Symbols

Symbol Description

$T(G; x, y)$ The Tutte polynomial of G $C_G(q)$ The flow polynomial of G

$T_G(x, y)$ The Tutte polynomial of G

$W_G(x, y)$ The rank generating function of G χ_G The chromatic polynomial of G

1

The Potts and random-cluster models

Geoffrey Grimmett

University of Cambridge
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge CB3 0WB, UK
g.r.grimmett@statslab.cam.ac.uk

CONTENTS

1.1   Synopsis

- Ising, Potts, and random-cluster models

- Basic properties of random-cluster measures

- Flow polynomial

- Limit as $q \downarrow 0$

- Zero-temperature limit

- Asymptotics of the Tutte polynomial on the complete graph

1.2   Introduction

The four principal elements in this chapter are the Ising model of 1925, the Tutte polynomial of 1947, the Potts model of 1952, and the random-cluster model of 1972.

The *Ising model*[1] [21] is the fundamental model for the ferromagnet, and it has generated enormous interest and activity in mathematics and physics over the intervening decades. The *Potts model* [26] extends the number of local states of the Ising model from 2 to a general number q. The *random-cluster model* of Fortuin and Kasteleyn [13] provides an overarching framework for the Ising/Potts models that incorporates percolation and electrical networks, together with certain other processes. The common aspect of importance for these three systems is the singularity that occurs at points of phase transition.

Whereas these three processes originated in mathematical physics, the *Tutte polynomial* [31] is an object from combinatorics, and it encapsulates a number of significant features of a graph or matroid. It turns out that the Tutte polynomial is equal (subject to a change of variables) to the partition function of the random-cluster model, and therefore to that of the Potts model. This connection is not a coincidence since both the Tutte and random-cluster functions arose in independent analyses of local graph operations such as deletion and contraction.

The Tutte polynomial originated in Tutte's exploration of deletion and contraction on a finite graph. The random-cluster model originated similarly in Kasteleyn's observation that the Ising, Potts, and percolation models, and also electrical networks, have a property of invariance under series and parallel operations on edges.

---

[1]The Ising model was proposed to Ising by Lenz, and the Potts model to Potts by Domb.

Combinatorial theory and statistical mechanics are areas of science which have much in common, while retaining their distinctive characteristics. Statistical mechanics is mostly concerned with the structure of phases and of singularities, and has developed appropriate methodology and language. Although, in principle, the properties of a physical model are encoded entirely within its partition function, the extraction of such properties is often challenging and hinges frequently on other factors such as the nature of the underlying graph.

The connection between the Tutte polynomial and statistical mechanics is summarised in this chapter, as follows. We aim: (i) to give a clear formulation of the relevant models, (ii) to explain the connection between their partition functions and the Tutte polynomial, (iii) to present some of the basic properties of the random-cluster model that are contingent on the partition function, and (iv) to present a selection of open problems concerning Potts and random-cluster models that may be related to the Tutte polynomial.

1.3 Probabilistic Models from Physics

1.3.1 The Ising and Potts ferromagnets

The Ising model for ferromagnetism was analysed in one dimension in Ising's thesis and 1925 paper [21]. It modelled the following physical experiment. A piece of iron is placed in a magnetic field, with an intensity that is increased from zero to a maximum, and then reduced to zero. The iron retains some residual magnetisation if and only if the temperature is sufficiently low, and the critical temperature for this phenomenon is called the *Curie point*.

The Ising model may be summarised as follows. Suppose that particles are placed at the vertices of a graph embedded in a Euclidean space. Each particle may be in either of two states: spin 'up' or spin 'down'. Spin-values are chosen at random according to a certain probability measure governed by interactions between neighbouring particles. This measure is described as follows.

Let $G = (V, E)$ be a finite, simple graph. Each vertex $v \in V$ is occupied by a particle with a random spin. Since spins are assumed to come in two basic types, we take as sample space the set $\Sigma = \{-1, +1\}^V$ of vectors $\sigma = (\sigma_x : x \in V)$ with entries ± 1.

Definition 1.3.1. Let $\beta \in (0, \infty)$ and $h \in \mathbb{R}$. The Ising probability measure $\lambda_{\beta,h}$ on G is given by

$$\lambda_{\beta,h}(\sigma) = \frac{1}{Z_{\mathrm{I}}} e^{-\beta H_{\mathrm{I}}(\sigma)}, \qquad \sigma \in \Sigma,$$

where the *partition function* Z_I and the *Hamiltonian* H_I are given by

$$Z_\mathrm{I}(\beta, h) = \sum_{\sigma \in \Sigma} e^{-\beta H_\mathrm{I}(\sigma)}, \qquad H_\mathrm{I}(\sigma) = -\sum_{e = \langle x,y \rangle \in E} \sigma_x \sigma_y - h \sum_{x \in V} \sigma_x.$$

The parameter β represents the reciprocal $1/T$ of temperature, and h is the *external field*. For reasons of simplicity, we assume usually that $h = 0$, and we write $\lambda_\beta = \lambda_{\beta,0}$. It is usual to include also an edge-interaction J, which we have chosen to include within the parameter β.

