LOCALITY OF CONNECTIVE CONSTANTS, I. TRANSITIVE GRAPHS

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Abstract. The connective constant $\mu(G)$ of a quasi-transitive graph G is the exponential growth rate of the number of self-avoiding walks from a given origin. We prove a locality theorem for connective constants, namely, that the connective constants of two graphs are close in value whenever the graphs agree on a large ball around the origin. The proof exploits a generalized bridge decomposition of self-avoiding walks, which is valid subject to the assumption that the underlying graph is quasi-transitive and possesses a so-called graph height function. It is proved that a broad category of transitive graphs have graph height functions.

1. Introduction, and summary of results

There is a rich theory of interacting systems on infinite graphs. The probability measure governing a process has, typically, a continuously varying parameter, p say, and there is a singularity at some 'critical point' p_c. The numerical value of p_c depends in general on the choice of underlying graph G, and a significant part of the associated literature is directed towards estimates of p_c for different graphs. In most cases of interest, p_c depends on the large-scale properties of G, rather than on the geometry of some bounded domain only. This leads to the question of 'locality': to what degree is the value of p_c determined by knowledge of a bounded domain of G?

The purpose of the current paper is to present a locality theorem (namely, Theorem 5.1) for the connective constant $\mu(G)$ of the graph G. A self-avoiding walk (SAW) is a path that visits no vertex more than once. SAWs were introduced in the study of long-chain polymers in chemistry (see, for example, the 1953 volume of Flory, [13]), and their theory has been much developed since (see the book of Madras and Slade, [27], and the recent review [2]). If the underlying graph G has some periodicity, the number of n-step SAWs from a given origin grows exponentially with some growth rate $\mu(G)$ called the *connective constant* of the graph G.

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There are only few graphs G for which the numerical value of $\mu(G)$ is known exactly, and a substantial part of the literature on SAWs is devoted to inequalities for $\mu(G)$. The current paper may be viewed in this light, as a continuation of the series of papers on the broad topic of connective constants of transitive graphs by the same authors, see [15, 16, 17, 19].

The main result (Theorem 5.1) of this paper is as follows. Let G, G' be infinite, vertex-transitive graphs, and write $S_K(v,G)$ for the K-ball around the vertex v in G. If $S_K(v,G)$ and $S_K(v',G')$ are isomorphic as rooted graphs, then

$$|\mu(G)-\mu(G')| \le \epsilon_K(G), \tag{1.1}$$

where $\epsilon_K(G)\to 0$ as $K\to\infty$. (A related result holds for quasi-transitive graphs.)

This is proved subject to a certain condition on the graphs G, G', namely that they support so-called 'graph height functions' (see Section 3 for the definition of a graph height function). The existence of graph height functions permits the use of a 'bridge decomposition' of SAWs (in the style of the work of Hammersley and Welsh [24]), and this leads in turn to computable sequences that converge to $\mu(G)$ from above and below, respectively. The locality result of (1.1) may be viewed as a resolution of a conjecture of Benjamini, [3, Conj. 2.3], made independently of the work reported here.

A class of vertex-transitive graphs of special interest is provided by the Cayley graphs of finitely presented groups. Cayley graphs have algebraic as well as geometric structure, and this allows a deeper investigation of locality and of graph height functions. The corresponding investigation is reported in the companion paper [18] where, in particular, we present a method for the construction of a graph height function via a suitable harmonic function on the graph.

The locality question for percolation was approached by Benjamini, Nachmias, and Peres [4] for tree-like graphs. Let G be vertex-transitive with degree $d+1$. It is elementary that the percolation critical point satisfies $p_c\ge 1/d$ (see [7, Thm 7]), and an asymptotically equivalent upper bound for p_c was developed in [4] for graphs with degree $d+1$ and large girth. In recent work of Martineau and Tassion [28], a locality result has been proved for percolation on abelian graphs. The proof extends the methods and conclusions of [20], where it is proved that the slab critical points converge to $p_c(\mathbb{Z}^d)$, in the limit as the slabs become infinitely 'fat'. (A related result for connective constants is included here at Example 5.3.)

