Spectral curves of periodic Fisher graphs

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Abstract

We study the spectral curves of dimer models on periodic Fisher graphs, obtained from periodic ferromagnetic Ising models on $\mathbb{Z}^2$. The spectral curve is defined by the zero locus of the determinant of a modified weighted adjacency matrix. We prove that either they are disjoint from the unit torus ($\mathbb{T}^2 = \{(z,w) : |z| = 1, |w| = 1\}$) or they intersect $\mathbb{T}^2$ at a single real point.

1 Introduction

A Fisher graph is a graph obtained from the hexagonal lattice by replacing each vertex with a triangle, see Figure 1.1. A perfect matching, or a dimer configuration on a graph is a choice of subset of edges satisfying the condition that each vertex is incident to exactly one edge in the subset. In this paper we study perfect matchings on Fisher graphs. The name "Fisher graph" comes from the correspondence between perfect matchings on the graph and the Ising configurations on $\mathbb{Z}^2$, discovered by Fisher in [5]. Ever since then, perfect matchings on Fisher graphs have been studied extensively by mathematicians and physicists as a method to study the celebrated 2D Ising model, see [18]. However, previous work focuses on graphs with homogeneous edge weights, yet in this paper, we study the Fisher graphs with periodic edge weights.

A cylindrical Fisher graph is a Fisher graph finite in one direction, and periodic in the perpendicular direction, so that the graph can be embedded into a cylinder. A toroidal graph is a bi-periodic graph, which can be embedded into a torus. We can also consider a toroidal graph as the quotient graph of an infinite bi-periodic planar graph under the action of the translation group, isomorphic to $\mathbb{Z}^2$.

To an edge-weighed, cylindrical (resp. toroidal) Fisher graph, one associates its spectral curve $P(z) = 0$ (resp. $P(z,w) = 0$). The real polynomial $P(z)$ (resp. $P(z,w)$) defining the spectral curve arises as the determinant of a modified weighted adjacency matrix of the graph.

The study of spectral curve for periodic Fisher graphs (which are non-bipartite), is partially inspired by the work of Kenyon, Okounkov and Sheffield [12, 10]. They prove that the spectral curves of bipartite dimer models with positive edge weights are always real curves of a special type, namely they are Harnack curves. Harnack curves are a class of algebraic curves which maximize the area of the amoeba among

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all curves with the same degree (see [19, 20]). Namely, If $f : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial, then its zero set in $(\mathbb{C}^*)^2$ is a curve $A = f^{-1}(0) \cap (\mathbb{C}^*)^2$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The amoeba $\mathcal{A} \setminus \mathbb{R}^2$ of f is $Log(A)$, where $Log : (\mathbb{C}^*)^2 \to \mathbb{R}^2$, $(z_1, z_2) \to (log|z_1|, log|z_2|)$. The surprising connection between spectral curves of bipartite dimer models and Harnack curves implies many qualitative and quantitative results about the phase transition of bipartite dimer models.

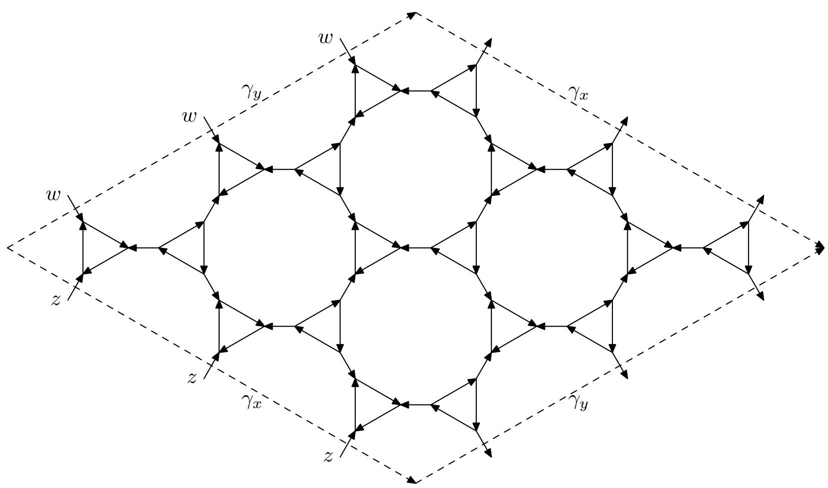

Figure 1.1: Fisher graph

A **planar graph** is one which can be embedded into the plane so that edges intersect only at vertices. Let $G = (V, E)$ be a planar graph. A **weight function** $\eta : E \to \mathbb{R}^+$ assigns each edge $e \in E$ a positive real number $\eta(e)$ (weight). An **edge-weighted planar graph** is a planar graph associated with a weight function $\eta : E \to \mathbb{R}^+$.

Now assume $G = (V, E)$ is an edge-weighted planar graph. Fix an embedding of G on the plane. A **clockwise-odd orientation** of G is an orientation of the edges, so that when each face (except the outer face) is traversed clockwise, an odd number of edges point along the given orientation. For a planar graph, such an orientation always exists, see [9]. The Kasteleyn matrix corresponding to G, with a given clock-wise odd orientation, is a $|V(G)| \times |V(G)|$ skew-symmetric matrix K defined by

$$K_{u,v} = \begin{cases} \eta(uv) & \text{if } u \sim v \text{ and } u \to v \\ -\eta(uv) & \text{if } u \sim v \text{ and } u \leftarrow v \\ 0 & \text{otherwise.} \end{cases}$$

where $\eta(uv) > 0$ is the weight associated to the edge uv.

Now let $G = (V, E)$ be a $\mathbb{Z}^2$-periodic edge-weighted planar graph. By this we mean that G can be embedded into the plane, so that there exists a subgroup Γ of the automorphism group of G (Aut(G)) satisfying

1. $\Gamma \simeq \mathbb{Z}^2$;

2. for any $\gamma \in \Gamma$, $e_1, e_2 \in E$, if $\gamma(e_1) = e_2$, then $\eta(e_1) = \eta(e_2)$.

In other words, each element in Γ is a weight preserving isomorphism of G. For simplicity, we will identify Γ and $\mathbb{Z}^2$, and consider $\mathbb{Z}^2$ as a subgroup of $\mathrm{Aut}(G)$. We assume elements of $\mathbb{Z}^2$ act on G by translations along the northeastern-southwestern direction and the northwestern-southeastern direction.

A period of G, or a fundamental domain of G, is a subset of edges $E_0 \subseteq E$, so that for all $e \in E$, there exist $e_0 \in E_0$ and $\gamma \in \Gamma$, satisfying $\gamma(e) = e_0$. Since we require Γ to be weight-preserving, the smallest period of an edge-weighted planar graph is in general larger than that of the corresponding unweighted graph.

Let $G_n = (V_n, E_n)$ be the quotient graph $G/(n\mathbb{Z} \times n\mathbb{Z})$. Namely, G_n is defined as follows. First of all, we define an equivalence relation on V. For $v_1, v_2 \in V$, v_1 is equivalent to v_2 if and only if there exists $(s,t) \in \mathbb{Z}^2$, so that $v_2 = (sn, tn)v_1$. In other words, if we translate the vertex v_1 by sn units along the horizontal direction, and by tn units along the vertical direction, we obtain the vertex v_2. To equivalence classes $\overline{v}_1$, $\overline{v}_2$ are incident in G_n if and only if there exists $v_1 \in \overline{v}_1 \subseteq V$, $v_2 \in \overline{v}_2 \subseteq V$, and v_1 is incident to v_2 in G.

The graph G_n is a finite graph, which can be embeded on a torus (toroidal graph). Let $\gamma_{x,n}$ (resp. $\gamma_{y,n}$) be a path in the dual graph of G_n winding once around the torus horizontally (resp. vertically). We can also consider $\gamma_{x,n}$ and $\gamma_{y,n}$ to be two homology generators of the torus. Let E_H (resp. E_V) be the set of edges crossed by $\gamma_{x,n}$ (resp. $\gamma_{y,n}$). For simplicity, we will also use γ_x (resp. γ_y) to denote $\gamma_{x,n}$ (resp. $\gamma_{y,n}$) throughout this paper when there is no ambiguity.

We give a **crossing orientation** for the toroidal graph G_n as follows. We orient all the edges of G_n clockwise-odd except those in $E_H \cup E_V$. This is possible since no other edges are crossing. Then we orient the edges of E_H clockwise-odd as if E_V did not exist. Again this is possible since $G - E_V$ is planar. To complete the orientation, we also orient the edges of E_V clockwise-odd as if E_H did not exist.

Let K_1 be the Kasteleyn matrix for the graph G_1. Given any parameters $z, w \in \mathbb{C}$, we construct a matrix $K(z,w)$ as follows. Let $\gamma_{x,1}$ and $\gamma_{y,1}$ be the paths introduced above. If an edge $uv \in E_H$ (resp. E_V) multiply $K_{u,v}$ by z (resp. w) if the orientation on that edge is from u to v, and multiply $K_{u,v}$ by $\frac{1}{z}$ (resp. $\frac{1}{w}$) if the orientation is from v to u. Define the **characteristic polynomial** $P(z,w) = \det K(z,w)$. The **spectral curve** is defined to be the zero locus $\{(z,w) \in \mathbb{C}^2 : P(z,w) = 0\}$.

Since each edge in G_1 represents a class of edges in the infinite, $\mathbb{Z}^2$-periodic, edge-weighted graph G, those edges with weights multiplied by z or $\frac{1}{z}$ in G_1 have corresponding edges in G and similar for those edges in G_1 with weights multiplied by w or $\frac{1}{w}$.

We also discuss cylindrical graphs in this paper. Let F be an infinite $\mathbb{Z}^2$-periodic, edge-weighted Fisher graph, and F_1 be the corresponding quotient graph. We consider a $\mathbb{Z}$-periodic subgraph of F, so that the subgraph is periodic in the direction parallel to edges of F whose projections to F_1 have weights multiplied by z or $(\frac{1}{z})$, finite in the direction parallel to edges of F whose projections to F_1 have weights multiplied by w or $\frac{1}{w}$. The quotient graph of this subgraph under the action of $\mathbb{Z}$ can be embedded into a cylinder, and is called a cylindrical graph. An example of a cylindrical graph is shown in Figure 1.2, in which we identify the two dashed lines along northeastern boundary and southwestern boundary to make a cylindrical graph. In other words, each edge crossed by the γ_x at the northeastern boundary has an identical copy drawn on the southwestern boundary crossed by γ_x; it is the same edge in the cylindrical graph. Similarly in Figure 1.1, we identify the northeastern dashed line and the southwestern dashed line, and also we identify the northwestern dashed line and southeastern dashed line, to make a toroidal graph (a graph embedded into a torus).

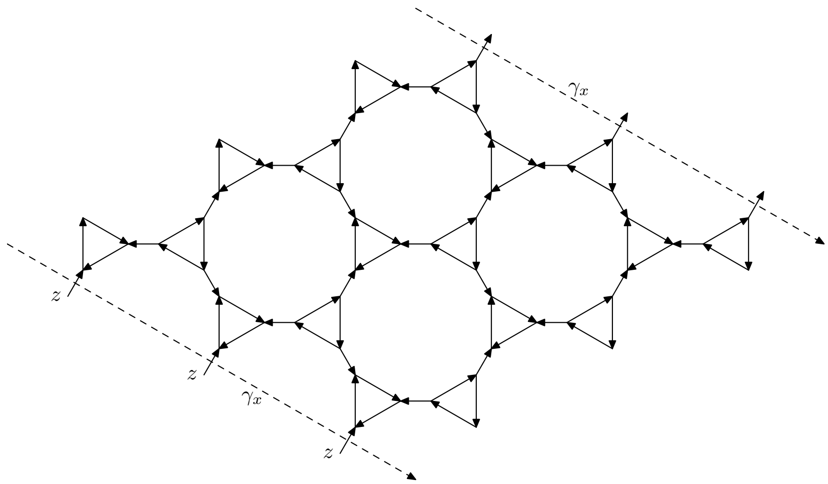

Figure 1.2: Fisher graph on a cylinder

Assume the size of the cylindrical graph is $m \times n$, meaning that it has width m with respect to z, but period n with respect to w. We embed this graph into a cylinder. Let $K(z)$ be the corresponding weighted adjacency matrix with an orientation shown in Figure 2. $K(z)$ is obtained from $K(z, w)$ by giving all edges crossed by γ_y weight 0. Let $P(z) = \det K(z)$.

