

## STRICT INEQUALITIES FOR CONNECTIVE CONSTANTS OF TRANSITIVE GRAPHS\*

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**Abstract.** The *connective constant* of a graph is the exponential growth rate of the number of self-avoiding walks starting at a given vertex. Strict inequalities are proved for connective constants of vertex-transitive graphs. First, the connective constant *decreases* strictly when the graph is replaced by a nontrivial quotient graph. Second, the connective constant *increases* strictly when a quasi-transitive family of new edges is added. These results have the following implications for Cayley graphs: the connective constant of a Cayley graph decreases strictly when a new relator is added to the group, and increases strictly when a nontrivial group element is declared to be a generator.

**Key words.** self-avoiding walk, connective constant, regular graph, vertex-transitive graph, quasi-transitive graph, Cayley graph

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**1. Introduction.** A self-avoiding walk (SAW) is a path that revisits no vertex. Self-avoiding walks were first introduced in the context of long-chain polymers in chemistry (see [10]), and they have been studied intensively since by mathematicians and physicists (see [25]). If the underlying graph  $G$  has some periodicity, the number of  $n$ -step SAWs with a given starting point grows (asymptotically) exponentially as  $n \rightarrow \infty$ , with some growth rate  $\mu(G)$  called the *connective constant* of the graph. There are few graphs  $G$  for which  $\mu(G)$  is known exactly, and a substantial part of the associated literature is devoted to inequalities for such constants. The purpose of the current work is to establish conditions under which a systematic change to  $G$  results in a *strict* change to  $\mu(G)$ .

We have two main results for an infinite, vertex-transitive graph  $G$ , as follows. The automorphism group of  $G$  is denoted by  $\text{Aut}(G)$ . Precise conditions are given in the formal statements of the theorems.

1. (Theorem 3.8) Let the subgroup  $\Gamma \subseteq \text{Aut}(G)$  act transitively on  $G$ , and let  $\mathcal{A} \subseteq \Gamma$  be a nontrivial, normal subgroup of  $\Gamma$  (satisfying a minor condition). The (directed) quotient graph  $\vec{G} = G/\mathcal{A}$  satisfies  $\mu(\vec{G}) < \mu(G)$ .
2. (Theorem 3.2) Suppose new edges are added in such a way that the resulting graph is quasi-transitive (subject to a certain algebraic condition). The connective constant of the new graph  $\overline{G}$  satisfies  $\mu(G) < \mu(\overline{G})$ .

These inequalities have the following implications for Cayley graphs. Let  $G$  be the Cayley graph of an infinite, finitely generated group  $\mathcal{G}$  with generator set  $S$  and relator set  $R$ .

3. (Corollary 4.1) Let  $G_\rho$  be the Cayley graph obtained by adding to  $\mathcal{G}$  a further nontrivial relator  $\rho$ . Then  $\mu(G_\rho) < \mu(G)$ .

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4. (Corollary 4.3) Let  $w$  be a nontrivial element of  $\mathcal{G}$  that is not a generator, and let  $\overline{G}_w$  be the Cayley graph obtained by declaring  $w$  to be a further generator. Then  $\mu(G) < \mu(\overline{G}_w)$ .

The proofs partly follow the general approach of Kesten's proof of the pattern theorem; see [21] and [25, sect. 7.2]. Any  $n$ -step SAW  $\pi$  in the smaller graph  $G$  lifts to a SAW  $\pi'$  in the larger graph  $G'$ . The idea is to show that "most" such  $\pi$  contain at least  $an$  subwalks for which the corresponding sections of  $\pi'$  may be replaced in at least two different ways by SAWs on  $G'$ . Different subsets of these subwalks give rise to different SAWs on  $G'$ . The number of such subsets grows exponentially in  $n$ , and this introduces an exponential "entropic" factor in the count of SAWs.

Whereas Kesten's proof and subsequent elaborations were directed at certain specific lattices, our results apply in the general setting of vertex-transitive graphs, and they require new algebraic and combinatorial techniques. Indeed, the work reported here may be the first systematic study of SAWs on general vertex-transitive graphs.

Related questions have been considered in the contexts of percolation and disordered systems. Consider a given model on a graph  $G$ , such as a percolation or an Ising/Potts model. There is generally a singularity at some parameter value called the "critical point." For percolation the parameter in question is the *density* of open sites or bonds, and for the models of statistical physics it is the *temperature*. It is important and useful to understand something of how the critical point varies with the choice of graph. In particular, under what conditions does a systematic change in the graph cause a *strict* change in the value of the critical point? A general approach to this issue was presented in [1] and developed further in [9, 12] and [13, Chap. 3].

Turning back to SAWs, the SAW generating function has radius of convergence  $1/\mu$  and is believed (for lattice-graphs at least) to have power-law behavior near its critical point; see [25]. The above theorems amount to strict inequalities for the critical point as the underlying graph  $G$  varies. Despite a similarity of the problem to that of disordered systems, the required techniques for SAWs are substantially different. We concentrate here on vertex-transitive graphs, and the required conditions are expressed in the language of algebra. There is another feasible approach to proving strict inequalities, namely the bridge-decomposition method introduced by Hammersley and Welsh in [19] and used more recently in various works including [6, 23] and [25, Thm. 8.2.1].

Basic notation and facts about SAWs and connective constants are presented in section 2. There is a large literature concerning SAWs, of which we mention [2, 5, 20, 25]. For accounts of algebraic graph theory, see [4, 11, 24, 28].

This paper has two companion papers, [15, 17]. In [15], we prove bounds on connective constants of vertex-transitive graphs, in particular  $\mu \geq \sqrt{\Delta - 1}$  when  $G$  is an infinite, connected,  $\Delta$ -regular, vertex-transitive, simple graph. In [17], we explore the effect on SAWs of the Fisher transformation, applied to a cubic or partially cubic graph. This work is reviewed in [16].

**2. Notation and definitions.** All graphs considered here are connected and infinite. Subject to a minor exception in section 3, they may not contain *loops* (that is, edges both of whose endpoints are the same vertex), but, in certain circumstances, they are permitted to have *multiple edges* (that is, two or more edges with the same pair of endpoints). A graph  $G = (V, E)$  is called *simple* if it has neither loops nor multiple edges. An edge  $e$  with endpoints  $u, v$  is written  $e = \langle u, v \rangle$ , and two distinct edges with the same endpoints are called *parallel*. If  $\langle u, v \rangle \in E$ , we call  $u$  and  $v$  *adjacent* and write  $u \sim v$ . Let  $\partial v = \{u : u \sim v\}$  denote the set of neighbors of  $v \in V$ .

The *degree* of vertex  $v$  is the number of edges incident to  $v$ . We assume that the vertex-degrees of a given graph  $G$  are finite with supremum  $\Delta < \infty$ . The *graph-distance* between two vertices  $u, v$  is the number of edges in the shortest path from  $u$  to  $v$ , denoted  $d_G(u, v)$ .

The automorphism group of the graph  $G = (V, E)$  is denoted  $\text{Aut}(G)$ . A subgroup  $\Gamma \subseteq \text{Aut}(G)$  is said to *act transitively* on  $G$  (or on the vertex-set  $V$ ) if, for  $v, w \in V$ , there exists  $\gamma \in \Gamma$  with  $\gamma v = w$ . It is said to *act quasi-transitively* if there is a finite set  $W$  of vertices (called a *fundamental domain*) such that, for  $v \in V$ , there exist  $w \in W$  and  $\gamma \in \Gamma$  with  $\gamma v = w$ . The graph is called *vertex-transitive* (respectively, *quasi-transitive*) if  $\text{Aut}(G)$  acts transitively (respectively, quasi-transitively). The identity element of any group is denoted by  $\iota$ .

A *walk*  $w$  on  $G$  is an alternating sequence  $w_0 e_0 w_1 e_1 \cdots e_{n-1} w_n$  of vertices  $w_i$  and edges  $e_i = \langle v_i, v_{i+1} \rangle$ . We write  $|w| = n$  for the *length* of  $w$ , that is, the number of edges in  $w$ . The walk  $w$  is called *closed* if  $w_0 = w_n$ . A *cycle* (or *n-cycle*) is a closed walk  $w$  with distinct edges and  $w_i \neq w_j$  for  $1 \leq i < j \leq n$ . Thus, two parallel edges form a 2-cycle.

Let  $n \in \{1, 2, \dots\} \cup \{\infty\}$ . An *n-step self-avoiding walk* (SAW) on  $G$  is a walk containing  $n$  edges no vertex of which appears more than once. Let  $\sigma_n(v)$  be the number of  $n$ -step SAWs starting at  $v \in V$ . We are interested here in the exponential growth rate of  $\sigma_n(v)$ . Note that, in the presence of parallel edges, two SAWs with identical vertex-sets but different edge-sets are considered as distinct SAWs.

**THEOREM 2.1** (see [18]). *Let  $G = (V, E)$  be an infinite, connected, quasi-transitive graph with finite vertex-degrees. There exists  $\mu = \mu(G) \in [1, \infty)$ , called the connective constant of  $G$ , such that*

$$(2.1) \quad \lim_{n \rightarrow \infty} \sigma_n(v)^{1/n} = \mu, \quad v \in V.$$

Subadditivity plays a key part in the proof of this theorem. It yields the inequality

$$(2.2) \quad \sup_{v \in V} \sigma_n(v) \geq \mu^n, \quad n \geq 0,$$

which will be useful later in this paper. A further proof of Theorem 2.1 may be found in [14].

See [17, Thm. 3.1] (and also [22, Prop. 1.1]) for an elaboration of Theorem 2.1 in the absence of quasi-transitivity. We note for use in section 3 that the above notation may be extended in a natural way to *directed* graphs, and we omit the details here. In particular, one may define the connective constant  $\vec{\mu} = \mu(\vec{G})$  of a *directed*, quasi-transitive graph  $\vec{G}$  by (2.1) with  $\sigma_n(v)$  replaced by the number of *directed*  $n$ -step SAWs (whenever the relevant limits exist).

### 3. Strict inequalities for vertex-transitive graphs.

**3.1. Introduction.** In this section, we present and discuss two strict inequalities for connective constants of vertex-transitive graphs. The first (Theorem 3.2 in section 3.2) deals with the effect of adding a quasi-transitive family of new edges, and the second (Theorem 3.8 in section 3.4) deals with quotient graphs. Proofs of these two principal theorems are found in sections 5 and 6. (Readers of these proofs will be aided by familiarity with the proof of the pattern theorem; see [25, sect. 7.2].) Two further results are presented in section 3.6 concerning, respectively, the question of whether the proof of Theorem 3.8 may be transformed into an algorithm, and the possible removal of the assumption of normality in Theorem 3.8. Interspersed with

the principal sections 3.2 and 3.4 are the discussions of sections 3.3 and 3.5, which may be omitted at first reading.

The implications of the above inequalities for Cayley graphs are presented in section 4; see Corollaries 4.1 and 4.3.

Let  $G = (V, E)$  be an infinite, connected, vertex-transitive, simple graph. We assume throughout that the vertex-degree  $\Delta$  of  $G$  satisfies  $\Delta < \infty$ .

**3.2. Quasi-transitive augmentation.** Let  $G = (V, E)$  be as above, and let  $\overline{G} = (V, \overline{E})$  be obtained from  $G$  by adding further edges, possibly in parallel to existing edges. We shall assume that  $E$  is a *proper* subset of  $\overline{E}$ , and that the additional edge-set  $\overline{E} \setminus E$  has a property of quasi-transitivity with respect to some automorphism-subgroup  $\mathcal{A}$ . We shall require some properties of this subgroup  $\mathcal{A}$ , and to this end we introduce next a certain definition, followed by the main result of this subsection.

**DEFINITION 3.1.** Let  $\Gamma \subseteq \text{Aut}(G)$  act transitively on  $G$ . A subgroup  $\mathcal{A} \subseteq \Gamma$  is said to have the finite coset property with root  $\rho \in V$  (in  $\Gamma$ ) if there exist  $\nu_0, \nu_1, \dots, \nu_s \in \Gamma$ , with  $\nu_0 = \iota$  and  $s < \infty$ , such that  $V$  is partitioned as  $\bigcup_{i=0}^s \nu_i \mathcal{A} \rho$ . It is said simply to have the finite coset property if it has this property with some root.

**THEOREM 3.2.** Let  $\Gamma$  act transitively on  $G$ , and let  $\mathcal{A}$  be a subgroup of  $\Gamma$  with the finite coset property. If  $\mathcal{A} \subseteq \text{Aut}(\overline{G})$ , then  $\mu(G) < \mu(\overline{G})$ .

The finite coset property is somewhat technical, but it is satisfied in two cases of practical interest. The following proposition is proved at the end of section 3.3.

**PROPOSITION 3.3.** Let  $\Gamma$  act transitively on  $G$ , and let  $\rho \in V$ . The subgroup  $\mathcal{A}$  of  $\Gamma$  has the finite coset property with root  $\rho$  if either of the following holds:

- (a)  $\mathcal{A}$  is a normal subgroup of  $\Gamma$  which acts quasi-transitively on  $G$ .
- (b) The index  $[\Gamma : \mathcal{A}]$  is finite.

We ask whether the condition of Theorem 3.2 may be relaxed. More generally, is it the case that  $\mu(G) < \mu(\overline{G})$  whenever  $G = (V, E)$  is transitive and  $\overline{G} = (V, \overline{E})$  is quasi-transitive, with  $E$  a proper subset of  $\overline{E}$ ?

**EXAMPLE 3.4.** The square/triangular lattices present an elementary example of Theorem 3.2 in action. Let  $\mathbb{Z}^2$  denote the square lattice, and let  $\Gamma$  be the group of its translations. The triangular lattice  $\mathbb{T}$  is obtained by adding the edge  $e = \langle \mathbf{0}, (1, 1) \rangle$  and its images under  $\Gamma$ , where  $\mathbf{0}$  is the origin  $(0, 0)$ . Since  $\Gamma$  is a normal subgroup of itself with the finite coset property, we deduce that  $\mu(\mathbb{Z}^2) < \mu(\mathbb{T})$ . This inequality is in fact an elementary consequence of known numerical bounds; see, for example, [2]. The current conclusion may, however, be extended beyond such known bounds by use of Theorem 3.2 as follows.

Let  $u, v \in \mathbb{Z}^2$  be linearly independent (nonzero) vectors, and let  $\mathcal{A}_{u,v}$  be the subgroup of  $\Gamma$  containing all translations of the form  $mu + nv$  as  $m$  and  $n$  range over  $\mathbb{Z}$ . It is easily seen that  $\mathcal{A}_{u,v}$  is a normal subgroup of  $\Gamma$  which acts quasi-transitively, and hence  $\mathcal{A}_{u,v}$  has the finite coset property.

Let  $w \in \mathbb{Z}^2$  satisfy  $w \neq \mathbf{0}$ , and let  $e$  be a new edge  $\langle \mathbf{0}, w \rangle$ . Let  $\mathbb{T}_{u,v,w}$  be the graph obtained from  $\mathbb{Z}^2$  by adding the set  $\mathcal{A}_{u,v}e$  of all images of  $e$  under  $\mathcal{A}_{u,v}$ . The regular triangular lattice  $\mathbb{T}$  is retrieved by appropriate choice of  $u, v$ , and  $w$ .

By Theorem 3.2,  $\mu(\mathbb{Z}^2) < \mu(\mathbb{T}_{u,v,w})$ . The corresponding percolation theorem states that the critical probabilities satisfy  $p_c(\mathbb{Z}^2) < p_c(\mathbb{T}')$  for certain  $\mathbb{T}'$  obtained by enhancing  $\mathbb{Z}^2$ ; see [1] and [13, Chap. 3]. This requires less symmetry on the distribution of new edges, and it suffices that there exists  $M < \infty$  such that every vertex is within distance  $M$  of a new edge.

The conclusion of Theorem 3.2 is generally invalid if  $G$  is assumed to be only quasi-transitive. Consider, for example, the pair  $G, \overline{G}$  of Figure 1, each of which has connective constant 1.

