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Markov Processes and **Related Fields**

Stochastic Geometry and Statistical Mechanics

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Stochastic geometry is an established field of mathematics that extends back into the nineteenth century at least, and arguably much earlier. Pioneers sometimes considered elementary problems in paradoxical ways, often deliberately. For example, in order to combat insomnia on the night of 20 January 1884, Charles Dodgson (otherwise known as Lewis Carroll) 'proved' that a triangle formed by three random points in the plane is obtuse with probability $(\pi/8)(\pi/3 - \sqrt{3}/4)$. On being invited to contribute the probability essay in the 9th (1885) edition of Encyclopædia Britannica, Morgan Crofton presented instead a summary of his research on integral geometry. Bertrand's paradox of 1888 about random chords is taught to undergraduates as a cautionary tale, as originally designed by Bertrand in his *Calcul des Probabilités*. Bertrand's purpose was to illustrate the now banal point that ambiguity in the probability measure gives birth to ill-posed problems.

Stochastic geometry has come a long way since, and is now is a rich and developed topic with many connections to theoretical and applied science. Aspects of stochastic geometry have emerged as key to the study of models of statistical mechanics. A wealth of examples were presented to the large audience at this one-day meeting, ranging from the more probabilistic models of percolation, self-avoiding walks, sandpiles, and exclusion processes, to models for the ferromagnet, and to dynamics for canonical Gibbs states.

The article of Arvind Ayyer, Joel Lebowitz, and Eugene Speer is a report on an investigation of certain one-dimensional two-type exclusion processes kept in equilibrium though contact with large reservoirs at different (fixed) temperatures. They have relaxed an assumption of earlier work according to which the type-2 particles may neither enter nor leave the system, thus obtaining results for a fully 'permiable' system.

Thierry Bodineau and Ben Graham have written about their work on the relaxation to equilibrium of conservative Ginzburg – Landau dynamics for a canonical Gibbs measure. They exploit a random-walk representation for the correlations derived by Helffer and Sjöstrand. For a certain class of one-dimensional Hamiltonians, the diffusive behaviour of the random walk may be connected to the diffusive relaxation of the dynamics.

The continuum scaling limit of the Ising magnetization field in two dimensions is the topic described by Federico Camia. Various issues are discussed that have arisen in ongoing joint work with Christophe Garban and Chuck Newman, including the existence of subsequential limits, and the uniqueness and conformality of the limiting field. This is part of a very significant project about that most classical object of statistical mechanics, the two-dimensional Ising model.

Antal Járai has presented an overview of the Abelian sandpile model on certain transitive graphs, with new results. A key technique is the bijection due to Majumdar and Dhar between recurrent states of the sandpile model and wired spanning trees of the underlying graph, and this enables new conclusions for infinite-volume limits on several families of graphs. Sandpiles have remarkable structure, and the current work identifies some of the significant features of the underlying graph.

Three speakers at IRS 2010 have not contributed to these written proceedings. Oliver Riordan described his joint work [1] with Béla Bollobás on generalized star-triangle transformations for two-dimensional percolation. They have been able to identify the exact critical point for certain bond percolation models with a property of generalized self-duality, thus verifying and extending a conjecture of Ziff and Scullard.

Stanislav Smirnov explained his beautiful recent proof with Hugo Duminil-Copin [2] that the connective constant for self-avoiding walks on the hexagonal lattice is the exact value $\sqrt{2+\sqrt{2}}$, as proposed by Nienhuis, [4]. This lecture, especially, attracted strong appreciation and reaction from the audience.

Finally, Bálint Tóth presented a survey of recent results about the long-time asymptotic behaviour of random processes with long memory arising from local self-interaction or self-repellence. Typical examples are the so-called myopic (or 'true') self-avoiding random walk and the self-repelling Brownian polymer models. The long-time asymptotics of the displacement are expected to be robust but dimension-dependent. See for example [3, 5].

References

[1] B. Bollobás and O. Riordan (2010) Percolation on self-dual polygon configurations. In: *An Irregular Mind*, Springer, Berlin, 131–217.

[2] H. Duminil-Copin and S. Smirnov (2010) The connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$. Preprint `arXiv:1007.0575`.

[3] I. Horváth, B. Tóth and B. Vető (2011) Diffusive limits for "true" (or myopic) self-avoiding random walks and self-repellent Brownian polymers in $d \geq 3$. To appear in *Probab. Theory and Relat. Fields*.

[4] B. Nienhuis (1982) Exact critical point and exponents of $\mathrm{O}(n)$ models in two dimensions. *Phys. Rev. Lett.* **49**, 1062–1065.