We are unaware of a locality theorem for the critical temperature T_c of the Ising model. Of the potentially relevant work on the Ising model to date, we mention [6, 8, 26].

This paper is organized as follows. Relevant background and notation is described in Section 2. The concept of a graph height function is presented in Section 3, where examples are included of infinite graphs with graph height functions, and a sufficient condition is presented in Theorem 3.4 for the existence of a graph height function. Bridges and the bridge constant are defined in Section 4, and it is proved in Theorem 4.2 that the bridge constant equals the connective constant whenever there exists a graph height function. The main 'locality theorem' is given at Theorem 5.1. Theorem 5.2 is an application of the locality theorem in the context of a sequence of quotient graphs; this parallels the Grimmett–Marstrand theorem [20] for percolation on slabs, but with the underlying lattice replaced by a transitive graph with a graph height function. Sections 6 and 7 contain the proofs of Theorems 3.4 and 4.2.

2. Notation and background

The graphs $G = (V, E)$ considered here are generally assumed to be infinite, connected, and also simple, in that they have neither loops nor multiple edges (we shall relax this for the quotient graph $G/\mathcal{H}$ of Section 3). An undirected edge e with endpoints u, v is written as $e = \langle u, v\rangle$, and if directed from u to v as $[u, v\rangle$. If $\langle u, v\rangle \in E$, we call u and v *adjacent* and write $u \sim v$. The set of neighbours of $v \in V$ is denoted ∂v.

Loops and multiple edges have been excluded for cosmetic reasons only. A SAW can traverse no loop, and thus loops may be removed without changing the connective constant. The same proofs are valid in the presence of multiple edges. When there are multiple edges, we are effectively considering SAWs on a weighted simple graph, and indeed our results are valid for edge-weighted graphs with strictly positive weights, and for counts of SAWs in which the contribution of a given SAW is the product of the weights of the edges therein.

The *degree* of vertex v is the number of edges incident to v, and G is called *locally finite* if every vertex-degree is finite. The *graph-distance* between two vertices u, v is the number of edges in the shortest path from u to v, denoted $d_G(u, v)$.

The automorphism group of the graph $G = (V, E)$ is denoted $\mathrm{Aut}(G)$. A subgroup $\Gamma \le \mathrm{Aut}(G)$ is said to *act transitively* on G (or on its vertex-set V) if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma v = w$. It is said to *act quasi-transitively* if there is a finite set W of vertices such that, for $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $\gamma v = w$. The graph is called *(vertex-)transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively). For a subgroup $\mathcal{H} \le \mathrm{Aut}(G)$ and a vertex $v \in V$, the orbit of v under $\mathcal{H}$ is written $\mathcal{H}v$. The number of such orbits is written as $|G/\mathcal{H}|$.

A *walk* w on G is an alternating sequence $w_0e_0w_1e_1 \cdots e_{n-1}w_n$ of vertices w_i and edges $e_i = \langle w_i, w_{i+1}\rangle$. We write $|w| = n$ for the *length* of w, that is, the number of edges in w. The walk w is called *closed* if $w_0 = w_n$. A *cycle* is a closed walk w satisfying $w_i \ne w_j$ for $1 \le i < j \le n$.

An *n-step self-avoiding walk* (SAW) on G is a walk containing n edges no vertex of which appears more than once. Let $\sigma_n(v)$ be the number of n-step SAWs starting at $v \in V$. Let $\Sigma_n(v)$ be the set of n-step SAWs starting at v, with cardinality $\sigma_n(v) := |\Sigma_n(v)|$, and let

$$\sigma_n = \sigma_n(G) := \sup\{\sigma_n(v) : v \in V\}. \tag{2.1}$$

We have in the usual way (see [23, 27]) that

$$\sigma_{m+n} \le \sigma_m \sigma_n, \tag{2.2}$$

whence the *connective constant*

$$\mu = \mu(G) := \lim_{n\to\infty} \sigma_n^{1/n}$$

exists, and furthermore

$$\sigma_n \ge \mu^n, \qquad n \ge 0. \tag{2.3}$$

Hammersley [22] proved that, if G is quasi-transitive, then

$$\lim_{n\to\infty} \sigma_n(v)^{1/n} = \mu, \qquad v \in V. \tag{2.4}$$

We select a vertex of G and call it the *identity* or *origin*, denoted $\mathbf{1}$. Further notation will be introduced when needed. The concept of a 'graph height function' is explained in the next section, and that leads in Section 4 to the definition of a 'bridge'.

