\Delta} \in E_\Delta$ of v are bisectors of angles of $\mathbb{H}$ at v. On each face of $\mathbb{H}$, we construct a gadget for $\mathbb{H}_\Delta$, which is a modified hexagon, with the topmost edge removed, and the four vertices except the top two replaced by a triangle, see Figure 4. At each angle of $\mathbb{H}$, if both sides of the angle of have the same configuration, that is, either both of them are present, or neither of them are present, then the bisector edge is present in the configuration of $\mathbb{H}_\Delta$. Otherwise the bisector is not present. For a 1-2 model configuration of $\mathbb{H}$, with a construction above, the corresponding configuration on

$\mathbb{H}_\Delta$ satisfies the condition that each vertex $v \in V$ has exactly one present incident edges. Moreover, such a configuration always has a unique extension to a dimer configuration (each vertex in V_Δ has exactly one present incident edges) of $\mathbb{H}_\Delta$, because around each face of $\mathbb{H}$, there is always an even number of bisector edges of $\mathbb{H}_\Delta$ present in the configuration. For such a construction, two 1-2 model configurations, the union of which is the graph $\mathbb{H}$, corresponds to the same dimer configuration of $\mathbb{H}_\Delta$.

Assigning edge weights for $\mathbb{H}_\Delta$ appropriately will ensure the correspondence to be measure-preserving. Namely, if ξ_v is a degree-1 local configuration at v (one incident edges of v is present) of $\mathbb{H}$, and ξ_v has weight a, we assign the same weight a to the bisector edge in E_Δ opposite to the present edge in ξ_v of E. This way, we can assign a weight to all edges in E_Δ incident to a vertex $v \in V$. For all the other edges in E_Δ, we assign weight 1. This way, if ξ, $\bar{\xi}$ are two 1-2 model configurations which are complement to each other, and $\mathcal{D}$ is the corresponding dimer configuration, by definition of the measures of the 1-2 model and the dimer model, we have

$$P(\xi) = P(\bar{\xi}) = \frac{1}{2} P(\mathcal{D})$$

Therefore, we can investigate the measure of the 1-2 model by investigating the measure of the dimer model. An important object in understanding the infinite volume limit of the periodic dimer model is the **characteristic polynomial**. To introduce the notation, first of all, we give $\mathbb{H}_\Delta$ a **clockwise-odd orientation** such that traversing each face of $\mathbb{H}_\Delta$ clockwise gives an odd number of edges with the same orientation with the traversal orientation. See Figure 4.

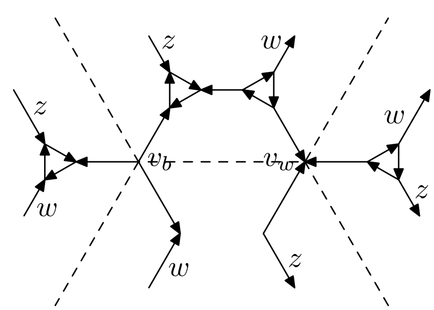

Figure 4: one fundamental domain

Let $\mathbb{H}_{\Delta,n}$ be the quotient graph of $\mathbb{H}_\Delta$ under the translation of $n\mathbb{Z} \times n\mathbb{Z}$. $\mathbb{H}_{\Delta,n}$ is a finite graph which can be embedded into an $n \times n$ torus. The orientation of $\mathbb{H}_{\Delta,n}$ is inherited from $\mathbb{H}_\Delta$. Let γ_x, γ_y be two homology generators of the torus. Multiply the weights of the edges crossed by γ_x by z (or $\frac{1}{z}$), according to its orientation, and similarly, multiply the weights of edges crossed by γ_y by w (or $\frac{1}{w}$) according to its orientation. This way we get a modified weighted adjacency matrix $K_n(z, w)$. The characteristic polynomial $P(z, w)$ is defined to be the $\det K_1(z, w)$, and the spectral curve is defined to be the zero locus $P(z, w) = 0$.

Lemma 8. *If $a, b, c > 0$, either the spectral curve of the dimer model does not intersect the unit torus $\mathbb{T}^2 := \{(z, w) \in \mathbb{C}^2 : |z| = 1, |w| = 1\}$, or the intersection is a single real point (1,1), and the intersection is of multiplicity 2.*

Proof. By definition, the characteristic polynomial

$$P(z,w) = \det K(z,w) \quad = \quad a^4+b^4+c^4+6a^2b^2+6a^2c^2+6b^2c^2+4ab(c^2-a^2-b^2)\cos\theta \quad (4)$$
$$+4ac(b^2-a^2-c^2)\cos\phi+4bc(a^2-b^2-c^2)\cos(\theta-\phi) \quad (5)$$

where $z=e^{i\theta}, w=e^{i\phi}$. First of all, note that if $a^2+b^2<c^2$, $a^2+c^2<b^2$ and $b^2+c^2<a^2$, we have for any $(z,w)\in\mathbb{T}^2\setminus\{(1,1)\}$,

$$P(z,w) > P(1,1) \geq 0,$$

Since P(1,1) is the determinant of an anti-symmetric matrix. If a^2,b^2,c^2 cannot be the lengths of sides of a triangle, we consider the minimal value of the characteristic polynomial in $\mathbb{T}^2$. Without loss of generality, assume $b^2\geq a^2+c^2$ It suffices to consider the minimal value of

$$G(\theta,\phi) = 4ab(c^2-a^2-b^2)\cos\theta+4ac(b^2-a^2-c^2)\cos\phi+4bc(a^2-b^2-c^2)\cos(\theta-\phi)$$

for $(\theta,\phi)\in[0,2\pi]^2$. In fact,

$$G(\theta,\phi) = A+B\cos\theta+C\sin\theta$$

where

$$\begin{aligned} A &= 4ac(b^2-a^2-c^2)\cos\phi \\ B &= 4ab(c^2-a^2-b^2)+4bc(a^2-b^2-c^2)\cos\phi \\ C &= 4bc(a^2-b^2-c^2)\sin\phi \end{aligned}$$

For any given ϕ, the minimal value of $G(\theta,\cdot)$ is

$$\begin{aligned} A^2-B^2-C^2 &= 16a^2c^2(a^2-b^2+c^2)^2\cos^2\phi-32ab^2c(-a^2+b^2+c^2)(a^2+b^2-c^2)\cos\phi \\ &\quad -16a^2b^2(a^2+b^2-c^2)^2-16b^2c^2(-a^2+b^2+c^2)^2 \end{aligned}$$

The minimal value of the quadratic polynomial with respect to $\cos\phi$ achieves at the point (as an extended real number)

$$t_0 = \frac{b^2(-a^2+b^2+c^2)(a^2+b^2-c^2)}{ac(a^2-b^2+c^2)^2} > 1$$

hence the minimal value of $A^2-B^2-C^2$ can only be achieved at $\cos\phi=1$, and the minimal value of $G(\theta,\phi)$ can only be achieved at $(0,0)$. It is trivial to check that the intersection of the spectral curve and $\mathbb{T}^2$, if exists, is of multiplicity 2 by computing the derivatives. □

Proposition 9. *Let P_n denote the probability measure of the dimer graph $\mathbb{H}_\Delta$ on the $n\times n$ torus. As $n\to\infty$, μ_n converges weakly to an ergodic translation invariant probability measure P of dimer configurations on $\mathbb{H}_\Delta$.*

Proof. The proof of convergence is exactly the same as Lemma 4.8 in [11]. The translation invariance of the measure P is obvious. P is an ergodic dimer measure because it is translation invariant and mixing. The mixing of P follows from the fact that the intersection of the spectral curve with $\mathbb{T}^2$ is either empty or a single real point of multiplicity 2, hence the covariance of any two event, measurable with respect to two finite set of vertices far away from each other, decays to zero either exponentially fast or polynomially fast. Note that P may not be ergodic as a measure for the 1-2 model configurations. □

Theorem 10. *Using a large torus to approximate the infinite hexagonal lattice, the probability that a path occurs in the 1-2 model is equal to the Pfaffian of an anti-symmetric matrix, multiplying one half of the product of the configuration weights.*

Proof. Consider an arbitrary path in the 1-2 model. The path occurs in a 1-2 model configuration corresponds to the condition that all the bisector edges along the path are present in the dimer configuration.