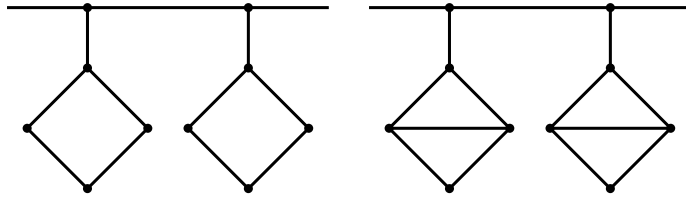


FIG. 1. The pair  $G, \overline{G}$ . The patterns are extended infinitely in both directions. Each graph is quasi-transitive with connective constant 1, and the second is obtained from the first by the systematic addition of edges.

**3.3. Algebraic discussion.** This section contains discussion of certain algebraic facts related to the finite coset property, and it may be omitted at first reading.

The assumption of normality will recur in this paper, and we recall the following “standard facts.”

REMARK 3.5. Let  $\Gamma$  act transitively on the infinite graph  $G = (V, E)$ . A partition  $\mathcal{P}$  of  $V$  is called  $\Gamma$ -invariant if, for  $u, v \in V$  belonging to the same set of the partition and for  $\gamma \in \Gamma$ ,  $\gamma u$  and  $\gamma v$  belong to the same set of the partition.

Let  $\mathcal{A}$  be a subgroup of  $\Gamma$ . The orbits of  $\mathcal{A}$  form a partition  $\mathcal{P}(\mathcal{A})$  of  $V$ . If  $\mathcal{A}$  is a normal subgroup, the partition  $\mathcal{P}(\mathcal{A})$  is  $\Gamma$ -invariant. If  $\mathcal{P}(\mathcal{A})$  is  $\Gamma$ -invariant, there exists a normal subgroup  $\mathcal{N}$  of  $\Gamma$  such that  $\mathcal{P}(\mathcal{N}) = \mathcal{P}(\mathcal{A})$ . In the latter case,  $\mathcal{N}$  may be taken as the normal closure of  $\mathcal{A}$ , that is, the intersection of all normal subgroups of  $\Gamma$  containing  $\mathcal{A}$ .

The proof of Proposition 3.3(b) makes use of the relationship between the index of  $\mathcal{A}$  and the number of its orbits when acting on  $V$ . Consider  $\Gamma$  as a group acting on  $V$  with orbit-set denoted by  $V/\Gamma$ . It is said to act *freely* if every stabilizer is trivial, that is, if

$$(3.1) \quad \text{Stab}_v := \{\gamma \in \Gamma : \gamma v = v\}$$

satisfies

$$(3.2) \quad \text{Stab}_v = \{\iota\}, \quad v \in V.$$

We abuse notation by saying that  $\Gamma \setminus \mathcal{A}$  acts freely on  $V$  if

$$(3.3) \quad \text{Stab}_v \subseteq \mathcal{A}, \quad v \in V.$$

The proof of the next proposition is at the end of this subsection.

PROPOSITION 3.6. Let  $\Gamma$  act transitively on the countable set  $V$ , and let  $\mathcal{A}$  be a subgroup of  $\Gamma$ . Then

$$(3.4) \quad |V/\mathcal{A}| \leq [\Gamma : \mathcal{A}].$$

If  $|V/\mathcal{A}| < \infty$ , equality holds in (3.4) if and only if  $\Gamma \setminus \mathcal{A}$  acts freely on the set  $V$ .

We make three comments concerning Theorem 3.3:

1. Let  $\mathcal{A} \subseteq \text{Aut}(\overline{G})$  act quasi-transitively on  $G$ , and suppose there exists  $\Gamma \subseteq \text{Aut}(G)$  such that  $\mathcal{A} \subseteq \Gamma$ ,  $\Gamma$  acts transitively on  $G$ , and  $\Gamma \setminus \mathcal{A}$  acts freely on  $V$ . We have by Proposition 3.6 that  $[\Gamma : \mathcal{A}] = |V/\mathcal{A}| < \infty$ , whence  $\mathcal{A}$  has the finite coset property (with arbitrary root) by Theorem 3.3(b).
2. By Remark 3.5, condition (a) of Theorem 3.3 may be replaced by the apparently weaker assumption that  $\mathcal{A}$  is a subgroup of  $\Gamma$  acting quasi-transitively on  $G$ , whose orbits form a  $\Gamma$ -invariant partition of  $V$ .

3. The *normal core* of  $\mathcal{A}$  is the intersection of the conjugate subgroups of  $\mathcal{A}$ . If  $[\Gamma : \mathcal{A}] < \infty$ , the normal core  $\mathcal{N}$  of  $\mathcal{A}$  satisfies  $[\Gamma : \mathcal{N}] < \infty$  (see [27, Prop. 1.6.9]). By Proposition 3.6,  $\mathcal{N}$  acts quasi-transitively on  $V$ .

This section closes with the proofs of Propositions 3.3 and 3.6.

*Proof of Proposition 3.3.* (a) Suppose  $\mathcal{A}$  is a normal subgroup of  $\Gamma$  acting quasi-transitively on  $G$ . With  $\rho \in V$ , the orbits of  $\mathcal{A}$  may be written as  $\mathcal{A}\nu_i\rho$  for suitable  $\nu_i \in \Gamma$  and  $0 \leq i \leq s < \infty$ . Since  $\mathcal{A}$  is normal,  $\mathcal{A}\nu_i = \nu_i\mathcal{A}$ , and the claim follows.

(b) Suppose  $[\Gamma : \mathcal{A}] < \infty$ . The finite coset property holds with the  $\nu_i$  chosen so that the  $\nu_i\mathcal{A}$  are the left cosets of  $\mathcal{A}$  in  $\Gamma$ .  $\square$

*Proof of Proposition 3.6.* Fix  $v_0 \in V$ , and choose  $v_1, v_2 \dots$  such that  $\mathcal{A}v_0, \mathcal{A}v_1, \dots$  are the (distinct) orbits of  $\mathcal{A}$ . Let  $\gamma_0 = \iota$  and, for  $i \geq 1$ , find  $\gamma_i \in \Gamma$  such that  $\gamma_iv_0 = v_i$ . Let  $\Gamma/\mathcal{A}$  denote the set of right cosets of  $\mathcal{A}$ , and let  $\phi : V/\mathcal{A} \rightarrow \Gamma/\mathcal{A}$  be given by  $\phi(\mathcal{A}v_i) = \mathcal{A}\gamma_i$ . We check next that  $\phi$  is an injection.

It suffices to show that  $i = j$  whenever  $\mathcal{A}\gamma_i = \mathcal{A}\gamma_j$ . Suppose  $\mathcal{A}\gamma_i = \mathcal{A}\gamma_j$ . There exists  $\alpha \in \mathcal{A}$  such that  $\gamma_i = \alpha\gamma_j$ . Therefore,

$$v_i = \gamma_iv_0 = \alpha\gamma_jv_0 = \alpha v_j,$$

whence  $\mathcal{A}v_i = \mathcal{A}v_j$  and  $i = j$ . Thus,  $\phi$  is an injection, and (3.4) follows.

Suppose  $|V/\mathcal{A}| < \infty$ , and write  $U := \bigcup_{i=0}^{\infty} \mathcal{A}\gamma_i$ . Equality holds in (3.4) if and only if  $\phi$  is a surjection, which is to say that  $\Gamma = U$ . Assume there exists  $\rho \in \Gamma \setminus U$ . Find  $j$  such that  $\rho v_0 \in \mathcal{A}v_j$ , say  $\rho v_0 = \alpha v_j$  with  $\alpha \in \mathcal{A}$ . Then  $\alpha^{-1}\rho\gamma_j^{-1} \in \text{Stab}_{v_j}$ . If  $\alpha^{-1}\rho\gamma_j^{-1} \in \mathcal{A}$ , then  $\rho \in \mathcal{A}\gamma_j$  in contradiction of the assumption  $\rho \notin U$ . Therefore,  $\alpha^{-1}\rho\gamma_j^{-1} \in \Gamma \setminus \mathcal{A}$ , and  $\Gamma \setminus \mathcal{A}$  does not act freely.

Suppose conversely that there exist  $\rho \in \Gamma \setminus \mathcal{A}$  and  $v \in V$  with  $\rho v = v$ . Set  $w_0 = v$  and find  $w_i \in V$  such that the (distinct) orbits of  $\mathcal{A}$  are  $\mathcal{A}w_0, \mathcal{A}w_1, \dots$ . Let  $\gamma_0 = \iota$  and, for  $i \geq 1$ , find  $\gamma_i \in \Gamma$  such that  $\gamma_iw_0 = w_i$ . If  $\rho \in \mathcal{A}\gamma_i$ , then  $\rho = \alpha\gamma_i$  for some  $\alpha \in \mathcal{A}$ , so that  $w_0 = \rho w_0 = \alpha\gamma_iw_0 = \alpha w_i$ . This implies that  $i = 0$ , and hence  $\rho \in \mathcal{A}$ , a contradiction. Therefore,  $\rho \in \Gamma \setminus U$ , where  $U = \bigcup_i \mathcal{A}\gamma_i$ . It follows as above that  $|V/\mathcal{A}| < [\Gamma : \mathcal{A}]$ .  $\square$

**3.4. Quotient graphs.** Let  $\Gamma$  be a subgroup of the automorphism group  $\text{Aut}(G)$  that acts transitively, and let  $\mathcal{A}$  be a subgroup of  $\Gamma$ . We denote by  $\vec{G} = (\vec{V}, \vec{E})$  the (directed) quotient graph  $G/\mathcal{A}$  constructed as follows. Let  $\approx$  be the equivalence relation on  $V$  given by  $v_1 \approx v_2$  if and only if there exists  $\alpha \in \mathcal{A}$  with  $\alpha v_1 = v_2$ . The vertex-set  $\vec{V}$  comprises the equivalence classes of  $(V, \approx)$ , that is, the orbits  $\vec{v} := \mathcal{A}v$  as  $v$  ranges over  $V$ . For  $v, w \in V$  we place  $|\partial v \cap \vec{w}|$  directed edges from  $\vec{v}$  to  $\vec{w}$  (if  $\vec{v} = \vec{w}$ , these edges are directed loops), and we write  $\vec{v} \sim \vec{w}$  if  $|\partial v \cap \vec{w}| \geq 1$  and  $\vec{v} \neq \vec{w}$ . (Thus  $\vec{v} \sim \vec{w}$  if and only if  $\vec{w} \sim \vec{v}$ .) By the next lemma, the number  $|\partial v \cap \vec{w}|$  is independent of the choice of  $v \in \vec{v}$ .

**LEMMA 3.7.** *Let  $\mathcal{A}$  be a subgroup of  $\Gamma$ , and let  $\vec{v}, \vec{w} \in \vec{V}$ . The number  $|\partial v \cap \vec{w}|$  is independent of the representative  $v \in \vec{v}$ .*

*Proof.* Let  $v, v' \in \vec{v}$  and  $v' \neq v$ . Choose  $\alpha \in \mathcal{A}$  such that  $\alpha v = v'$ . Then  $x \in \partial v \cap \vec{w}$  is mapped to  $\alpha x \in \partial v' \cap \vec{w}$ , whence  $\alpha$  acts as an injection from  $\partial v \cap \vec{w}$  to  $\partial v' \cap \vec{w}$ . Therefore,  $|\partial v \cap \vec{w}| \leq |\partial v' \cap \vec{w}|$ , and the claim follows by symmetry.  $\square$

We call  $\mathcal{A}$  *symmetric* if

$$(3.5) \quad |\partial v \cap \vec{w}| = |\partial w \cap \vec{v}|, \quad v, w \in V.$$

Two sufficient conditions for symmetry are presented in the forthcoming Lemma 3.10.



We recall that the orbits of  $\mathcal{A}$  are invariant under  $\Gamma$  if and only if they are the orbits of some normal subgroup of  $\Gamma$  (see Remark 3.5). Assume henceforth that  $\mathcal{A}$  is a normal subgroup of  $\Gamma$ . It is standard that  $\alpha \in \Gamma$  acts on  $\vec{G}$  by  $\alpha(\mathcal{A}v) = \mathcal{A}(\alpha v)$ , and that  $\Gamma$  acts transitively on  $\vec{G}$ . Furthermore, for  $v, w \in V$ ,

$$(3.6) \quad \bar{v} = \bar{w} \iff \forall \gamma \in \Gamma, \gamma \bar{v} = \gamma \bar{w}.$$

Any walk  $\pi$  on  $G$  induces a (directed) walk  $\vec{\pi}$  on  $\vec{G}$ , and we say that  $\pi$  projects onto  $\vec{\pi}$ . For a walk  $\vec{\pi}$  on  $\vec{G}$ , there exists a walk  $\pi$  on  $G$  that projects onto  $\vec{\pi}$ , and we say that  $\vec{\pi}$  lifts to  $\pi$ . There are generally many choices for such  $\pi$ , and we fix such a choice as follows. For  $\bar{v}_1, \bar{v}_2 \in \bar{V}$ , we label the  $N(v_1, v_2) := |\partial v_1 \cap \bar{v}_2|$  directed edges from  $\bar{v}_1$  to  $\bar{v}_2$  in a fixed but arbitrary way with the integers  $1, 2, \dots, N(v_1, v_2)$ . For  $v \in \bar{v}_1$ , we label similarly the edges from  $v$  to vertices in the set  $\bar{v}_2$ . For  $v \in \bar{v}$ , any  $\vec{\pi}$  from  $\bar{v}$  lifts to a unique  $\pi$  from  $v$  that conserves edge-labellings, and thus walks from a given  $v$  on  $G$  are in one-to-one correspondence with walks from  $\bar{v}$  on  $\vec{G}$ . Since a SAW on  $\vec{G}$  lifts to a SAW on  $G$ ,  $\vec{\mu} := \mu(\vec{G})$  satisfies  $\vec{\mu} \leq \mu(G)$ . Our task in this subsection is to identify sufficient conditions for the strict inequality  $\vec{\mu} < \mu(G)$ .

We introduce next the so-called *type* of the subgroup  $\mathcal{A}$ , in terms of the length of the shortest SAW of  $G$  with endpoints in the same orbit. Let  $v \in V$ , and let  $w \neq v$  satisfy that  $\bar{w} = \bar{v}$  and  $d_G(v, w)$  is minimal with this property. By the transitive action of  $\Gamma$ ,

$$(3.7) \quad d_G(x, y) \geq d_G(v, w), \quad x \neq y, \bar{x} = \bar{y}.$$

We say that  $v$  is of the types given thus:

- type 1 if  $d_G(v, w) = 1$ ,
- type 2 if  $d_G(v, w) = 2$ ,
- type 3 if  $d_G(v, w) \geq 3$ .

By (3.6), every vertex has the same type, and thus we shall speak of the *type* of  $\mathcal{A}$ .

Given next is the main theorem of this section. A group is called *trivial* if it comprises the identity  $\iota$  only. An automorphism  $\beta$  is said to *fix* a vertex  $w$  if  $\beta w = w$ .

**THEOREM 3.8.** *Let  $\mathcal{A}$  be a nontrivial, normal subgroup of  $\Gamma$ . The connective constant  $\vec{\mu} = \mu(\vec{G})$  satisfies  $\vec{\mu} < \mu(G)$  if either (a) or (b) holds:*

- (a) *the type of  $\mathcal{A}$  is 1 or 3, or*
- (b)  *$\mathcal{A}$  has type 2 and either of the following holds:*
  - (i)  *$G$  contains a SAW  $v_0, w, v'$  satisfying  $\bar{v}_0 = \bar{v}'$  and  $|\partial v_0 \cap \bar{w}| \geq 2$ ,*
  - (ii)  *$G$  contains a SAW  $v_0 (= w_0), w_1, w_2, \dots, w_l (= v')$  satisfying  $\bar{v}_0 = \bar{v}'$ ,  $\bar{w}_i \neq \bar{w}_j$  for  $0 \leq i < j < l$ , and furthermore  $v' = \beta v_0$  for some  $\beta \in \mathcal{A}$  which fixes no  $w_i$ .*

In the special case when  $\mu(G) = 1$ , by [15, Thm. 1.1],  $G$  has degree 2 and is therefore the line  $\mathbb{Z}$ . It is easily seen that  $\bar{V}$  is finite, so that  $\vec{\mu} = 0$ .