The Ising model has two admissible spin-values, and a very rich theory. In his 1952 paper [26], Potts developed an extension of the Ising model to a general number of spin-values.

Let q be an integer satisfying $q \geq 2$, and consider the sample space $\Sigma = \{1, 2, \ldots, q\}^V$. Each vertex of G may now be in any of q states.

Definition 1.3.2. Let $\beta \in (0, \infty)$ and $q \in \{2, 3, \ldots\}$. The Potts probability measure on G is given by

$$\pi_{\beta,q}(\sigma) = \frac{1}{Z_\mathrm{P}} e^{-\beta H_\mathrm{P}(\sigma)}, \qquad \sigma \in \Sigma,$$

where the Potts partition function Z_P and Hamiltonian H_P are given by

$$Z_\mathrm{P}(\beta, q) = \sum_{\sigma \in \Sigma} e^{-\beta H_\mathrm{P}(\sigma)}, \qquad H_\mathrm{P}(\sigma) = -\sum_{e = \langle x,y \rangle} \delta_{\sigma_x, \sigma_y},$$

and $\delta_{u,v}$ is the Kronecker delta.

When $q = 2$, we have that

$$\delta_{\sigma_x, \sigma_y} = \tfrac{1}{2}(1 + \sigma_x \sigma_y),$$

from which it follows that the $q = 2$ Potts model is simply the Ising model with β replaced by $\frac{1}{2}\beta$.

In a more general definition, one may include a non-zero external field h and a vector **J** of edge-parameters. See Section 1.3.5.

1.3.2   The random-cluster model

The random-cluster model was formulated in a series of papers [11, 12, 13] by Fortuin and Kasteleyn. It is described next, and its relationship to the Potts model is explained in Section 1.3.3.

Let $G = (V, E)$ be a finite, simple graph. The relevant state space is the set $\Omega = \{0, 1\}^E$ of vectors $\omega = (\omega(e) : e \in E)$ with 0/1 entries. An edge e as said to be *open* in $\omega \in \Omega$ if $\omega(e) = 1$, and *closed* if $\omega(e) = 0$. For $\omega \in \Omega$, let $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ denote the set of open edges, and let $k(\omega)$ be the number of connected components (or 'open clusters') of the graph $(V, \eta(\omega))$; the count $k(\omega)$ includes the number of isolated vertices.

Definition 1.3.3. Let $p \in (0,1)$ and $q \in (0, \infty)$. The *random-cluster measure* $\phi_{p,q}$ on G is given by

$$\phi_{p,q}(\omega) = \frac{1}{Z_{\mathrm{RC}}} \left\{ \prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \qquad \omega \in \Omega,$$

where the partition function Z_{RC} is given by

$$Z_{\mathrm{RC}}(p,q) = \sum_{\omega \in \Omega} \left\{ \prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega)}. \qquad (1.1)$$

The most important values of q are arguably the positive integers. When $q = 1$, $\phi_p := \phi_{p,1}$ is a product measure, and the words 'percolation' and 'random graph' are often used in this context, see [15, 22]. The random-cluster model with $q \in \{2, 3, \dots\}$ corresponds, as sketched in the next section, to the Potts model with q local states.

See [16] for the general theory of the random-cluster model.

1.3.3 Coupling of the Potts and random-cluster measures

Let $q \in \{2, 3, \dots\}$, $p \in (0,1)$, and let $G = (V, E)$ be a finite, simple graph. We consider the product sample space $\Sigma \times \Omega$ where $\Sigma = \{1, 2, \dots, q\}^V$ and $\Omega = \{0,1\}^E$ as above. Let μ be the probability measure on $\Sigma \times \Omega$ given by

$$\mu(\sigma, \omega) \propto \psi(\sigma)\phi_p(\omega)1_F(\sigma, \omega), \qquad (\sigma, \omega) \in \Sigma \times \Omega,$$

where ψ is uniform measure on Σ, $\phi_p = \phi_{p,1}$ is product measure on Ω with density p, and 1_F is the indicator function of the event that σ is constant on each open cluster of ω, that is,

$$F = \left\{ (\sigma, \omega) \in \Sigma \times \Omega : \sigma_x = \sigma_y \text{ for every } e = \langle x, y \rangle \text{ satisfying } \omega(e) = 1 \right\}.$$

The measure μ may be viewed as the product measure $\psi \times \phi_p$ conditioned on the event F.

Theorem 1.3.4. *Let $q \in \{2, 3, \dots\}$ and $p \in (0,1)$.*

(a) *Marginal on Σ. The first marginal of μ (on Σ) is the Potts measure $\pi_{\beta,q}$ where $p = 1 - e^{-\beta}$.*

(b) *Marginal on Ω. The second marginal of μ (on Ω) is the random-cluster measure $\phi_{p,q}$.*

(c) *Conditional measures.*

  (i) *Given $\omega \in \Omega$, the conditional measure on Σ is obtained by putting (uniformly) random spins on entire clusters of ω. These spins are constant on given clusters, and are independent between clusters.*

(ii) *Given $\sigma \in \Sigma$, the conditional measure on Ω is obtained as follows. For $e = \langle x, y \rangle \in E$, we set $\omega(e) = 0$ if $\sigma_x \neq \sigma_y$, and otherwise $\omega(e) = 1$ with probability p (independently of other edges).*

(c) Partition functions. *We have that*

$$Z_{\mathrm{RC}}(p, q) = e^{-\beta|E|} Z_{\mathrm{P}}(\beta, q). \tag{1.2}$$

This coupling may be used to show that correlations in Potts models correspond to connection probabilities in random-cluster models. In this way, one may harness methods of stochastic geometry in order to understand the correlation structure of the Potts system. The basic theorem of this type is Theorem 1.3.5, following.