Given an arbitrary 1-2 model configuration $\mathcal{C}$. The configuration $\mathcal{C}^*$, which occupies all the unoccupied edges of $\mathcal{C}$ and leaves all the occupied edges of $\mathcal{C}$ unoccupied, has the same probability as the configuration $\mathcal{C}$, and corresponds to the same dimer configuration of $\mathbb{H}_\Delta$. Hence we have

$$P(path\ \mathfrak{L}) = \frac{1}{2}Pr(\text{all the bisector edges along } \mathfrak{L} \text{ are present in the dimer configurations of } \mathbb{H}_\Delta).$$

Using a large torus to approximate the infinite graph, we can actually compute the probability on the right explicitly. First we consider a finite $n \times n$ torus, where n is even. Let $K_n(z,w)$ be the corresponding modified adjacency matrix, then the partition function of the dimer model on $\mathbb{H}_\Delta$ is given by

$$2Z_{\Delta,n} = |-\mathrm{Pf}K_n(1,1) + \mathrm{Pf}K_n(1,-1) + \mathrm{Pf}K_n(-1,1) + \mathrm{Pf}K_n(-1,-1)| := 2\hat{Z}_n$$

In fact, under the given clockwise odd orientation, each term in $-\mathrm{Pf}K_n(1,1)+\mathrm{Pf}K_n(1,-1)+\mathrm{Pf}K_n(-1,1)+\mathrm{Pf}K_n(-1,-1)$ has the same sign. Either the sign is negative or positive depends on the ordering of vertices in the weighted adjacency matrix. However, the sign does not matter when we compute the probability, since we are taking the quotient of two such polynomials.

Let $\tilde{K}_n(z,w)$ be the modified weighted adjacency matrix of a graph by removing all the bisector edges along the path as well as their ending vertices, Let $E_{\mathfrak{L}}$ be the set of all vertices of bisector edges along the path. Namely,

$$E_{\mathfrak{L}} = \{u_1v_1, u_2v_2, \cdots, u_kv_k\}$$

Let $|u_i|$ be the index of the column,corresponding to the vertex u, in the weighted adjacency matrix. Then

$$\begin{aligned}
&P(\text{all the edges in } \mathrm{E}_{\mathfrak{L}} \text{ arepresent}) \\
= \quad &\frac{(-1)^{(\sum_{i=1}^{k}|u_i|+|v_i|)+k}\prod_{1\leq i\leq k}\mathrm{Pf}K_{u_i,v_i}}{2\hat{Z}_n}[-\mathrm{Pf}\tilde{K}_n(1,1)+\mathrm{Pf}\tilde{K}_n(1,-1)+\mathrm{Pf}\tilde{K}_n(-1,1)+\mathrm{Pf}\tilde{K}_n(-1,-1)]
\end{aligned}$$

where $\mathrm{Pf}K_{u_i,v_i}$ is the Pfaffian of the 2×2 submatrix of K_n with rows and columns corresponding to u_i and v_i, the ordering of u_i and v_i in the submatrix is the same as the relative order of u_i and v_i in K_n. It is obvious that the numerator in the above equation are exactly those terms in $2\hat{Z}_n$ with all edge weights of $E_{\mathfrak{L}}$. Moreover,

$$\mathrm{Pf}\tilde{K}_n = (-1)^{\sum_i |u_i|+|v_i|}\mathrm{Pf}K_{E,n}^{-1}\mathrm{Pf}K_n$$

Let (p_1,q_1,u) and (p_2,q_2,v) be two vertices, where (p_1,q_1) and (p_2,q_2) are, indices of the fundamental domains, and u,v are indices of vertices in one fundamental domain. Then we have

$$\begin{aligned}
K^{-1}_{(p_1,q_1,u),(p_2,q_2,v)} &= \lim_{n\to\infty} K^{-1}_{n,(p_1,q_1,u),(p_2,q_2,v)}((-1)^\theta,(-1)^\tau) \\
&= \frac{1}{2\pi}\iint_{\mathbb{T}^2} z^{p_1-q_1}w^{p_2-q_2}\frac{\mathrm{Cofactor}K_1(z,w)_{u,v}}{\det K_1(z,w)}\frac{dz}{\sqrt{-1}z}\frac{dw}{\sqrt{-1}w}
\end{aligned}$$

for any θ, τ in $\{0,1\}$. The convergence follows from the fact that, the intersection of $\det K_1(z,w)$ and $\mathbb{T}^2$ is either empty of a single real point of multiplicity 2, and the machinery described in [2, 11]. Since we are only interested in the bisector edges, we can always give indices of vertices in the weighted adjacent matrix such that the index of the starting vertex of each bisector edge is always less than or equal to the index of the ending vertex of the bisector edge, under the given clockwise odd orientation. Hence, as the size of the torus goes to infinity, we have

$$P(\text{Path } \mathfrak{L}) = \frac{1}{2}(-1)^{|E_{\mathfrak{L}}|} \prod_{1 \leq i \leq k} w_{u_i v_i} \text{Pf} K_{E_{\mathfrak{L}}}^{-1}$$

where the entries of K^{-1} are described as above, and $w_{u_i v_i}$ is the weight of the edge $u_i v_i$. □

5 Infinite Clusters

In this section, we provide an evidence of the existence of phase transition by exploring the behavior of infinite clusters. At each vertex, a 1-2 model configuration has 3 possible weights, a, b or c. Let us consider all local configurations at one vertex according to their weights: a-type, b-type and c-type. Note that each type of configurations actually consists of 2 configurations, namely, they are complement to each other and the occupying degree of the vertex is either 1 or 2. An a-cluster (resp. b-cluster or c-cluster) is a connected set of vertices such that each vertex in it has an a-type (resp. b-type or c-type) configuration. Note that although an a-type configuration consists of 2 configurations, each a-cluster can have only one configuration for all the vertices, similarly for b-clusters and c-clusters. We fix the value of b and c, and are interested in the behavior of a-clusters as we vary the value of a. We will always use P to denote the measure obtained by torus approximation, and μ to denote an arbitrary translation invariant Gibbs measure.

Let t_n denote the number of connected sets of vertices including the origin with total number of vertices n. Then

Lemma 11. *$t_n \leq 10^n$, for all n.*

Proof. We consider site percolation on the hexagonal lattice, we have

$$P(\text{the open cluster at the origin has size n}) = \sum_b t_{n,b} p^n (1-p)^b$$

where b is the number of boundary vertices, and $t_{n,b}$ is the number of all the connected sets including the origin with n interior vertices and b boundary vertices. Note that $t_{n,b} = 0$ unless $1 \leq b \leq 3n$ We have

$$1 \geq \sum_b t_{n,b} p^n (1-p)^b \geq \sum_b t_{n,b} p^n (1-p)^{3n}$$

Hence

$$\sum_b t_{n,b} \leq [p(1-p)^3]^{-n}$$

for all p. We choose p to maximize $p(1-p)^3$, and obtain

$$t_n = \sum_b t_{n,b} \leq \left(\frac{256}{27}\right)^n$$

and the lemma follows. □

Let $\mathcal{C}_{a,v_0}$ denote the a-cluster passing a fixed vertex v_0, and $|\mathcal{C}_{a,v_0}|$ denote the number of vertices in $\mathcal{C}_{v,a_0}$. Then we have the following theorem:

Theorem 12. *Let μ be an arbitrary translation-invariant Gibbs measure for 1-2 model configurations on $\mathbb{H}$ with parameters a, b, c. Fix b, c, when a is sufficiently small, there exists $\rho > 0$, such that*

$$P(|\mathcal{C}_{a,v_0}| \geq n) \leq e^{-n\rho}$$

Moreover, $\rho \to \infty$, as $a \to 0$.