The conclusion of Theorem 3.8 is generally invalid for *quasi-transitive* graphs. Consider, for example, the graph  $G$  of Figure 2, with  $\mathcal{A} = \{\iota, \rho\}$ , where  $\rho$  is a reflection in the horizontal axis. Both  $G$  and its quotient graph have connective constant 1.

Theorem 3.8 is related to but distinct from the strict inequalities of [6, 20, 29] and [25, Thm. 8.2.1], where specific examples are considered of graphs that are not vertex-transitive.

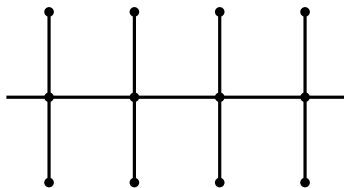


FIG. 2. The pattern is extended infinitely in both directions.

Suppose that  $\mathcal{A}$  has type 2. Condition (i) of Theorem 3.8(b) holds if  $\mathcal{A}$  is symmetric, since  $|\partial w \cap \bar{v}| \geq 2$ . While symmetry is sufficient for Theorem 3.8, we shall see in the next example that it is not necessary.

EXAMPLE 3.9. Conditions (i)–(ii) of Theorem 3.8 are necessary in the type-2 case, in the sense illustrated by the following example. Let  $G$  be the infinite 3-regular tree with a distinguished end  $\omega$ . Let  $\Gamma$  be the set of automorphisms that preserve  $\omega$ , and let  $\mathcal{A}$  be the normal subgroup generated by the interchange of two children of a given vertex  $v$  (and the associated relabeling of their descendants). The graph  $\vec{G}$  is obtained from  $\mathbb{Z}$  by placing two directed edges between consecutive integers in one direction, and one directed edge in the reverse direction. The type of  $\mathcal{A}$  is 2. It is easily seen that neither (i) nor (ii) holds, and indeed  $\mu(\vec{G}) = \mu(G) = 2$ . We develop this example as follows.

Let  $k \geq 0$ , and let  $\mathcal{A}_k$  be the normal subgroup generated by  $\mathcal{A}$  together with the map that shifts  $v$  to its ancestor  $k$  generations earlier. Note that  $\mathcal{A}_k$  has type 2 for  $k \neq 1$ . The case of  $\mathcal{A}_1$  is trivial since  $\vec{G}$  has a unique vertex.

We have that  $\mathcal{A}_2$  is symmetric, and condition (i) of Theorem 3.8(b) applies. In contrast,  $\mathcal{A}_3$  is asymmetric (and therefore nonunimodular; see [30] and also [26, 28] as well as Lemma 3.10), and condition (ii) applies. In either case,  $\bar{\mu} < \mu(G)$ . The situation is in fact trivial since  $\vec{G}$  is a directed  $k$ -cycle with two directed edges clockwise and one anticlockwise. Thus  $\bar{\mu} = 0$  and  $\mu(G) = 2$ . The same argument shows  $\bar{\mu} < \mu(G)$  in the less trivial case with  $G$  the direct product of  $\mathbb{Z}^d$  and the tree.

A further example is provided later in Example 4.2.

It is sometimes convenient to work with an undirected graph derived from  $\vec{G}$ . There are two such graphs, depending on whether or not the multiplicities of edges are retained. The first is the simple graph, denoted  $\overline{G}_0$ , derived from  $\vec{G}$  by declaring two distinct vertices  $\bar{v}, \bar{w}$  to be adjacent if and only if there is a directed edge between  $\bar{v}$  and  $\bar{w}$  (this property is symmetric in  $\bar{v}, \bar{w}$ ).

The second such graph, denoted  $\overline{G} = (\overline{V}, \overline{E})$ , is a multigraph derived from  $\overline{G}_0$  by retaining the multiplicities of parallel edges of  $\vec{G}$ . If  $\mathcal{A}$  is symmetric (in that (3.8) holds), we obtain  $\overline{G}$  by placing  $|\partial v \cap \bar{w}|$  (undirected) parallel edges between each distinct pair  $\bar{v}, \bar{w} \in \overline{V}$  and adding  $|\partial v \cap \bar{v}|$  (undirected) loops at each  $\bar{v} \in \overline{V}$ .

The relationship between  $\bar{\mu}$  and the connective constants of the undirected graphs derived from  $\vec{G}$  is as follows. It is easily seen that  $\mu(\overline{G}_0) \leq \bar{\mu}$ . The graph  $\overline{G}$  is defined whenever  $\mathcal{A}$  is symmetric, and in this case  $\mu(\overline{G}) = \bar{\mu}$ .

Finally, we discuss the assumption of normality in Theorem 3.8. By Remark 3.5, this assumption may be replaced by the following: there exists an automorphism group  $\Gamma$  acting transitively on  $G$ , and a subgroup  $\mathcal{A}$  whose partition of  $V$  is  $\Gamma$ -invariant.

**3.5. Unimodularity.** The concept of unimodularity is relevant to the type of  $\mathcal{A}$ , and we discuss this briefly in this optional section. First recall the stabilizer  $\text{Stab}_v$



of a vertex  $v$  given at (3.1). As before, we say that  $\gamma \in \Gamma$  fixes vertex  $v$  if  $\gamma \in \text{Stab}_v$ . Note that  $|\text{Stab}_v w| < \infty$  for  $v, w \in V$ , since  $G$  is locally finite and all elements of  $\text{Stab}_v w$  are at the same distance from  $v$ .

Let  $\text{Stab}_v^0 = \text{Stab}_v \cap \mathcal{A}$ . As shown in [30] (see also [7, 28]), when viewed as a topological group with the usual topology,  $\mathcal{A}$  is unimodular if and only if

$$(3.8) \quad |\text{Stab}_u^0 v| = |\text{Stab}_v^0 u|, \quad u, v \in V.$$

Since all groups considered here are subgroups of  $\text{Aut}(G)$ , we may follow [24, Chap. 8] by defining  $\mathcal{A}$  to be unimodular if (3.8) holds.

It turns out that  $\mathcal{A}$  is symmetric if either it is unimodular or it has type 3.

LEMMA 3.10.

- (a) If  $\mathcal{A}$  is unimodular, then it is symmetric (in that (3.5) holds).
- (b) If  $\mathcal{A}$  has type 3, then  $|\partial v \cap \bar{v}| = 0$  for  $v \in V$ , and  $|\partial v \cap \bar{w}| = |\partial w \cap \bar{v}| = 1$  whenever  $\bar{v} \sim \bar{w}$ . In particular,  $\mathcal{A}$  is symmetric.

*Proof.* (a) Suppose that  $\mathcal{A}$  is unimodular, and let  $\bar{v}, \bar{w} \in \bar{V}$  with  $\bar{v} \neq \bar{w}$ . For  $x, y \in V$ , let

$$f(x, y) = \begin{cases} 1 & \text{if } x \in \bar{v}, y \in \bar{w}, \text{ and } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

The function  $f$  is invariant under the diagonal action of  $\mathcal{A}$ , in that  $f(\alpha x, \alpha y) = f(x, y)$  for  $\alpha \in \mathcal{A}$ . By the mass-transport principle as enunciated in, for example, [24, Thm. 8.7],

$$\sum_{w' \in \bar{w}} f(v, w') = \sum_{v' \in \bar{v}} f(v', w) \frac{|\text{Stab}_{v'}^0 w|}{|\text{Stab}_w^0 v'|}, \quad v, w \in V.$$

Equation (3.5) follows by (3.8).

- (b) When  $\mathcal{A}$  has type 3, the claim is a consequence of (3.7).  $\square$

**3.6. Two further results.** The proof of Theorem 3.8 does not appear to yield an explicit nontrivial upper bound for the ratio  $\bar{\mu}/\mu(G)$ . One may, however, show the following theorem, which is subject to the same assumptions as Theorem 3.8.

THEOREM 3.11. *Suppose there exists a real sequence  $(b_n : n = 1, 2, \dots)$ , each term of which may be calculated in finite time, such that  $b_n \leq \mu(G)$  and  $\lim_{n \rightarrow \infty} b_n = \mu(G)$ . There exists an algorithm which terminates in finite time and, on termination, yields a number  $R = R(G, \mathcal{A}) < 1$  such that  $\bar{\mu}/\mu(G) \leq R$ .*

If  $\mu$  is known, we may set  $b_n \equiv \mu(G)$ . In certain other cases, such a sequence  $(b_n)$  may be found; for example, when  $G = \mathbb{Z}^d$ , we may take  $b_n$  to be the  $n$ th root of the number of  $n$ -step bridges from the origin (see [19, 21]). Theorem 3.11 is unlikely to be useful in practice since the algorithm in question relies on successive enumerations of the numbers of  $n$ -step SAWs on  $G$  and  $\bar{G}$ .

A result similar to Theorem 3.11 is valid in the context of Theorem 3.2 also.

The relationship between the percolation critical points of a graph  $G$  and a version of the quotient graph  $\bar{G}$  is the topic of a conjecture of Benjamini and Schramm [8, Quest. 1]. As observed above (see also [8]), it is not necessary for the definition of quotient graph to assume that  $\mathcal{A}$  is a normal subgroup of  $\Gamma$ . However, [8] appears to make the additional assumption that  $\mathcal{A}$  acts freely on  $V$ . This is a stronger assumption than unimodularity.

Following [8], we ask whether  $\mu(\vec{G}) < \mu(G)$ , where  $\vec{G} = G/\mathcal{A}$ , under the weaker assumption that  $\mathcal{A}$  is a nontrivial (not necessarily normal) subgroup of  $\Gamma$  acting freely on  $V$ , such that  $\vec{G}$  is vertex-transitive. The proof of Theorem 3.8 may be adapted to give an affirmative answer to this question under an extra condition, as follows. An outline proof is included at the end of section 5.

**THEOREM 3.12.** *Let  $G$  be an infinite, locally finite graph on which the automorphism group  $\Gamma$  acts transitively. Let  $\mathcal{A}$  be a nontrivial subgroup of  $\Gamma$  acting freely on  $V$  such that the quotient graph  $\vec{G} := G/\mathcal{A}$  is vertex-transitive. Assume that there exists  $l \geq 1$  such that  $\vec{G}$  possesses an  $l$ -cycle but  $G$  does not. Then  $\mu(\vec{G}) < \mu(G)$ .*

**4. Strict inequalities for Cayley graphs.** We turn to the special case of Cayley graphs. Let  $\mathcal{G}$  be an infinite group with a finite set  $S$  of generators, where  $S$  is assumed symmetric in that  $S = S^{-1}$ , and the identity  $\iota$  satisfies  $\iota \notin S$ . Thus  $\mathcal{G}$  has a presentation as  $\mathcal{G} = \langle S \mid R \rangle$ , where  $R$  is a set of relators. The Cayley graph  $G = G(\mathcal{G}, S)$  is the simple graph defined as follows. The vertex-set  $V$  of  $G$  is the set of elements of  $\mathcal{G}$ . Distinct elements  $g, h \in V$  are connected by an edge if and only if there exists  $s \in S$  such that  $h = gs$ . It is easily seen that  $G$  is connected and vertex-transitive.

The group  $\mathcal{G}$  may be viewed as a subgroup of the automorphism group of  $G = (V, E)$  by:  $\gamma \in \mathcal{G}$  acts on  $V$  by  $v \mapsto \gamma v$ . Thus,  $\mathcal{G}$  acts transitively.

The reader is reminded of an elementary property of Cayley graphs, namely that  $\mathcal{G}$  acts freely on  $V$ . This is seen as follows. Suppose  $\gamma \in \text{Stab}_v$ . For  $w \in V$  there exists  $h \in \mathcal{G}$  with  $w = vh$ , so that  $\gamma$  fixes every  $w \in V$ , whence  $\gamma = \iota$ . See [4] for a general account of the theory of Cayley graphs, and [24, sect. 3.4] for a brief account.

Suppose that a product  $s_1 s_2 \cdots s_l$  of generators is a relator. The relation  $s_1 s_2 \cdots s_l = \iota$  corresponds to the closed walk  $(\iota, s_1, s_1 s_2, \dots, s_1 s_2 \cdots s_l = \iota)$  of  $G$  passing through the identity  $\iota$ . Consider now the effect of adding a further relator. Let  $s_1, s_2, \dots, s_l \in S$  be such that  $\rho := s_1 s_2 \cdots s_l \neq \iota$ , and write  $\mathcal{G}_\rho = \langle S \mid R \cup \{\rho\} \rangle$ . Then  $\mathcal{G}_\rho$  is isomorphic to the quotient group  $\mathcal{G}/\mathcal{N}$ , where  $\mathcal{N}$  is the normal subgroup of  $\mathcal{G}$  generated by  $\rho$ . Let  $G(\mathcal{G}_\rho, S)$  be the Cayley graph of  $\mathcal{G}_\rho$ .

**COROLLARY 4.1.** *Let  $G = G(\mathcal{G}, S)$  be the Cayley graph of the infinite, finitely presented group  $\mathcal{G} = \langle S \mid R \rangle$ . Let  $\rho \in \mathcal{G}$ ,  $\rho \notin R \cup \{\iota\}$ , and let  $G_\rho = G(\mathcal{G}_\rho, S)$ . The connective constants of  $G$  and  $G_\rho$  satisfy  $\mu(G_\rho) < \mu(G)$ .*

A complementary result may be found in [23, Thm. 2].

*Proof.* The Cayley graph  $G_\rho$  is obtained from the quotient graph  $G/\mathcal{G}_\rho$  by replacing every set of parallel edges by a single edge. Since  $\mathcal{G}_\rho$  acts freely on  $V$ , it is unimodular, and the claim follows by Lemma 3.10 and Theorem 3.8 (see also the statement about symmetry before Example 3.9).  $\square$

**EXAMPLE 4.2.** *The square/octagon lattice, otherwise known as the Archimedean lattice  $(4, 8^2)$ , is illustrated in Figure 3. It is the Cayley graph of the group with generator set  $S = \{s_1, s_2, s_3\}$  and relators  $\{s_1^2, s_2^2, s_3^2, s_1 s_2 s_1 s_2, s_1 s_3 s_2 s_3 s_1 s_3 s_2 s_3\}$ . The horizontal edges correspond to  $s_1$ , the vertical edges to  $s_2$ , and the other edges to  $s_3$ . Adding the further relator  $s_2 s_3 s_2 s_3$ , we obtain a graph isomorphic to the ladder graph of Figure 4, whose connective constant is the golden mean  $\phi := \frac{1}{2}(\sqrt{5} + 1)$ .*

*By Corollary 4.1, the connective constant  $\mu$  of the square/octagon lattice is strictly greater than  $\phi = 1.618\dots$ . The best lower bound currently known appears to be  $\mu > 1.804\dots$ ; see [20].*

*We ask whether  $\mu(G) \geq \phi$  for all infinite, vertex-transitive, simple, cubic graphs  $G$ ; see [15, 17].*

Our second inequality for Cayley graphs concerns the addition of a generator.

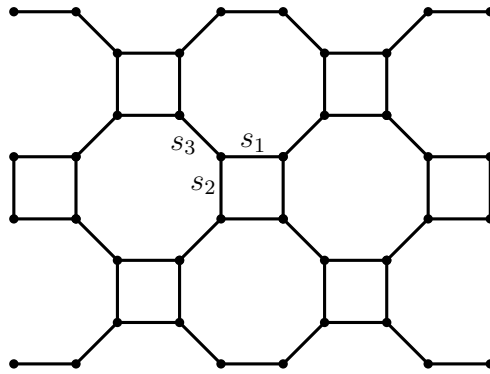


FIG. 3. The square/octagon lattice, also denoted  $(4, 8^2)$ .

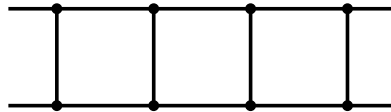


FIG. 4. The (doubly infinite) ladder graph has as connective constant the golden mean  $\phi := \frac{1}{2}(\sqrt{5} + 1)$ ; see [3, p. 184] and [17].