Kenyon and Okounkov ([12]) proved that the spectral curves of periodic bipartite graphs with positive edge weights are **Harnack curves** [19, 20], whose intersection with $\mathbb{T}_{a,b} = \{(z, w) : |z| = e^a, |w| = e^b\}$ can only be: *i*. no intersection; *ii*. a pair of conjugate points, each of which is of multiplicity 1; *iii*. a single real zero of multiplicity 2 (real node). For the non-bipartite Fisher graph, we prove the following theorems

Theorem 1.1. *Consider a positively-weighted cylindrical Fisher graph, which is periodic in one direction, and finite in the other direction. Its quotient graph under the translation can be embedded into a cylinder, as illustrated in Figure 1.2. The intersection of $P(z) = 0$ and $\mathbb{T}$ is either empty or a single real point.*

In particular, if the circumference of the cylinder is even, the only possible intersection of $P(z) = 0$ with the unit circle $\mathbb{T}$ is 1.

In Figure 1.1, each edge of the graph is either a side of the triangle (triangular edge), or an edge connecting different triangles (non-triangle edge). We call all the horizontal non-triangle edges in Figure 1.1, *a*-edges.

Theorem 1.2. *Consider a bi-periodic Fisher graph, given edge-orientations as illustrated in Figure 1.1. Assume the edge weights satisfy the following conditions*

1. *all the edge weights are positive;*

2. *all the triangular edges have weight 1;*

3. *around each triangle, there are an even number of non-triangular edges with weights in* $(0,1)$*;*

then the only possible intersection of $P(z,w)=0$ *with the unit torus* $\mathbb{T}^2=\{(z,w): |z|=1,|w|=1\}$ *is a single real point of multiplicity 2.*

The proof of Theorem 1.2 is, first to prove that $P(z,w)\geq 0$, then use a known correspondence ([4]) between dimers on the Fisher graph and dimers on a bipartite graph, as well as the result in [12] that the spectral curve of dimers on any bipartite graph with positive edge weights is Harnack. Note also that in [1], the authors studied the special case when the spectral curve rises from an isoradial, Z-invariant Ising model, and proved that it is Harnack, again by making connections with the spectral curve of bipartite dimer model.

Note that the restrictions of edge weights in Theorem 1.2, namely, Conditions 1-3, correspond to ferromagnetic Ising weights. More precisely, we can construct an Ising model on the triangular lattice, in which we place a spin at each non-triangular face of the Fisher graph; two spins are connected by a bond if the two non-triangular faces are adjacent across a non-triangular edge. There is a two-to-one correspondence between the Ising spin configurations on the triangular lattice and the dimer configurations on the Fisher graph, i.e., a non-triangular edge is present if and only if one of the following cases is true

1. the non-triangular edge separates opposite Ising spins and has weight in $(0,1)$;

2. the non-triangular edge separates Ising spins with the same sign, and has weight in $[1,\infty)$.

For each non-triangular edge with weight in $(0,1)$, we have

$$\eta_e=\exp(-2J_e);$$

and for each non-triangular edge with weight in $[1,\infty)$, we have

$$\eta_e=\exp(2J_e);$$

where η_e is the dimer edge weight for an edge e, and J_e is the Ising interaction for the edge e. It is not hard to see that if we require that the edge weights of the Fisher graph satisfy Conditions 1-3 as in Theorem 1.2, the dimer configurations correspond to ferromagnetic Ising configurations, i.e. the Ising interaction $J_e\geq 0$ on each edge e, and around each triangle, the spins change signs for an even number of times.

Our result has many applications on various statistical mechanical models. Firstly, it leads to important properties of the dimer model on cylindrical graphs, such as weak convergence of Boltzmann measures and convergence rate of correlations for an infinite, periodic graph with finite width along one direction, see Proposition 3.4. Secondly, since the dimer model on the Fisher graph is closely related to the Ising model

[18] and the vertex model [15], our result leads to a quantitative characterization of the critical temperature of the arbitrary periodic ferromagnetic, two-dimensional Ising model, as the solution of an algebraic equation [16]. Thirdly, Fisher graphs and spectral curves are also important in studying other statistical mechanical models, such as the self-avoiding walk [7, 6], and the 1-2 model [17].

The outline of the paper is as follows. In Sect. 2, we explain the connection between the Kasteleyn operator, characteristic polynomial, spectral curve with the dimer model. In Sect. 3, we prove Theorem 1.1, as well as a phase transition characterized by the decay rate of edge-edge correlation for the dimer model on the cylindrical Fisher graph, resulting from Theorem 1.1. In Sect. 4, we prove Theorem 1.2. The proof consists of two critical components, one is that $P(z, w) \geq 0$ for any $(z, w) \in \mathbb{T}^2$, the other is an explicit correspondence between the Kasteleyn operator on the Fisher graph, and the Kasteleyn operator in the square-octagon lattice, introduced in [4].

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2 Background

A **perfect matching**, or a **dimer cover**, of a graph is a collection of edges with the property that each vertex is incident to exactly one edge. A graph is **bipartite** if the vertices can be 2-colored, that is, colored black and white so that black vertices are adjacent only to white vertices and vice versa.

To a weighted finite graph $G = (V, E, \eta)$, the weight $\eta : E \to \mathbb{R}^+$ is a function from the set of edges to positive real numbers. We define a probability measure, called the **Boltzmann measure** μ with sample space the set of dimer covers. Namely, for a dimer cover D

$$\mu(D) = \frac{1}{Z} \prod_{e \in D} \eta(e)$$

where the product is over all edges present in D, and Z is a normalizing constant called the **partition function**, defined to be

$$Z = \sum_{D} \prod_{e \in D} \eta(e),$$

the sum over all dimer configurations of G.

If we change the weight function η by multiplying the edge weights of all edges incident to a single vertex v by the same constant, the probability measure defined above does not change. So we define two weight functions η, η' to be **gauge equivalent** if one can be obtained from the other by a sequence of such multiplications.

The key objects used to obtain explicit expressions for the dimer model are **Kasteleyn matrices**. They are weighted, oriented adjacency matrices of the graph G defined as in Sect. 1.

Before discussing how to compute the partition function and local statistics of perfect matchings via the Kasteleyn matrix, we first define the Pfaffian of an anti-symmetric matrix. Let $A = \{a_{ij}\}$ be a $2n \times 2n$ anti-symmetric matrix, (i.e., $a_{ij} = -a_{ji}$, for $1 \leq i, j \leq 2n$). The Pfaffian of A is defined as follows

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1), \sigma(2i)}.$$

Moreover, it is not hard to check that when A is a $2n \times 2n$ anti-symmetric matrix,

$$\det(A) = [\mathrm{Pf}(A)]^2,$$

where $\det(A)$ is the determinant of A.

It is known [9] that for a planar graph with a clockwise odd orientation, the partition function of dimers satisfies

$$Z = \sqrt{\det K}.$$

Given a Fisher graph with an orientation as illustrated in Figure 1, the quotient graph can be embedded into an $n \times n$ torus. When n is even, if we reverse the orientations of all the edges crossed by $\gamma_{x,n}$ and all the edges crossed by $\gamma_{y,n}$, the resulting orientation is a crossing orientation. For $\theta, \tau \in \{0, 1\}$, given the orientation as in Figure 1, let $K_n^{\theta,\tau}$ be the Kasteleyn matrix K_n in which the weights of edges in E_H are multiplied by $(-1)^\theta$, and those in E_V are multiplied by $(-1)^\tau$. Using the result proved in [8, 21], we can derive that when n is even, the partition function Z_n of the graph G_n is

$$Z_n = \frac{1}{2} | - \mathrm{Pf}(K_n^{00}) + \mathrm{Pf}(K_n^{10}) + \mathrm{Pf}(K_n^{01}) + \mathrm{Pf}(K_n^{11})|.$$

Let $E_m = \{e_1 = u_1 v_1, ..., e_m = u_m v_m\}$ be a subset of edges of G_n. It is proved in [11, 1] that the probability of these edges occurring in a dimer configuration of G_n with respect to the Boltzmann measure P_n, denoted by $P_n(e_1, ..., e_m)$, is

$$P_n(e_1, ..., e_m) = \frac{\prod_{i=1}^m \eta(u_i v_i)}{2 Z_n} | - \mathrm{Pf}(K_n^{00})_{E_m^c} + \mathrm{Pf}(K_n^{10})_{E_m^c} + \mathrm{Pf}(K_n^{01})_{E_m^c} + \mathrm{Pf}(K_n^{11})_{E_m^c}|$$

where $E_m^c = V(G_n) \setminus \{u_1, v_1, ..., u_m, v_m\}$, and $(K_n^{\theta\tau})_{E_m^c}$ is the submatrix of $K_n^{\theta\tau}$ whose lines and columns are indexed by E_m^c.

The asymptotic behavior of Z_n when n is large is an interesting subject. One important concept is the free energy per fundamental domain, which is defined to be

$$-\lim_{n \to \infty} \frac{1}{n^2} \log Z_n.$$

Gauge equivalent dimer weights give the same spectral curve. That is because after gauge transformation, the determinant is scaled by a nonzero constant, thus not changing the zero locus of $P(z, w)$.

A formula for enlarging the fundamental domain is proved in [3, 10]. Let $P_n(z, w)$ be the characteristic polynomial of G_n, and $P_1(z, w)$ be the characteristic polynomial of G_1, then

$$P_n(z, w) = \prod_{\{u:u^n=z\}} \prod_{\{v:v^n=w\}} P_1(u, v). \tag{1}$$

3 Graph on a Cylinder

In this section, we study dimer models on cylindrical Fisher graphs using the technique of spectral curves. The spectral curve has a unique real zero on the unit circle $\mathbb{T}$, or is disjoint from the unit circle $\mathbb{T}$. Unlike some other 1d statistical mechanical models, there is a phase transition even in the cylindrical graph case when we consider the edge-edge correlation. In particular, at criticality, the edge-edge correlation, as the distance of two edges goes to infinity, converges to a nonzero constant.

3.1 Zeros of the Characteristic Polynomial

In this section, we consider characteristic polynomials for cylindrical Fisher graphs and prove Theorem 1.1. We begin with a lemma.

Lemma 3.1. *Let $P_\ell(z)$ denote the characteristic polynomial of the cylindrical Fisher graph consisting of ℓ periods, i.e., the quotient graph of the infinite Fisher graph, which is periodic in one direction, and finite in the perpendicular direction, under the action of $\ell\mathbb{Z}$. Let n be the size of one period. If ℓn is even, then*

$$P_\ell(-1) = Z_\ell^2, \tag{2}$$

where Z_ℓ is the partition function of perfect matchings on the cylindrical Fisher graph consisting of ℓ periods.

Proof. A cylindrical Fisher graph is also a planar graph, i.e., it can be drawn on the plane so that edges intersect only at vertices, see Figure 1.2, where each edge at the southwestern boundary crossed by γ_x is identified with an edge on the northeastern boundary. For each edge crossed by γ_x, we choose the opposite orientation to the orientation given in Figure 1.2. This way we obtain a clockwise-odd orientation when ℓn is even. Given a clockwise-odd orientation for a planar graph, the partition function of perfect matchings is exactly the Pfaffian of the weighted adjacency matrix. Since $P_\ell(-1)$ is the determinant of the weighted adjacency matrix for the graph with clockwise-odd orientation obtained from the orientation given in Figure 1.2 by reversing the orientations on all edges crossed by γ_x, and the determinant is the square of the Pfaffian, we conclude the identity (2), when ℓn is even.

□

In the proof of Theorem 1.1 below, we will use the term "loop configuration". A loop configuration on a Fisher graph is a subset of edges, consisting of doubled edges and cycles, so that each vertex has exactly two incident present edges (where doubled edges are counted twice). Here a cycle is a finite alternating sequence of vertices and edges $v_1 e_1 v_2 e_2 ... e_n v_{n+1}$, so that $v_{n+1} = v_1$, and $v_i \neq v_j$ for $1 \leq i, j \leq n$, $i \neq j$. The superposition of two perfect matchings on a Fisher graph always gives a loop configuration, in which each cycle consists of an even number of edges (an even loop). However, if in a loop configuration, there are cycles consisting of an odd number of edges (odd loops), the loop configuration cannot decompose into two perfect matchings.

An essential loop, or a non-contractible loop in a loop configuration on a cylindrical graph is a cycle which is not contractible to a point by moving on the surface of the cylinder. It is not hard to see that for a loop configuration on a cylindrical graph, whose total number of vertices is even, there are always an even number of odd loops.