The 'two-point correlation function' of the Potts measure $\pi_{\beta,q}$ on the finite graph $G = (V, E)$ is the function

$$\tau_{\beta,q}(x, y) := \pi_{\beta,q}(\sigma_x = \sigma_y) - \frac{1}{q}, \qquad x, y \in V. \tag{1.3}$$

Let $\{x \leftrightarrow y\}$ be the event of Ω on which there exists an open path joining vertex x to vertex y. The 'two-point connectivity function' of the random-cluster measure $\phi_{p,q}$ is the function $\phi_{p,q}(x \leftrightarrow y)$ for $x, y \in V$, that is, the probability that x and y are joined by a path of open edges. It turns out that these two-point functions the same up to a constant factor.

Theorem 1.3.5 (Correlation/connection theorem). *If $q \in \{2, 3, \ldots\}$ and $p = 1 - e^{-\beta} \in (0, 1)$, then*

$$\tau_{\beta,q}(x, y) = (1 - q^{-1})\phi_{p,q}(x \leftrightarrow y), \qquad x, y \in V. \tag{1.4}$$

The Potts models considered above have zero external field. Some complications arise when an external field is added; see Section 1.3.5.

1.3.4   Partition functions and the Tutte polynomial

The Potts and random-cluster partition functions may be viewed as evaluations of rank-generating functions and the Tutte polynomial, defined as follows. The (Whitney) *rank-generating function* of $G = (V, E)$ is the function

$$W_G(u, v) = \sum_{A \subseteq E} u^{r(A)} v^{c(A)}, \qquad u, v \in \mathbb{R},$$

where

$$r(A) = |V| - k(A), \qquad c(A) = |A| - |V| + k(A),$$

are the rank and co-rank of $G' = (V, A)$, respectively. Here, $k(A)$ denotes the number of components of G'. The Tutte polynomial may be written as

$$T_G(u, v) = (u - 1)^{|V|-1} W_G\big((u - 1)^{-1}, v - 1\big).$$

Theorem 1.3.6. *Let* $p \in (0, 1)$, $q \in (0, \infty)$, *and*

$$u - 1 = \frac{q(1 - p)}{p}, \qquad v - 1 = \frac{p}{1 - p}.$$

(a) *The partition function* $Z_{\mathrm{RC}} = Z_{\mathrm{RC}}(G)$ *of the random-cluster measure on G with parameters p, q satisfies*

$$Z_{\mathrm{RC}}(G) = \frac{(u - 1)(v - 1)^{|V|}}{v^{|E|}} T_G(u - 1, v - 1).$$

(b) *If $q \in \{2, 3, \dots\}$ and $p = 1 - e^{-\beta}$, the partition function of the q-state Potts model satisfies*

$$Z_{\mathrm{P}}(\beta, q) = (u - 1)(v - 1)^{|V|} T_G(u - 1, v - 1).$$

1.3.5 Potts extensions

The Potts model of Definition 1.3.2 is in its simplest form in three senses: (i) each edge plays an equal (deterministic) role, (ii) the external field satisfies $h = 0$, and (iii) the model is *ferromagnetic*. More generally, one may consider the partition function $Z_{\mathrm{P}}(\beta, \mathbf{J}, h) = \sum_{\sigma \in \Sigma} e^{-\beta H_{\mathrm{P}}(\sigma)}$ where the Hamiltonian is given by

$$H_{\mathrm{P}}(\sigma) = - \sum_{e = \langle x, y \rangle} J_e \delta_{\sigma_x, \sigma_y} - \sum_{j=1}^{q} \sum_{x \in V} h_j \delta_{\sigma_x, j}. \tag{1.5}$$

Here, $\mathbf{J} = (J_e : e \in E)$ is a family of edge-parameters assumed to satisfy $J_e \neq 0$, and $\mathbf{h} = (h_x : x \in V)$ is a vector of external fields, one for each possible local spin $1, 2, \dots, q$. The model is termed *ferromagnetic* if $J_e > 0$ for $e \in E$, and *purely antiferromagnetic* if $J_e < 0$ for all $e \in E$. The general Potts partition function of (1.5) poses some new difficulties.

Assume first that $\mathbf{h} = \mathbf{0}$. The associated random-cluster formula yields a function $\phi_{\mathbf{p}, q}$ where $p_e = 1 - e^{-\beta J_e}$. If $J_e < 0$ for some e, this does not define a probability measure. In addition, the Potts model does not satisfy the range of correlation inequalities that hold in the ferromagnetic case. On the other hand, Theorem 1.3.6(b) is easily extended for general $\mathbf{J}$ to a multivariate Tutte polynomial on $G = (V, E)$ which may be written in the form

$$T_G(q, \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e, \tag{1.6}$$

where q and $\mathbf{v} = (v_e : e \in E)$ are viewed as parameters. See [9, 30] for recent accounts.