Proof. Let $\mathcal{B}$ be any one of the three types of boxes consisting of two hexagons (including the boundary of the hexagons), and ℓ_1, ℓ_2 be two parts on the boundary of the box, as illustrated in Figure 5.

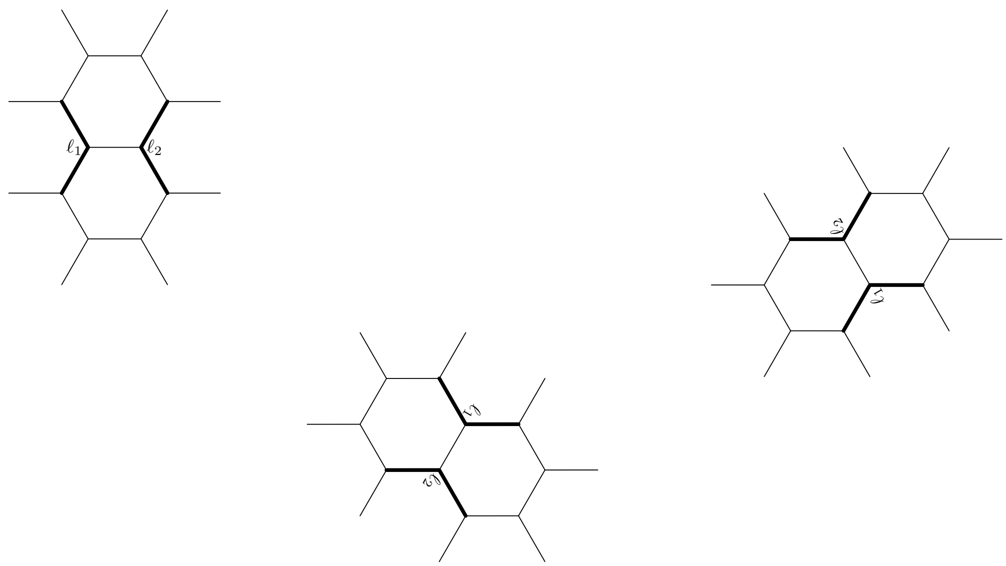

Figure 5: Boxes

Let E be the event that there exists a connected set of vertices in $\mathcal{B}$ with a-configurations connecting an endpoint of ℓ_1 to an endpoint of ℓ_2. Assume $\mathcal{C}_{a,v_0} = N$, then we can find at least kN non-intersecting (without common boundaries either) $\mathcal{B}$'s, such that E occurs at each $\mathcal{B}$, where $k > 0$ is a constant independent of N. Let us denote these boxes by $B_1, B_2, \cdots, B_{kN}$. If E happens at $\mathcal{B}$, then $\mathcal{B}$ has at least 4 vertices with a-configurations. We use the term boundary condition of a box $\mathcal{B}$ to denote any specific configuration for the 8 outer edges connecting one vertex in $\mathcal{B}$ and one vertex out of $\mathcal{B}$. It is easy to check that what boundary condition given to $\mathcal{B}$, we can always construct a configuration of $\mathcal{B}$, which has at most 3 vertices with a-configurations. Figure 5 is an example of such a configuration, in which an a-configuration at a vertex is the presence of a single horizontal incident edge or the presence of both non-horizontal incident edges.

Let $\mathcal{A}_{n,v_0}$ be the number of animals (connected set of vertices) of size n on $\mathbb{H}$ passing a fixed vertex v_0. By Lemma 11, $\mathcal{A}_{n,v_0} \leq 10^n$. If a-configurations appear at each vertex of an animal of size n, we can find at least kn non-intersecting boxes $B_1, \cdots, B_{kn}$, such that E happens at each $B_i, 1 \leq i \leq kn$. Let b_i denote a boundary condition for the box B_i.

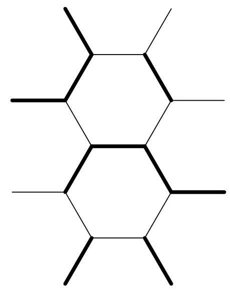

Figure 6: Configurations on Boxes

Then we have

$$
\begin{aligned}
&\mu(|\mathcal{C}_{a,v_0}| \geq n)\\
\leq &\sum_{A\in\mathcal{A}_{n,v_0}} \mu(all\ vertices\ in\ A\ have\ a-configurations)\\
\leq &\sum_{A\in\mathcal{A}_{n,v_0}} \mu(E\ happens\ at\ B_1,\cdots,B_{kn})\\
\leq &\sum_{A\in\mathcal{A}_{n,v_0}} \max_{b_1,\cdots,b_{kn}} \mu(E\ happens\ at\ B_1,\cdots,B_{kn}|b_1,\cdots,b_{kn}).
\end{aligned}
$$

By domain Markov property,

$$
\begin{aligned}
&\sum_{A\in\mathcal{A}_{n,v_0}} \max_{b_1,\cdots,b_{kn}} \mu(E\ happens\ at\ B_1,\cdots,B_{kn}|b_1,\cdots,b_{kn})\\
\leq &\sum_{A\in\mathcal{A}_{n,v_0}} \prod_{i=1}^{kn} \max_{b_i} \mu(E\ happens\ at\ B_i|b_i)\\
\leq &\sum_{A\in\mathcal{A}_{n,v_0}} \prod_{i=1}^{kn} 3^{10} a \frac{(\max\{b,c\})^6}{(\min\{b,c\})^7}\\
\leq &\left(10\cdot 3^{10k}\frac{(\max\{b,c\})^{6k}}{(\min\{b,c\})^{7k}} a^k\right)^n
\end{aligned}
$$

where $k > 0$ is a constant independent of n, a, b, c. Let

$$
\rho = -\log\left(10\cdot 3^{10k}\frac{(\max\{b,c\})^{6k}}{(\min\{b,c\})^{7k}} a^k\right).
$$

Fix b, c, $\lim_{a\to 0}\rho = \infty$, so the theorem follows. □

Corollary 13. *When a is sufficiently small, almost surely there is no infinite a-clusters.*

Lemma 14. *P_a, the probability that an a-configuration appears at a vertex for the measure P, is continuous in a, for any $a > 0$. Moreover,*

$$
\begin{aligned}
\lim_{a\to\infty} P_a = 1 \qquad &\text{for any fixed b, c} > 0\\
\lim_{a\to 0} P_a = 0 \qquad &\text{for any fixed b, c} > 0,\ \text{b} \neq \text{c}
\end{aligned}
$$

Proof. By definition

$$P_a = \frac{a}{4\pi^2} \int_{|w|=1} \int_{|z|=1} \frac{Q(z,w)}{P(z,w)} \frac{dz}{iz} \frac{dw}{iw}$$

where $P(z,w)$ is the characteristic polynomial as in (5), and $Q(z,w)$ is the cofactor of $K(z,w)$ by removing rows and columns corresponding to endpoints of the edge e. Namely,

$$Q(z,w) = 3ab^2 + 3ac^2 - c^3w - b^3z + a^3 - \frac{a^2c}{w} - 2a^2cw + b^2cw - 2a^2bz + bc^2z - \frac{ab(a-cw)}{z} + \frac{abcz}{w}$$