[5] B. Tóth and B. Vető (2011) Continuous time 'true' self-avoiding random walk on Z. *ALEA, Lat. Am. J. Probab. Math. Stat.* **8**, 59–75.

Abstracts of the talks on "Stochastic geometry and statistical mechanics", January 27, 2010

Thierry Bodineau (Paris): On the equivalence of ensembles

Equivalence of ensembles plays a crucial role in equilibrium statistical mechanics and in the justification of the Gibbs measures. We will first review some facts on the equivalence of ensembles and then address the question of its validity for non-equilibrium statistical mechanics.

Federico Camia (Amsterdam): Ising Euclidean fields and (conformal) measure ensembles

The two-dimensional Ising model is one of the most studied models of statistical mechanics and has played a fundamental role in the theory of phase transitions. In this talk, I will focus on the magnetization field, which describes the spatial fluctuations of the local magnetic field generated by the spins and is one of the main objects in the Ising field theory.

In the scaling limit, as the lattice spacing is sent to zero, the magnetization field can be desribed as a random generalized function. Above the critical temperature, for instance, the scaling limit of the lattice magnetization field is Gaussian white noise. The situation is more interesting at the critical point where thermal fluctuations extend over all scales, leading to scale invariance and a conformally covariant field. One also expects cluster boundaries to converge in the scaling limit to SLE-type curves, and a proof of this fact has been recently announced by S. Smirnov.

I will introduce a representation for the magnetization field which leads to a simple proof of the existence of subsequential scaling limits, provides a connection between the field-theoretic and the SLE approach, and can be used to prove uniqueness of the scaling limit and its expected conformal covariance properties. The key ingredient is an ensemble of measures of fractal support coming from the scaling limit of the rescaled areas of critical FK clusters.

(Based on Joint work with C.M. Newman and with C. Garban and Newman.)

Antal A. Járai (Bath): Abelian sandpiles: an overview and results on transitive graphs

The Abelian sandpile was introduced in the physics literature as a model for self-organized criticality. Its dynamics is defined in terms of simple local rules that result, due to a separation of time scales, in a non-local dynamics. The stationary distribution of the model has a description in terms of the uniform spanning tree. In this talk I will give an introduction to the sandpile model, explain the connection to the uniform spanning tree, and consider the infinite volume limit on certain transitive graphs.

Joel L. Lebowitz (Rutgers University): Properties of systems with non reflection invariant interactions: variations on the ABC model

We consider q-component systems with q greater or equal to three on a d-dimensional lattice. The energy of the system is given by the sum of pair-interactions which are not invariant under spatial reflections, e.g. in one dimension the interaction between an A and B particle depends on whether A is to the left or right of B. We have obtained the exact phase diagram for such a system when the interactions are of mean field type, `Arxiv: 0905.4849`. It is very different from the standard mean field model with reflection symmetric interactions. Various generalization of this model will be discussed.

Oliver Riordan (Oxford): The generalized triangle-triangle transformation for percolation

One of the main aims in the theory of percolation is to find the 'critical probability' above which long range connections emerge from random local connections with a given pattern and certain individual probabilities. The quintessential example is Kesten's result from 1980 that if the edges of the square lattice are selected independently with probability p, then long range connections appear if and only if $p > 1/2$. The starting point is a certain self-duality property, observed already in the early 60s; the difficulty is not in this observation, but in proving that self-duality does imply criticality in this setting.

Since Kesten's result, more complicated duality properties have been used to determine a variety of other critical probabilities. Recently, Scullard and Ziff have described a very general class of self-dual planar percolation models; we show that for the entire class (in fact, a larger class), self-duality does imply criticality.

Stanislav Smirnov (Genève): On the scaling limits of 2D lattice models

Balint Tóth (Budapest): Long-time asymptotic behaviour of self-repelling random processes

I will present a survey of recent results about the long-time asymptotic behaviour of random processes with long memory due to some rather natural local self-interaction (self-repellence) of the trajectories. Typical examples are the so-called myopic (or "true") self-avoiding random walk and the self-repelling Brownian polymer models. The long-time asymptotics of the displacement is expected to be robust (not depending on some microscopic details), but dimension-dependent. It is expected that: in 1d, the motion is strongly superdiffusive with time-to-the-two-thirds scaling; in 2d, the motion is marginally superdiffusive with logarithmic multiplicative correction in the scaling; in three and more dimensions the displacement is diffusive. Some of these have been recently proved, at least for some particular models.