Loop configurations are important because they correspond to the monomials in the expansion of $\det K(z)$, where $K(z)$ is the modified weighted adjacency matrix for the cylindrical Fisher graph. In fact,

$$P(z) = \det K(z) = \sum_{\sigma \in S_{6mn}} (-1)^{\text{sgn}(\sigma)} [K(z)]_{1,\sigma(1)} [K(z)]_{2,\sigma(2)} \cdot \ldots \cdot [K(z)]_{6mn,\sigma(6mn)}. \quad (3)$$

Here m is the height of the cylinder, and it is not hard to see that for a cylindrical Fisher graph with circumference n and height m, the total number of vertices is $6mn$. Moreover, each non-vanishing term in the expansion of (3) corresponds to an oriented loop configuration; i.e., each cycle in a loop configuration has a fixed orientation, chosen from one of the two possible orientations of the cycle. The cycle with a given orientation corresponds to a cycle in the permutation σ.

We also observe that an odd loop in a loop configuration corresponding to an non-vanishing monomial in (3) must be essential and wind around the cylinder exactly once. Because for a contractible odd loop, when we reverse the orientation of only that odd loop, and keep the orientation on all the other loops in the configuration, all the edge weights along the odd loop are negated, since there are an odd number of edges along the loop. The new orientation will cancel with the original one if we sum up their products of edge weights.

Proof. (Proof of Theorem 1.1) Without loss of generality, assume the period n, or the circumference of the cylinder, is even. We will prove that either $P(z) = 0$ does not intersect $\mathbb{T}$, or the intersection is the real point 1.

In fact, if the period n is odd, we can always enlarge the fundamental domain by two, and study the spectral curve for a cylindrical graph consisting of two periods. Or equivalently, instead of considering the smallest period, we consider a period of size $2n$, which is always even. Let $P(z) = P_1(z)$ be the spectral curve of the cylindrical graph consisting of one period of size n, and let $P_2(z)$ be the spectral curve of the cylindrical graph consisting of two periods, each of which is of size n. A similar formula to (1) for the cylindrical graph gives

$$P_2(z^2) = P(z)P(-z).$$

If the only possible zero of $P_2(z)$ on $\mathbb{T}$ is 1, then the only possible zero of $P(z)$ is real.

Now let the period be even, and let $P(z)$ be the corresponding spectral curve. Assume $P(z) = 0$ has a non-real zero $z_0 \in \mathbb{T}$. Assume

$$z_0 = e^{i\alpha_0\pi}, \qquad \alpha_0 \in (0,1) \cup (1,2).$$

We classify all the real numbers in $(0,1) \cup (1,2)$ into 3 types

1. $\alpha_0 = \frac{p}{q}$ where p, q are positive integers with no common factors, p is odd
2. α_0 is irrational
3. $\alpha_0 = \frac{p}{q}$ where p, q are positive integers with no common factors, p is even

First let us consider Case 1 and Case 2. There exists a sequence $\ell_k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} z_0^{\ell_k} = -1. \tag{4}$$

The sequence $\{\ell_k\}_{k=1}^{\infty}$ can be constructed in both Case 1 and Case 2; in Case 1, things are much simpler, since we can choose $\ell_k = 3^{k-1}q$, then for any $k \in \mathbb{N}$, $z_0^{\ell_k} = -1$. If we assume $z_0^{\ell_k} = e^{i\alpha_k\pi}$ where $\alpha_k \in [0,2)$, then the identity (4) can be rephrased as follows

$$\lim_{k\to\infty} \alpha_k = 1.$$

According to the formula of enlarging the fundamental domain,

$$P(z_0) = 0$$

implies

$$P_{\ell_k}(z_0^{\ell_k}) = 0, \qquad \forall k.$$

Since the size of a period, n, is even, by Lemma 3.1, we have

$$P_{\ell_k}(-1) = Z_{\ell_k}^2,$$

where Z_{ℓ_k} is the partition function of dimer configurations of the cylinder with circumference $n\ell_k$ and height m. Therefore we have

$$\begin{aligned} 1 &= \lim_{k\to\infty} \left| \frac{P_{\ell_k}(z_0^{\ell_k}) - P_{\ell_k}(-1)}{P_{\ell_k}(-1)} \right| \\ &= \lim_{k\to\infty} \frac{1}{Z_{\ell_k}^2} |P_{\ell_k}(e^{i\alpha_k\pi}) - P_{\ell_k}(e^{i\pi})| \\ &= \lim_{k\to\infty} \frac{1}{Z_{\ell_k}^2} \left| \sum_{t=1}^{\infty} \frac{[\pi(\alpha_k - 1)]^{2t}}{(2t)!} \frac{\partial^{2t} P_{\ell_k}(e^{i\theta})}{\partial \theta^{2t}} |_{\theta=\pi} \right|. \end{aligned} \tag{5}$$

Note that the odd order terms in (5) vanish because $P_{\ell_k}(e^{i\theta})$ is symmetric with $e^{i\theta}$ and $e^{-i\theta}$. Since

$$P_{\ell_k}(z) = \sum_{0\leq j\leq m} P_{\ell_k}^{(j)}\left(z^j + \frac{1}{z^j}\right),$$

where $P_{\ell_k}^{(j)}$ is the signed sum of oriented loop configurations winding exactly j times around the cylinder (where if two cycles are winding around the cylinder in opposite directions, we count them as a configuration winding 0 times around the cylinder). We have

$$\left.\frac{\partial^{2t} P_{\ell_k}(e^{i\theta})}{\partial\theta^{2t}}\right|_{\theta=\pi} = \sum_{1\leq j\leq m} 2j^{2t}(-1)^t P_j^{(\ell_k)},$$

and

$$\lim_{k\to\infty}\frac{1}{Z_{\ell_k}^2}\left|\sum_{t=1}^{\infty}\frac{[\pi(\alpha_k-1)]^{2t}}{(2t)!}\frac{\partial^{2t}P_{\ell_k}(e^{i\theta})}{\partial\theta^{2t}}|_{\theta=\pi}\right|$$
$$\leq \lim_{k\to\infty}\frac{1}{Z_{\ell_k}^2}\sum_{t=1}^{\infty}\frac{[\pi m(\alpha_k-1)]^{2t}}{(2t)!}2\sum_{1\leq j\leq m}|P_j^{(\ell_k)}|$$

To estimate $|P_j^{(\ell_k)}|$, let us divide the $m\times\ell_k n$ cylinder into $\ell_k n$, $m\times 1$ slices. Connecting edges between two slices may be occupied once, unoccupied, or occupied twice (appear as doubled edges). Choose one slice L_0, we construct an equivalence class of loop configurations. Two loop configurations are equivalent if they differ from each other only in L_0, and coincide on all the other slices and boundary edges of L_0.

We claim that for any given equivalence class, there is at least one configuration including only even loops. To see that we choose an arbitrary configuration in that equivalence class including odd loops. Let $\mathcal{T}$ be the set of all triangles in L_0 which have one or two edges belonging to some loop crossing L_0. We choose an odd loop s_1. Choose a triangle $\Delta_1 \in s_1\cap L_0$. Move along L_0 until we find a triangle Δ_2 with one or two edges belonging to a different essential odd loop s_2. This is always possible because any loop configuration arising from the determinant and including odd loops always has an even number of odd loops, and all the odd loops are winding once along the cylinder, as explained before. Change path through Δ_1. Starting from Δ_1, we change the doubled edge configuration to alternating edges along a path in L_0, and change the path through all triangles, belonging to even loops, or s_1, between Δ_1 and Δ_2, then we change the path through Δ_2. For any even loop between Δ_1 and Δ_2, we change paths through an even number of triangles of that even loop, hence it is still an even loop. However, for s_1 and s_2, we change paths through an odd number of triangles, so both of them become even. We can continue this process until we eliminate all the odd loops, because there are always an even number of odd loops in the configuration. An example of such a path change process is illustrated in Figures 3.1 and 3.2.

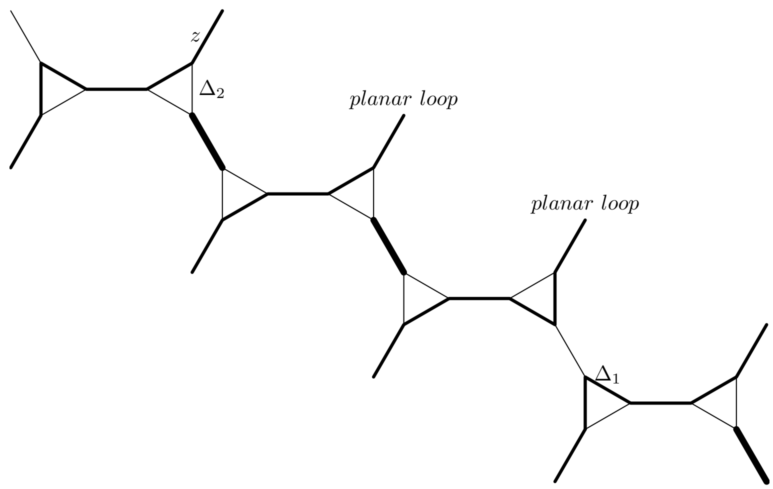

Figure 3.1: before path change

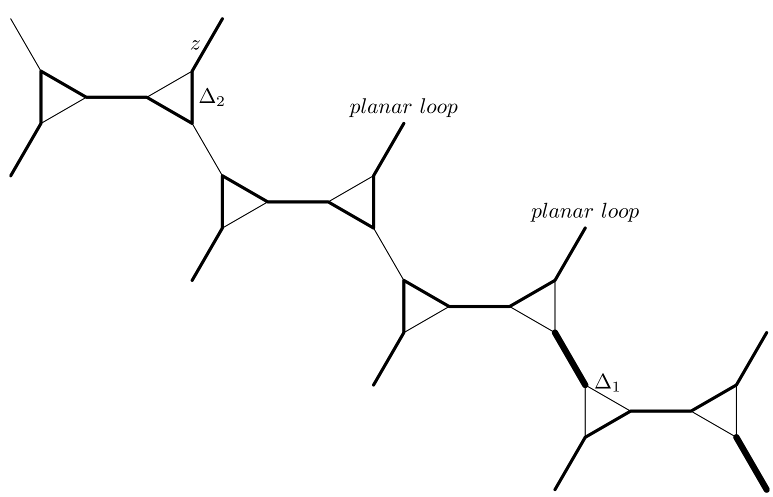

Figure 3.2: after path change

Since we have at most C_0^m different configurations in each equivalence class, where C_0 is a constant, and each equivalence class has at least one configuration including only even loops, we have

$$\frac{\#\text{ of loop configurations including odd loops}}{\#\text{ of even loop configurations}} < C_0^m$$

We have a finite number of different edge weights, each of which is positive, hence the quotient of any two weights is bounded by a constant C_1, then

$$\sum_{1\leq j\leq m} |P_j^{(\ell_k)}| \leq \text{Partition function of configurations including essential odd loops}$$
$$+\text{Partition function of configurations with only even loops}$$
$$\leq (C_0^m C_1^{6m}+1)\text{Partition function of even loop configurations} \leq C_2^m Z_{\ell_k}^2$$

As a result

$$\lim_{k\to\infty} \frac{1}{Z_{\ell_k}^2}\left|\sum_{t=1}^{\infty}\frac{[\pi(\alpha_k-1)]^{2t}}{(2t)!}\frac{\partial^{2t}P_k(e^{i\theta})}{\partial\theta^{2t}}|_{\theta=\pi}\right|$$
$$\leq \lim_{k\to\infty}\frac{1}{Z_{\ell_k}^2}\sum_{t=1}^{\infty}\frac{[m\pi(\alpha_k-1)]^{2t}}{(2t)!}C_2^m Z_{\ell_k}^2$$

Since m is a constant, and $\lim_{k\to\infty}\alpha_k = 1$, we have

$$\lim_{k\to\infty}\sum_{t=1}^{\infty}\frac{[m\pi(\alpha_k-1)]^{2t}}{(2t)!}C_2^m = 0$$

which is a contradiction to (5). Hence for all α_0 in Case 1 and Case 2, $P(e^{i\alpha_0\pi}) \neq 0$.