Let  $\mathcal{G} = \langle S \mid R \rangle$  be a finitely generated group as above, and let  $w$  be a group element satisfying  $w \notin S$  and  $w \neq \iota$ . Let  $\overline{\mathcal{G}}_w = \langle S \cup \{w, w^{-1}\} \mid R \rangle$ .

COROLLARY 4.3. *The connective constants of the Cayley graphs  $G, \overline{G}_w$  of the above presentations of the groups  $\mathcal{G}, \overline{\mathcal{G}}_w$  satisfy  $\mu(G) < \mu(\overline{G}_w)$ .*

*Proof.* This is an immediate corollary of Theorem 3.2, on noting that  $\mathcal{G}$  acts transitively on both  $G$  and  $\overline{G}_w$ .  $\square$

In the special case when  $\mu(G) = 1$ , we have that  $G = \mathbb{Z}$  (as noted after Theorem 3.8). Therefore,  $\overline{G}_w$  has degree either 3 or 4, whence  $\mu(\overline{G}_w) \geq \sqrt{2}$  by [15, Thm. 1.1].

EXAMPLE 4.4. *Consider  $\mathbb{Z}^2$  as the Cayley graph of the abelian group with  $S = \{a, b\}$  and  $R = \{aba^{-1}b^{-1}\}$ . Adding the generator  $ab$  (and its inverse) amounts to adding a diagonal to each square of  $\mathbb{Z}^2$ . Recall Example 3.4. One may easily construct more interesting examples based on, for example, the square/octagon lattice of Figure 3.*

For both Corollaries 4.1 and 4.3, there exists an algorithm which, under a certain condition, terminates in finite time and, on termination, yields an explicit nontrivial bound for the ratio of the connective constants under consideration. This holds just as in Theorem 3.11, and the details are omitted.

**5. Proofs of Theorems 3.8 and 3.11.** The proof of Theorem 3.8 is inspired by Kesten’s proof of the pattern theorem in [21] (see also [25, sect. 7.2]), and a familiarity with the latter proof will aid the reader of the following proofs. The overall shape of the argument from [21] recurs more than once in this paper; at later occurrences we shall outline any necessary adaptation rather than attempt to systematize the method. We begin with an elementary lemma.

LEMMA 5.1. *Let  $v \in V$ , and let  $\mathcal{A}$  be a normal subgroup of  $\Gamma$ . We have that  $\mathcal{A} \subseteq \text{Stab}_v$  if and only if  $\mathcal{A}$  is trivial.*

*Proof.* If  $\mathcal{A}$  is trivial, then  $\mathcal{A} = \{v\} \subseteq \text{Stab}_v$ . Conversely, suppose  $\mathcal{A} \subseteq \text{Stab}_v$  and let  $w \in V$ . Since  $G$  is vertex-transitive, there exists  $\gamma \in \Gamma$  such that  $\gamma v = w$ . Since  $\mathcal{A}$  is normal,

$$\mathcal{A}w = \mathcal{A}\gamma v = \gamma\mathcal{A}v = \{\gamma v\} = \{w\}.$$

Therefore,  $\mathcal{A} \subseteq \text{Stab}_w$  for all  $w \in V$ , and hence  $\mathcal{A}$  is trivial.  $\square$

Let  $G, \mathcal{A} \subseteq \Gamma, \vec{G} = G/\mathcal{A}$ , etc., be given as for Theorem 3.8, and fix  $v_0 \in V$ . Let  $\Sigma_n$  (respectively,  $\vec{\Sigma}_n$ ) be the set of  $n$ -step SAWs of  $G$  (respectively,  $\vec{G}$ ) starting from  $v_0$  (respectively,  $\vec{v}_0$ ), and write  $\sigma_n = |\Sigma_n|$  (respectively,  $\vec{\sigma}_n = |\vec{\Sigma}_n|$ ).

ASSUMPTION 5.2. *We assume henceforth that either condition (a) of Theorem 3.8 holds, or condition (b)(i). An explanation of the sufficiency of condition (b)(ii) for type 2 is given in Remark 5.5 at the end of this proof.*

Any walk  $\pi$  on  $G$  induces a walk  $\vec{\pi}$  on  $\vec{G}$ , and we say that  $\pi$  projects onto  $\vec{\pi}$ . For  $\vec{\pi} \in \vec{\Sigma}_n$ , there exists a SAW  $\pi \in \Sigma_n$  that projects onto  $\vec{\pi}$ , and we say that  $\vec{\pi}$  lifts to  $\pi$ . There are generally many choices for such  $\pi$ , and we fix such a choice as explained after Lemma 3.7. Let  $\mu = \mu(G)$  and  $\vec{\mu} = \mu(\vec{G})$ . By the above remarks,  $\vec{\mu} \leq \mu$ .

The idea of the proof is to replace certain subwalks of  $\vec{\pi} \in \vec{\Sigma}_n$  by new walks that lift to SAWs on  $G$ . Such replacements are given in terms of a certain SAW on  $G$  that we introduce in the next paragraph. We shall show that “most”  $n$ -step SAWs on  $\vec{G}$  contain at least  $An$  subwalks with the special property that they lift to at least two SAWs on  $G$ . This introduces a factor of  $2^{An}$  in the relative counts of SAWs on  $\vec{G}$  and  $G$ , and hence the required strict inequality  $\vec{\mu} < \mu$ . The word “most” must be interpreted in terms of an exponential growth rate, and the constant  $A > 0$  chosen appropriately. The principal point of divergence of the current proof from that of the pattern theorem lies in the multiplicity of liftings: whereas the pattern theorem is combinatorial at this point, the current proof is algebraic.

Let  $\ell_{v_0}$  be a shortest SAW from  $v_0$  to some  $w \neq v_0$  satisfying  $\vec{v}_0 = \vec{w}$ . Such a walk exists by Lemma 5.1 and the nontriviality of  $\mathcal{A}$ . We consider  $\ell_{v_0}$  as a directed walk from  $v_0$  to  $w$ . Let  $\vec{\ell}_{v_0}$  be the projection of  $\ell_{v_0}$  in  $\vec{G}$ . Thus, if  $G$  has type 1,  $\vec{\ell}_{v_0}$  is a loop; if type 2,  $\vec{\ell}_{v_0}$  traverses a directed edge from  $\vec{v}_0$  to  $\vec{w}$ , and then returns along a directed edge from  $\vec{w}$  to  $\vec{v}_0$ ; if type 3,  $\vec{\ell}_{v_0}$  is a (directed) cycle of  $\vec{G}$  of length 3 or more.

When  $\mathcal{A}$  has type 2, we take  $\ell_{v_0}$  to be the path given in condition (b)(i) of the theorem.

For  $w \in V$ , let  $L_w$  (or  $L(w)$ ) denote the set of all walks in  $G$  starting from  $w$  which are images of  $\ell_{v_0}$  under elements of  $\Gamma$ , and let  $\vec{L}_{\vec{w}}$  (or  $\vec{L}(\vec{w})$ ) be the corresponding collection of walks in  $\vec{G}$ . Let  $L = \bigcup\{L_w : w \in V\}$  and  $\vec{L} = \bigcup\{\vec{L}_{\vec{w}} : \vec{w} \in \vec{V}\}$ . All walks in  $L$  have equal length, written  $\Lambda$ . Thus,  $\Lambda$  is the number of distinct vertices in  $\vec{\ell}$ . We shall refer to elements of  $\vec{L}$  as *cycles*, and later we shall use the fact that the graph  $G$ , “decorated” with members of  $L$ , is preserved under the action of  $\Gamma$ .

For  $u \in V$  and a positive integer  $r$ , we define the *ball*  $B_r(u) = \{v \in V : d_G(u, v) \leq r\}$ . Since  $G$  is  $\Delta$ -regular,

$$(5.1) \quad |B_r(u)| \leq 1 + \Delta + \Delta(\Delta - 1) + \dots + \Delta(\Delta - 1)^{r-1} < \Delta^{r+1}.$$

Let  $\vec{\pi}$  be an  $r$ -step SAW on  $\vec{G}$ , and write  $\vec{\pi}_0, \vec{\pi}_1, \dots, \vec{\pi}_r$  for the vertices traversed by  $\vec{\pi}$  in order. For  $k \in \mathbb{N}$ , we say that  $E_k = E_k(\vec{\pi})$  occurs at the  $j$ th step of  $\vec{\pi}$  if there exists  $\vec{\ell} \in \vec{L}_{\vec{\pi}_j}$ , denoted  $\vec{\ell} = \vec{\ell}(\vec{\pi}_j)$ , such that at least  $k$  vertices of  $\vec{\ell}$  are visited by  $\vec{\pi}$ .

Now let  $\vec{\pi}$  be an  $n$ -step SAW on  $\vec{G}$ . Let  $j, m \in \mathbb{N}$ , and let  $\vec{\nu}$  be the sub-SAW of  $\vec{\pi}$  from  $\vec{\pi}_{j-m}$  to  $\vec{\pi}_{j+m}$ . (If  $j - m < 0$  or  $j + m > n$ , a modification must be made in this definition: if  $j - m < 0$ , we take  $\vec{\nu}$  from  $\vec{\pi}_0$  to  $\vec{\pi}_{j+m}$ , and similarly if  $j + m > n$ .) We say that  $E_k^m$  occurs at the  $j$ th step of  $\vec{\pi}$  if  $E_k(\vec{\nu})$  occurs at the  $m$ th step of  $\vec{\nu}$ . (If  $j - m < 0$ , we require that  $E_k(\vec{\nu})$  occur at the  $j$ th step, and similarly if  $j + m > n$ .) In particular, if  $E_k^m$  occurs at the  $j$ th step of  $\vec{\pi}$ , then  $E_k$  occurs at the  $j$ th step of  $\vec{\pi}$ .

For  $r \geq 0$ , let  $\vec{\sigma}_n(r, E_k)$  (respectively,  $\vec{\sigma}_n(r, E_k^m)$ ) be the number of SAWs in  $\vec{\Sigma}_n$  for which  $E_k$  (respectively,  $E_k^m$ ) occurs at no more than  $r$  different steps. Observe that, for given  $n, r$ , the count  $\vec{\sigma}_n(r, E_k^m)$  is nonincreasing in  $m$ .

It is easily seen that  $\vec{\sigma}_{m+n}(0, E_k) \leq \vec{\sigma}_m(0, E_k)\vec{\sigma}_n(0, E_k)$ . By the subadditive limit theorem, the limit

$$(5.2) \quad \lambda_k := \lim_{n \rightarrow \infty} \vec{\sigma}_n(0, E_k)^{1/n}$$

exists and satisfies

$$(5.3) \quad \lambda_k \leq \vec{\sigma}_n(0, E_k)^{1/n}, \quad n \geq 1.$$

Thus,  $\lambda_k < \mu$  if and only if

$$(5.4) \quad \exists \epsilon > 0, M \in \mathbb{N} \text{ such that } \vec{\sigma}_m(0, E_k) < [\mu(1 - \epsilon)]^m \text{ for } m \geq M.$$

Our target in the next two lemmas is to show that  $\lambda_\Lambda < \mu$ , and to deduce that, for suitable  $a, m$ , the number of  $n$ -step SAWs on  $\vec{G}$  in which  $E_\Lambda^m$  occurs fewer than  $an$  times grows at a smaller exponential rate than the total number of SAWs. Once the last is proved, the proof of the theorem is fairly immediately completed in the manner of the sketch near the start of this section.

LEMMA 5.3. *Let  $k$  satisfy  $1 \leq k \leq \Lambda$  and*

$$(5.5) \quad \lambda_k < \mu.$$

*Let  $\epsilon, M$  satisfy (5.4), and let  $m \geq M$  satisfy*

$$(5.6) \quad \vec{\sigma}_m \leq [\mu(1 + \epsilon)]^m.$$

*There exist  $a = a(\epsilon, m) > 0$  and  $R = R(\epsilon, m) \in (0, 1)$  such that*

$$(5.7) \quad \limsup_{n \rightarrow \infty} \vec{\sigma}_n(an, E_k^m)^{1/n} < R\mu.$$

*Proof.* Assume that  $k$  is such that (5.5) holds, and let  $\epsilon, M, m$  satisfy (5.4) and (5.6). Since  $\vec{\sigma}_m(0, E_k) = \vec{\sigma}_m(0, E_k^m)$ ,

$$(5.8) \quad \vec{\sigma}_m(0, E_k^m) < [\mu(1 - \epsilon)]^m.$$

Let  $\vec{\pi} \in \vec{\Sigma}_n$  and  $N = \lfloor n/m \rfloor$ . If  $E_k^m$  occurs at no more than  $r$  steps in  $\vec{\pi}$ , then  $E_k^m$  occurs at no more than  $r$  of the  $N$   $m$ -step subwalks

$$(\vec{\pi}_{(j-1)m}, \vec{\pi}_{(j-1)m+1}, \dots, \vec{\pi}_{jm}), \quad 1 \leq j \leq N.$$

Counting the number of ways in which  $k$  or fewer of these subwalks can contain an occurrence of  $E_k^m$ , we have by (5.6) and (5.8) that

$$(5.9) \quad \begin{aligned} \vec{\sigma}_n(r, E_k^m) &\leq \sum_{i=0}^r \binom{N}{i} (\vec{\sigma}_m)^i \vec{\sigma}_m(0, E_k^m)^{N-i} \vec{\sigma}_{n-Nm} \\ &\leq \mu^{Nm} \vec{\sigma}_{n-Nm} \sum_{i=0}^r \binom{N}{i} (1 + \epsilon)^{im} (1 - \epsilon)^{(N-i)m}. \end{aligned}$$

It suffices to show that there exist  $\zeta > 0$  and  $t < 1$ , depending on  $\epsilon, m$  only, such that

$$(5.10) \quad \bar{\sigma}_n(\zeta N, E_k^m)^{1/N} < t\mu^m$$

for all sufficiently large  $n$ , since this yields (5.7) with  $0 < a < \zeta/m$  and  $R = t^{1/m}$ . For  $\zeta$  small and positive,

$$(5.11) \quad \sum_{i=0}^{\zeta N} \binom{N}{i} (1+\epsilon)^{im} (1-\epsilon)^{(N-i)m} \leq (\zeta N + 1) \binom{N}{\zeta N} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\zeta Nm} (1-\epsilon)^{Nm}.$$

The  $N$ th root of the right side converges as  $N \rightarrow \infty$  to

$$\frac{1}{\zeta^\zeta (1-\zeta)^{1-\zeta}} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\zeta m} (1-\epsilon)^m,$$

which is strictly less than 1 for  $0 < \zeta < \zeta_0$  and some  $\zeta_0 = \zeta_0(\epsilon, m) > 0$ . Combining this with (5.9), we obtain (5.10) for  $0 < \zeta < \zeta_0$ , suitable  $t = t(\epsilon, m)$ , and  $n$  sufficiently large.  $\square$

LEMMA 5.4. *We have that  $\lambda_\Lambda < \mu$ .*

*Proof.* Since each vertex  $\pi_j$  is visited by  $\pi$ ,

$$(5.12) \quad \bar{\sigma}_n(0, E_1) = 0, \quad n \geq 1.$$

*Type 1.* If  $G$  has type 1, then  $\Lambda = 1$ , and  $\lambda_\Lambda = 0$  by (5.12).