The set of integers is written $\mathbb{Z}$, the natural numbers as $\mathbb{N}$, and the rationals as $\mathbb{Q}$.

3. Quasi-transitive graphs and graph height functions

We consider quasi-transitive graphs with certain properties, and begin with some notation. Let $G = (V, E)$ be an infinite, connected, quasi-transitive, locally finite, simple graph.

Definition 3.1. *A* graph height function *on G is a function $h : V \to \mathbb{Z}$ such that:*

(a) *$h(\mathbf{1}) = 0$, and there exists $v \in V$ with $h(v) \ne 0$,*
(b) *there exists a subgroup $\mathcal{H} \le \mathrm{Aut}(G)$ acting quasi-transitively on G such that h is $\mathcal{H}$-*difference-invariant *in the sense that*

$$h(\alpha v) - h(\alpha u) = h(v) - h(u), \qquad \alpha \in \mathcal{H},\ u, v \in V,$$

(c) *for $v \in V$, there exist $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$.*

The expression 'graph height function' is chosen in contrast to the 'group height function' of [18]. It is explained in [18] that a group height function of a finitely presented group is a graph height function of its Cayley graph, but not *vice versa*.

For a given graph height function h, we may select a corresponding group of automorphisms denoted $\mathcal{H} = \mathcal{H}(h)$, and we shall sometimes refer to the pair $(h, \mathcal{H})$ as the graph height function. Associated with the pair $(h, \mathcal{H})$ are two integers d, r which will play roles in the following sections and which we define next. Let

$$d = d(h) = \max\big\{|h(u) - h(v)| : u, v \in V,\ u \sim v\big\}. \tag{3.1}$$

If $\mathcal{H}$ acts transitively, we set $r = 0$. Assume $\mathcal{H}$ does not act transitively, and let $r = r(h, \mathcal{H})$ be the least integer such that the following holds. For $u, v \in V$ in different orbits of $\mathcal{H}$, there exist $u' \in \mathcal{H}u$, $v' \in \mathcal{H}v$ with $|u' - v'| \le r$ such that: (i) $h(u') < h(v')$, and (ii) there is a SAW $\nu(u', v')$ from u' to v' all of whose vertices x, other than its endvertices, satisfy $h(u') < h(x) < h(v')$. The following proposition is proved at the end of this section.

Proposition 3.2. *Let $(h, \mathcal{H})$ be a graph height function on the graph G. Then*

$$0 \le r(h, \mathcal{H}) \le (N - 1)(2d + 1) + 2,$$

where $N = |G/\mathcal{H}|$ and d is given by (3.1).

Not every quasi-transitive graph has a graph height function, since (c), above, fails if G has a cut-vertex whose removal breaks G into an infinite and a finite part. We do not have a useful necessary and sufficient condition for the existence of a graph height function, and we pose the following more restrictive question.

Question 3.3. *Does every infinite, connected,* transitive*, locally finite, simple graph have a graph height function?*

In [18] a related (but different) question is answered in the negative in the context of Cayley graphs of finitely generated groups, by consideration of the Higman group. It may be the case that the Cayley graph of the Higman group has no graph height function.