Now let us consider α_0 in Case 3. From the argument above we derive that as long as

$$|\alpha - 1| < \delta,$$

where δ is a small positive number depending only on m,

$$P(e^{i\alpha\pi}) \neq 0.$$

Consider $\alpha_0 = \frac{p}{q}$, where p, q have no common factors and p is even. We claim that as long as the denominator q is sufficiently large, $P(z_0) \neq 0$. To see that, since p, q are relatively prime, there exist integers k, k', such that

$$k'q + kp = 1. \tag{6}$$

Since p is even, q must be odd, therefore (6) implies that k' is odd. If

$$\frac{1}{q} < \delta,$$

by (6), we have

$$\left|\frac{1}{\pi}\text{Arg}[e^{ik\alpha_0\pi}] - 1\right| < \delta.$$

Hence after enlarging the fundamental domain to an $m \times kn$ cylinder, we have

$$P_k(e^{ik\alpha_0\pi}) \neq 0,$$

hence

$$P(e^{i\alpha_0\pi}) \neq 0.$$

Now we consider $\frac{1}{q} \geq \delta$, only finitely many q's satisfy this condition. Let ℓ be a prime number satisfying $\ell > \left[\frac{1}{\delta}\right] + 1$, then after enlarging the fundamental domain to an $m \times \ell n$ cylinder, the corresponding δ will not change because it depends only on m. For any

$$1 \geq \frac{1}{q} \geq \delta,$$

we have

$$\delta > \frac{1}{\ell q}.$$

Let $\ell\alpha_1(\mod 2) = \alpha_0$. Namely,

$$\alpha_1 = \frac{p}{\ell q}$$

in reduced form. By the previous argument $P(e^{\alpha_1 i\pi}) \neq 0$, hence $P_\ell(e^{i\alpha_0\pi}) \neq 0$. Hence when α_0 is in Case 3, $P_\ell(z_0) \neq 0$ after enlarging the fundamental domain to $m \times n\ell$, where ℓ depends only on m, and is independent of α_0. If $P(z_0) = 0$, z_0 is not real, then we enlarge the fundamental domain to $m \times n\ell$, where ℓ is a big prime number depending only on m, we derive that $P_\ell(z_0^\ell) = 0$, however this is impossible since $P_\ell(z) = 0$ can have only real root on $\mathbb{T}$.

In particular, if n is even, then $P(-1) = Z^2 > 0$, where Z is the partition function of dimer configurations on the cylindrical graph, since the cylindrical graph is planar, and reversing the orientations of edges crossed by γ_x, gives us a clockwise-odd orientation. Hence the only possible zero of $P(z)$ on the unit circle $\mathbb{T}$ is 1, when n is even. □

Corollary 3.2. *For any non-real $z \in \mathbb{T}$, all eigenvalues of $K(z)$ are of the form $i\lambda_j, j = 1, ..., 6mn$, where i is the imaginary unit. $3mn$ of the λ_j's are positive, the other λ_j's are negative.*

Proof. From the definition of $K(z)$, $iK(z)$ is a Hermitian matrix. We claim that $iK(z)$ has $3mn$ positive and $3mn$ negative eigenvalues for any non-real $z \in \mathbb{T}$. Let K_0 be the Kasteleyn matrix of the graph formed from the cylinder by removing all $2m$ vertices, which are endpoints of an edge crossed by γ_x (z-type edge), see Figure 1.2. Similarly, let K_r be the Kasteleyn matrix of the graph formed from the cylinder by removing $(m-r)$ z-type edges as well as their incident vertices. Here the $(m-r)$ removed z-type edges are required to be a subset of the $(m-r+1)$ removed z-type edges when constructing K_{r-1} in the previous step.

Note that $\det K_r$ is real-valued for any z on the unit circle, and is strictly positive when $z = -1$, n is even. K_0 is an antisymmetric real matrix with eigenvalues $\pm i\lambda_j$, $1 \leq j \leq 3mn - m$, and $\det K_0 > 0$, because $|\mathrm{Pf} K_0|$ is the partition function of dimer configurations with positive edge weights, and $\det K_0 = (\mathrm{Pf} K_0)^2$. Every time we add a pair of boundary vertices connected by a z edge, we have an anti-Hermitian matrix K_{r+1} of order $6mn-2m+2(r+1)$ with K_r as a principal minor. By induction hypothesis iK_r has the same number of positive and negative eigenvalues, the Cauchy's interlacing theorem implies that iK_{r+1} has at least $3mn-m+r$ positive and $3mn-m+r$ negative eigenvalues. By Theorem 1.1, $\det K_{r+1} > 0$ for non-real z, so the other two eigenvalues of iK_{r+1} can only be one positive and one negative. □

3.2 Limit Measure of Cylindrical Approximation

In this subsection we consider the limit measure on the infinite Fisher graph, which is periodic in one direction, and finite in the perpendicular direction. The limit is obtained by using larger and larger cylinders to approximate the infinite periodic graph. Although this is essentially a one-dimensional model, the surprising part is there exists a phase transition and at criticality, the limit measure is not mixing. Example 3.6 is a simple illustration of this fact. The phase transition is obtained by studying the edge-edge correlation, using the result proved in the previous subsection regarding the intersection of the spectral curve with the unit circle $\mathbb{T}$.

We will start with the limit of the free energy, by using larger and larger cylinder to approximate the infinite periodic graph.

From the proof we know that all terms in $P_{m\times n}(-1)$ are positive, if n is even. We can always enlarge the fundamental domain in the z direction without changing edge weights to get

$$P_{m\times 2n}(-1) = P_{m\times n}(i)P_{m\times n}(-i) = P^2_{m\times n}(i) = |Pf_{m\times 2n}K(-1)|^2$$

Therefore $P_{m\times n}(i)$ is the partition function of dimer configurations of the $m \times 2n$ cylinder graph. According to the formula of enlarging the fundamental domain,

$$P_{m\times 2ln}(-1) = \prod_{z^{2l}=-1} P_{m\times n}(z),$$

we have the free energy

$$\begin{aligned} -\lim_{l\to\infty}\frac{1}{4l}\log P_{m\times 2ln}(-1) &= -\lim_{l\to\infty}\frac{1}{4\pi}\frac{2\pi}{2l}\sum_{z^{2l}=-1}\log P_{m\times n}(z) \\ &= -\frac{1}{4\pi}\int_{\mathbb{T}}\log P_{m\times n}(z)\frac{dz}{iz} \qquad (7) \end{aligned}$$

The convergence of the Riemann sums to the integral follows from the fact that the only possible zero of $P(z)$ on $\mathbb{T}$ is a single real node, see [3] for a proof.

Any probability measure on the infinite banded graph, with depth m in one direction, and period n in the other direction, is determined by the probability of cylinder events. Namely, we choose a finite number of edges $e_1, e_2, \cdots, e_k$ arbitrarily, and the probabilities

$$Pr(e_1\&e_2\&\cdots\&e_k)$$

that $e_1, e_2, \cdots, e_k$ occur in the dimer configuration simultaneously for all finite edge sets determines the probability measure. We consider the measures on the infinite graph as weak limits of measures on cylindrical graphs. First of all, we prove a lemma about the entries of the inverse Kasteleyn matrix using the cylindrical approximation.

Lemma 3.3.

$$\lim_{l\to\infty} K^{-1}_{m\times 2ln}(-1)_{(k_v,s_v),(k_w,s_w)} = \frac{1}{2\pi}p.v.\int_{\mathbb{T}} z^{k_v-k_w}\frac{\text{cofactor}[K_{m\times n}(s_v,s_w)(z)]}{P_{m\times n}(z)}\frac{dz}{iz}$$

Proof. The proof is inspired by [3], the difference is that we are dealing with one-dimensional non-bipartite graphs here. Although in the two dimensional periodic Fisher graph, the entry of the infinite inverse Kasteleyn matrix is a well-defined integral, using larger and larger tori to approximate the infinite bi-periodic graph (see [1] for a proof), in the one-dimensional case, the integral itself is improper, and the limit of the entry of the inverse Kasteleyn matrices on cylindrical graphs is the principal value of the integral. We construct a transition matrix S to make $S^{-1}K_{m\times 2ln}S$ block diagonal, with each block corresponding to a $m\times n$ quotient graph. Define

$$S = (e_0^1, ..., e_0^{6mn}, e_1^1, ..., e_1^{6mn}, ..., e_{2l-1}^1, ..., e_{2l-1}^{6mn})$$

where

$$e_k^s(j,t) = \begin{cases} e^{\frac{i\pi(2j+1)(2k+1)}{4l}} & s=t \\ 0 & s\neq t \end{cases}$$

then

$$S^{-1}K_{m\times 2ln}S = \begin{pmatrix} K_{m\times n}(e^{\frac{i\pi}{2l}}) & & & 0 \\ & K_{m\times n}(e^{\frac{3i\pi}{2l}}) & & \\ & 0 & \ddots & \\ & & & K_{m\times n}(e^{\frac{i(4l-1)\pi}{2l}}) \end{pmatrix}$$

Since $S^{-1} = \frac{1}{2l}\bar{S}^t$, we have

$$K^{-1}_{m\times 2ln}(-1)_{(k_v,s_v),(k_w,s_w)} = \frac{1}{2l}\sum_{j=0}^{2l-1}\frac{\text{cofactor}[K_{m\times n}(s_v,s_w)]}{P_{m\times n}(e^{\frac{(2j+1)i\pi}{2l}})}e^{\frac{i(2j+1)\pi(k_v-k_w)}{2l}}$$

where k_v is the index of the fundamental domain for vertex v and s_v is the index of vertex in the fundamental domain.

If $P_{m\times n}(z)$ has no zero on $\mathbb{T}$, we have

$$\lim_{l\longrightarrow\infty} K^{-1}_{m\times 2ln}(-1)_{(k_v,s_v),(k_w,s_w)} = \frac{1}{2\pi}\int_{\mathbb{T}} z^{k_v-k_w}\frac{\text{cofactor}[K_{m\times n}(s_v,s_w)(z)]}{P_{m\times n}(z)}\frac{dz}{iz} \tag{8}$$

If 1 is an order-2 zero of $P_{m\times n}(z)$, let

$$Q(z) = z^{k_v-k_w}\frac{\text{cofactor}[K_{m\times n}(s_v,s_w)(z)]}{P_{m\times n}(z)},$$

Since $\det K(1)$ is an anti-symmetric real matrix of even order, and non-invertible, the dimension of its null space is non-zero and even. Hence $Adj K(1)$ is a zero matrix, and 1 is at least a zero of order 1 for $cofactor K_{m\times n}(s_v, s_w)(z)$. Then in a neighborhood of 1,

$$Q(z) = \frac{Res_{z=1}Q(z)}{z-1} + R(z),$$

where $R(z)$ is analytic at 1.

$$\begin{aligned}
&\lim_{l\to\infty} K^{-1}_{m\times 2ln}(-1)_{(k_v,s_v),(k_w,s_w)} \\
= &\lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}\left(\sum_{0\leq j<\frac{l\delta}{\pi}-\frac{1}{2}} + \sum_{\frac{l\delta}{\pi}-\frac{1}{2}\leq j\leq 2l-\frac{1}{2}-\frac{l\delta}{\pi}} + \sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\leq 2l-1}\right) Q(e^{\frac{(2j+1)i\pi}{2l}}) \qquad (9)
\end{aligned}$$

For the second term,

$$\begin{aligned}
&\lim_{\delta\to 0+}\lim_{l\to 0}\frac{1}{2l}\sum_{\frac{l\delta}{\pi}-\frac{1}{2}\leq j\leq 2l-\frac{1}{2}-\frac{l\delta}{\pi}} Q(e^{\frac{(2j+1)i\pi}{2l}}) \\
= &\lim_{\delta\to 0+}\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} Q(e^{i\theta})d\theta \\
= &\ p.v.\frac{1}{2\pi}\int_{\mathbb{T}} Q(z)\frac{dz}{iz} \\
= &\ \frac{1}{2}Res_{z=1}Q(z) + \sum Res_{|z|<1}\frac{Q(z)}{z} \qquad (10)
\end{aligned}$$

For the first and the third term

$$\begin{aligned}
&\lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}\left(\sum_{0\leq j<\frac{l\delta}{\pi}-\frac{1}{2}} + \sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\leq 2l-1}\right) Q(e^{\frac{(2j+1)i\pi}{2l}}) \\
= &\lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}\left(\sum_{0\leq j<\frac{l\delta}{\pi}-\frac{1}{2}} + \sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\leq 2l-1}\right)\left(\frac{Res_{z=1}Q(z)}{e^{\frac{(2j+1)i\pi}{2l}}-1} + R(e^{\frac{(2j+1)i\pi}{2l}})\right)
\end{aligned}$$

Since $R(z)$ is analytic in a neighborhood of 1,

$$\lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}\left(\sum_{0\leq j<\frac{l\delta}{\pi}-\frac{1}{2}} + \sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\leq 2l-1}\right) R(e^{\frac{(2j+1)i\pi}{2l}}) = 0 \qquad (11)$$

Moreover,

$$\begin{aligned}
&\lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}Res_{z=1}Q(z)\left(\sum_{0\leq j<\frac{l\delta}{\pi}-\frac{1}{2}} + \sum_{2l-\frac{1}{2}-\frac{l\delta}{\pi}<j\leq 2l-1}\right)\frac{1}{e^{\frac{(2j+1)i\pi}{2l}}-1} \\
= &\lim_{\delta\to 0+}\lim_{l\to\infty}\frac{1}{2l}Res_{z=1}Q(z)\sum_{0\leq j<\frac{l\delta}{\pi}-\frac{1}{2}}\left(\frac{1}{e^{\frac{(2j+1)i\pi}{2l}}-1} + \frac{1}{e^{-\frac{(2j+1)i\pi}{2l}}-1}\right) \\
= &-\lim_{\delta\to 0+}\frac{1}{2l}Res_{z=1}Q(z)\left[\frac{l\delta}{\pi}+\frac{1}{2}\right] = 0 \qquad (12)
\end{aligned}$$

And the lemma follows from (8), (9), (10),(11), (12). □

Proposition 3.4. *Using a large cylinder to approximate the infinite periodic banded graph, we derive that the weak limit of probability measures of the dimer model exists. The probability of a cylindrical set under this limit measure is*

$$Pr(e_1\&e_2\&\cdots\&e_k) = \lim_{l\to\infty}\prod_{j=1}^{k} w_{e_j}\sqrt{\det K^{-1}_{m\times 2ln}\begin{pmatrix} u_1 & v_1 & \cdots & u_k & v_k \\ u_1 & v_1 & \cdots & u_k & v_k \end{pmatrix}}$$

where w_{e_j} is the weight of e_j, and u_j, v_j are the two vertices of e_j.