The J_e may themselves be random, in which case the model is termed *quenched*, in contrast to the *annealed* case in which one averages initially over

the J_e. If the probability distribution of the J_e allocates mass to both positive and negative values, the system is a 'spin glass'. See [24].

When $\mathbf{h} \neq \mathbf{0}$, a form of the Tutte–Potts correspondence may be found in [6], where positive association and infinite-volume limits are explored, and also in a slightly more general setting in [9] (see also [25]).

1.4   Basic Properties of Random-Cluster Measures

This section includes some of the basic properties of a random-cluster measure on the finite graph $G = (V, E)$.

1.4.1   Stochastic ordering

The state space $\Omega = \{0, 1\}^E$ is a partially ordered set with partial order given by: $\omega_1 \leq \omega_2$ if $\omega_1(e) \leq \omega_2(e)$ for all $e \in E$. This partial order is extremely useful in the analysis of Potts and random-cluster models, and it induces partial orderings on the spaces of associated functions and measures.

Definition 1.4.1.

(a) A random variable $f : \Omega \to \mathbb{R}$ is called *increasing* if $f(\omega_1) \leq f(\omega_2)$ whenever $\omega_1 \leq \omega_2$.

(b) An event $A \subseteq \Omega$ is called *increasing* if its indicator function 1_A is increasing.

(c) Given two probability measures μ_1, μ_2 on Ω, we write $\mu_1 \leq_{\mathrm{st}} \mu_2$, and say that μ_1 is stochastically smaller than μ_2, if $\mu_1(f) \leq \mu_2(f)$ for all increasing random variables f on Ω.

Arguably the most useful approach to stochastic ordering is due to Holley. We obtain the following comparison inequalities as corollaries of Holley's inequality, see [20] and [16, Thm 2.1].

Theorem 1.4.2 (Comparison inequalities)**.** *It is the case that*

$$\phi_{p',q'} \leq_{\mathrm{st}} \phi_{p,q} \qquad if \quad q' \geq q, \ q' \geq 1, \ and \ p' \leq p,$$

$$\phi_{p',q'} \geq_{\mathrm{st}} \phi_{p,q} \qquad if \quad q' \geq q, \ q' \geq 1, \ and \ \frac{p'}{q'(1-p')} \geq \frac{p}{q(1-p)}.$$

1.4.2   Positive association

Holley's inequality admits a neat proof of the FKG inequality of [14]. This amounts to the following in the case of random-cluster measures.

Theorem 1.4.3 (Positive association). *Let $p \in (0,1)$ and $q \in [1,\infty)$. If f and g are increasing functions on Ω, then*

$$\phi_{p,q}(fg) \geq \phi_{p,q}(f)\phi_{p,q}(g).$$

Specializing to indicator functions $f = 1_A$, $g = 1_B$, we obtain that

$$\phi_{p,q}(A \cap B) \geq \phi_{p,q}(A)\phi_{p,q}(B) \qquad \text{for increasing events } A, B,$$

whenever $q \geq 1$. Positive association is generally false when $0 < q < 1$.

1.5 The Limit as $q \downarrow 0$

1.5.1 UST, USF, and USC

Some interesting limits with combinatorial flavours arise from consideration of $\phi_{p,q}$ as $q \downarrow 0$. Write Ω_{for}, Ω_{st}, Ω_{cs} for the subsets of Ω containing all forests, spanning trees, and connected subgraphs, respectively, and write USF, UST, and UCS for the uniform probability measures on the respective sets Ω_{for}, Ω_{st}, Ω_{cs}. An account of the following limits and their history may be found at [16, Thm 1.2].

Theorem 1.5.1. *We have in the limit as $q \downarrow 0$ that:*

$$\phi_{p,q} \Rightarrow \begin{cases} \text{UCS} & \text{if } p = \frac{1}{2}, \\ \text{UST} & \text{if } p \to 0 \text{ and } q/p \to 0, \\ \text{USF} & \text{if } p = q. \end{cases}$$

The spanning tree limit UST is especially interesting for historical and mathematical reasons. The random-cluster model originated in a systematic study by Fortuin and Kasteleyn of systems of a certain type which satisfy certain parallel and series laws. Electrical networks are the best known such systems: two resistors of resistances r_1 and r_2 in parallel (respectively, in series) may be replaced by a single resistor with resistance $(r_1^{-1}+r_2^{-1})^{-1}$ (respectively, $r_1 + r_2$). Fortuin and Kasteleyn realized that the electrical-network theory of a graph G is related to the limit as $q \downarrow 0$ of the random-cluster model on G, where p is given by $p = \sqrt{q}/(1 + \sqrt{q})$. It has been known since Kirchhoff's theorem [23] that the electrical currents which flow in a network may be expressed in terms of counts of spanning trees.

The theory of the uniform spanning tree measure UST is beautiful in its own right, and is linked in an important way to the emerging field of stochastic growth processes of 'stochastic Löwner evolution' (SLE) type. See [18, Chap. 2].