*Types 2 and 3.* Assume  $G$  has type 2 or 3, and that the conclusion of the lemma is false in that

$$(5.13) \quad \lambda_\Lambda = \mu.$$

Now,  $\bar{\sigma}_n(0, E_k)$  (and hence  $\lambda_k$  also) is nondecreasing in  $k$ . By (5.12), we may choose  $k$  with  $1 \leq k < \Lambda$  such that

$$(5.14) \quad \lambda_k < \mu, \quad \lambda_{k+1} = \mu.$$

Let  $\epsilon, M$  satisfy (5.4), and let  $m \geq M$  satisfy (5.6). By Lemma 5.3, there exists  $a = a(\epsilon, m) > 0$  such that

$$(5.15) \quad \limsup_{n \rightarrow \infty} \bar{\sigma}_n(an, E_k^m)^{1/n} < \mu.$$

Let  $T_n$  be the subset of  $\bar{\Sigma}_n$  comprising SAWs for which  $E_{k+1}$  never occurs but  $E_k^m$  occurs at least  $an$  times. We have that

$$|T_n| \geq \bar{\sigma}_n(0, E_{k+1}) - \bar{\sigma}_n(an, E_k^m),$$

whence, by (5.14) and (5.15),

$$(5.16) \quad \lim_{n \rightarrow \infty} |T_n|^{1/n} = \mu.$$

Thus, under (5.13), for “most” SAWs  $\bar{\pi}$ , there exist many cycles in  $\bar{L}$  having exactly  $k$  ( $< \Lambda$ ) vertices visited by  $\bar{\pi}$  and none with more than  $k$  such vertices. The rest of the proof is devoted to showing the existence of  $S = S(\epsilon, m, \Delta, \Lambda) < 1$  such that

$$(5.17) \quad \limsup_{n \rightarrow \infty} |T_n|^{1/n} \leq S\mu.$$



This contradicts (5.16), and the claim follows.

The idea is as follows. Let  $\vec{\pi} \in T_n$ , and consider the set of cycles  $\vec{\ell}(\vec{\pi}_j)$  with exactly  $k$  vertices visited by  $\vec{\pi}$ . Where such a cycle is met by  $\vec{\pi}$ , we may augment  $\vec{\pi}$  with an entire copy of it. The ensuing transformation is not one-to-one, and the length of the resulting walk differs from that of  $\vec{\pi}$ . By selecting the places where the new elements of  $\vec{L}$  are added, we shall show that the number of resulting walks exceeds  $|T_n|$  by an exponential factor. It is key that such augmented walks lift to SAWs on  $G$  while traversing cycles in  $\vec{G}$ , and thus we shall contradict (5.16).

Let  $\vec{\pi} \in T_n$ , so that  $\vec{\pi}$  contains at least  $an$  occurrences of  $E_k^m$ . We can find  $j_1 < \dots < j_u$  with  $u = \lfloor \kappa n \rfloor - 2$ , where

$$(5.18) \quad \kappa = \frac{a}{(2m + 2)\Delta^{2\Lambda+1}},$$

such that

$$(5.19) \quad E_k^m \text{ occurs at the } j_1\text{th, } j_2\text{th, } \dots, j_u\text{th steps of } \vec{\pi}$$

(and perhaps at other steps as well), and in addition

$$(5.20) \quad 0 < j_1 - m, \quad j_u + m < n,$$

$$(5.21) \quad j_t + m < j_{t+1} - m, \quad 1 \leq t < u,$$

$$(5.22) \quad \forall \vec{\ell}_t \in \vec{L}(\vec{\pi}_t), \quad \text{the } \vec{\ell}_1, \vec{\ell}_2, \dots, \vec{\ell}_u \text{ are pairwise vertex-disjoint.}$$

Such  $j_t$  may be found by the following iterative construction. First,  $j_1$  is the smallest  $j > m$  such that  $E_k^m$  occurs at the  $j$ th step of  $\vec{\pi}$ . Having found  $j_1, j_2, \dots, j_r$ , let  $j_{r+1}$  be the smallest  $j$  such that

1.  $j_r + m < j - m$ ,
2. every element of  $\vec{L}(\vec{\pi}_j)$  is disjoint from every element of  $\vec{L}(\vec{\pi}_{j_1}), \vec{L}(\vec{\pi}_{j_2}), \dots, \vec{L}(\vec{\pi}_{j_r})$ ,
3.  $E_k^m$  occurs at the  $j$ th step of  $\vec{\pi}$ .

Condition 1 gives rise to the factor  $2m + 2$  in the denominator of (5.18), and, by (5.1), condition 2 gives rise to the factor  $\Delta^{2k+1}$  ( $\leq \Delta^{2\Lambda+1}$ ).

Let  $t \in \{1, 2, \dots, u\}$ . Since  $E_k^m$  but not  $E_{k+1}$  occurs at the  $j_t$ th step,  $\vec{\pi}$  visits at most  $k$  vertices in each cycle of  $\vec{L}(\vec{\pi}_{j_t})$ . Let  $\vec{L}(\vec{\pi}_{j_t}, k)$  be the subset of  $\vec{L}(\vec{\pi}_{j_t})$  containing all such cycles with exactly  $k$  vertices visited by  $\vec{\pi}$ , and such that these  $k$  vertices lie between  $\vec{\pi}_{j_t-m}$  and  $\vec{\pi}_{j_t+m}$  on  $\vec{\pi}$ . Choose a specific cycle  $\vec{\ell}(\vec{\pi}_{j_t}) \in \vec{L}(\vec{\pi}_{j_t}, k)$ . For  $t = 1, 2, \dots, u$ , let

$$(5.23) \quad \alpha_t = \min\{i : \vec{\pi}_i \in \vec{\ell}(\vec{\pi}_{j_t})\}, \quad \omega_t = \max\{i : \vec{\pi}_i \in \vec{\ell}(\vec{\pi}_{j_t})\},$$

so that

$$j_t - m \leq \alpha_t \leq j_t \leq \omega_t \leq j_t + m, \quad 1 \leq t \leq u.$$

We describe next the strategy for replacement of the subwalk  $(\vec{\pi}_{\alpha_t}, \vec{\pi}_{\alpha_t+1}, \dots, \vec{\pi}_{\omega_t})$ . Starting from  $\vec{\pi}_{\alpha_t}$ , the new walk winds once around the cycle  $\vec{\ell} := \vec{\ell}(\vec{\pi}_{j_t})$ ; having returned to the vertex  $\vec{\pi}_{\alpha_t}$ , it continues around the same cycle  $\vec{\ell}$  until it reaches the vertex  $\vec{\pi}_{\omega_t}$ . This new subwalk is inserted into  $\vec{\pi}$  at the appropriate place. The resulting walk  $\vec{\pi}^{(t)}$  is evidently not self-avoiding in  $\vec{G}$  (it includes a unique cycle, namely  $\vec{\ell}$  in

some order), but we shall see that it lifts to a SAW  $\pi_*^{(t)}$  on  $G$ . The precise definition and properties of  $\tilde{\pi}^{(t)}$  and  $\pi_*^{(t)}$  are described next.

Suppose  $\vec{\pi} \in T_n$  lifts to  $\pi \in \Sigma_n$ . The initial segment  $(\vec{\pi}_0, \dots, \vec{\pi}_{j_t})$  of  $\vec{\pi}$  lifts to a SAW of  $G$  that traverses the vertices  $\pi_0, \pi_1, \dots, \pi_{j_t}$ . We write  $v := \pi_{j_t}$ , and we shall consider graphs of type 2 and 3 separately.

*Type 3.* Assume that  $G$  has type 3, and think of  $\vec{\ell}$  as a rooted, directed cycle of  $\vec{G}$  with root  $\bar{v}$ . The cycle  $\vec{\ell}$  lifts to the SAW  $\ell := \ell_v$  of  $G$ , traversing the vertices  $v (= w_0), w_1, w_2, \dots, w_\Lambda$ . We have that

$$(5.24) \quad v (= w_0), w_1, \dots, w_{\Lambda-1} \text{ belong to different equivalence classes}$$

of  $(V, \approx)$ , and we may choose  $\beta \in \mathcal{A}$  such that  $w_\Lambda = \beta v$ .

We prove next that

$$(5.25) \quad w_r \neq \beta w_r, \quad 1 \leq r < \Lambda.$$

Suppose first that  $w_1 = \beta w_1$ , and consider the walk

$$v (= w_0), w_1, \dots, w_\Lambda (= \beta v), \beta w_1 (= w_1).$$

By the triangle inequality,

$$(5.26) \quad \Lambda = d_G(v, \beta v) \leq d_G(v, w_1) + d_G(\beta w_1, \beta v) = 2,$$

a contradiction since  $\Lambda \geq 3$ . Applying the same argument to the walk

$$w_1, w_2, \dots, w_\Lambda (= \beta v), \beta w_1, \beta w_2,$$

we obtain by (3.7) that  $w_2 \neq \beta w_2$ , and (5.25) follows by iteration.

Suppose that  $\vec{\pi}_{\alpha_t}$  lifts to vertex  $x \in V$  with  $\bar{x} = \bar{w}_i$  and  $i \geq 1$ ; similarly, suppose  $\vec{\pi}_{\omega_t} = \bar{w}_j$ . We show next that the replacement of the subwalk  $(\vec{\pi}_{\alpha_t}, \dots, \vec{\pi}_{\omega_t})$  lifts to some SAW of  $G$ . Find  $\gamma \in \mathcal{A}$  such that  $x = \gamma w_i$ .

*Case 3.1:*  $i < j$ . Consider the walk

$$(5.27) \quad x (= \gamma w_i), \gamma w_{i+1}, \dots, \gamma w_\Lambda (= \gamma \beta v), \gamma \beta w_1, \dots, \gamma \beta w_i,$$

followed by  $\gamma \beta w_{i+1}, \gamma \beta w_{i+2}, \dots, \gamma \beta w_j$ . By (5.24), two vertices of this walk are equal if and only if there exists  $r$  such that  $1 \leq r < \Lambda$  and  $w_r = \beta w_r$ . By (5.25), this is not the case. We have proved that  $(\vec{\pi}_0, \dots, \vec{\pi}_{\alpha_t})$ , followed by the above walk, lifts to a SAW  $\nu$  on  $G$ . Thus,  $\tilde{\pi}^{(t)}$  lifts to the SAW  $\nu$  followed by the image of  $(\pi_{\omega_t}, \dots, \pi_n)$  under the map that sends  $\pi_{\omega_t}$  to  $\gamma \beta w_j$ . Since  $\vec{\pi} \in T_n$ ,  $\tilde{\pi}^{(t)}$  lifts to a SAW  $\pi_*^{(t)}$  of  $G$ .

*Case 3.2:*  $i > j$ . Consider the walk

$$(5.28) \quad x (= \gamma w_i), \gamma w_{i-1}, \dots, \gamma w_0 (= \gamma v), \gamma \beta^{-1} w_{\Lambda-1}, \dots, \gamma \beta^{-1} w_i,$$

followed by  $\gamma \beta^{-1} w_{i-1}, \gamma \beta^{-1} w_{i-2}, \dots, \gamma \beta^{-1} w_j$ . By (5.25), this is a SAW on  $G$ , and the step is completed as above.

*Type 2.* Assume that  $G$  has type 2, so that  $\Lambda = 2$  and  $k = 1$ . The required argument is slightly different, since (5.26) is no longer a contradiction. Let  $j_t$  be as above, and let  $\vec{\ell} = \vec{\ell}(\vec{\pi}_{j_t}) \in \vec{L}(\vec{\pi}_{j_t})$  (with corresponding  $\ell \in L$ ) be a witness to the occurrence of  $E_1^m$  at  $\vec{\pi}_{j_t}$ . As in (5.23),

$$(5.29) \quad \begin{aligned} \alpha_t &= \alpha_t(\vec{\pi}) = \min\{i : \vec{\pi}_i \in \vec{\ell}\}, \\ \omega_t &= \omega_t(\vec{\pi}) = \max\{i : \vec{\pi}_i \in \vec{\ell}\}. \end{aligned}$$

Let  $\pi$  be the lift of  $\tilde{\pi}$  to a SAW in  $G$  from  $v_0$ , and write  $v = \pi_{j_t}$ . Recall that  $\ell$  is a SAW visiting  $v, w, \beta v$  in  $G$  for some  $\beta = \beta_t \in \mathcal{A}$  with  $\beta v \neq v$ , and the pair  $v, w$  are visited (in some order) at the  $\alpha_t$ th and  $\omega_t$ th steps of  $\pi$ . We may assume that  $\beta w \neq v$ , since otherwise  $w$  is adjacent to  $\beta w$ , and  $G$  is of type 1. We next describe the required substitution.

*Case 2.1:*  $\beta w \neq w$ . As in Cases 3.1 and 3.2 above, both  $v, w, \beta v, \beta w$  and  $w, v, \beta^{-1}w, \beta^{-1}v$  are SAWs on  $G$ . We write  $\pi_*^{(t)}$  for the SAW on  $G$  obtained by replacing the segment of  $\pi$  between  $\pi_{\alpha_t}$  and  $\pi_{\omega_t}$  by

$$\begin{aligned} v, w, \beta v, \beta w & \text{ if } \pi \text{ visits } v \text{ before } w, \\ w, v, \beta^{-1}w, \beta^{-1}v & \text{ if } \pi \text{ visits } w \text{ before } v, \end{aligned}$$

and the walk after  $\beta w$  (respectively,  $\beta^{-1}v$ ) by the image under  $\beta$  (respectively,  $\beta^{-1}$ ) of  $\pi$  after  $w$  (respectively,  $v$ ).

*Case 2.2.1:*  $\beta w = w$ , and  $\pi$  visits  $w$  before  $v$ . Only in the following shall we use the assumed condition (b)(i) of Theorem 3.8. Since  $|\partial v \cap \overline{w}|, |\partial w \cap \overline{v}| \geq 2$ , there exist  $w' \in \overline{w}$  and  $v' \in \overline{v}$  such that

$$(5.30) \quad w, v, w', v'$$

is a SAW of  $G$ . Let  $\gamma \in \mathcal{A}$  be such that  $v' = \gamma v$ . We write  $\pi_*^{(t)}$  for the SAW on  $G$  obtained by replacing the segment of  $\pi$  between  $\pi_{\alpha_t}$  and  $\pi_{\omega_t}$  by (5.30), and the walk after  $v'$  ( $= \gamma v$ ) by the image under  $\gamma$  of  $\pi$  after  $v$ .

*Case 2.2.2:*  $\beta w = w$ , and  $\pi$  visits  $v$  before  $w$ . Since  $|\partial v \cap \overline{w}| \geq 2$ , there exists  $w' \in \overline{w}$  such that

$$(5.31) \quad v, w, \beta v, w'$$

is a SAW on  $G$ . We find  $\gamma \in \mathcal{A}$  such that  $w' = \gamma w$ , and we write  $\pi_*^{(t)}$  for the SAW on  $G$  obtained by replacing the segment of  $\pi$  between  $\pi_{\alpha_t}$  and  $\pi_{\omega_t}$  by (5.31), and the walk after  $w'$  ( $= \gamma w$ ) by the image under  $\gamma$  of  $\pi$  after  $w$ .

This ends the definitions of the required substitutions, and we consider next their enactment. Let  $\delta > 0$ , to be chosen later, and set  $s = \delta n$ . (Here and later, for simplicity of notation, we omit the integer-part symbols.) Let  $H = (h_1, h_2, \dots, h_s)$  be an ordered subset of  $\{j_1, j_2, \dots, j_u\}$ . We shall make an appropriate substitution in the neighborhood of each  $\tilde{\pi}_{h_t}$ , by an iterative construction.

We consider the cases  $t = 1, 2, \dots, u$  in order. First, let  $t = 1$ . If  $j_1 \notin H$ , we do nothing, and we set  $\pi^{(1)} = \pi$ . If  $j_1 \in H$ , we make the appropriate substitution around the point  $\tilde{\pi}_{j_1}$  and write  $\pi^{(1)}$  for the resulting SAW on  $G$ . Now let  $t = 2$ . Once again, if  $j_2 \notin H$ , we do nothing and set  $\pi^{(2)} = \pi^{(1)}$ . Otherwise, let  $\alpha'_2 = \alpha_2(\tilde{\pi}^{(1)})$ ,  $\omega'_2 = \omega_2(\tilde{\pi}^{(1)})$ . The sub-SAW of  $\pi^{(1)}$  between steps  $\alpha'_2$  and  $\omega'_2$  is the image of  $\pi$  between steps  $\alpha_2$  and  $\omega_2$  under some  $\gamma' \in \mathcal{A}$ . Therefore, we may perform operations on  $\pi^{(1)}$  after the  $\alpha'_2$ th step as discussed above for the  $\alpha_2$ th step of  $\pi$ , thus obtaining a SAW  $\pi^{(2)}$ . This process may be iterated to obtain a sequence  $\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(u)}$  of SAWs on  $G$ , and we set  $\pi_* = \pi^{(u)}$ . The role of  $H$  is emphasized by writing  $\pi_* = \pi_*(\tilde{\pi}, H)$ . It follows from the construction that  $\pi_*(\tilde{\pi}, H) \neq \pi_*(\tilde{\pi}, H')$  if  $H \neq H'$ .