Proof. On a finite cylindrical graph of $m \times 2ln$

$$Pr(e_1\&e_2\&\cdots\&e_k) = \frac{Z_{e_1,\cdots,e_k}}{Z}$$

where Z is the partition function of dimer configurations on that graph, and $Z_{e_1,\cdots,e_k}$ is the partition function of dimer configurations for which $e_1,\cdots e_k$ appear simultaneously. Since

$$\begin{aligned} Z^2 &= \det K_{m\times 2ln}(-1) \\ Z^2_{e_1,\cdots,e_k} &= \prod_{j=1}^{k} w_{e_j}\text{cofactor}[K_{m\times 2ln,(u_1,v_1,\cdots,u_k,v_k)}(-1)] \end{aligned} \quad (13)$$

(13) follows from the fact that if originally we have a clockwise-odd orientation, we still have a clockwise-odd orientation when removing edges and ending vertices, while keeping the orientation on the rest of the graph. The proposition follows from Jacobi's formula for the determinant of minor matrices. □

We consider two edges e_1 and e_2 with weight x_1 and x_2 on $m \times 2ln$ cylinder, and compute the covariance.

$$\begin{aligned} & Pr_{m\times 2ln}(e_1\&e_2) - Pr_{m\times 2ln}(e_1)Pr_{m\times 2ln}(e_2) \\ = \; & x_1x_2\sqrt{\det K^{-1}_{m\times 2ln}\begin{pmatrix} v_1 & w_1 & v_2 & w_2 \\ v_1 & w_1 & v_2 & w_2 \end{pmatrix}(-1)} \\ & -x_1x_2\sqrt{\det K^{-1}_{m\times 2ln}\begin{pmatrix} v_1 & w_1 \\ v_1 & w_1 \end{pmatrix}(-1)\cdot \det K^{-1}_{m\times 2ln}\begin{pmatrix} v_2 & w_2 \\ v_2 & w_2 \end{pmatrix}(-1)} \\ = \; & x_1x_2(|K^{-1}_{m\times 2ln}(v_1,w_1)K^{-1}_{m\times 2ln}(v_2,w_2) + K^{-1}_{m\times 2ln}(v_1,w_2)K^{-1}_{m\times 2ln}(w_1,v_2) \\ & -K^{-1}_{m\times 2ln}(v_1,v_2)K^{-1}_{m\times 2ln}(w_1,w_2)| - |K^{-1}_{m\times 2ln}(v_1,w_1)K^{-1}_{m\times 2ln}(v_2,w_2)|) \end{aligned} \quad (14)$$

In order to compute the covariance as $l \nearrow \infty$, we only need to compute the entries of $K^{-1}_{m\times 2ln}(-1)$ as $l \nearrow \infty$.

Then we have the following proposition

Proposition 3.5. *Consider the dimer model on a Fisher graph, embedded into an $m \times ln$ cylinder, as illustrated in Figure 4. Let the circumference of the cylinder go to infinity, i.e. $l \to \infty$, and keep the height of the cylinder unchanged. Consider two edges e_1 and e_2. As $|e_1 - e_2| \nearrow \infty$: if $P(z) = 0$ does not intersect $\mathbb{T}$, the edge-edge correlation decays exponentially ; if $P(z) = 0$ has a node at 1, the edge-edge correlation tends to a constant.*

Proof. The theorem follows from formula (14), and the estimates of the entries of the inverse Kasteleyn matrix. By (10), the second term goes to 0 as $|e_1 - e_2| \to \infty$, the first term is 0, if no zero exists on $\mathbb{T}$, see also [10]; the first term is a nonzero constant if the spectral curve has a real node at 1. □

Example 3.6. *(1×2 cylindrical graph)Assume we have a dimer model on a Fisher graph embedded into an infinite cylinder of height 1 and period 2. One period of the graph is illustrated in Figure 3.3.*

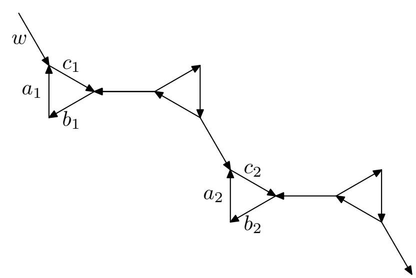

Figure 3.3: 1×2 cylindrical graph

The characteristic polynomial of the model is

$$P(z) = (b_1 b_2 w - a_1 a_2)\left(\frac{b_1 b_2}{w} - a_1 a_2\right).$$

The probability that an a_1-edge occurs is

$$\begin{aligned} Pr(a_1) &= \frac{a_1 a_2}{4\pi}\left(p.v. \int_{\mathbb{T}} \frac{1}{a_1 a_2 - b_1 b_2 w}\frac{dw}{iw} + p.v. \int_{\mathbb{T}} \frac{1}{a_1 a_2 - \frac{b_1 b_2}{w}}\frac{dw}{iw}\right) \\ &= \begin{cases} 1 & \text{if } a_1 a_2 > b_1 b_2 \\ 0 & \text{if } a_1 a_2 < b_1 b_2 \\ \frac{1}{2} & \text{if } a_1 a_2 = b_1 b_2 \end{cases} \end{aligned}$$

$P(z) = 0$ has a real node at 1 if and only if $a_1 a_2 = b_1 b_2$. In the critical case, the covariance of an a_1 edge and a b_2 edge is

$$Pr(a_1 \& b_2) - Pr(a_1)Pr(b_2) = 0 - \frac{1}{4} = -\frac{1}{4}$$

4 Graph on a Torus

4.1 Combinatorial and Analytic Properties

This subsection is devoted to the exploration of combinatorial and analytical properties of the characteristic polynomial $P(z, w)$. The loop configuration interpretation of the coefficients of $P(z, w)$ (proof of Lemma 4.1), and the results about odd loop configurations (Lemma 4.2) proved in this subsection are important in proving Theorem 1.2.

To compute the characteristic polynomial $P(z, w)$ of the Fisher graph, we give an orientation to edges as illustrated in Figure 1.1.

Lemma 4.1. *The characteristic polynomial P(z,w) for a periodic Fisher graph with period $(m, 0)$ and $(0, n)$ is a Laurent polynomial of the following form:*

$$P(z, w) = \sum_{i,j} P_{ij} \left(z^i w^j + \frac{1}{w^j z^i} \right),$$

where (i, j) are integral points contained in the Newton polygon with vertices $(\pm m, 0)$, $(0, \pm n)$, $(\pm m, \mp n)$:

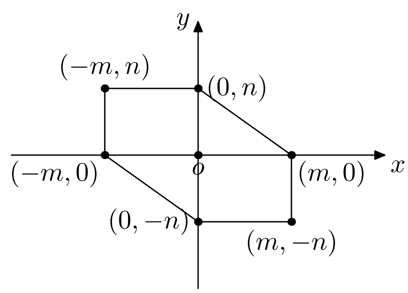

Figure 4.1: Newton polygon

Proof. Let $p = 6mn$. By definition,

$$P(z, w) = \det K(z, w) = \sum_{i_1,...,i_p} (-1)^{\sigma(i_1,...,i_p)} k_{1,i_1} \times \cdots \times k_{p,i_p}.$$

Here σ is the number of even cycles of the permutation $(i_1, ..., i_p)$, and $k_{i,j}$ is the entry of $K(z, w)$ with row index i and column index j. In fact $k_{i,j} = 0$ if ij does not form an edge. The sum is over all possible permutations of p elements.

Each term of $P(z, w)$ corresponds to an oriented loop configuration occupying each vertex exactly twice. In other words, a loop configuration consists of cycles and doubled edges, and each vertex has exactly two incident present edges in the loop configuration, where a doubled edge is counted twice.

For the graph G_1 with m z-edges and n w-edges, $P(z, w)$ is a Laurent polynomial with leading terms z^m,$\frac{1}{z^m}$,w^n,$\frac{1}{w^n}$,$\frac{z^m}{w^n}$,$\frac{w^n}{z^m}$. $z^i w^j$ corresponds to loops of homology class (i, j). $P(z, w)$ is symmetric with respect to $z^i w^j$ and $\frac{1}{w^j z^i}$. That is because for each term of $z^i w^j$, if we reverse the orientation of all loops, we get a term of $\frac{1}{w^j z^i}$ with coefficients of the same absolute value, corresponding to the product of weights of edges included in the configuration. The sign of the term is multiplied by $(-1)^p = 1$.

Now we use b-edges to denote the edges parallel to the edges crossed by γ_x; and c-edges to denote the edges parallel to the edges crossed by γ_y. To show that all the powers (i, j) lie in the polygon, we multiply all the b-edges by z (or $\frac{1}{z}$), and all the c-edges by w(or $\frac{1}{w}$), according to their orientation. This way the corresponding characteristic polynomial becomes $P(z^n, w^m)$. That is because each time an oriented cycle passes an edge crossed by γ_x (resp. γ_y) satisfying the condition that the orientation of the cycle is the same as (the opposite to) the orientation of the edge, it passes exactly n b-edges (resp. m w-edges), satisfying the condition that the orientation of the cycle is the same as (the opposite to) the orientation of the edge. Let $(\tilde{i}, \tilde{j})$ be a power of monomial in $P(z^n, w^m)$. At each triangle, all the possible contributions of local configurations to the power of the monomial can only be $(0, 0), (0, \pm 1), (\pm 1, 0), (\pm 1, \mp 1)$. Examples are illustrated in the following Figure 4.2. The left graph has two doubled edges, and the contribution to the power of the monomial is $(0, 0)$, the right graph has a loop winding from the z-edge to the w-edge, and the contribution to the power of the monomial is $(1, -1)$.

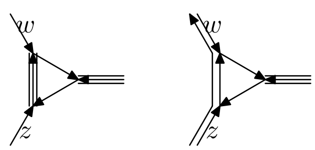

Figure 4.2: local configurations

Consider all the $2mn$ triangles. For each edge, we considered its contribution twice, since we counted it from both triangles it connected. Hence we have

$$
\begin{aligned}
-2mn \leq 2\tilde{i} \leq 2mn \\
-2mn \leq 2\tilde{j} \leq 2mn \\
-2mn \leq 2(\tilde{i} + \tilde{j}) \leq 2mn
\end{aligned}
$$

Since $\tilde{i} = ni$, $\tilde{j} = mj$, and the Newton polygon $N(P)$ is defined to be

$$N(P) = \text{convex hull}\{(i, j) \in \mathbb{Z}^2 | z^i w^j \text{ is a monomial in } P(z, w)\},$$

the lemma follows. □

Lemma 4.2. *For configurations with odd loops corresponding to a non-vanishing term in $P(z, w)$, all odd loops have non-trivial homology, and the total number of odd loops is even.*

Proof. For any loop configuration on the Fisher graph embedded on a torus, the number of odd loops is always even. That is because the total number of vertices is even, while odd loops always involve odd number of vertices. Any term in P_{ij} including odd loops can appear only when odd loops have non-trivial homology.