1.5.2   Negative association

Let E be a finite set, and let μ be a probability measure on the space $\Omega = \{0,1\}^E$. There are several concepts of negative association, of which we present three here.

For $\omega \in \Omega$ and $F \subseteq E$, the cylinder event $\Omega_{F,\omega}$ generated by ω on F is given by

$$\Omega_{F,\omega} = \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for } e \in F\}.$$

For $E' \subseteq E$ and an event $A \subseteq \Omega$, we say that A *is defined on* E' if, for all $\omega \in \Omega$, we have that $\omega \in A$ if and only if $\Omega_{E',\omega} \subseteq A$. Let A and B be events in Ω. We define $A \square B$ to be the set of all vectors $\omega \in \Omega$ for which there exists $F \subseteq E$ such that $\Omega_{F,\omega} \subseteq A$ and $\Omega_{\overline{F},\omega} \subseteq B$, where $\overline{F} = E \setminus F$. Note that the choice of F is allowed to depend on the vector ω. The operator $\square$ originated in the work of van den Berg and Kesten [32] on the well known BK inequality.

Definition 1.5.2.

(a) The measure μ is *edge negatively associated* if

$$\mu(J_e \cap J_f) \leq \mu(J_e)\mu(J_f), \qquad e, f \in E, \ e \neq f,$$

   where J_e is the event that e is open.

(b) We call μ *negatively associated* if

$$\mu(A \cap B) \leq \mu(A)\mu(B)$$

   for all pairs (A, B) of increasing events with the property that there exists $E' \subseteq E$ such that A is defined on E' and B is defined on its complement $E \setminus E'$.

(c) We say that μ has the *disjoint occurrence property* if

$$\mu(A \square B) \leq \mu(A)\mu(B), \qquad A, B \subseteq \Omega.$$

Proposition 1.5.3. *We have that*

$$\mu \text{ has the disjoint occurrence property}$$
$$\Rightarrow \mu \text{ is negatively associated}$$
$$\Rightarrow \mu \text{ is edge negatively associated.}$$

It was proved by Reimer [27] that the product measures $\phi_{p,1}$ have the disjoint occurrence property. The random-cluster measure $\phi_{p,q}$ cannot (generally) be edge negatively associated when $q > 1$. It may be conjectured that $\phi_{p,q}$ satisfies some form of negative association when $q < 1$. Such a property would be very useful in studying random-cluster measures when $q < 1$.

In the absence of a satisfactory approach to the general case of random-cluster measures with $q < 1$, we turn next to the issue of negative association of $\phi_{p,q}$ in the limit as $q \downarrow 0$.

Conjecture 1.5.4. For any finite graph $G = (V, E)$, the uniform spanning forest measure USF and the uniform connected subgraph measure UCS are edge negatively associated.

A stronger version of this conjecture is that USF and UCS are negatively associated in one or both of the further senses described above. Numerical evidence for the conjecture is found in [19].

The UST measure is, in contrast, much better understood, owing to the theory of electrical networks and, more particularly, Kirchhoff's matrix–tree theorem, [23] and its ramifications. The following was proved by Feder and Mihail, [10].

Theorem 1.5.5. *The uniform spanning tree measure* UST *is negatively associated.*

The material in this section may be found in expanded form in [16, 18].

1.6 Flow Polynomial

1.6.1 Definition of the flow polynomial

We turn G into an oriented graph by allocating an orientation to each edge $e \in E$, and the resulting digraph is denoted $\vec{G} = (V, \vec{E})$. If the edge $e = \langle u, v \rangle \in E$ is oriented from u to v, we say that e *leaves* u and *arrives* at v.

Let $q \in \{2, 3, \dots\}$. A function $f : E \to \{0, 1, 2, \dots, q-1\}$ is called a *mod-q flow* on $\vec{G}$ if

$$\sum_{\substack{\vec{e} \in \vec{E}\,:\\ \vec{e}\ \text{leaves}\ v}} f(\vec{e}) - \sum_{\substack{\vec{e} \in \vec{E}\,:\\ \vec{e}\ \text{arrives at}\ v}} f(\vec{e}) = 0 \quad \text{modulo } q, \qquad v \in V, \qquad (1.1)$$

so that flow is conserved at every vertex. A mod-q flow f is called *non-zero* if $f(\vec{e}) \neq 0$ for all $\vec{e} \in \vec{E}$. Let $C_G(q)$ be the number of non-zero mod-q flows on $\vec{G}$.

It is standard that $C_G(q)$ does not depend on the orientations of the edges of G. The function $C_G(q)$, viewed as a function of q, is called the *flow polynomial* of G. We adopt the convention that $C_G(q) = 1$ if $E = \varnothing$.

The flow polynomial may be obtained as evaluations of the Tutte and rank generating polynomial with two particular parameter values, as follows,

$$\begin{aligned} C_G(q) &= (-1)^{|E|} W_G(-1, -q) \\ &= (-1)^{|E|-|V|+1} T_G(0, 1-q), \qquad q \in \{2, 3, \dots\}, \end{aligned} \qquad (1.2)$$

and thus C_G is indeed a polynomial. We shall later write $C(G; q)$ for $C_G(q)$, and similarly for other polynomials when the need arises.