A sufficient condition for the existence of a graph height function on a transitive graph G is provided in the forthcoming Theorem 3.4. The cycle space $\mathcal{C} = \mathcal{C}(G)$ of G is the vector space over the field $\mathbb{Z}_2$ generated by the cycles (see, for example, [9]). Let $\mathcal{H} \le \mathrm{Aut}(G)$ act quasi-transitively on G. The cycle space is said to be *finitely generated* (*with respect to* $\mathcal{H}$) if there is a finite set $\mathcal{B}(\mathcal{C})$ of cycles which, taken together with their images under $\mathcal{H}$, form a basis for $\mathcal{C}(G)$. If this holds, $\mathcal{C}$ may be viewed as a finitely generated $\mathcal{H}$-module (see [21]). It is elementary that the Cayley graph of any finitely presented group has this property, since its cycle space is generated by the cycles derived from the action of the group on the conjugates of the relators. As examples of locally finite, transitive graphs whose cycle spaces are not finitely generated, we propose the Cayley graphs of groups that are finitely generated but not finitely presented, such as the interesting examples of Grigorchuk [14] and Dunwoody [11].

We remind the reader of the definition of a quotient graph. Let $\mathcal{H}$ be a subgroup of $\mathrm{Aut}(G)$. We denote by $\vec{G} = (\overline{V}, \vec{E})$ the (directed) quotient graph $G/\mathcal{H}$ constructed as follows. The vertex-set $\overline{V}$ comprises the orbits $\overline{v} := \mathcal{H}v$ as v ranges over V. For $v, w \in V$, we place $|\partial v \cap \overline{w}|$ directed edges from $\overline{v}$ to $\overline{w}$, and we write $\overline{v} \sim \overline{w}$ if $|\partial v \cap \overline{w}| \geq 1$ and $\overline{v} \neq \overline{w}$. If $\overline{v} = \overline{w}$, an edge from $\overline{v}$ to $\overline{w}$ is a directed 'loop', and the word 'loop' is used only in this context in this paper. By [19, Lemma 3.6], the number $|\partial v \cap \overline{w}|$ is independent of the choice of $v \in \overline{v}$. When convenient, we shall work with the undirected simple graph derived from this directed multigraph by ignoring any loops, orientations, and multiplicities of edges.

Any (directed) walk π on G induces a (directed) walk $\vec{\pi}$ on $\vec{G}$, and we say that π *projects* onto $\vec{\pi}$. For a walk $\vec{\pi}$ on $\vec{G}$, there exists a walk π on G that projects onto $\vec{\pi}$, and we say that $\vec{\pi}$ *lifts* to π. There may be many choices for such π, and we fix the choice as follows. For $\overline{v}_1, \overline{v}_2 \in \overline{V}$, we label the $D := |\partial v_1 \cap \overline{v}_2|$ directed edges from $\overline{v}_1$ to $\overline{v}_2$ with the integers $1, 2, \dots, D$. For $v \in \overline{v}_1$, we label similarly the edges from v to vertices $v' \in \overline{v}_2$. For $v \in \overline{v}$, any $\vec{\pi}$ from $\overline{v}$ lifts to a unique π from v that conserves edge-labellings, and in this way walks from a given $v \in V$ on G are in one-to-one correspondence with walks from $\overline{v}$ on $\vec{G}$. Note that a cycle of G projects onto a closed walk of $\vec{G}$, which may or may not be a cycle.

Theorem 3.4. *Let $G = (V, E)$ be an infinite, connected, transitive, locally finite, simple graph. Suppose there exists a subgroup $\mathcal{H} \leq \mathrm{Aut}(G)$ acting quasi-transitively on G. For $v \in V$, let l_v be a shortest path connecting v and $\mathcal{H}v \setminus \{v\}$. If*

(a) *$\mathcal{C}(G)$ is finitely generated (with respect to $\mathcal{H}$) by a (finite) set $\mathcal{B}$ of cycles,*
(b) *every $B \in \mathcal{B}$ projects onto a cycle of $\vec{G}$,*
(c) *$\min_{v \in V} |l_v| \geq 3$, and for each $v \in V$, l_v projects onto a cycle of $\vec{G}$,*

then G has a graph height function of the form $(h, \mathcal{H})$.

Suppose, in addition, that $\mathcal{H}$ is a normal subgroup of some Γ that acts transitively. Under this assumption, $\vec{G}$ is transitive (see [19, Remark 3.5]). Therefore, the length l of l_v is independent of the choice of $v \in V$, and each l_v projects onto a cycle of $\vec{G}$. In this situation, Theorem 3.4 may be simplified as follows in this case.