That is because for a contractible odd loop, we can reverse the orientation of that loop to negate the sign of that term. The term with reversed orientation on the odd loop cancels with the original term, because the homology class $(0,0)$ of the configurations is not changed given the odd loop is contractible. □

4.2 Fisher Correspondence

Inspired by the original Fisher correspondence introduced in [5], we discuss a generalized Fisher correspondence between the ferromagnetic Ising model on the triangular lattice and the perfect matchings on the Fisher graph, obtained by replacing each vertex of the hexagonal lattice by a triangle. The generalization is that instead of requiring all edge weights to be greater than 1, the Fisher graph could have edge weights less than 1 and still corresponds to a ferromagnetic Ising model. Note that the ferromagnetic Ising model on the square grid can also be considered as a ferromagnetic Ising model on the triangular grid with coupling constants on all the oblique edges to be 0, the correspondence discussed in this subsection applies to the ferromagnetic Ising model on the square grid.

Consider the Fisher graph obtained by replacing each vertex of the whole-plane, planar honeycomb lattice by a triangle. Assume all the triangle edges have weight 1, and all the non-triangle edges have positive weights. Furthermore, we assume that the following condition on non-triangle edge weights:

Condition 4.3. *At each triangle, there are an even number of adjacent edges with weights less than 1.*

We claim that the dimer configurations on such a Fisher graph correspond to ferromagnetic Ising configurations. Indeed, under this condition, there is a "ground state", which is a spin configuration where adjacent spins are different exactly when the corresponding edge has weight less than 1. For a perfect matching M on the Fisher graph, we produce an Ising spin configuration $I_1(M)$ by making a pair of adjacent spin values to be the same (resp. opposite) across a present (resp. non-present) non-triangle edge in the Fisher graph. Taking the XOR of $I_1(M)$ and the ground state produces an Ising spin configuration $I_2(M)$ whose distribution is the same as the ferromagnetic Ising model. Here by XOR we mean each spin value in $I_2(M)$ is the product of the corresponding spin values in $I_1(M)$ and the ground state.

This correspondence is 2-to-1 since negating the spins at all vertices corresponds to the same dimer configuration. Figure 4.3 is an example of the generalized Fisher correspondence given $a > 1$, $b > 1$, $c < 1$, $d > 1$, $e < 1$.

Since under Condition 4.3, the dimer model corresponds to a ferromagnetic Ising model, as well as when we assume all the non-triangle edge weights are no less than 1, we can, without loss of generality, assume all the non-triangle edge weights are no less than 1. Choose the interaction J_e associated to a bond as follows:

$$J_e = \frac{1}{2} \log \eta_e$$

where $\eta_e \geq 1$ is the weight of the corresponding non-triangle edge. When $\eta_e \geq 1$, $J_e \geq 0$ corresponds to the ferromagnetic interaction.

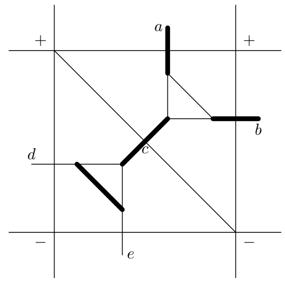

Figure 4.3: Generalized Fisher Correspondence

4.3 Duality Transformation

In this subsection, we consider a duality relation for perfect matchings on toroidal Fisher graphs. This duality relation is obtained from the Fisher correspondence between perfect matchings and Ising spin configurations, as well as high-temperature and low-temperature expansions for Ising partition functions, see [13, 14, 18]. We will also discuss a different Fisher correspondence in Sect. 4.3 from the correspondence discussed in Sect. 4.2, those two correspondences give us a duality relation of perfect matchings on two different Fisher graphs. This result of duality for the Fisher dimer model is one of the key ingredients of the proof of Theorem 1.2.

We call a geometric figure built with a certain number of bonds of a lattice an even-degree subgraph if at every lattice point an even number of bonds occur. It is clear that every configuration of "+" and "−" spins on a lattice can be associated an even-degree subgraph of the dual lattice in the following way. A dual bond belongs to the subgraph if it separates different spins and does not belong to the subgraph if it separates equal spins. The same even-degree subgraph is associated to two symmetric configurations in which the "+" and "−" spins are interchanged.

Let $T_{m\times n} = (V(T_{m\times n}), E(T_{m\times n}))$ be the quotient graph of the whole-plane triangular lattice, as defined on Sect. 2. Let $H_{m\times n}$ be the dual graph of $T_{m\times n}$, $H_{m\times n}$ is a honeycomb lattice which can be embedded into an $m \times n$ torus. Without loss of generality, assume both m and n are even. In fact, if m or n is odd, we can always enlarge the fundamental domain by 2, so that in the enlarged graph, both of them are even.

Define an Ising model on $T_{m\times n}$ with interactions $\{J_e\}_{e\in E(T_{m\times n})}$. Assume the Ising model on $T_{m\times n}$ has partition function $Z^{\text{Ising}}_{T_{m\times n}}$. Let $Z_{F_{m\times n},D_{00}}$ be the partition function of certain dimer configurations on a Fisher graph $F_{m\times n}$, which will be explained in detail later. Then $Z^{\text{Ising}}_{T_{m\times n}}$ can be written as,

$$Z^{\text{Ising}}_{T_{m\times n}} = 2\prod_{e\in E(T_{m\times n})}\exp(J_e)\sum_{C^*\in S^*_{00}}\prod_{e\in C^*}\exp(-2J_e) := 2\prod_{e\in E(T_{m\times n})}\exp(-J_e)Z_{F_{m\times n},D_{00}},$$

where S^*_{00} is the set of even-degree subgraphs of $H_{m\times n}$, with an even number of occupied bonds crossed by both γ_x and γ_y. The sum is over all configurations in S^*_{00}. Similarly, we can define $S^*_{01}(S^*_{10}, S^*_{11})$ to be the set of even-degree subgraphs of $H_{m\times n}$, with an even(odd,odd) number of occupied bonds crossed by γ_x, and an odd(even,odd) number of occupied bonds crossed by γ_y. This equality is obtained by using the Fisher correspondence.

Let $F_{m\times n}$ be the Fisher graph obtained by replacing each vertex of $H_{m\times n}$ by a triangle, with weights on all the non-triangle edges given by

$$\eta_e = \exp(2J_e).$$

Let $Z_{F_{m\times n},D}$ be the partition function of dimer configurations on $F_{m\times n}$, a Fisher graph embedded into an $m \times n$ torus, with weights $\exp(2J_e)$ on edges of $H_{m\times n}$, and weight 1 on all the other edges. Then

$$Z_{F_{m\times n},D} = Z_{F_{m\times n},D_{0,0}} + Z_{F_{m\times n},D_{0,1}} + Z_{F_{m\times n},D_{1,0}} + Z_{F_{m\times n},D_{1,1}},$$

where

$$Z_{F_{m\times n},D_{\theta,\tau}} = \prod_{e\in E(T_{m\times n})} \exp(2J_e) \sum_{C^*\in S^*_{\theta,\tau}} \prod_{e\in C^*} \exp(-2J_e).$$

For example, $Z_{F_{m\times n},D_{0,1}}$ is the dimer partition function on $F_{m\times n}$ with an even number of occupied edges crossed by γ_x, and an odd number of occupied edges crossed by γ_y. It also corresponds to an Ising model which has the same configuration on the two boundaries parallel to γ_y, and the opposite configurations on the two boundaries parallel to γ_x. Similar results hold for all the $Z_{F_{m\times n},D_{\theta,\tau}}$, $\theta, \tau \in \{0, 1\}$.

On the other hand, if we consider the high temperature expansion of the Ising model on $T_{m\times n}$, we have

$$\begin{aligned}
Z^{\text{Ising}}_{T_{m\times n}} &= \sum_{\sigma} \prod_{e=uv\in E(T_{m\times n})} \exp(J_e\sigma_u\sigma_v) \\
&= \sum_{\sigma} \prod_{e=uv\in E(T_{m\times n})} (\cosh J_e + \sigma_u\sigma_v \sinh J_e) \\
&= \prod_{e=uv\in E(T_{m\times n})} \cosh J_e \sum_{\sigma} \prod_{e=uv\in E(T_{m\times n})} (1 + \sigma_u\sigma_v \tanh J_e) \\
&= \prod_{e=uv\in E(T_{m\times n})} \cosh J_e \sum_{C\in S} 2^{mn} \prod_{e\in C} \tanh J_e,
\end{aligned}$$

where S is the set of all even-degree subgraphs of $T_{m\times n}$. Let $\tilde{F}_{m\times n}$ be a Fisher graph embedded into an $m \times n$ torus, with each vertex of $T_{m\times n}$ replaced by a gadget, as illustrated in Figure 4.4. Assume each edge in $\tilde{F}_{m\times n}$ corresponding to an edge in $T_{m\times n}$ has weight $\tanh J_e$, and any other edge in $\tilde{F}_{m\times n}$ has weight 1. Note that the gadget here is not the same as the gadget in [5].

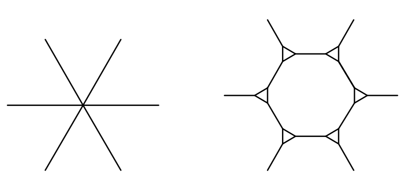

Figure 4.4: Fisher Correspondence

There is a 1-to-2 correspondence between even-degree sugraphs on the left graph and the dimer configurations on the right graph. An edge is present in the even-degree subgraph of the left graph if and only if it is present in the dimer configuration of the right graph. Recall that $\tilde{F}_{m\times n}$ has weight $\tanh J_e$ on edges of $T_{m\times n}$, and weight 1 on all the other edges. In fact, the edge with weight $\tanh J_e$ of $\tilde{F}_{m\times n}$ and the edge with weight e^{2J_e} of $F_{m\times n}$ are dual edges. Then we have

$$Z_{T_{m\times n}}^{\text{Ising}} = 2Z_{F_{m\times n},D_{0,0}} \prod_{e\in E(T_{m\times n})} \exp(-J_e) = Z_{\tilde{F}_{m\times n},D} \prod_{e\in E(T_{m\times n})} \cosh J_e$$

where $Z_{\tilde{F}_{m\times n},D}$ is the partition function of dimer configurations on $\tilde{F}_{m\times n}$. Hence we have

$$Z_{F_{m\times n},D_{0,0}} = \frac{1}{2^{mn+1}} Z_{\tilde{F}_{m\times n},D} \prod_{e\in E_{T_{mn}}} (1+\exp(2J_e)).$$

More generally, we can expand all the $Z_{F_{m\times n},D_{\theta\tau}}$ as follows:

$$Z_{F_{m\times n},D_{\theta,\tau}} = \frac{1}{2^{mn+1}} \prod_{e\in E_{T_{m\times n}}} (1+\exp(2J_e)) Z_{\tilde{F}_{m\times n},D}((-1)^{\tau},(-1)^{\theta}).$$

$Z_{\tilde{F}_{m\times n},D}(-1,1)$ is the dimer partition function of $\tilde{F}_{m\times n}$ with weights of edges crossed by γ_x multiplied by -1. Similarly for $Z_{\tilde{F}_{m\times n},D}(1,-1)$ and $Z_{\tilde{F}_{m\times n},D}(-1,-1)$. Therefore

$$Z_{F_{m\times n},D_{0,0}} = \max_{\theta,\tau\in\{0,1\}} Z_{F_{m\times n},D_{\theta,\tau}} \tag{15}$$

We see from (15) that $Z_{F_{m\times n},D_{0,0}}$ is the largest one of the four $Z_{F_{m\times n},D_{\theta,\tau}}$, $\theta,\tau\in\{0,1\}$ when edge weights satisfy $\eta_e = \exp(2J_e)$, and $J_e > 0$; i.e., all the non-triangle edges have weights strictly greater than 1. Now we will see that (15) is also true if we relax the assumption a little bit, i.e., if around each vertex there exist an even number of edges with weights strictly less than one.