1.6.2   Potts correlations and flow counts

The Potts correlation functions (1.3) may be expressed in terms of flow polynomials associated with a certain Poissonian random graph derived from G by replacing each edge by a random number of copies.

For a vector $\mathbf{m} = (m_e : e \in E)$ of non-negative integers, let $G_{\mathbf{m}} = (V, E_{\mathbf{m}})$ be the graph with vertex set V and, for each $e \in E$, with exactly m_e edges in parallel joining the endvertices of the edge e; the original edge e is removed.

Let $\beta \geq 0$, and let $\mathbf{P} = (P_e : e \in E)$ be a family of independent random variables such that P_e has the Poisson distribution with parameter β. The random graph $G_{\mathbf{P}} = (V, E_{\mathbf{P}})$ is called a *Poisson graph with intensity* β. Let $\mathbb{P}_\beta$ and $\mathbb{E}_\beta$ denote the corresponding probability measure and expectation.

For $x, y \in V$, let $G_{\mathbf{P}}^{x,y}$ denote the graph obtained from $G_{\mathbf{P}}$ by adding an edge with endvertices x, y. If x and y are adjacent in the original graph $G_{\mathbf{P}}$, we add a further edge between them. Potts correlations are related to flow counts as follows.

Theorem 1.6.1. *Let $q \in \{2, 3, \dots\}$ and $\beta \geq 0$. Then*

$$q\tau_{\beta,q}(x, y) = \frac{\mathbb{E}_\beta(C(G_{\mathbf{P}}^{x,y}; q))}{\mathbb{E}_\beta(C(G_{\mathbf{P}}; q))}, \qquad x, y \in V. \tag{1.3}$$

This formula is particularly striking when $q = 2$, since non-zero mod-2 flows take only the value 1. A finite graph $H = (W, F)$ is said to be *even* if every vertex has even degree. Evidently $C_H(2) = 1$ if H is even, and $C_H(2) = 0$ otherwise. By (1.3), for any graph G,

$$q\tau_{\beta,q}(x, y) = \frac{\mathbb{P}_\beta(G_{\mathbf{P}}^{x,y} \text{ is even})}{\mathbb{P}_\beta(G_{\mathbf{P}} \text{ is even})}, \tag{1.4}$$

when $q = 2$. This observation is central to the so called 'random-current expansion' of the Ising model, which has proved very powerful in the study of both classical and quantum Ising models. See [1, 2, 3, 4, 7].

Theorem 1.6.1 may be extended via (1.3.5) to the random-cluster model. The following is obtained by expressing the flow polynomial in terms of the Tutte polynomial T, and allowing q to vary continuously.

Theorem 1.6.2. *Let $p \in (0, 1)$, $q \in (0, \infty)$, and let β satisfy $p = 1 - e^{-\beta q}$.*

(i) *For $x, y \in V$,*

$$(q - 1)\phi_{G,p,q}(x \leftrightarrow y) = \frac{\mathbb{E}_\beta\big((-1)^{1+|E_{\mathbf{P}}|} T(G_{\mathbf{P}}^{x,y}; 0, 1 - q)\big)}{\mathbb{E}_\beta\big((-1)^{|E_{\mathbf{P}}|} T(G_{\mathbf{P}}; 0, 1 - q)\big)}. \tag{1.5}$$

(ii) *For $q \in \{2, 3, \dots\}$,*

$$Z_{\mathrm{RC}}(p, q) = \phi_{G,p}(q^{k(\omega)}) = (1 - p)^{|E|(q-2)/q} q^{|V|} \mathbb{E}_\beta(C(G_{\mathbf{P}}; q)). \tag{1.6}$$

Further details may be found in [17].

1.6.3 The random-current expansion when $q = 2$

Unlike the Potts model, there is a fairly complete analysis of the Ising model. A principal part in this analysis is played by Theorem 1.6.1 with $q = 2$ under the heading 'random-current expansion'. This has permitted proofs amongst other things of the exponential decay of correlations in the low-β regime on the cubic lattice $\mathbb{L}^d$ with $d \geq 2$. It has not so far been possible to extend this work to general Potts models, but Theorem 1.6.1 could play a part in such an extension.

Let $G = (V, E)$ be a finite graph and set $q = 2$. By Theorem 1.6.1,

$$2\tau_{\beta,2}(x,y) = \frac{\mathbb{P}_\beta(G_{\mathbf{P}}^{x,y} \text{ is even})}{\mathbb{P}_\beta(G_{\mathbf{P}} \text{ is even})}, \qquad 0 \leq \beta < \infty. \tag{1.7}$$

There is an important correlation inequality known as Simon's inequality, [28]. Let $x, z \in V$ be distinct vertices. A set W of vertices is said to *separate* x and z if $x, z \notin W$ and every path from x to z contains some vertex of W.