One small issue arises during the iteration, namely that the projections of the SAWs  $\pi^{(t)}$  are not generally self-avoiding. However, by (5.20)–(5.22), the subwalk of  $\tilde{\pi}$  under inspection at any given step is disjoint from all previously inspected subwalks, and thus the current substitution is unaffected by the past.

By (5.22) and the discussion above,  $\pi_*$  is self-avoiding on  $G$ . By inspection of such  $\pi_*$  and its projection, one may reconstruct the places at which cycles have been added to  $\vec{\pi}$ . The length of  $\pi_*$  does not exceed  $n + 2\Lambda s$ .

We estimate the number of pairs  $(\vec{\pi}, H)$  as follows. First, the number  $|(\vec{\pi}, H)|$  is at least the cardinality of  $T_n$  multiplied by the minimum number of possible choices of  $H$  as  $\vec{\pi}$  ranges over  $T_n$ . Any subset of  $\{j_1, j_2, \dots, j_u\}$  with cardinality  $s = \delta n$  may be chosen for  $H$ , whence

$$(5.32) \quad |(\vec{\pi}, H)| \geq |T_n| \binom{\kappa n - 2}{\delta n}.$$

We bound  $|(\vec{\pi}, H)|$  above by counting the number of SAWs  $\pi_*$  of  $G$  with length not exceeding  $n + 2\Lambda\delta n$  and multiplying by an upper bound for the number of pairs  $(\vec{\pi}, H)$  giving rise to a particular  $\pi_*$ . The number of possible choices for  $\pi_*$  is no greater than  $\sum_{i=0}^{n+2\Lambda\delta n} \sigma_i$ . A given  $\pi_*$  contains  $|H| = \delta n$  elements of  $L$ . At the  $t$ th such occurrence,  $\vec{\pi}_{h_t}$  is a point on the corresponding cycle, and there are no more than  $2\Lambda$  different choices for  $\vec{\pi}_{h_t}$ . For given  $\pi$  and  $(\vec{\pi}_{h_t} : t = 1, 2, \dots, s)$  there are at most  $(\sum_{i=1}^{2m} \vec{\sigma}_i)^{\delta n}$  corresponding SAWs  $\vec{\pi}$  of  $\vec{G}$ . Therefore,

$$(5.33) \quad |(\vec{\pi}, H)| \leq \left(2\Lambda \sum_{i=1}^{2m} \vec{\sigma}_i\right)^{\delta n} \left(\sum_{i=0}^{n+2\Lambda\delta n} \sigma_i\right).$$

Let  $\tau := \limsup |T_n|^{1/n}$ . We combine (5.32)–(5.33), and take  $n$ th roots and the limit as  $n \rightarrow \infty$ , to obtain, by the fact that  $\sigma_N^{1/N} \rightarrow \mu$ ,

$$\tau \frac{\kappa^\kappa}{\delta^\delta (\kappa - \delta)^{\kappa - \delta}} \leq \left(2\Lambda \sum_{i=1}^{2m} \vec{\sigma}_i\right)^\delta \mu^{1+2\Lambda\delta}.$$

There exists  $Z = Z(\epsilon, m, \Delta, \Lambda) < \infty$  such that

$$2\Lambda\mu^{2\Lambda} \sum_{i=1}^{2m} \vec{\sigma}_i \leq Z.$$

Therefore,

$$\tau \leq f(\eta)^\kappa \mu,$$

where  $f(\eta) = Z^\eta \eta^\eta (1 - \eta)^{1-\eta}$  and  $\eta = \delta/\kappa$ . Since

$$\lim_{\eta \downarrow 0} f(\eta) = 1, \quad \lim_{\eta \downarrow 0} f'(\eta) = -\infty,$$

we have that  $f(\eta) < 1$  for sufficiently small  $\eta = \eta(Z) > 0$ , and (5.17) follows for suitable  $S < 1$ . The proof is complete.  $\square$

*Proof of Theorem 3.8 under Assumption 5.2.* By Lemma (5.4),  $\lambda_\Lambda < \mu$ . Let  $\epsilon, M$  satisfy (5.4), and let  $m \geq M$  satisfy (5.6), with  $k = \Lambda$ . By Lemma 5.3, there exist  $a = a(\epsilon, m) > 0$  and  $R = R(\epsilon, m) \in (0, 1)$  such that

$$(5.34) \quad \limsup_{n \rightarrow \infty} \vec{\sigma}_n(an, E_\Lambda^m)^{1/n} < R\mu.$$

Let  $T_n$  be the subset of  $\vec{\Sigma}_n$  comprising SAWs for which  $E_\Lambda^m$  occurs at least  $an$  times. Thus

$$(5.35) \quad |T_n| \geq \vec{\sigma}_n - \vec{\sigma}_n(an, E_\Lambda^m).$$

We follow the route of the previous proof. For  $\vec{\pi} \in T_n$  and  $\kappa$  as in (5.18), we may find  $j_1 < \dots < j_u$  with  $u = \kappa n - 2$  such that (5.19)–(5.22) hold with  $k = \Lambda$ .

Let  $\delta > 0$ , set  $s = \delta n$ , and let  $\vec{\pi} \in T_n$ . We choose a subset  $H$  of  $\{j_1, j_2, \dots, j_u\}$  with cardinality  $s$ , and we construct a SAW  $\pi_* = \pi_*(\vec{\pi}, H)$  on  $G$  accordingly. This may be done exactly as in the previous proof if  $G$  has type 2 or 3 to obtain as in (5.17) that there exists  $S = S(\epsilon, m, \Delta, \Lambda) < 1$  such that

$$(5.36) \quad \limsup_{n \rightarrow \infty} |T_n|^{1/n} \leq S\mu.$$

By (5.34)–(5.35),

$$\vec{\mu} = \lim_{n \rightarrow \infty} \vec{\sigma}_n^{1/n} \leq \max\{R, S\}\mu,$$

and the claim of the theorem is proved in this case.

Assume that  $G$  has type 1. This case is easier than the others. Suppose  $\vec{\pi}_{j_t} = \vec{v}$  and that  $\vec{\pi}$  lifts to  $\pi$  with  $\pi_{j_t} = v$ . We obtain  $\pi_*^{(t)}$  by replacing  $v$  by the pair  $v, \gamma v$  for some  $\gamma \in \mathcal{A}$  with  $\gamma \neq \iota$ , and translating by  $\gamma$  the subwalk of  $\pi$  starting at  $v$ . The above counting argument yields the result.  $\square$

REMARK 5.5. *Theorem 3.8 has been proved subject to Assumption 5.2, and it remains to prove it when  $\mathcal{A}$  has type 2 and condition (b)(ii) holds. Suppose the latter holds, so that  $l \geq 2$ . If  $l = 2$ , the argument in the proof of Lemma 5.4 is valid (this is Case 2.1 of the proof, which does not use condition (i)), and the theorem follows similarly. Assume  $l \geq 3$ . By vertex-transitivity, we may set  $v_0 = v$ , and replace  $\ell_{v_0}$  by the path of (ii), so that  $\Lambda = l$ . The proof for type 3 in Lemma 5.4 may be followed with one difference, namely that (5.25) holds by the assumption that  $\beta$  fixes no  $w_i$ .*

*Proof of Theorem 3.11.* We use the notation of the previous proofs. The constants  $R, S$  of the last proof may be calculated explicitly in terms of  $\epsilon, m, \Delta, \Lambda$ , and so it suffices to describe how to choose  $\epsilon$  and  $m$ .

Let  $a_n = \sigma_n^{1/n}$ , noting by (2.2) that  $a_n \geq \mu$  and  $a_n \rightarrow \mu$ . By changing  $(b_n)$  if necessary, we may assume for convenience that  $(b_n)$  is a nondecreasing sequence, so that  $b_n \uparrow \mu$ .

By Lemma 5.4,  $\lambda_\Lambda < \mu$ . Since  $b_r(1 - r^{-1}) \uparrow \mu$ , we may find the earliest  $r$  such that

$$\vec{\sigma}_r(0, E_\Lambda)^{1/r} < b_r(1 - r^{-1}).$$

Let  $\epsilon = r^{-1}$ , and let  $s \geq r$  be such that  $b_s(1 + \epsilon) \geq a_s(1 + \frac{1}{2}\epsilon)$ . Since  $b_s \geq b_r$ ,

$$\vec{\sigma}_r(0, E_\Lambda)^{1/r} < b_s(1 - \epsilon).$$

By (5.3),  $\lambda_\Lambda < b_s(1 - \epsilon)$ , so that

$$(5.37) \quad \vec{\sigma}_m(0, E_\Lambda)^{1/m} < b_s(1 - \epsilon)$$

for infinitely many values of  $m$ .

Since  $a_s \geq \mu$ , we may find the earliest  $m$  such that (5.37) holds and in addition  $\vec{\sigma}_m^{1/m} \leq b_s(1 + \epsilon)$ . With  $\epsilon$  and  $m$  defined thus, (5.6) and (5.8) are valid as required.

Each of the above computations requires a finite enumeration of a suitable family of SAWs.  $\square$

*Outline proof of Theorem 3.12.* Let  $v_0 \in V$  with orbit  $\bar{v}_0 = \mathcal{A}v_0$ . With  $l \geq 1$  as given, let  $\vec{\ell}$  be a (directed) cycle of  $\vec{G}$  of length  $l$ . Since  $\vec{G}$  is vertex-transitive, we may assume that  $\vec{\ell}$  goes through  $\bar{v}_0$ , and thus we write  $\vec{\ell}_{\bar{v}_0}$  for  $\vec{\ell}$ , considered as a (directed) walk from  $\bar{v}_0$  to  $\bar{v}_0$ . Now,  $\vec{\ell}_{\bar{v}_0}$  lifts to a walk  $\ell_{v_0}$  from  $v_0$  on  $G$ . Since all vertices of  $\vec{\ell}_{\bar{v}_0}$  other than its endpoints are distinct, and since  $G$  possesses no cycle of length  $l$ ,  $\ell_{v_0}$  is a SAW from  $v_0$  on  $G$ .

Let  $\mathcal{B}$  be a subgroup of  $\text{Aut}(\vec{G})$  acting transitively on  $\vec{G}$ . For  $w \in V$ , let  $\vec{L}(w)$  be the set of images of  $\vec{\ell}_{\bar{v}_0}$  under  $\mathcal{B}$ , and let  $L(w)$  be the set of lifts of such images. Since  $G$  possesses no cycle of length  $l$ , every member of  $L(w)$  is a SAW on  $G$ .

We now follow the above proof of Theorem 3.8. No assumption is made on the relationship between the groups  $\Gamma$  and  $\mathcal{B}$ . If  $l \neq 2$ , the argument is as presented previously, except insofar as (5.25) holds (since  $\mathcal{A}$  acts freely). If  $l = 2$ , we follow Case 2.1 as in Remark 5.5.  $\square$

**6. Proof of Theorem 3.2.** The general idea of the proof of Theorem 3.2 is similar to that of Theorem 3.8, but with some differences. The graphs  $G, \bar{G}$  have the same vertex-set  $V$ , but  $\bar{G}$  possesses cycles not present in  $G$ . Let  $v_0, w_0 \in V$  be such that there are strictly more edges of the form  $\langle v_0, w_0 \rangle$  in  $\bar{E}$  than in  $E$ . Since  $G$  is connected, it has a shortest path from  $v_0$  to  $w_0$ , written

$$(6.1) \quad v_0 f_1 v_1 f_2 v_2 \cdots f_\Lambda v_\Lambda (= w_0), \quad \text{where } f_t = \langle v_{t-1}, v_t \rangle.$$

The path (6.1), followed by the new edge  $f := \langle w_0, v_0 \rangle$ , forms a cycle in  $\bar{G}$  but not in  $G$ . Let  $C_1$  denote this cycle of  $\bar{G}$ , and note that  $C_1$  has length  $\Lambda + 1$ . As usual,  $\Delta$  denotes the vertex-degree of  $G$ .

Let  $C$  be a finite set of vertices of  $G$  with  $v_0 \in C$ , and denote by  $\gamma C = \{\gamma z : z \in C\}$  the image of  $C$  under the automorphism  $\gamma \in \Gamma$ . Let  $\pi = (\pi_0, \pi_1, \dots, \pi_n)$  be an  $n$ -step SAW on  $G$  from  $v_0$ . For  $k \in \mathbb{N}$  we say that  $E_k(C)$  occurs at the  $j$ th step of  $\pi$  if there exists  $\gamma \in \Gamma$  with  $\gamma v_0 = \pi_j$  such that at least  $k$  vertices of  $\gamma C$  are visited by  $\pi$ . Let  $\sigma_n(r, E_k(C))$  (respectively,  $\sigma_n(r, E_*(C))$ ) be the number of  $n$ -step SAWs on  $G$  from  $v_0$  in which  $E_k(C)$  (respectively,  $E_{|C|}(C)$ ) occurs at no more than  $r$  steps. When  $C$  is the vertex-set of  $C_1$ , we write  $E_k^1 = E_k(C_1)$  for  $E_k(C)$ , and so on.

The main step of the current proof is the following lemma.

LEMMA 6.1. *With  $\mu = \mu(G)$ , we have that*

$$(6.2) \quad \liminf_{n \rightarrow \infty} \sigma_n(0, E_2^1)^{1/n} < \mu.$$

As in (5.2), the  $\liminf$  is in fact a limit (here and later). An automorphism  $\gamma \in \Gamma$  acts on the edge  $e = \langle x, y \rangle$  by  $\gamma e = \langle \gamma x, \gamma y \rangle$ . Let  $\asymp$  be the equivalence relation on the edge-set of  $\bar{G}$  given by  $e \asymp e'$  if there exists  $\gamma \in \Gamma$  such that  $\gamma e = e'$ . We write  $\bar{e}$  for the equivalence class containing edge  $e$ , and  $\{\bar{e} : e \in E\}$  for the set of such classes. Since  $\Gamma$  acts transitively on  $\bar{G}$ , for every edge  $e$  and vertex  $v$ , there exists a member of  $\bar{e}$  that is incident to  $v$ .

The proof of Lemma 6.1 will make use of the intermediate Lemmas 6.2–6.3.

LEMMA 6.2. *Assume that*

$$(6.3) \quad \lim_{n \rightarrow \infty} \sigma_n(0, E_2^1)^{1/n} = \mu.$$

*There exist  $h \geq 2$  and a sequence  $(\bar{e}_r : r = 1, 2, \dots, h - 1)$  of distinct equivalence classes of edges of  $G$ , with  $\bar{e}_1 = \bar{f}_1$ , such that the following holds:*



For  $r \geq 1$ , let

$$F_r = \bar{e}_1 \cup \bar{e}_2 \cup \dots \cup \bar{e}_r,$$

and let  $G_r$  be the graph obtained from  $G$  by the deletion of  $F_r$ .

There exists a sequence  $C_2, C_3, \dots, C_h$  of cycles in  $G$  such that

(6.4)  $\quad$  for  $2 \leq r \leq h$ , there exists  $e_{r-1} \in \bar{e}_{r-1}$  that is incident to  $v_0$   
 and belongs to both  $C_{r-1}$  and  $C_r$ ,

(6.5)  $\quad \lim_{n \rightarrow \infty} \sigma_{n,r-1}(0, E_2(C_r))^{1/n} = \mu, \quad 2 \leq r < h,$

(6.6)  $\quad \liminf_{n \rightarrow \infty} \sigma_{n,h-1}(0, E_2(C_h))^{1/n} < \mu,$

where  $\sigma_{n,r-1}(0, E_2(C_r))$  is the number of  $n$ -step SAWs on  $G_{r-1}$  from  $v_0$  for which  $E_2(C_r)$  never occurs.