We consider a Fisher graph $\hat{F}_{m\times n}$. $\hat{F}_{m\times n}$ is the same graph as $F_{m\times n}$ except edge weights. Namely, $\hat{F}_{m\times n}$ has weight 1 on all the triangle edges, and around each triangle, we have an even number of connecting edges satisfying

$$\hat{\eta}_e = \frac{1}{\eta_e},$$

where η_e(resp. $\hat{\eta}_e$) is the weight of edge e for the graph $F_{m\times n}$(resp. $\hat{F}_{m\times n}$). All the other edge weights satisfy

$$\hat{\eta}_e = \eta_e.$$

Without loss of generality, we assume that an even number of edges with weight strictly less than 1 are crossed by γ_x, and an even number of edges with weight less than 1 are crossed by γ_y. Then there is a 1-to-1 correspondence between configurations in $Z_{F_{m\times n},D_{\theta\tau}}$ and $Z_{\hat{F}_{m\times n},D_{\theta\tau}}$ by changing the configurations on all the edges with weight strictly less than 1. Hence we have

$$Z_{F_{m\times n},D_{\theta,\tau}} = Z_{\hat{F}_{m\times n},D_{\theta,\tau}} \prod_{\{e:w_e<1\}} \eta_e.$$

As a result

$$Z_{\hat{F}_{m\times n},D_{0,0}} = \max_{\theta,\tau\in\{0,1\}} Z_{\hat{F}_{m\times n},D_{\theta,\tau}} \tag{16}$$

Proposition 4.4. *Assume all the triangle edges have weight 1, and all the non-triangle edges have weight not equal to 1. Assume around each triangle, an even number of edges have weight strictly less than 1. Assume the size of the graph m and n are even, and the number of edges crossed by γ_x and γ_y with weight strictly less than 1 are both even. Then $P(z,-1)=0$ has no solution on the unit circle $\mathbb{T}$.*

Proof. In this proof we use $Z_{\theta,\tau}$ to denote $Z_{\hat{F}_{m\times n},D_{\theta,\tau}}$ for simplicity, when $\theta,\tau\in\{0,1\}$.

When both m and n are even, we have (see [8, 21])

$$\begin{aligned}
\mathrm{Pf}K(1,1) &= Z_{0,0}-Z_{0,1}-Z_{1,0}-Z_{1,1}\\
\mathrm{Pf}K(1,-1) &= Z_{0,0}+Z_{0,1}-Z_{1,0}+Z_{1,1}\\
\mathrm{Pf}K(-1,1) &= Z_{0,0}-Z_{0,1}+Z_{1,0}+Z_{1,1}\\
\mathrm{Pf}K(-1,-1) &= Z_{0,0}+Z_{0,1}+Z_{1,0}-Z_{1,1}
\end{aligned}$$

By (16), $\mathrm{Pf}K(1,-1)>0$, $\mathrm{Pf}K(-1,1)>0$, $\mathrm{Pf}K(-1,-1)>0$, given all the edge weights are strictly positive. Hence $P(z,-1)$ has no real roots on $\mathbb{T}$. In fact, if $P(z,-1)$ has real roots on $\mathbb{T}$, then either $P(1,-1)=0$, or $P(-1,-1)=0$. But $P(1,-1)=[\mathrm{Pf}K(1,-1)]^2>0$, and $P(-1,-1)=[\mathrm{Pf}K(-1,-1)]^2>0$.

Assume $P(z,-1)=0$ has a non-real zero $z_0\in\mathbb{T}$. Let $\alpha_0\in(0,1)\cup(1,2)$ satisfy

$$z_0=e^{i\alpha_0\pi}.$$

If α_0 is rational, namely $\alpha_0=\frac{p}{q}$, then after enlarging the fundamental domain to $m\times qn$, $P_q(z^q,-1)=0$, while z^q is real, which is impossible.

Now let us consider the case of irrational α_0. There exists a sequence $\ell_k\in\mathbb{N}$, such that

$$\lim_{k\to\infty} z_0^{\ell_k}=1$$

In other words, if we assume $z_0^{\ell_k}=e^{i\alpha_k\pi}$ where $\alpha_k\in(-1,1)$, then

$$\lim_{k\to\infty}\alpha_k=0.$$

According to the formula of enlarging the fundamental domain,

$$P_{\ell_k}(z_0^{\ell_k},-1)=0 \qquad \forall k$$

By (16),

$$P(1,-1)=(Z_{0,0}+Z_{0,1}-Z_{1,0}+Z_{1,1})^2\geq(Z_{0,1}+Z_{1,1})^2$$

Therefore we have

$$1 \leq \liminf_{k\to\infty}\left|\frac{P_{\ell_k}(z_0^{\ell_k},-1)-P_{\ell_k}(1,-1)}{(Z_{\ell_k,01}+Z_{\ell_k,11})^2}\right| \tag{17}$$

On the other hand

$$
\begin{aligned}
&\liminf_{k\to\infty}\left|\frac{P_{\ell_k}(z_0^{\ell_k},-1)-P_{\ell_k}(1,-1)}{(Z_{\ell_k,01}+Z_{\ell_k,11})^2}\right| \\
= &\liminf_{k\to\infty}\frac{1}{(Z_{\ell_k,01}+Z_{\ell_k,11})^2}|P_{\ell_k}(e^{i\alpha_k\pi},-1)-P_{\ell_k}(e^{i0\pi},-1)| \\
= &\liminf_{k\to\infty}\frac{1}{(Z_{\ell_k,01}+Z_{\ell_k,11})^2}\left|\sum_{t=1}^{\infty}\frac{[\pi\alpha_k]^{2t}}{(2t)!}\frac{\partial^{2t}P_{\ell_k}(e^{i\theta},-1)}{\partial\theta^{2t}}|_{\theta=0}\right| \qquad (18)
\end{aligned}
$$

Since

$$
P_{\ell_k}(z,-1)=\sum_{0\leq j\leq m}P_j^{(\ell_k)}\left(z^j+\frac{1}{z^j}\right).
$$

where $P_j^{(\ell_k)}$ is the signed sum of oriented loop configurations winding exactly j times in the z-direction (where if two cycles are winding around the cylinder in opposite directions, we count them as a configuration winding 0 times). We have

$$
\frac{\partial^{2t}P_{\ell_k}(e^{i\theta},-1)}{\partial\theta^{2t}}|_{\theta=0}=\sum_{1\leq j\leq m}2j^{2t}(-1)^tP_j^{(\ell_k)},
$$

and

$$
\begin{aligned}
&\liminf_{k\to\infty}\frac{1}{(Z_{\ell_k,01}+Z_{\ell_k,11})^2}\left|\sum_{t=1}^{\infty}\frac{(\pi\alpha_k)^{2t}}{(2t)!}\frac{\partial^{2t}P_{\ell_k}(e^{i\theta},-1)}{\partial\theta^{2t}}|_{\theta=0}\right| \\
&\leq\liminf_{k\to\infty}\frac{1}{(Z_{\ell_k,01}+Z_{\ell_k,11})^2}\sum_{t=1}^{\infty}\frac{(\pi m\alpha_k)^{2t}}{(2t)!}2\sum_{1\leq j\leq m}|P_j^{(\ell_k)}|.
\end{aligned}
$$

Using the same technique as described in the proof of Theorem 1.1, we have

$$
\begin{aligned}
\sum_{1\leq j\leq m}|P_j^{(\ell_k)}| &\leq \text{Partition function of configurations including essential odd loops} \\
&\quad+\text{Partition function of configurations with only even loops} \\
&\leq (C_0^mC_1^{6m}+1)\text{Partition function of even loop configurations}\leq C_2^mZ_{\ell_k}^2.
\end{aligned}
$$

Moreover, there is a one-to-one correspondence between dimer configurations in $Z_{0,0}$ and $Z_{0,1}$, similarly between dimer configurations in $Z_{1,0}$ and $Z_{1,1}$. We divide the $m\times\ell_kn$ torus into ℓ_kn bands, each band has circumference m. Fix one band $\mathcal{C}_0$, and fix configurations outside the band and on the boundary of the band. For each dimer configuration in $Z_{0,0}$, if we rotate the configuration to alternating edges along $\mathcal{C}_0$, we get a dimer configuration in $Z_{0,1}$. An example of such an transformation is illustrated in the Figures 4.5 and 4.6.

Since we have finitely many edge weights and each edge weight is strictly positive, we have

$$
\frac{1}{C^m}Z_{\theta,1}\leq Z_{\theta,0}\leq C^mZ_{\theta,1},\qquad \text{for } \theta\in\{0,1\},
$$

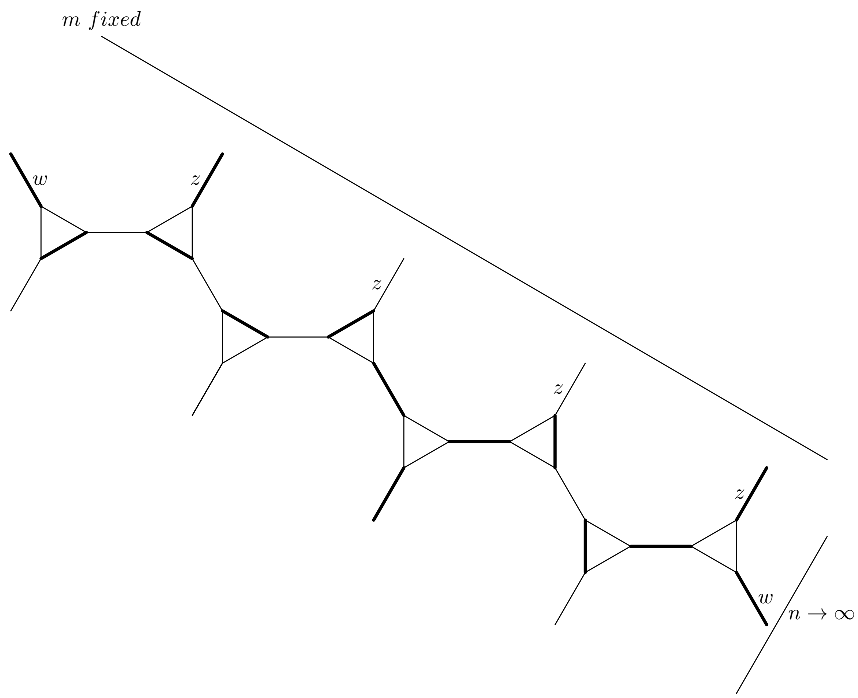

Figure 4.5: Before change of configurations on $\mathcal{C}_0$

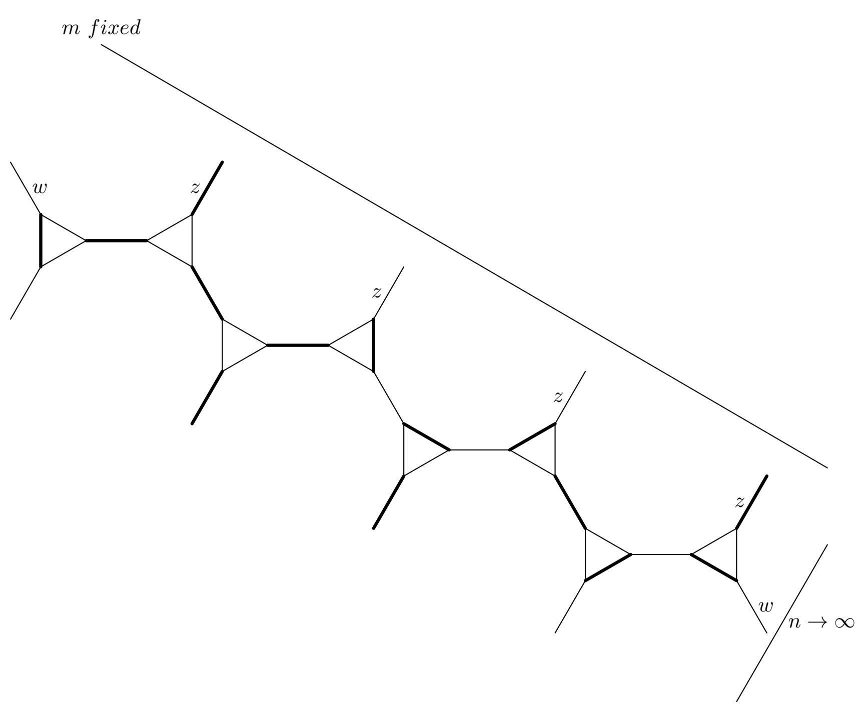

Figure 4.6: After change of configurations on $\mathcal{C}_0$ to alternating edges

as a result,

$$(Z_{\ell_k,00}+Z_{\ell_k,01}-Z_{\ell_k,10}+Z_{\ell_k,11})^2 \le C^m (Z_{\ell_k,01}+Z_{\ell_k,11})^2$$

where C is a constant independent of k. Hence

$$\begin{aligned}
& \liminf_{k\to\infty} \frac{1}{(Z_{\ell_k,01}+Z_{\ell_k,11})^2} \left| \sum_{t=1}^{\infty} \frac{(\pi\alpha_k)^{2t}}{(2t)!} \frac{\partial^{2t} P_k(e^{i\theta},-1)}{\partial\theta^{2t}} \right|_{\theta=0} \right| \\
\le\ & \liminf_{k\to\infty} \frac{1}{(Z_{\ell_k,01}+Z_{\ell_k,11})^2} \sum_{t=1}^{\infty} \frac{(m\pi\alpha_k)^{2t}}{(2t)!} C_2^m Z_{\ell_k}^2 \\
\le\ & \liminf_{k\to\infty} C_3^m \sum_{t=1}^{\infty} \frac{(m\pi\alpha_k)^{2t}}{(2t)!}
\end{aligned}$$

Since m is a constant, and $\lim_{k\to\infty}\alpha_k = 0$, we have

$$\lim_{k\to\infty} \sum_{t=1}^{\infty} \frac{(m\pi\alpha_k)^{2t}}{(2t)!} C_3^m = 0$$

which is a contradiction to (18). Therefore the proposition follows.