Theorem 1.6.3. *Let $x, z \in V$ be distinct vertices, and let W separate x and z. Then $\kappa_{\beta,2}(x,y) := 2\tau_{\beta,2}(x,y)$ satisfies*

$$\kappa_{\beta,2}(x,z) \leq \sum_{y \in W} \kappa_{\beta,2}(x,y)\kappa_{\beta,2}(y,z).$$

The Ising model corresponds to a random-cluster measure $\phi_{p,q}$ with $q = 2$. By (1.4),

$$\kappa_{\beta,q}(x,y) = \phi_{p,q}(x \leftrightarrow y),$$

where $p = 1 - e^{-\beta q}$ and $q = 2$. The Simon inequality may be written in the form

$$\phi_{p,q}(x \leftrightarrow z) \leq \sum_{y \in W} \phi_{p,q}(x \leftrightarrow y)\phi_{p,q}(y \leftrightarrow z) \tag{1.8}$$

whenever W separates x and z. It is well known that this inequality is valid also when $q = 1$, see [15, Chap. 6]. One may conjecture that it holds for any $q \in [1, 2]$.

1.7 The Limit of Zero Temperature

The physical interpretation of the constant β is as $\beta = 1/(kT)$ where k is Boltzmann's constant and T denotes (absolute) temperature. The limit $T \downarrow 0$ corresponds to the limit $\beta \to \infty$. The ferromagnetic Potts measure $\pi_{\beta,q}$ on a finite graph $G = (V, E)$ converges weakly to the probability measure that allocates a uniform random spin to each connected component of G, this being constant on each component and independent between components. A realization of this recipe is called a 'ground state' of the system.

The situation is more interesting in the presence of a vector **J** of edge-parameters, some of which are negative. The ground states in this case are colourings κ of V with the colour palette $\{1, 2, \ldots, q\}$ and the property that, for any each $e = \langle x, y \rangle$,

$$\kappa(x) \begin{cases} = \kappa(y) & \text{if } J_e > 0, \\ \neq \kappa(y) & \text{if } J_e < 0. \end{cases} \tag{1.1}$$

In the purely antiferromagnetic case, such a colouring κ is has the property that any two neighbours have different colours. There exist graphs G for which (1.1) has no solution, and such graphs are called 'frustrated'.

Theorem 1.7.1. *For the purely antiferromagnetic Potts model,*

$$Z_{\mathrm{P}}(\beta, q) \to \chi_G(q) \qquad as \ \beta \to \infty,$$

where χ_G is the chromatic polynomial of G.

1.8   The Random-Cluster Model on the Complete Graph

When the underlying graph is the complete graph K_n, the asymptotic behaviour of the corresponding random-cluster partition function $Z_{\mathrm{RC}}(n, p, q)$ may be studied using a mixture of combinatorics and probability, within the regime $q \geq 1$, $p = \lambda/n$. Here is some notation and explanation, in preparation for the main theorem.

Let $q \geq 1$ and $p = \lambda/n$. It turns out that there is a critical value of λ that marks the arrival of a giant cluster in the random-cluster model on K_n, and this value is given by

$$\lambda_{\mathrm{c}}(q) = \begin{cases} q & \text{if } q \in (0, 2], \\ 2 \left(\dfrac{q-1}{q-2} \right) \log(q-1) & \text{if } q \in (2, \infty). \end{cases}$$

As λ increases through the value λ_{c}, a giant cluster of size approximately $\theta(\lambda, q)n$ is created, where

$$\theta(\lambda, q) = \begin{cases} 0 & \text{if } \lambda < \lambda_{\mathrm{c}}(q), \\ \theta_{\max} & \text{if } \lambda \geq \lambda_{\mathrm{c}}(q), \end{cases}$$

and $\theta_{\max}$ is the largest root of the equation

$$e^{-\lambda\theta} = \frac{1-\theta}{1 + (q-1)\theta}.$$

Theorem 1.8.1. *Let $q \in (0, \infty)$ and $\lambda \in (0, \infty)$. We have that*

$$\frac{1}{n} \log Z_{\mathrm{RC}}(n, \lambda/n, q) \to \eta(\lambda) \qquad as\ n \to \infty,$$

where

$$\eta(\lambda) = \frac{g(\theta(\lambda))}{2q} - \frac{q-1}{2q}\lambda + \log q,$$
$$g(\theta) = -(q-1)(2-\theta)\log(1-\theta) - \big[2 + (q-1)\theta\big]\log\big[1 + (q-1)\theta\big].$$

By Theorem 1.3.6, this provides an asymptotic evaluation of the Tutte polynomial $T_{n,\lambda/n,q}(u-1, v-1)$ within the quadrant $u, v \in (0, \infty)$.

See [8] and [16, Chap. 10] for further details.

1.9 Open Problems

There is an enormous range of open problems associated with Ising, Potts, and random-cluster models. Of these, there follows a brief selection some of which may be allied in part to the Tutte polynomial.

1. Prove or disprove some version of negative association for the uniform forest measure USF or the uniform connected subgraph measure UCS. See Conjecture 1.5.4.

2. Prove or disprove some version of negative association for the random-cluster measure $\phi_{p,q}$ with $0 < q < 1$.

3. Prove or disprove a version of Simon's inequality for a random-cluster measure $\phi_{p,q}$ with $q \in [1, 2]$, as in (1.8).

4. Establish a version of Simon's inequality, Theorem 1.6.3, for the Potts model with $q \geq 3$.

5. More generally, find an application of mod-q flows to the q-state Potts model with $q \geq 3$, as in Theorem 1.6.1. Develop such an application for real $q > 0$, as in Theorem 1.6.2.