*Proof of Lemma 6.2.* We construct the  $\bar{e}_r$  and  $C_r$  by iteration. Let  $G'_r$  be obtained from  $G_{r-1}$  (where  $G_0 := G$ ) by the deletion of all edges in  $\bar{e}_r$  incident to  $v_0$ . Assume that (6.3) holds, and let  $\bar{e}_1 = \bar{f}_1$ .

The case  $r = 2$ . By (6.3), for “most”  $n$ -step SAWs  $\pi$  on  $G$  from  $v_0$ , and for all  $j$  and all  $\gamma C_1$  with  $\gamma \in \Gamma$  and  $\gamma v_0 = \pi_j$ ,  $\pi_j$  is the unique vertex of  $\gamma C_1$  visited by  $\pi$ . Since no edge in  $F_1$  is traversed by any path contributing to  $\sigma_n(0, E_2^1)$ , by (6.3),

(6.7)  $\quad \mu(G_1) = \mu.$

Let  $v_0^{(1)}$  be the neighbor of  $v_0$  such that  $f_1 = \langle v_0, v_0^{(1)} \rangle$ . We claim that, subject to (6.3),

(6.8)  $\quad v_0$  and  $v_0^{(1)}$  are connected in  $G'_1$ ,

and shall prove this by contradiction. Assume that (6.3) holds and that  $v_0$  and  $v_0^{(1)}$  are not connected in  $G'_1$ .

Let  $\pi^{(1)}, \pi^{(2)}$  be two SAWs from  $v_0$  on  $G_1$ , and consider a walk  $\nu$  on  $G$  given as follows:  $\nu$  follows  $\pi^{(1)}$  from  $v_0$  to its other endvertex  $z$ , then traverses an edge of the form  $f = \langle z, w \rangle \in \bar{e}_1$ , and then follows  $\gamma \pi^{(2)}$  for some  $\gamma \in \Gamma$  satisfying  $\gamma v_0 = w$ . By the above assumption and vertex-transitivity, the removal of all edges of  $\bar{e}_1$  incident to  $z$  disconnects  $z$  and  $w$ , and thus any such  $\nu$  is a SAW on  $G$ .

This process of concatenation may be iterated as follows. Let  $\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(N)}$  be SAWs on  $G_1$  from  $v_0$ . We aim to construct a SAW  $\nu$  on  $G$  as follows. Suppose we have concatenated  $\pi^{(1)}, \dots, \pi^{(r)}$  to obtain a SAW  $\nu^{(r)}$  on  $G$  from  $v_0$  to some vertex  $z$ . We now traverse an edge of the form  $f = \langle z, w \rangle \in \bar{e}_1$ , followed by the SAW  $\gamma \pi^{(2)}$  for some  $\gamma \in \Gamma$  with  $\gamma v_0 = w$ . Unless (i) there is a unique edge of  $\bar{e}_1$  incident to  $z$  and (ii)  $\pi^{(r)}$  is the SAW with zero length, then there exists a choice of  $w$  such that the resulting walk  $\nu^{(r+1)}$  is a SAW. Therefore, if every  $\pi^{(r)}$  has length 1 or more, the sequence  $(\pi^{(r)})$  may be concatenated to obtain a SAW  $\nu = \nu^{(N)}$  on  $G$ .

Let  $\sigma_{n,1}$  be the number of  $n$ -step SAWs from  $v_0$  on  $G_1$ , and let  $\mu_1 = \mu(G_1)$ . By (2.2),

(6.9)  $\quad \sigma_{2n,1} \geq \mu_1^{2n}, \quad n \geq 1.$

Let  $\Sigma_{2n}^\circ$  be the set of  $2n$ -step SAWs from  $v_0$  on  $G$  that traverse edges in  $F_1$  (in either direction) at the odd-numbered steps only, and write  $\sigma_{2n}^\circ = |\Sigma_{2n}^\circ|$ . Our purpose in considering only the odd steps is to avoid component walks of zero length.

Let  $0 \leq j \leq n$ , and consider the set of all  $\pi \in \Sigma_{2n}^o$  that traverse exactly  $j$  edges of  $F_1$ . There are  $\binom{n}{j}$  ways of choosing the indices of these edges within  $\pi$ . By the construction described above, and (6.9), there are at least  $\mu_1^{2n-j}$  choices for the sequence of SAWs on  $G_1$  whose concatenation makes  $\pi$ . Therefore,

$$(6.10) \quad \sigma_{2n}^o \geq \sum_{j=0}^n \binom{n}{j} \mu_1^{2n-j} = \mu_1^{2n} (1 + \mu_1^{-1})^n.$$

This implies

$$(6.11) \quad \sigma_{2n} \geq \sigma_{2n}^o \geq \mu_1^{2n} (1 + \mu_1^{-1})^n,$$

whence  $\mu \geq \mu_1 \sqrt{1 + \mu_1^{-1}} > \mu_1$ , in contradiction of (6.7). Statement (6.8) is proved.

By (6.8),  $G$  has a shortest (directed) walk  $W_2$  from  $v_0$  to  $v_0^{(1)}$  using no edge of  $\bar{e}_1$  incident to  $v_0$ . The walk  $W_2$ , followed by the (directed) edge  $\langle v_0^{(1)}, v_0 \rangle$ , forms the required (directed) cycle  $C_2$  of  $G$ , having  $e_1$  in common with  $C_1$ . Let  $e_2 = \langle v_0, v_0^{(2)} \rangle$  be the first edge of  $W_2$ , and write  $E_k^2 = E_k(C_2)$  and so on.

If

$$(6.12) \quad \liminf_{n \rightarrow \infty} \sigma_{n,1}(0, E_2^2)^{1/n} < \mu,$$

we have proved the claim with  $h = 2$ . Suppose conversely that

$$(6.13) \quad \lim_{n \rightarrow \infty} \sigma_{n,1}(0, E_2^2)^{1/n} = \mu,$$

whence  $\mu(G_2) = \mu$  as in (6.7).

*The general case.* Suppose  $u \geq 2$  and that cycles  $C_2, C_3, \dots, C_u$  from  $v_0$  have been found, with corresponding classes  $\bar{e}_2, \bar{e}_3, \dots, \bar{e}_u$  and edges  $e_i = \langle v_0, v_0^{(i)} \rangle \in \bar{e}_i$ , such that (6.4) holds with  $h$  replaced by  $u$  and

$$(6.14) \quad \lim_{n \rightarrow \infty} \sigma_{n,i-1}(0, E_2^i)^{1/n} = \mu, \quad 1 \leq i \leq u,$$

where  $E_2^i = E_2(C_i)$ . By (6.14) with  $r = u$ , we have as above that  $\mu(G_u) = \mu$ .

Let  $\sigma_{2n,u-1}^o$  be the number of  $2n$ -step SAWs on  $G_{u-1}$  from  $v_0$  that traverse edges in  $F_u$  only at odd steps. If  $v_0$  and  $v_0^{(u)}$  are disconnected in  $G'_u$ , we have

$$(6.15) \quad \sigma_{2n} \geq \sigma_{2n,u-1}^o \geq \sum_{j=0}^n \binom{n}{j} \mu_u^{2n-j} = \mu_u^{2n} (1 + \mu_u^{-1})^n,$$

where  $\mu_u = \mu(G_u)$ . This contradicts  $\mu_u = \mu$ , whence  $v_0$  and  $v_0^{(u)}$  are connected in  $G'_u$ .

Therefore,  $v_0$  and  $v_0^{(u)}$  are connected by a shortest (directed) walk  $W_{u+1}$  of  $G_{u-1}$  using no edge of  $\bar{e}_u$  incident to  $v_0$ , and we write  $e_{u+1} = \langle v_0, v_0^{(u+1)} \rangle$  for the first edge of  $W_{u+1}$ , and  $C_{u+1}$  for the cycle in  $G_{u-1}$  formed by  $W_{u+1}$  followed by  $\langle v_0^{(u)}, v_0 \rangle$ . If

$$(6.16) \quad \liminf_{n \rightarrow \infty} \sigma_{n,u}(0, E_2^{u+1})^{1/n} < \mu,$$

we stop and set  $h = u + 1$ , and otherwise we continue as above.

Either this iterative process terminates at some earliest  $u$  satisfying (6.16) or it does not. In the step from  $G_u$  to  $G_{u+1}$ , the degree of  $v_0$  is reduced strictly. If the process does not terminate in the manner of (6.16), all edges incident to  $v_0$  in the last nontrivial graph encountered,  $G_t$  say, lie in the same equivalence class. Furthermore, (6.14) holds with  $i$  replaced by  $t + 1$ , and hence  $\mu_{t+1} = \mu$ . This is impossible since  $v_0$  is isolated in  $G_{t+1}$ . The proof is complete.  $\square$

If (6.3) fails, we set  $h = 1$  and  $G_0 = G$ . If (6.3) holds, we adopt the notation of Lemma 6.3 and write  $E_k^i = E_k(C_i)$  and so on. For an  $n$ -step SAW  $\pi = (\pi_0, \pi_1, \dots, \pi_n)$  on  $G$ , we say that  $E_k^i(m)$  occurs at the  $j$ th step of  $\pi$  if  $E_k^i$  occurs at the  $m$ th step of the  $2m$ -step SAW  $(\pi_{j-m}, \dots, \pi_{j+m})$  (with a modified definition if either  $j - m < 0$  or  $j + m > n$ , as in the second paragraph after (5.1)).

LEMMA 6.3. *Let  $1 \leq i \leq h$  and  $1 \leq k \leq |C_i|$ . If*

$$\liminf_{n \rightarrow \infty} \sigma_{n,i-1}(0, E_k^i) < \mu,$$

then there exist  $a > 0$  and  $m \in \mathbb{N}$  such that

$$\limsup_{n \rightarrow \infty} \sigma_{n,i-1}(an, E_k^i(m))^{1/n} < \mu.$$

*Proof.* This is proved in the same manner as was Lemma 5.3.  $\square$

*Proof of Lemma 6.1.* Assume the converse of (6.2), namely (6.3), and adopt the notation of Lemma 6.2. Recall the notation  $E_*$  from the beginning of section 6. Exactly one of the following holds,

$$(6.17) \quad \liminf_{n \rightarrow \infty} \sigma_{n,h-1}(0, E_*^h)^{1/n} < \mu,$$

$$(6.18) \quad \lim_{n \rightarrow \infty} \sigma_{n,h-1}(0, E_*^h)^{1/n} = \mu,$$

and we shall derive a contradiction in each case.

Assume first that (6.17) holds. No SAW counted in  $\sigma_{n,h-2}(0, E_2^{h-1})$  traverses any edge of  $\bar{e}_{h-1}$ , whence  $\sigma_{n,h-2}(0, E_2^{h-1}) = \sigma_{n,h-1}(0, E_2^{h-1})$ . Since  $C_h$  and  $C_{h-1}$  have at least one common edge, we deduce that

$$\sigma_{n,h-2}(0, E_2^{h-1}) \leq \sigma_{n,h-1}(0, E_*^h),$$

and hence

$$\liminf_{n \rightarrow \infty} \sigma_{n,h-2}(0, E_2^{h-1})^{1/n} < \mu,$$

in contradiction of the minimality of  $h$ .

Assume now that (6.18) holds. By (6.6) and the monotonicity of  $\sigma_{n,h-1}(0, E_k^h)$  in  $k$ , there exists  $k \in \mathbb{N}$  satisfying  $2 \leq k < \ell := |C_h|$  such that

$$(6.19) \quad \liminf_{n \rightarrow \infty} \sigma_{n,h-1}(0, E_k^h)^{1/n} < \mu,$$

$$(6.20) \quad \lim_{n \rightarrow \infty} \sigma_{n,h-1}(0, E_{k+1}^h)^{1/n} = \mu.$$

By (6.19) and Lemma 6.3, there exist  $a > 0$  and  $m \in \mathbb{N}$  such that

$$(6.21) \quad \limsup_{n \rightarrow \infty} \sigma_{n,h-1}(an, E_k^h(m))^{1/n} < \mu.$$

We shall derive a contradiction from (6.20)–(6.21) in a manner similar to the proof of (5.17).

Let  $T_n$  be the set of  $n$ -step SAWs on  $G_{h-1}$  from  $v_0$  such that  $E_{k+1}^h$  never occurs and  $E_k^h(m)$  occurs at least  $an$  times. By (6.20)–(6.21),

$$(6.22) \quad \lim_{n \rightarrow \infty} |T_n|^{1/n} = \mu.$$

Let  $\pi = (\pi_0, \pi_1, \dots, \pi_n) \in T_n$  and  $u = \lfloor \kappa n \rfloor - 2$ , where

$$\kappa = \frac{a}{(2m+2)\ell}.$$

As after (5.18), we may find  $j_1 < j_2 < \dots < j_u$  and  $\gamma_1, \dots, \gamma_u \in \Gamma$  such that

$$(6.23) \quad \gamma_t(v_0) = \pi_{j_t}, \quad 1 \leq t \leq u,$$

$$(6.24) \quad \text{each } 2m\text{-step SAW } (\pi_{j_t-m}, \dots, \pi_{j_t+m}) \text{ visits at least } k \text{ vertices of } \gamma_t C_h,$$

$$(6.25) \quad 0 < j_1 - m, \quad j_u + m < n,$$

$$(6.26) \quad j_t + m < j_{t+1} - m, \quad 1 \leq t < u,$$

$$(6.27) \quad \gamma_1 C_h, \gamma_2 C_h, \dots, \gamma_u C_h \text{ are pairwise vertex-disjoint.}$$

For  $t = 1, 2, \dots, u$ , let

$$(6.28) \quad \alpha_t = \min\{i : \pi_i \in \gamma_t C_h\}, \quad \omega_t = \max\{i : \pi_i \in \gamma_t C_h\}.$$

Since  $E_k^h(m)$  occurs at the  $j_t$ th step but not  $E_{k+1}^h$ , there are exactly  $k$  points of  $\gamma_t C_h$  that are visited by  $\pi$ , and these points lie on  $\pi$  between positions  $j_t - m$  and  $j_t + m$ . Therefore,

$$(6.29) \quad j_t - m \leq \alpha_t < \omega_t \leq j_t + m, \quad 1 \leq t \leq u.$$

We propose to replace the subwalk  $(\pi_{\alpha_t}, \dots, \pi_{\omega_t})$  by the part of the cycle  $\gamma_t C_h$  with the same endpoints and using at least one edge in  $\gamma_t C_{h-1}$ ; this may be done since  $\gamma_t C_h$  and  $\gamma_t C_{h-1}$  have at one edge in common, namely  $\gamma_t e_{h-1}$ . By (6.27), such a replacement may be performed simultaneously for all  $t$ . The resulting walk  $\psi$  is a SAW on  $G$  with length  $n'$  satisfying

$$n' < n + u\ell.$$

Furthermore, since  $\pi \in T_n$ , the only edges of  $\bar{e}_{h-1}$  in  $\psi$  are those introduced during a substitution.

Let  $\delta > 0$  and  $s = \delta n$ , where  $\delta$  will be chosen later (and we omit integer-part symbols as before). Consider the set of pairs  $(\pi, H)$ , where  $\pi \in T_n$  and  $H = (h_1, h_2, \dots, h_s)$  is an ordered subset of  $\{j_1, j_2, \dots, j_u\}$ . We may make the above replacement around each  $\pi_{h_i}$  to obtain a SAW  $\psi = \psi(\pi, H)$  on  $G$ .