The continuity of P_a with respect to a follows from the fact that the intersection of $P(z,w) = 0$ with $\mathbb{T}^2$ can either be empty of a single real point, and the intersection is of multiplicity 2. For any $(z,w) \in \mathbb{T}^2$, and fixed b,c,

$$\lim_{a\to\infty} \frac{Q(z,w)}{P(z,w)} = 1$$

and $\lim_{a\to\infty} P_a = 1$ follows from the dominated convergence Theorem. When $b \neq c$,

$$\lim_{a\to\infty} \frac{Q(z,w)}{P(z,w)} = \frac{-zb^3 + wb^2c + zbc^2 - wc^3}{((wb^2 - 2zbc + wc^2)(zb^2 - 2wbc + zc^2))/(wz)} \tag{6}$$

If $b \neq c$, the denominator of (6) has no zeros on $\mathbb{T}^2$. Again the order of the integral and limit can change according to the dominated convergence Theorem. As a result, when $b \neq c$, $\lim_{a\to 0} P_a = 0$. □

Define the mean size of the cluster at the origin as follows

$$\chi = \sum_{n=1}^{\infty} nP(|\mathcal{C}| = n).$$

We have the following proposition

Proposition 15. *Then mean size χ of the open cluster at the origin is an continuous function of a, when a is sufficiently small.*

Proof.

$$P(|\mathcal{C}| = n) = \sum_{\mathcal{C}} P(\mathcal{C})$$

, where the sum is over all the choices of size-n a-clusters at the origin, $P(\mathcal{C})$ is the probability of the event that all vertices in $\mathcal{C}$ have an a-configurations, while all the vertices on the boundary of $\mathcal{C}$ does not have an a-configuration. Hence

$$P(|\mathcal{C}| = n) = \sum_{\mathcal{C}} \sum b_{\mathcal{C}} P(\mathcal{C}, b_{\mathcal{C}}) \tag{7}$$

where $b_{\mathcal{C}}$ denote any possible configuration (either a b-configuration or a c-configuration at each vertex) on the boundary of C. By Lemma 14, for each fixed n, (7) is a finite sum. According to the dimer representation of the 1-2 model, $P(\mathcal{C}, b_{\mathcal{C}})$ is the same as the probability that an a-bisector edge is present in the perfect matching at all the vertices of $\mathcal{C}$, while a b-bisector edge, or a c-bisector edge is present in the perfect matching is present in the perfect matching at all the vertices on the boundary of $\mathcal{C}$. This is the product of edge weights, multiplied by the Pfaffian of a submatrix $K_{\mathcal{C}}^{-1}$ of the inverse Kasteleyn matrix, and $K_{\mathcal{C}}^{-1}$ is indexed by the endpoints of the specified a, b, c edges in the dimer graph. When the spectral curve does not intersect the unit torus $\mathbb{T}^2$, each entry of the inverse matrix is an analytic function in a. For fixed n, $K_{\mathcal{C}}^{-1}$ is a matrix of finite order, hence $\mathrm{Pf} K_{\mathcal{C}}^{-1}$ is analytic in a, so is $P(|C| = n)$, since there are only finitely many configurations for each fixed n, by Lemma 14. By definition,

$$\chi = \sum_{n=1}^{\infty} nP(|\mathcal{C}| = n),$$

which is the limit of a sequence of analytic functions. When $0 < a \leq a_0 < a_c$, there exists a positive number α, such that

$$nP(|\mathcal{C}| = n) \leq nP(|\mathcal{C}| \geq n) \leq e^{-\alpha n},$$

by Theorem 12. Hence the sequence of continuous functions converges uniformly in any closed interval , as a result, χ is analytic if a is sufficiently small such that the spectral curve does not intersect $\mathbb{T}^2$ and that the probability of a size-n a-cluster decays exponentially at the origin. □

Lemma 16. *Let $\mathcal{S}$ be a set of N vertices of $\mathbb{H}$,where N is a large integer. Then*

1. *assume $\frac{b}{c} = \gamma > 0$. Fix a, γ, when b, c are sufficiently small, there exists $\eta > 0$, such that*

$$P(no\ vertices\ in\ \mathcal{S}\ have\ a - configurations) \leq e^{-\eta N},$$

moreover, $\eta \to \infty$, as $b, c \to 0$;

2. *fix b, c, when a is sufficiently small, there exists $\beta > 0$, such that*

$$P(no\ vertices\ in\ \mathcal{S}\ have\ a - configurations) \leq e^{-\beta N},$$

moreover, $\beta \to \infty$, as $a \to 0$;

Proof. We prove Part 1 here; Part 2 is very similar. If no vertices in $\mathcal{S}$ have a-configurations, each vertex in $\mathcal{S}$ can have either a b-configuration or a c-configuration. Let Δ_N be the set of all such configurations on $\mathcal{S}$, then $|\delta_N| = 2^N$, moreover,

$$P(no\ vertices\ in\ \mathcal{S}\ have\ a - configurations) = \sum_{\mathcal{C} \in \Delta_N} P(\mathcal{C}).$$

Let $\mathcal{C}$ be an arbitrary fixed configuration in Δ_N. Without loss of generality, assume N is even and $\mathcal{S}$ contains $\frac{N}{2}$ black vertices and $\frac{N}{2}$ white vertices. We can label all the black vertices in $\mathcal{S}$ by $v_{i,j}$, where i, j are integers. Let $u_{i,j}^{(1)}$, $u_{i,j}^{(2)}$ be the endpoints of the corresponding edge of $\mathbb{H}_\Delta$ at $v_{i,j}$. Namely, if $v_{i,j}$ has a b-configuration (resp. c-configuration) in $\mathcal{C}$, then $u_{i,j}^{(1)u_{i,j}^2}$ is a b-edge (resp. c-edge). Then the configuration $\mathcal{C}$ occurs in the 1-2 model only if all the edges $e_{i,j} = u_{i,j}^{(1)} u_{i,j}^{(2)}$ are present in the dimer configuration of $\mathbb{H}_\Delta$. For simplicity, we denote these edges by $e_k = u_k^{(1)} u_k^{(2)}$ in lexicographical order, where $1 \leq k \leq \left[\frac{N}{2}\right]$. Define D to be a square matrix with rows and columns labeled by all the u_k^1, u_k^2's as follows

$$D(u_k^{(p)}, u_l^{(q)}) = \begin{cases} bK^{-1}(u_k^{(p)}, u_l^{(q)}) & If\ both\ e_k\ and e_l\ are\ b-edges \\ \sqrt{bc}K^{-1}(u_k^{(p)}, u_l^{(q)}) & If\ exactly\ one\ of\ e_k\ and\ e_l\ is\ a\ b-edge \\ cK^{-1}(u_k^{(p)}, u_l^{(q)}) & If\ both\ e_k\ and e_l\ are\ c-edges \end{cases}$$

Then according to the results in Section 3,

$$P(\mathcal{C}) \leq |\mathrm{Pf}D|$$

Note that $|D(u_k^{(1)}, u_k^{(2)})| = P_b, or P_c$, depending on whether the edge e_k is a b-edge or a c-edge. P_b (P_c) is the probability that a b-configuration (c-configuration) appears at a vertex. By definition, the probability measure on 1-2 model depends only on $a; b : c$, hence fixing a, γ, and letting $b, c \to 0$, is the same as fixing b, c, and letting $a \to \infty$. Since $P_a + P_b + P_c = 1$, by Lemma 14, we have

$$\lim_{b,c \to 0} P_b + P_c = 0$$

Moreover, since when we fix γ, a, and b, c are sufficiently small, the spectral curve has no zeros on $\mathbb{T}^2$, the entries $D(u, v)$ decay exponentially to 0 when $|u - v| \to \infty$. Namely,

$$|D(u, v)| \leq e^{-\beta|u-v|}, where\ \beta > 0, and\ |u - v|\ is\ large.$$

Now let us consider

$$\mathrm{PfD} = \sum_{\substack{\sigma \in S_N \\ \sigma(1) < \sigma(3) < \cdots < \sigma(N-1) \\ \sigma(2i-1) < \sigma(2i)}} sgn(\sigma) \prod_{i=1}^{\frac{N}{2}} D(w_{\sigma(2i-1)}, w_{\sigma(2i)}),$$