6. A problem of great current interest to probabilists and mathematical physicists is to understand the geometry of interfaces in the 3-state Potts model on the square lattice $\mathbb{Z}^2$, and more generally the random-cluster model on $\mathbb{Z}^2$ with $1 < q < 4$. See [29], and [5] for a recent reference.

Acknowledgement

Resource List

[1] M. Aizenman. Geometric analysis of φ^4 fields and Ising models. In *Mathematical problems in theoretical physics (Berlin, 1981)*, volume 153 of *Lecture Notes in Phys.*, pages 37–46. Springer, Berlin, 1982.

[2] M. Aizenman, D. J. Barsky, and R. Fernández. The phase transition in a general class of Ising-type models is sharp. *J. Statist. Phys.*, 47:343–374, 1987.

[3] M. Aizenman, H. Duminil-Copin, and V. Sidoravicius. Random currents and continuity of Ising model's spontaneous magnetization. 2013. `http://arxiv.org/abs/1311.1937`.

[4] M. Aizenman and R. Fernández. On the critical behavior of the magnetization in high-dimensional Ising models. *J. Statist. Phys.*, 44:393–454, 1986.

[5] V. Beffara, H. Duminil-Copin, and S. Smirnov. On the critical parameters of the $q \geq 4$ random-cluster model on isoradial graphs. 2013. preprint.

[6] M. Biskup, C. Borgs, J. T. Chayes, and R. Kotecký. Gibbs states of graphical representations of the Potts model with external fields. *J. Math. Phys.*, 41:1170–1210, 2000.

[7] J. E. Björnberg. Vanishing critical magnetization in the quantum Ising model. 2014. `http://arxiv.org/abs/1405.6946`.

[8] B. Bollobás, G. Grimmett, and S. Janson. The random-cluster model on the complete graph. *Probab. Theory Related Fields*, 104:283–317, 1996.

[9] J. A. Ellis-Monaghan and I. Moffatt. The Tutte–Potts connection in the presence of an external magnetic field. *Adv. in Appl. Math.*, 47:772–782, 2011.

[10] T. Feder and M. Mihail. Balanced matroids. In *24th ACM Symposum on the Theory of Computing*, pages 26–38, 1992.

[11] C. M. Fortuin. On the random-cluster model. II. The percolation model. *Physica*, 58:393–418, 1972.

[12] C. M. Fortuin. On the random-cluster model. III. The simple random-cluster model. *Physica*, 59:545–570, 1972.

[13] C. M. Fortuin and P. W. Kasteleyn. On the random-cluster model. I. Introduction and relation to other models. *Physica*, 57:536–564, 1972.

[14] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, 22:89–103, 1971.

[15] G. R. Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.

[16] G. R. Grimmett. *The Random-Cluster Model*, volume 333 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[17] G. R. Grimmett. Flows and ferromagnets. In *Combinatorics, Complexity, and Chance*, volume 34 of *Oxford Lecture Ser. Math. Appl.*, pages 130–143. Oxford Univ. Press, Oxford, 2007.

[18] G. R. Grimmett. *Probability on Graphs*, volume 1 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, Cambridge, 2010.

[19] G. R. Grimmett and S. N. Winkler. Negative association in uniform forests and connected graphs. *Random Structures Algorithms*, 24:444–460, 2004.

[20] R. Holley. Remarks on the FKG inequalities. *Comm. Math. Phys.*, 36:227–231, 1974.

[21] E. Ising. Beitrag zur Theorie des Ferromagnetismus. *Zeitschrift für Physik*, 31:253–258, 1925.

[22] S. Janson, T. Łuczak, and A. Rucinski. *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[23] G. Kirchhoff. Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Strome gefuhrt wird. *Annalen der Physik und Chemie*, 2:497–508, 1847.

[24] C. M. Newman and D. L. Stein. *Spin Glasses and Complexity*. Princeton University Press, Princeton, N.J., 2013.

[25] S. D. Noble and D. J. A. Welsh. A weighted graph polynomial from chromatic invariants of knots. *Ann. Inst. Fourier (Grenoble)*, 49:1057–1087, 1999. Symposium à la Mémoire de François Jaeger (Grenoble, 1998).

[26] R. B. Potts. Some generalized order-disorder transformations. *Proc. Cambridge Philos. Soc.*, 48:106–109, 1952.

[27] D. Reimer. Proof of the van den Berg–Kesten conjecture. *Combin. Probab. Comput.*, 9:27–32, 2000.

[28] B. Simon. Correlation inequalities and the decay of correlations in ferromagnets. *Comm. Math. Phys.*, 77:111–126, 1980.

[29] S. Smirnov. Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. *Ann. Math.*, 172:1435–1467, 2010.

[30] A. D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In *Surveys in Combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 173–226. Cambridge Univ. Press, Cambridge, 2005.

[31] W. T. Tutte. A ring in graph theory. *Proc. Cambridge Philos. Soc.*, 43:26–40, 1947.

[32] J. van den Berg and H. Kesten. Inequalities with applications to percolation and reliability. *J. Appl. Probab.*, 22:556–569, 1985.