As in (5.32),

$$(6.30) \quad |(\pi, H)| \geq |T_n| \binom{\kappa n - 2}{\delta n}.$$

For an upper bound, consider a given pair  $(\pi, H)$ . Edges in  $\bar{e}_{h-1}$  are traversed between  $|H| = \delta n$  and  $\ell \delta n$  times on  $\psi$ . Therefore, given  $\psi$ , there are at most  $2m \binom{\ell \delta n}{\delta n}$  ( $\leq m 2^{\ell \delta n + 1}$ ) possibilities for the location of the earliest point of  $\psi$  in  $\gamma_t C_h$ . Given  $\psi$  and the locations of these earliest points, there are at most  $\Delta^\ell$  different choices for

each  $\gamma_t C_h$ . Once these are determined, each such  $\gamma_t C_h$  determines a subwalk of  $\psi$  that replaces some subwalk of  $\pi$ . Since each of the replaced subwalks of  $\pi$  has length not exceeding  $2m$ , there are at most  $Y^{\delta N}$  possibilities for  $\pi$ , where  $Y = \sum_{i=0}^{2m} \sigma_i$ . Therefore,

$$(6.31) \quad |(\pi, H)| \leq m 2^{\ell \delta n + 1} \Delta^\ell Y^{\delta n} \sum_{i=0}^{n + \ell \delta n} \sigma_i.$$

We combine (6.30) and (6.31), take the  $n$ th root and let  $n \rightarrow \infty$ , obtaining by (6.22) that

$$\mu \frac{\kappa^\kappa}{\delta^\delta (\kappa - \delta)^{\kappa - \delta}} \leq 2^{\ell \delta} Y^\delta \mu^{1 + \ell \delta}.$$

Setting  $Z = (2\mu)^\ell Y$  and  $\eta = \delta/\kappa$ , we deduce that

$$1 \leq [Z^\eta \eta^\eta (1 - \eta)^{1 - \eta}]^\kappa.$$

As at the end of the proof of Lemma 5.4, this is a contradiction for small  $\eta > 0$ .

In conclusion, if (6.3) and (6.18) hold, we have a contradiction, and the lemma is proved.  $\square$

*Proof of Theorem 3.2.* Write  $\bar{\mu} := \mu(\bar{G})$ , and let  $F$  be the set of edges of  $\bar{G}$  not in  $G$ . Let  $f$  and  $C_1$  be given as around (6.1), and let  $\ell = \Lambda + 1$  be the number of vertices in  $C_1$  (or, equivalently, the number of edges in  $C_1$  viewed as a cycle of  $\bar{G}$ ). It suffices to make the following assumption.

ASSUMPTION 6.4. *We have that  $F = \mathcal{A}f$ .*

For  $v, w \in V$  and a graph  $H$  with vertex-set  $V$ , let  $N(v, w; H)$  denote the number of edges between  $v$  and  $w$  in  $H$ . For  $\gamma \in \Gamma$ , we construct the graph denoted  $\gamma\bar{G}$  as follows. First,  $\gamma\bar{G}$  has vertex-set  $V$ . For  $v, w \in V$ , we place  $N(v, w; \bar{G})$  edges between  $\gamma v$  and  $\gamma w$  in  $\gamma\bar{G}$ . Thus,  $\gamma\bar{G}$  is obtained from  $\gamma G$  by adding the edges of  $\gamma F$ . Two elementary properties of  $\gamma\bar{G}$  are as follows.

LEMMA 6.5. *Let  $\gamma \in \Gamma$ .*

- (a)  $\gamma\bar{G}$  is isomorphic to  $\bar{G}$ .
- (b)  $\gamma\mathcal{A}\gamma^{-1} \subseteq \text{Aut}(\gamma\bar{G})$ .

*Proof.* (a) This is an immediate consequence of the definition of  $\gamma\bar{G}$ .

(b) By the definition of  $\gamma\bar{G}$ , for  $\alpha \in \mathcal{A}$ ,

$$\begin{aligned} N(\gamma\alpha\gamma^{-1}(\gamma v), \gamma\alpha\gamma^{-1}(\gamma w); \gamma\bar{G}) &= N(\gamma\alpha v, \gamma\alpha w; \gamma\bar{G}) \\ &= N(\alpha v, \alpha w; \bar{G}) \\ &= N(v, w; \bar{G}) \\ &= N(\gamma v, \gamma w; \gamma\bar{G}), \end{aligned}$$

and the claim follows.  $\square$

Assume  $\bar{\mu} = \mu$ . By Lemma 6.1 with  $E_k := E_k^1$ ,

$$(6.32) \quad \liminf_{n \rightarrow \infty} \sigma_n(0, E_2)^{1/n} < \mu (= \bar{\mu}).$$

Exactly one of the following holds:

$$(6.33) \quad \liminf_{n \rightarrow \infty} \sigma_n(0, E_\ell)^{1/n} < \mu,$$

$$(6.34) \quad \lim_{n \rightarrow \infty} \sigma_n(0, E_\ell)^{1/n} = \mu.$$

Assume first that (6.33) holds. By Lemma 6.3, there exist  $a > 0$  and  $m \in \mathbb{N}$  such that

$$(6.35) \quad \limsup_{n \rightarrow \infty} \sigma_n(an, E_\ell(m))^{1/n} < \mu.$$

Let  $R_n$  be the set of  $n$ -step SAWs from  $v_0$  on  $G$  on which  $E_\ell(m)$  occurs at least  $an$  times. By (6.35),

$$(6.36) \quad \lim_{n \rightarrow \infty} |R_n|^{1/n} = \mu.$$

Let  $\rho \in V$  be such that  $\mathcal{A}$  has the finite coset property with root  $\rho$ , and let  $\nu_0, \nu_1, \dots, \nu_s$  be given accordingly, as in Definition 3.1. Let  $\mathcal{O} = \Gamma C_1$  be the orbit under  $\Gamma$  of  $C_1$  viewed as a set of labeled vertices, and let  $\mathcal{O}_i = \nu_i \mathcal{A} C_1$ . Let  $0 \leq i \leq s$ . We say that  $E_\ell^i(m)$  occurs at the  $j$ th step of a SAW  $\pi$  on  $G$  if there exists  $\gamma \in \Gamma$  with  $\gamma \pi_0 = \pi_j$  and  $\gamma C_1 \in \mathcal{O}_i$  such that all the vertices of  $\gamma C_1$  are visited by the  $2m$ -step SAW  $(\pi_{j-m}, \dots, \pi_{j+m})$  (subject to the usual amendment if  $j - m < 0$  or  $j + m > n$ ).

Let  $R_n^{(i)}$  be the set of  $n$ -step SAWs from  $v_0$  on  $G$  for which  $E_\ell^i(m)$  occurs at least  $an/(s + 1)$  times. Thus,

$$(6.37) \quad R_n \subseteq \bigcup_{i=0}^s R_n^{(i)}.$$

By (6.36)–(6.37), there exists  $i$  satisfying  $0 \leq i \leq s$  such that

$$(6.38) \quad \limsup_{n \rightarrow \infty} |R_n^{(i)}|^{1/n} = \mu,$$

and we choose  $i$  accordingly.

We now apply the argument in the proof of Lemma 6.1 with the constant  $a$  replaced by  $a/(s + 1)$ . Let  $\pi \in R_n^{(i)}$  be such that  $E_\ell^i(m)$  occurs at steps  $j_1 < j_2 < \dots < j_u$ , and in addition (6.25)–(6.27) hold with  $h$  replaced by 1 and with each  $\gamma_t C_1 \in \mathcal{O}_i$ . Since  $\gamma_t C_1 \in \mathcal{O}_i = \nu_i \mathcal{A} C_1$  there exists  $\alpha_t \in \mathcal{A}$  such that  $\gamma_t C_1 = [\nu_i \alpha_t \nu_i^{-1}] \nu_i C_1$ . By Lemma 6.5(b),  $[\nu_i \alpha_t \nu_i^{-1}] \nu_i \overline{G}$  is isomorphic to  $\nu_i \overline{G}$ . Therefore,  $\gamma_t f$  is an edge of  $\nu_i \overline{G}$  but not of  $\nu_i G$ . We think of  $\pi$  as a SAW on the graph  $\nu_i G$ .

Let  $\alpha_t, \omega_t$  be as in (6.28), and note that (6.29) holds as before. Consider the replacement of the subwalk  $(\pi_{\alpha_t}, \dots, \pi_{\omega_t})$  by a walk that goes along that part of  $\gamma_t C_1$  ( $= \gamma'_t \nu_i C_1$ , where  $\gamma'_t = \nu_i \alpha_t \nu_i^{-1}$ ), viewed as a cycle, that includes an edge of  $\gamma_t F$  ( $= \gamma'_t \nu_i F$ ). By Lemma 6.5(b), the new walk is a SAW on  $\nu_i \overline{G}$ . Furthermore it uses edges of  $\nu_i \overline{G}$  not belonging to  $\nu_i G$ . Lower and upper bounds may be derived as in the proof of Lemma 6.1, and these lead to a contradiction when working along the subsequence implicit in (6.38). This implies  $\mu(G) < \mu(\nu_i \overline{G})$  subject to (6.33). By Lemma 6.5(a),  $\mu(\nu_i \overline{G}) = \bar{\mu}$  and hence  $\mu(G) < \bar{\mu}$ .

Assume finally that (6.34) holds. By (6.32)–(6.34), there exists  $k \in \mathbb{N}$  satisfying  $2 \leq k < \ell$  such that

$$(6.39) \quad \begin{aligned} & \liminf_{n \rightarrow \infty} \sigma_n(0, E_k)^{1/n} < \mu, \\ & \lim_{n \rightarrow \infty} \sigma_n(0, E_{k+1})^{1/n} = \mu. \end{aligned}$$

By Lemma 6.3, there exist  $a > 0$  and  $m \in \mathbb{N}$  such that

$$(6.40) \quad \limsup_{n \rightarrow \infty} \sigma_n(an, E_k(m))^{1/n} < \mu.$$



Let  $S_n$  be the set of  $n$ -step SAWs from  $v_0$  on  $G$  such that  $E_{k+1}$  never occurs and  $E_k(m)$  occurs at least  $an$  times. By (6.39)–(6.40),

$$(6.41) \quad \lim_{n \rightarrow \infty} |S_n|^{1/n} = \mu.$$

For  $0 \leq i \leq s$ , let  $S_n^{(i)}$  be the set of  $n$ -step SAWs from  $v_0$  on  $G$  such that  $E_{k+1}$  never occurs and  $E_k^i(m)$  occurs at least  $an/s$  times. Thus,

$$(6.42) \quad S_n \subseteq \bigcup_{i=0}^s S_n^{(i)}.$$

By (6.41)–(6.42), there exists  $i$  satisfying  $0 \leq i \leq s$  such that

$$\limsup_{n \rightarrow \infty} |S_n^{(i)}|^{1/n} = \mu,$$

and we choose  $i$  accordingly.

We apply the argument of the proof of Lemma 6.1 once again. Let  $\pi \in S_n^{(i)}$  be such that  $E_k^i(m)$  occurs at steps  $j_1 < j_2 < \dots < j_u$ , and in addition (6.25)–(6.27) hold with  $\gamma_i C_1 \in \mathcal{O}_i$ . The argument is as above, and we do not repeat the details. This implies  $\mu(G) < \mu(\overline{G})$  when (6.34) holds, and the proof is complete.  $\square$

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#### REFERENCES

- [1] M. AIZENMAN AND G. R. GRIMMETT, *Strict monotonicity of critical points in percolation and ferromagnetic models*, J. Statist. Phys., 63 (1991), pp. 817–835.
- [2] S. E. ALM, *Upper and lower bounds for the connective constants of self-avoiding walks on the Archimedean and Laves lattices*, J. Phys. A, 38 (2005), pp. 2055–2080.
- [3] S. E. ALM AND S. JANSON, *Random self-avoiding walks on one-dimensional lattices*, Comm. Statist. Stochastic Models, 6 (1990), pp. 169–212.
- [4] L. BABAI, *Automorphism groups, isomorphism, reconstruction*, in Handbook of Combinatorics, Vol. II, Elsevier, Amsterdam, 1995, pp. 1447–1540.
- [5] R. BAUERSCHMIDT, H. DUMINIL-COPIN, J. GOODMAN, AND G. SLADE, *Lectures on self-avoiding walks*, in Probability and Statistical Physics in Two and More Dimensions, D. Ellwood, C. M. Newman, V. Sidoravicius, and W. Werner, eds., Clay Mathematics Institute Proceedings 15, CMI/AMS, New York, 2012, pp. 395–476.
- [6] N. R. BEATON, M. BOUSQUET-MÉLOU, J. DE GIER, H. DUMINIL-COPIN, AND A. G. GUTTMANN, *The critical fugacity for surface adsorption of self-avoiding walks on the honeycomb lattice is  $1 + \sqrt{2}$* , Comm. Math. Phys., 326 (2014), pp. 727–754.
- [7] I. BENJAMINI, R. LYONS, Y. PERES, AND O. SCHRAMM, *Group-invariant percolation on graphs*, Geom. Funct. Anal., 9 (1999), pp. 29–66.
- [8] I. BENJAMINI AND O. SCHRAMM, *Percolation beyond  $\mathbb{Z}^d$ , Many questions and a few answers*, Electron. Comm. Probab., 1 (1996), pp. 71–82.
- [9] C. BEZUIDENHOUT, G. R. GRIMMETT, AND H. KESTEN, *Strict inequality for critical values of Potts models and random-cluster processes*, Comm. Math. Phys., 158 (1993), pp. 1–16.
- [10] P. FLORY, *Principles of Polymer Chemistry*, Cornell University Press, Ithaca, NY, 1953.
- [11] C. GODSIL AND G. ROYLE, *Algebraic Graph Theory*, Springer, Berlin, 2001.
- [12] G. R. GRIMMETT, *Potts models and random-cluster processes with many-body interactions*, J. Statist. Phys., 75 (1994), pp. 67–121.
- [13] G. R. GRIMMETT, *Percolation*, 2nd ed., Springer, Berlin, 1999.
- [14] G. R. GRIMMETT, A. E. HOLROYD, AND Y. PERES, *Extendable self-avoiding walks*, Ann. Inst. Henri Poincaré D, 1 (2014), pp. 61–75.
- [15] G. R. GRIMMETT AND Z. LI, *Bounds on the connective constants of regular graphs*, Combinatorica, submitted, 2012; <http://arxiv.org/abs/1210.6277>.

- [16] G. R. GRIMMETT AND Z. LI, *Counting self-avoiding walks*, <http://arxiv.org/abs/1304.7216>, 2013.
- [17] G. R. GRIMMETT AND Z. LI, *Self-avoiding walks and the Fisher transformation*, *Electron. J. Combin.*, 20 (2013), P47.
- [18] J. M. HAMMERSLEY, *Percolation processes II. The connective constant*, *Proc. Camb. Phil. Soc.*, 53 (1957), pp. 642–645.
- [19] J. M. HAMMERSLEY AND D. J. A. WELSH, *Further results on the rate of convergence to the connective constant of the hypercubical lattice*, *Quart. J. Math. Oxford*, 13 (1962), pp. 108–110.
- [20] I. JENSEN, *Improved lower bounds on the connective constants for two-dimensional self-avoiding walks*, *J. Phys. A*, 37 (2004), pp. 11521–11529.
- [21] H. KESTEN, *On the number of self-avoiding walks*, *J. Math. Phys.*, 4 (1963), pp. 960–969.
- [22] H. LACOIN, *Non-coincidence of quenched and annealed connective constants on the supercritical planar percolation cluster*, *Probab. Theory Rel. Fields*, 159 (2014), pp. 777–808.
- [23] Z. LI, *Locality of the Connective Constant on Cayley Graphs*, preprint, Cambridge University, 2013.
- [24] R. LYONS WITH Y. PERES, *Probability on Trees and Networks*, Cambridge University Press, Cambridge, UK, 2012, in preparation; <http://mypage.iu.edu/~rdlyons/>.
- [25] N. MADRAS AND G. SLADE, *Self-Avoiding Walks*, Birkhäuser Boston, Cambridge, MA, 1993.
- [26] C. NEBBIA, *On the amenability and Kunze–Stein property for groups acting on a tree*, *Pacific J. Math.*, 135 (1988), pp. 371–380.
- [27] D. J. S. ROBINSON, *A Course in the Theory of Groups*, Springer, New York, 1982.
- [28] P. M. SOARDI AND W. WOESS, *Amenability, unimodularity, and the spectral radius of random walks on infinite graphs*, *Math. Z.*, 205 (1990), pp. 471–486.
- [29] C. E. SOTEROS AND S. G. WHITTINGTON, *Lattice models of branched polymers: Effects of geometrical constraints*, *J. Phys. A*, 22 (1989), pp. 5259–5270.
- [30] V. I. TROFIMOV, *Automorphism groups of graphs as topological groups*, *Math. Notes*, 38 (1985), pp. 717–720.