where S_N is the symmetric group, and $sgn(\sigma)$ is the sign of the permutation σ, and $w_{2i-1} = u_i^{(1)}$, $w_{2i} = u_i^{(2)}$. On the other hand, let V_0 denote the set of vertices in $\mathbb{H}_\Delta$, such that if $v \in V_0$, if v is not a vertex of $\mathbb{H}$, then v is not an endpoint of an a-edge, we have

$$\sum_{\substack{\sigma \in S_N \\ \sigma(1) < \sigma(3) < \cdots < \sigma(N-1) \\ \sigma(2i-1) < \sigma(2i)}} \left| \prod_{i=1}^{\frac{N}{2}} D(w_{\sigma(2i-1)}, w_{\sigma(2i)}) \right|$$

$$\leq \left(\sum_{w \in V_0} |D(w_1, w)| \right)^{\frac{N}{2}} \leq (C \max\{P_b, P_c\})^N,$$

where C is a constant independent of b, c and N, Let $\eta = -\log 2C \max\{P_b, P_c\}$, then $\eta \to \infty$, as $b, c \to 0$, and the lemma follows. □

Now we consider connected sets of vertices such that none of those vertices in the set has an a-configuration. We give such set of vertices a name $\bar{a}$-cluster.

Proposition 17. *Fix $b \neq c$, when a is small, the probability that an infinite $\bar{a}$-cluster appears at the origin is strictly positive.*

Proof. Consider $\mathcal{C}_{\bar{a}}$, the connected set of vertices of $\bar{a}$-configuration at the origin. We define boundary vertices of a connected vertex set $\mathcal{S}$ to be those vertices that are adjacent to vertices in $\mathcal{S}$ but not in $\mathcal{S}$ themselves. First of all, for any fixed integer k, there is a positive probability that the $k \times k$ box of $\bar{a}$-cluster centered at the origin appears. Let us call such an event $\mathcal{U}_k$. Conditional on $\mathcal{U}_k$, we consider the event that an infinite $\bar{a}$-cluster does not exist at the origin. The conditional probability is less than or equal to the probability that there is a set $\mathcal{B}$ outside the $k \times k$ box, of which all the vertices have an a-configuration, and can be considered as all the boundary vertices of a simply-connected set $\mathcal{S}$.

Let $\mathcal{T}$ be the dual lattice of the hexagonal lattice $\mathbb{H}$. For any simply connected set $\mathcal{S}$ including the $k \times k$ box centered at the origin, we have a connected set $T_{\mathcal{S}}$ of edges in $\mathcal{T}$, each of which is a dual edge of an edge in $\mathbb{H}$ connecting a vertex of $\mathcal{S}$ and a vertex not in $\mathcal{S}$. Since $\mathcal{B}$ is uniquely determined by $T_{\mathcal{S}}$, and the total number of connected set of $|T_{\mathcal{S}}|$ edges with distance to the origin at most $|T_{\mathcal{S}}|$, is less than $4|T_{\mathcal{S}}|^2\beta^{|T_{\mathcal{S}}|}$, where $\beta > 1$ is a constant depending only on the structure of the triangular lattice. As a result, the total number of $\mathcal{B}$, with fixed number of edges in $T_{\mathcal{S}}$, is bounded by

$$4|T_{\mathcal{S}}|^2\beta^{|T_{\mathcal{S}}|}$$

Obviously $|T_{\mathcal{S}}| \geq 2k$(Since any for simply connected region with area k^2, the perimeter is at least $2k$). Moreover, since the hexagonal lattice is a degree-3 graph and $\mathcal{S}$ is simply connected, we have $2|T_{\mathcal{S}}| \geq |\mathcal{B}| \geq \frac{|T_{\mathcal{S}}|}{2}$. Hence

$$\begin{aligned}
& P(\text{an infinite } \bar{\text{a}} - \text{cluster appears at the origin}) \\
\geq\; & P(\text{an infinite } \bar{\text{a}} - \text{cluster appears at the origin}|\mathcal{U}_k)P(\mathcal{U}_k) \\
=\; & [1 - P(\text{no infinite } \bar{\text{a}} - \text{cluster at the origin}|\mathcal{U}_k)]P(\mathcal{U}_k) \\
\geq\; & [1 - \sum_{h \geq 2k} \sum_{\{\mathcal{B}: |T_{\mathcal{S}}| = h\}} P(\text{all the vertices in } \mathcal{B} \text{ have a} - \text{configurations}|\mathcal{U}_k)]P(\mathcal{U}_k) \\
\geq\; & P(\mathcal{U}_k) - \sum_{h \geq 2k} \sum_{\{\mathcal{B}: |T_{\mathcal{S}}| = h\}} P(\text{all vertices in } \mathcal{B} \text{ have a} - \text{configurations}) \\
\geq\; & P(\mathcal{U}_k) - \sum_{\{h: h \geq 2k\}} 4h^2 e^{-\beta h}
\end{aligned}$$

where the last inequality follows from Lemma 12. For fixed k, when a is small, $P(\mathcal{U}_k)$ is continuous in a, and is 1 if $a = 0$. Hence we can choose a sufficiently small such that, $P(\mathcal{U}_k) > \frac{3}{4}$, and $e^{-\beta}4h^{\frac{2}{h}} < \frac{1}{2}$, then

$$P(\text{an infinite } \bar{\text{a}} - \text{cluster appears at the origin}) \geq \frac{3}{4} - \left(\frac{1}{2}\right)^{2k-1} > 0,$$

and the proposition follows. ∎

Proposition 18. *When a is sufficiently large, the probability that an infinite a-cluster appears at the origin is strictly positive.*

Proof. First of all, let Λ be a fixed finite set of vertices, and consider the probability that none of vertices in Λ have a-configurations. Since each vertex of Λ could either take a

b-configuration or a c-configuration, there are $2^{|\Lambda|}$ possible (dimer)configurations at most. For each configuration ξ_Λ, applying the same technique as in Theorem 8, we obtain

$$Pr(\xi_\Lambda) \leq (\max\{P_b, P_c\})^{|\Lambda|}$$

where $|\Lambda|$ is the total number of vertices in the finite set Λ. Given $\lim_{a\to\infty} P_a = 1$, and $P_a + P_b + P_c = 1$, we obtain $\lim_{a\to\infty} \max\{P_b, P_c\} = 0$. Then using a similar technique as in Proposition 17, Proposition 18 can be proved. □

Let $\mathcal{T}$ be the dual triangular lattice of the hexagonal lattice $\mathbb{H}$. An **a-interface** is a connected set of edges of $\mathcal{T}$, in which every edge separates a pair of vertices in $\mathbb{H}$; one has an a-configuration and the other does not have an a-configuration. The union of all interfaces on the plane (or on the torus) for the 1-2 model configuration forms a closed polygon configuration for the triangular lattice $\mathcal{T}$, i.e., at each vertex there are an even number of incident present edges, see Figure 5 - an illustration of interfaces for a 1-2 model configuration on a 3×3 torus.