□

Lemma 4.5. *Let $P(z,w)$ be the characteristic polynomial as in Theorem 1.2, then*

$$P(z,w) \ge 0, \qquad \forall (z,w) \in \mathbb{T}^2.$$

Proof. Assume there exists $(z_0,w_0)\in\mathbb{T}^2$ such that $P(z_0,w_0)<0$. Since $P(z,-1)>0$ and $P(-1,w)>0$, if we consider $z=e^{i\theta}, w=e^{i\phi}$, $z_0=e^{i\theta_0}$, $w_0=e^{i\phi_0}$, and $(\theta,\phi)\in[-\pi,\pi]^2$, on the boundary of the domain $[-\pi,\pi]^2$, $P(e^{i\theta},e^{i\phi})$ is strictly positive, while $P(e^{i\theta_0},e^{i\phi_0})<0$, by continuity there exists a neighborhood $O_h=(\phi_0-h,\phi_0+h)$, such that for any $\phi\in O_h$, $P(e^{i\theta_0},e^{i\phi})<0$. Consider straight lines in $[-\pi,\pi]^2$, connecting (θ_0,ϕ) to (π,ϕ), namely

$$\gamma_\phi(t) = ((\pi-\theta_0)t+\theta_0,\phi)$$

for $\phi\in(\phi_0-h,\phi_0+h)$, there exists t_ϕ, such that

$$P(e^{i[(\pi-\theta_0)t_\phi+\theta_0]}, e^{i\phi}) = 0$$

By continuity, there exists a ϕ in the open interval (ϕ_0-h,ϕ_0+h) such that $\exp(i\phi)$ a root of -1, i.e. there exists an integer s, such that $\exp(is\phi)=-1$. For instance $\phi=\frac{p}{q}\pi$ in reduced form where p is odd, and $\phi=\frac{p}{q}\pi\in(\phi_0-h,\phi_0+h)$. Then after enlarging the fundamental domain to $qm\times n$, we have $P(e^{i[(\pi-\theta_0)t_\phi+\theta_0]},-1)=0$, which is a contradiction.

□

There is a simpler argument to go from $P(-1,-1)>0$ to $P(z,w)\ge 0$ on $\mathbb{T}^2$ in [2].

4.4 Correspondence to Perfect Matchings on a Bipartite Graph

In this section we discuss an explicit correspondence between the Kasteleyn operators on the square-octagon lattice and the Fisher graph associated to the Ising model, see also [22, 4].

The square-octagon lattice is constructed from $\mathbb{Z}^2$ as follows. Assume $\mathbb{Z}^2 = (V, E)$, where V is the vertex set, and E is the edge set. We first introduce a medial graph, $M = (V_M, E_M)$ of $\mathbb{Z}^2$: the vertex set V_M of M consists of the midpoints of edges of $\mathbb{Z}^2$; two vertices of M are adjacent if they correspond to edges sharing both a vertex and a face of $\mathbb{Z}^2$. Note that M is 4-regular (all vertices have degree 4), and can also be constructed as the medial graph of $\mathbb{Z}_d^2$ (the dual graph of $\mathbb{Z}^2$). Let F be the set of edges of $\mathbb{Z}^2$. Every edge e of M corresponds to two edges of $\mathbb{Z}^2$ that have a common endpoint $v \in V$ and lie on the boundary of a common face $f \in F$. This establishes a natural correspondence between edges of M and pairs $(v, f) \in V \times F$, where v is a vertex on the boundary of F.

We can color the faces of the 4-regular graph M by black and white such that black faces are incident only to white faces and vice versa. Starting from M, we can construct a square-octagon lattice C by replacing each vertex v with a city, consisting of a rectangle with a vertex on each edge of M incident to v. Each city has four internal edges and two adjacent cities are connected by a road edge. In fact, vertices of C correspond to pairs (v, e), $v \in V_M, e \in E_M$ incident to v, $((v, e), (v', e'))$ is an edge of C iff $e = e' = (vv')$ (road) or $v = v'$ and e, e' are two consecutive (in cyclic order) edges around v (city street), as illustrated in Figure 4.7, where the Ising graph $\mathbb{Z}^2$ is represented by dashed lines, and the 4-regular graph M is represented by dotted lines.

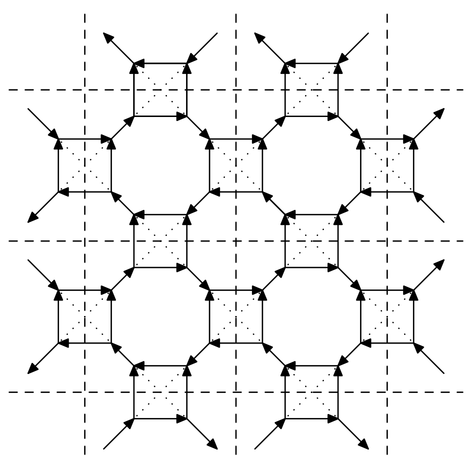

Figure 4.7: Ising model and bipartite graph

We assign edge weights to the graph C as follows. Each road edge connecting different cities has weight 1. Each horizontal internal edge of a city has weight $(2e^{-2J_e})/(1+e^{-4J_e})$, while each vertical edge of a city has weight $(1-e^{-4J_e})/(1+e^{-4J_e})$.

Starting from $\mathbb{Z}^2$, we also construct a Fisher graph $\mathcal{FF} = (V_{\mathcal{FF}}, E_{\mathcal{FF}})$. Namely, we replace each vertex by a gadget, see Figure 4.8. The weights of edges of $\mathcal{FF}$ connecting different gadgets inherit the edge weight e^{-2J_e} — the weight of the corresponding edge of $\mathbb{Z}^2$. All the other edges of $\mathcal{FF}$ have weight 1. It is known that there is a correspondence between Ising configurations on $\mathbb{Z}^2_d$, the dual square grid, and dimer configurations on $\mathcal{FF}$. Namely, if the two neighbors have the opposite sign, then the edge separating $\mathcal{FF}$ is present in the dimer configuration, otherwise it is not present.

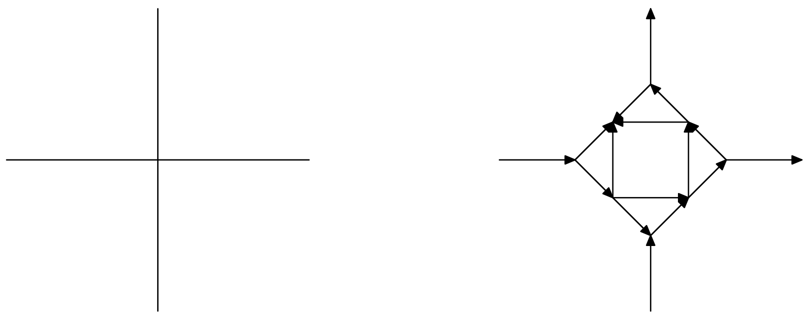

Figure 4.8: Ising model and Fisher graph

The clockwise-odd orientation on $\mathcal{FF}$ is illustrated in Figure 4.8.

The following lemma is proved in [4].

Lemma 4.6. *Let $K_C^{BW} : \mathbb{R}^{B(V_C)} \to \mathbb{R}^{W(V_C)}$ and $K_{\mathcal{FF}} : \mathbb{R}^{V_{\mathcal{FF}}} \to \mathbb{R}^{V_{\mathcal{FF}}}$ be the corresponding Kasteleyn operators, where $B(V_C)$ (resp. $W(V_C)$) is the set of black (resp. white) vertices in V_C, since C is a bipartite graph. Assume $\mathbb{Z}^2$ is endowed with periodic weights, and $P_C(z,w) = \det K_C^{BW}(z,w)$, $P_{\mathcal{FF}}(z,w) = \det K_{\mathcal{FF}}(z,w)$ be the corresponding characteristic polynomials, then*

$$P_{\mathcal{FF}}(z,w) = cP_C(z,w),$$

where $c > 0$ is a constant independent of (z,w).

4.5 Proof of Theorem 1.2

In this subsection, we prove that the intersection of the spectral curve of Fisher graphs associated to any periodic, ferromagnetic Ising model, with the unit torus $\mathbb{T}^2 = \{(z,w) : |z| = 1, |w| = 1\}$, is either empty or a single real intersection of multiplicity 2. The fact that the spectral curve is Harnack follows from the correspondence as in Sect. 4.4.

The Fisher graph we consider in this paper is the one obtained by replacing each vertex of the hexagonal lattice with a triangle, see Figure 1. It can also be obtained by replacing each vertex of $\mathbb{Z}^2$ by a gadget as in Figure 4.9. Let $\mathcal{F}$ denote the resulting Fisher graph. We consider the expansion of $\det K_{\mathcal{F}}(z,w)$. Each term corresponds to an oriented loop configuration on the torus, consisting of oriented loops and doubled edges such that each vertex is occupied exactly twice. We consider the gadget on the right graph of Figure 13 and the right graph of Figure 14. Obviously there is a natural correspondence between outer edges of each gadget of $\mathcal{FF}$ and the outer edges of

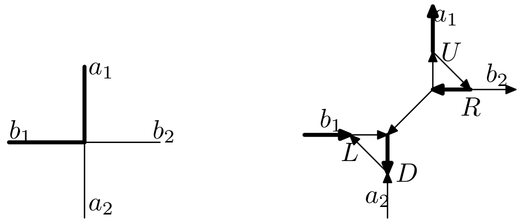

Figure 4.9: vertex of Ising model and corresponding dimer gadget

each gadget of $\mathcal{F}$. Assume each pair of corresponding outer edges in $\mathcal{F}\mathcal{F}$ and $\mathcal{F}$ have the configuration. Fixing the configurations on outer edges, we enumerate all the possible oriented loop configurations on each gadget, and compute the weighted sum. Given the orientations as illustrated in the figure, for each fixed configuration on the outer edges, the weighted sum of all possible configurations for a gadget of $\mathcal{F}\mathcal{F}$ is always 2 times the weighted sum of all possible configurations for the corresponding gadget of $\mathcal{F}$. In particular, both of them have the same sign. As a result, if we have an $m \times n$ torus, then

$$\det K_{\mathcal{F}\mathcal{F}}(z,w) = 2^{mn} \det K_{\mathcal{F}}(z,w) = 2^{mn} P_{\mathcal{F}}(z,w)$$

Hence $P_{\mathcal{F}}(z,w)$ and $P_C(z,w)$ have the same zero locus. $P_C(z,w)$ is the characteristic polynomial of a bipartite graph with positive edge weights and clockwise-odd orientation, it is proved in [12] that $P_C(z,w) = 0$ is a Harnack curve, i.e., for any fixed $x, y > 0$ the number of zeros of $P_C(z,w)$ on the torus $\mathbb{T}_{x,y} = \{(z,w) : |z| = x, |w| = y\}$ is at most 2 (counting multiplicities). Hence the number of zeros of $P_{\mathcal{F}}(z,w)$ on the unit torus is at most 2. Since $P_{\mathcal{F}}(z,w) \geq 0$, $\forall (z,w) \in \mathbb{T}^2$, each zero $P_{\mathcal{F}}(z,w)$ on $\mathbb{T}^2$ is at least of multiplicity 2. If $P_{\mathcal{F}}(z,w)$ has a non-real zero (z_0, w_0) on $\mathbb{T}^2$, then $(\bar{z}_0, \bar{w}_0) \neq (z_0, w_0)$ is also a zero of $P_{\mathcal{F}}(z,w)$, and each of them is at least of multiplicity 2. Then $P_{\mathcal{F}}(z,w)$ has at least 4 zeros on $\mathbb{T}^2$, which is a contradiction to the fact that $P_{\mathcal{F}}(z,w) = 0$ is a Harnack curve. Hence the intersection of $P_{\mathcal{F}}(z,w) = 0$ with $\mathbb{T}^2$ is either empty or a single real intersection of multiplicity 2.

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