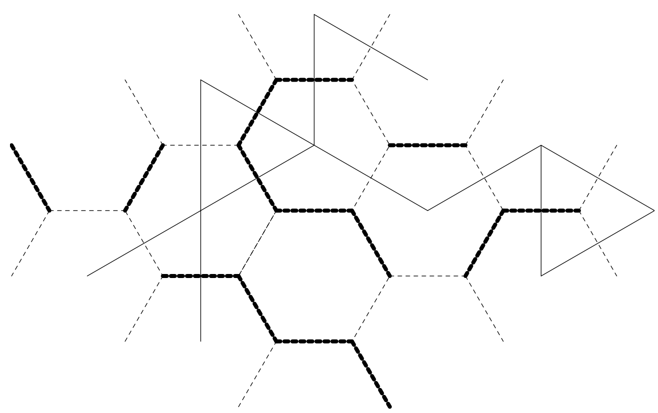

Figure 7: Interface

Lemma 19. *When a is sufficiently large, almost surely there are no infinite a-interfaces.*

Proof. Since the triangular lattice $\mathcal{T}$ has countably many vertices, by translation invariance, it suffices to prove that almost surely there are no infinite a-interfaces passing the origin, when a is sufficiently large. Let $\mathcal{S}_k$ be the family of all connected set of k edges of $\mathcal{T}$ passing the origin, and let $s_k \in \mathcal{S}_k$. We use **nearest vertices of** s_k to denote the endpoints of dual edges of edges in s_k. The nearest vertices of s_k are vertices of the hexagonal lattice which has either an a-, a b-, or a c-configuration. Let $\mathcal{C}_{s_k}$ be the set of all configurations on the nearest vertices of s_k such that s_k is an a-interface. Then we have

$$|\mathcal{C}_{s_k}| \leq 4^k,$$

because each pair of vertices v_1, v_2 separating by an edge of s_k have 4 possible configurations

$$\begin{cases} \mathcal{C}_{s_k}(v_1) = a \\ \mathcal{C}_{s_k}(v_2) = b \end{cases} \quad \begin{cases} \mathcal{C}_{s_k}(v_1) = a \\ \mathcal{C}_{s_k}(v_2) = c \end{cases} \quad \begin{cases} \mathcal{C}_{s_k}(v_1) = b \\ \mathcal{C}_{s_k}(v_2) = a \end{cases} \quad \begin{cases} \mathcal{C}_{s_k}(v_1) = c \\ \mathcal{C}_{s_k}(v_2) = a \end{cases},$$

and there are at most k pairs of independent vertices. Let $A_\mathcal{C}(\overline{A}_\mathcal{C})$ be the set of nearest vertices of $\mathcal{S}_k$ which have (do not have) a-configurations given the configuration $\mathcal{C}$. Then according to the structure of the triangular lattice we have,

$$\frac{k}{3} \leq |\overline{A}_\mathcal{C}| \leq k, \qquad \text{if } \mathcal{C} \in \mathcal{C}_{s_k}.$$

Therefore

$$\begin{aligned}
Pr(\mathrm{s_k\ is\ an\ a-interface}) &= Pr(\bigcup_{\mathcal{C}\in\mathcal{C}_{s_k}} \mathcal{C}) \leq \sum_{\mathcal{C}\in\mathcal{C}_{s_k}} Pr(\mathcal{C}) \\
&\leq \sum_{\mathcal{C}\in\mathcal{C}_{s_k}} Pr(\mathrm{None\ of\ vertices\ in\ \overline{A}_{\mathcal{C}}\ have\ a-configurations}) \\
&\leq 4^k(e^{-\eta})^{\frac{k}{3}}.
\end{aligned}$$

As a result,

$$\begin{aligned}
& Pr(\mathrm{there\ exists\ an\ infinite\ a-interface\ passing\ the\ origin}) \\
= & Pr(\bigcap_k \bigcup_{s_k\in\mathcal{S}_k} \mathrm{s_k\ is\ an\ a-interface\ passing\ the\ origin}) \\
\leq & \lim_{k\to\infty} \sum_{s_k\in\mathcal{S}_k} Pr(\mathcal{S}_\mathrm{k}\ \mathrm{is\ an\ a-interface}) \\
\leq & \lim_{k\to\infty} \sum_{s_k\in\mathcal{S}_k} 4^k(e^{-\eta})^{\frac{k}{3}}
\end{aligned}$$

However, $|\mathcal{S}_k| \leq C^k$, where C is a constant depending only on the structure of the triangular lattice. Fix b, c, $\eta \to \infty$, as $a \to \infty$, then the lemma follows. □

Theorem 20. *When a is sufficiently large, almost surely there exists exactly one infinite a-cluster; in particular, the two types of infinite a-clusters, $\{100\}$ and $\{011\}$ cannot coexist.*

Proof. First of all, we prove that conditional on the existence of infinite a-clusters, it is almost surely the case that the number of infinite a-clusters is exactly 1. Let $\mathcal{S}$ be an maximal infinite a-cluster, i.e., none of the vertices in $V(\mathbb{H}) \setminus \mathcal{S}$ adjacent to a vertex in $\mathcal{S}$ have a-configurations. Define $\partial\mathcal{S}$ be a set of edges of $\mathcal{T}$, each of which is the dual edge of an edge of $\mathbb{H}$ connecting a vertex in $\mathcal{S}$ and a vertex not in $\mathcal{S}$. Then $\partial\mathcal{S}$ is an a-interface by definition. By lemma 15, $\partial\mathcal{S}$ has no infinite connected components, hence $V(\mathbb{H}) \setminus \mathcal{S}$ is a union of disconnected finite sets. Hence if infinite a-clusters exists, there is exactly one infinite a-cluster almost surely.

Assume infinite a-clusters do not exist. When a-is sufficiently large, there are no infinite $\overline{a}$-clusters either. Moreover, there are no infinite connected interface. Let $\mathcal{I}_0$ be a finite maximal connected a-interface passing the origin, i.e. all the edges in $E(\mathcal{T}) \setminus \mathcal{I}_0$, sharing an endpoint with an edge in $\mathcal{I}_0$ is not an a-interface. Let $F_{\mathcal{I}_0}$ be the set of all faces of $\mathbb{H}$ intersecting $\mathcal{I}_0$. Let $E_{\mathcal{I}_0}$ be the edges of $F_{\mathcal{I}_0}$ which do not intersect $\mathcal{I}_0$. On each connected component of $E_{\mathcal{I}_0}$, either all the vertices have the configuration a, or none of the vertices have the configuration a. Choose a connected component of $E_{\mathcal{I}_0}$ surrounding $\mathcal{I}_0$ from outside, denoted by A_0. Because there are no infinite a-clusters, $\overline{a}$-clusters or a-interfaces, there exists a finite connected a-interface $\mathcal{I}_1$, surrounding A_0 from outside. Repeating the process we can find a sequence of a-interfaces $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \cdots$, such that $\lim_{k\to\infty} |\mathcal{I}_k| = \infty$. Hence when a is sufficiently large we have

$$\begin{aligned}
& Pr(\mathrm{infinite\ a-clusters\ do\ not\ exist}) \\
\leq & Pr(\bigcup_{v_0\in V(\mathcal{T})} \bigcap_n \bigcup_{k\geq n} \{\mathrm{there\ exists\ an\ interface\ of\ size\ k\ surrounding\ v_0}\}) \\
\leq & \sum_{v_0:v_0\in V(\mathcal{T})} \lim_{n\to\infty} \sum_{k\geq n} Pr(\mathrm{there\ exists\ an\ interface\ of\ size\ k\ surrounding\ v_0}).
\end{aligned}$$

It is trivial to check that the right side is 0 using the technique as in Proposition 17.

□

Remark. Corollary 13 and Theorem 20 imply that the system undergoes a phase transition. Figure 5 is a picture for 1-2 model configurations with large a. It is obtained using the Markov chain Morte Carlo simulation with parameter $a = 5, b = c = 1$.

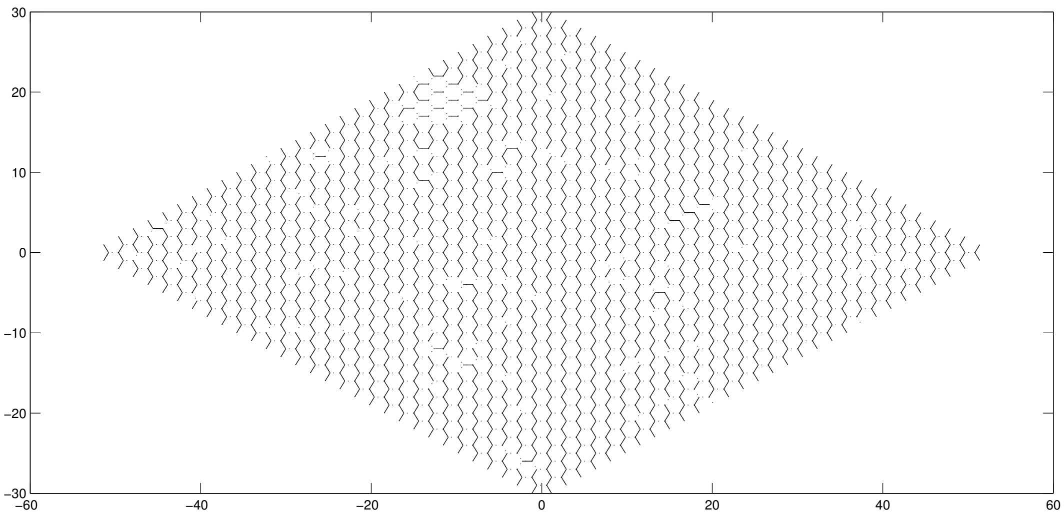

Figure 8: Large a configuration

6 Law of Large Numbers

In this section, we prove a law of large numbers regarding the number of a-configurations on the hexagonal graph embedded into an $n \times n$ torus. Let $\mathcal{A}_n$ be the random variable denoting the mean number of a-configurations on the $n \times n$ torus, minus its expectation. Let μ_n be the measure on the $n \times n$ torus. Let $P_{a,n}$ denote the probability that an a-configuration appears at a vertex of the $n \times n$ torus, namely,

$$\mathcal{A}_n = \frac{\mathcal{N}_a}{2n^2} - P_{a,n}.$$

We have the following lemma

Lemma 21. *Let ϵ be an arbitrary number in $(0, 1)$, then*

$$\mu_n(|\mathcal{A}_n| \geq \epsilon) \leq \frac{C}{\epsilon^2 n^2}.$$

where $C > 0$ is a constant independent of ϵ and n.

Proof. According to the Chebyshev's inequality,

$$\mu_n(|\mathcal{A}_n| \geq \epsilon) \leq \frac{\mathbb{E}(\mathcal{N}_a - 2n^2 P_{a,n})^2}{4\epsilon^2 n^4} = \frac{1}{4\epsilon^2 n^4}\mathbb{E}[\sum_{k=1}^{2n^2}(1_{a,k} - P_{a,n})]^2$$

where $1_{a,k}$ is the indicator that the kth vertex has an a-configuration. Hence we have

$$\mu_n(|\mathcal{A}_n| \geq \epsilon) \leq \frac{1}{4\epsilon^2 n^4}\left[\sum_{k=1}^{2n^2}\mathbb{E}(1_{a,k}-P_{a,n})^2 + \sum_{1\leq j,\ell\leq 2n^2, j\neq \ell}\mathbb{E}(1_{a,j}-P_{a,n})(1_{a,\ell}-P_{a,n})\right].$$

The first term

$$\sum_{k=1}^{2n^2}\mathbb{E}(1_{a,k}-P_{a,n})^2 = 2n^2(P_{a,n}-P_{a,n}^2).$$

By translation invariance, the second term

$$\sum_{1\leq j,\ell\leq 2n^2, j\neq\ell}\mathbb{E}(1_{a,j}-P_{a.n})(1_{a,\ell}-P_{a,n}) = 2n^2\sum_{j=2}^{2n^2}\mathbb{E}(1_{a,1}-P_{a,n})(1_{a,j}-P_{a,n})$$

Next we will prove that $\sum_{j=2}^{2n^2}\mathbb{E}(1_{a,1}-P_{a,n})(1_{a,j}-P_{a,n})$ is bounded by a uniform constant independent of n. We divide the $n\times n$ torus into n^2 fundamental domains, each of which contains two vertices, one black (v_b) and one white (v_w), as illustrated in Figure 4. We use $1_{a,w}^{(i,j)}$ ($1_{a,b}^{(i,j)}$) to denote the indicator of the event that an a-configuration appears at the white (black) vertex of the fundamental domain (i,j). Then

$$\begin{aligned}\sum_{j=2}^{2n^2}\mathbb{E}(1_{a,1}-P_{a,n})(1_{a,j}-P_{a,n}) &= \sum_{(i,j)\in[0,n-1]\times[0,n-1]}\mathbb{E}(1_{a,b}^{(0,0)}-P_{a,n})(1_{a,w}^{(i,j)}-P_{a,n})\\ &+\sum_{(i,j)\in[0,n-1]\times[0,n-1],(i,j)\neq(0,0)}\mathbb{E}(1_{a,b}^{(0,0)}-P_{a,n})(1_{a,b}^{(i,j)}-P_{a,n})\end{aligned}$$

Moreover, assume $u_b^{(i,j)}, v_b^{(i,j)}$ ($u_w^{(i,j)}, v_w^{(i,j)}$) are end vertices of the bisector edge with weight a at the black (white) vertex of the (i,j) fundamental domain. If we label the weighted adjacency matrix such that the starting vertex of each bisector edge always has a smaller index than the ending vertex of the bisector edge, we have

$$\begin{aligned}&\mathbb{E}(1_{a,b}^{(0,0)}-P_{a,n})(1_{a,w}^{(i,j)}-P_{a,n})\\ =\ & a^2(-1)^{|u_b^{(0,0)}|+|v_b^{(0,0)}|+|u_w^{(i,j)}|+|v_w^{(i,j)}|}\left[\frac{\sum_{\theta,\tau\in\{0,1\}}(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}\tilde{K}_n((-1)^\theta,(-1)^\tau)}{\sum_{\theta,\tau\in\{0,1\}}(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)}\right.\\ &\left.-\frac{(\sum_{\theta,\tau\in\{0,1\}}(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}\hat{K}_n((-1)^\theta,(-1)^\tau))\cdot(\sum_{\theta,\tau\in\{0,1\}}(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}K_n^*((-1)^\theta,(-1)^\tau))}{(\sum_{\theta,\tau\in\{0,1\}}(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau))^2}\right]\end{aligned}$$

where

$$\kappa(\theta,\tau)=\begin{cases}1 & \text{if } (\theta,\tau)=(0,0)\\ 0 & \text{otherwise}\end{cases},$$

and $\tilde{K}_n((-1)^\theta,(-1)^\tau)$ ($\hat{K}_n((-1)^\theta,(-1)^\tau)$, $K_n^*((-1)^\theta,(-1)^\tau)$) is the submatrix of $K_n((-1)^\theta,(-1)^\tau)$ by removing rows and columns indexed by endpoints of a-edges at vertices $b_{(0,0)}$ and $w_{(i,j)}$(the vertex $b_{(0,0)}$, the vertex $w_{(i,j)}$). Let Z_n be the partition function of the dimer model on the $n\times n$ torus, then

$$\sum_{\theta,\tau\in\{0,1\}}(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau) = 2Z_n$$

If $\det K_1(z,w)$ has no zeros on the unit torus $\mathbb{T}^2$,

$$\begin{aligned}
&\frac{\mathrm{Pf}\tilde{K}_n((-1)^\theta,(-1)^\tau)}{\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)}\\
= &(-1)^{|u_b^{(0,0)}|+|v_b^{(0,0)}|+|u_w^{(i,j)}|+|v_w^{(i,j)}|}\mathrm{Pf}K^{-1}\left(\begin{array}{cccc} u_b^{(0,0)} & v_b^{(0,0)} & u_w^{(i,j)} & v_w^{(i,j)} \\ u_b^{(0,0)} & v_b^{(0,0)} & v_w^{(i,j)} & v_w^{(i,j)}\end{array}\right)((-1)^\theta,(-1)^\tau)
\end{aligned}$$

and

$$\frac{\mathrm{Pf}\hat{K}_n((-1)^\theta,(-1)^\tau)}{\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)}=(-1)^{|u_b^{(0,0)}|+|v_b^{(0,0)}|}\mathrm{Pf}K_n^{-1}\left(\begin{array}{cc} u_b^{(0,0)} & v_b^{(0,0)} \\ u_b^{(0,0)} & v_b^{(0,0)}\end{array}\right)((-1)^\theta,(-1)^\tau)$$

and

$$\frac{\mathrm{Pf}K_n^*((-1)^\theta,(-1)^\tau)}{\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)}=(-1)^{|u_w^{(i,j)}|+|v_w^{(i,j)}|}\mathrm{Pf}K_n^{-1}\left(\begin{array}{cc} u_w^{(i,j)} & v_w^{(i,j)} \\ v_w^{(i,j)} & v_w^{(i,j)}\end{array}\right)((-1)^\theta,(-1)^\tau).$$

By translation invariance, for any (i,j),

$$\mathrm{Pf}K_n^{-1}\left(\begin{array}{cc} u_b^{(0,0)} & v_b^{(0,0)} \\ u_b^{(0,0)} & v_b^{(0,0)}\end{array}\right)((-1)^\theta,(-1)^\tau)=\mathrm{Pf}K_n^{-1}\left(\begin{array}{cc} u_w^{(i,j)} & v_w^{(i,j)} \\ u_w^{(i,j)} & v_w^{(i,j)}\end{array}\right)((-1)^\theta,(-1)^\tau):=t_{n,\theta,\tau}$$

As a result,

$$\begin{aligned}
&\sum_{(i,j)}\mathbb{E}(1_{a,b}^{(0,0)}-P_{a,n})(1_{a,w}^{(i,j)}-P_{a,n})\\
= \ &a^2\sum_{(i,j)}\sum_{\theta,\tau}\frac{(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)}{2Z_n}t_{n,\theta,\tau}^2-\left[\sum_{\theta,\tau}\frac{(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)}{2Z_n}t_{n,\theta,\tau}\right]^2\\
&-\sum_{\theta,\tau}\frac{(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)}{2Z_n}\det K_n^{-1}\left(\begin{array}{cc} u_b^{(0,0)} & v_b^{(0,0)} \\ u_w^{(i,j)} & v_w^{(i,j)}\end{array}\right)((-1)^\theta,(-1)^\tau)
\end{aligned}$$

When the spectral curve has no zeros on the unit torus $\mathbb{T}^2$, there exists a constant $-\infty<t<\infty$, such that for any $\theta,\tau\in\{0,1\}$,

$$t_{n,\theta,\tau}=t+O\left(\frac{1}{n^2}\right)$$

As a result,

$$\left|a^2\sum_{(i,j)}\sum_{\theta,\tau}\frac{(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)}{2Z_n}t_{n,\theta,\tau}^2-\left[\sum_{\theta,\tau}\frac{(-1)^{\kappa(\theta,\tau)}\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)}{2Z_n}t_{n,\theta,\tau}\right]^2\right|\leq C$$

where $C > 0$ is a constant. Moreover,

$$\sum_{i,j} \det K_n^{-1} \begin{pmatrix} u_b^{(0,0)} & v_b^{(0,0)} \\ u_w^{(i,j)} & v_w^{(i,j)} \end{pmatrix} ((-1)^\theta, (-1)^\tau)$$

$$= \sum_{i,j} K_n^{-1}(u_b^{(0,0)}, u_w^{(i,j)}) K_n^{-1}(v_b^{(0,0)}, v_w^{(i,j)}) - K_n^{-1}(u_b^{(0,0)}, v_w^{(i,j)}) K_n^{-1}(v_b^{(0,0)}, u_w^{(i,j)})$$

$$= \frac{1}{n^4} \sum_{i,j} [\sum_{z,w,z^n=(-1)^\theta, w^n=(-1)^\tau} z^i w^j K_1^{-1}(u_b^{(0,0)}, u_w^{(0,0)})][\sum_{z,w,z^n=(-1)^\theta, w^n=(-1)^\tau} z^i w^j K_1^{-1}(v_b^{(0,0)}, v_w^{(0,0)})]$$

$$-[\sum_{z,w,z^n=(-1)^\theta, w^n=(-1)^\tau} z^i w^j K_1^{-1}(u_b^{(0,0)}, v_w^{(0,0)})][\sum_{z,w,z^n=(-1)^\theta, w^n=(-1)^\tau} z^i w^j K_1^{-1}(v_b^{(0,0)}, u_w^{(0,0)})] \mid_{(z,w)}$$

$$= \frac{1}{n^4} \sum_{z_1,z_2,w_1,w_2,z_1^n=z_2^n=(-1)^\theta, w_1^n=w_2^n=(-1)^\tau} \sum_{i,j} (z_1 z_2)^i (w_1 w_2)^j [K_1^{-1}(u_b, u_w)(z_1, w_1) K_1^{-1}(v_b, v_w)(z_2, w_2)$$

$$-K_1^{-1}(u_b, v_w)(z_1, w_1) K_1^{-1}(v_b, u_w)(z_2, w_2)]$$

If $z_1 z_2 \neq 1$, $\sum_i (z_1 z_2)^i = 0$. As a result,

$$\sum_{i,j} \det K_n^{-1} \begin{pmatrix} u_b^{(0,0)} & v_b^{(0,0)} \\ u_w^{(i,j)} & v_w^{(i,j)} \end{pmatrix} ((-1)^\theta, (-1)^\tau) \leq C$$

Similarly we can prove that

$$\sum_{(i,j)\neq(0,0)} \mathbb{E}(1_{a,b}^{(0,0)} - P_{a,n})(1_{a,b}^{(i,j)} - P_{a,n}) \leq C,$$

and we proved the lemma when the spectral curve does not intersect the unit torus $\mathbb{T}^2$.

If the spectral curve intersects the unit torus $\mathbb{T}^2$ at $(1,1)$ with multiplicity 2, it is proved in [2] that

$$\frac{\mathrm{Pf}\hat{K}_n(1,1)}{Z_n} = O\left(\frac{1}{n}\right)$$

and the same estimate holds for $\frac{\mathrm{Pf}\tilde{K}_n(1,1)}{Z_n}$ and $\frac{\mathrm{Pf}K_n^*(1,1)}{Z_n}$. Using the same technique as before to estimate the terms in $\mathrm{Pf}K_n^{-1}((-1)^\theta, (-1)^\tau)$ for $(\theta,\tau) \neq (0,0)$, we can derive that the same result holds also in the case that the spectral curve intersects $\mathbb{T}^2$ at $(1,1)$ with multiplicity 2. □

Theorem 22. *There exists a positive constant P_a, such that if $\mathcal{N}_{a,n}$ is the number of a-configurations on the $n \times n$ torus, then*

$$\lim_{n\to\infty} \frac{\mathcal{N}_{a,n}}{2n^2} = P_a, \qquad \text{in probability}$$

Moreover, P_a is increasing in a.

Proof. The law of large numbers follows from Lemma 21 and the fact that $\lim_{n\to\infty} P_{a,n} = P_a$. For each fixed n, $P_{a,n}$ is increasing in a because by translation invariance, $P_{a,n} = \frac{\mathbb{E}\mathcal{N}_a}{n^2}$, and $\mathbb{E}\mathcal{N}_a$ is increasing in a by Proposition 3. As a result, P_a is increasing a as the limit of $P_{a,n}$ when $n \to \infty$. □

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