Corollary 3.5. *Let $G = (V, E)$ be an infinite, connected, transitive, locally finite, simple graph. Suppose there exist a subgroup $\Gamma \leq \mathrm{Aut}(G)$ acting transitively on V, and a normal subgroup $\mathcal{H} \trianglelefteq \Gamma$ with index satisfying $[\Gamma : \mathcal{H}] < \infty$. If*

(a) *$\mathcal{C}(G)$ is finitely generated (with respect to $\mathcal{H}$) by a (finite) set $\mathcal{B}$ of cycles,*
(b) *every $B \in \mathcal{B}$ projects onto a cycle of $\vec{G}$,*
(c) *$l \geq 3$,*

then G has a graph height function of the form $(h, \mathcal{H})$.

The idea of the proof, which is found in Section 6 and illustrated in Example 6.3, is to construct a suitable function on the quotient graph $\vec{G}$, and to export this to a graph height function h on G via the action of $\mathcal{H}$. For given $N = |G/\mathcal{H}|$, there are only finitely many possibilities for the quotient graph, and therefore there exist $D_N, R_N < \infty$ such that $d(h) \le D_N$ and $r(h, \mathcal{H}) \le R_N$.

In [18] is presented an alternative version of Theorem 3.4 which, on the one hand, is more restrictive in that it makes the further assumption that $\mathcal{H}$ is unimodular, but, on the other hand, has an interesting and informative proof using the language of harmonic functions and random walk. The restriction of unimodularity is satisfied automatically in the context of Cayley graphs, which are the objects of study of the associated article [18].

Here are several examples of transitive graphs with graph height functions.

(a) The *hypercubic lattice* $\mathbb{Z}^n$ with, say, $h(x_1, x_2, \dots, x_n) = x_1$. With $\mathcal{H}$ the set of translations of $\mathbb{Z}^n$, we have $d(h) = 1$ and $r(h) = 0$.

(b) The *d-regular tree* T may be drawn as in Figure 3.1. We fix a ray ω of T, and we 'suspend' T from ω as in the figure. A vertex on ω is designated as identity $\mathbf{1}$ and is given height 0, and other vertices have heights as indicated. The set $\mathcal{H}$ is the subgroup of automorphisms that fix ω, and it acts transitively. We have $d = 1$ and $r = 0$.

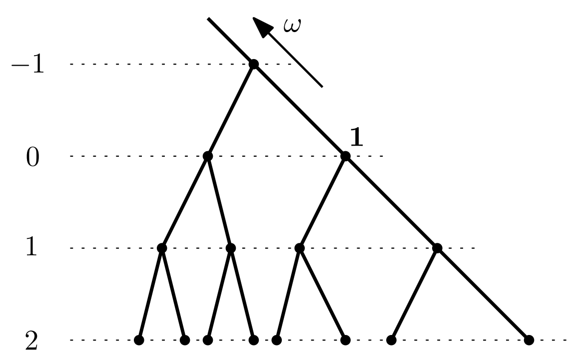

FIGURE 3.1. The 3-regular tree with the 'horocyclic' height function.

(c) There follow three examples of Cayley graphs of finitely presented groups (readers are referred to [18] for further information on Cayley graphs). The *discrete Heisenberg group*

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

has generator set $S = \{s_1, s_2, s_3, s'_1, s'_2, s'_3\}$ where

$$s_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad s_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and relator set

$$R = \{s_1s'_1, s_2s'_2, s_3s'_3\} \cup \{s_1s_2s'_1s'_2s'_3, s_1s_3s'_1s'_3, s_2s_3s'_2s'_3\}.$$

Consider its Cayley graph. To a directed edge of the form $[v, vs_1\rangle$ (respectively, $[v, vs'_1\rangle$) we associate the height difference 1 (respectively, -1), and to all other edges height difference 0. We have $d = 1$ and $r = 0$. The height $h(v)$ of vertex v is given by adding the height differences along any directed path from $\mathbf{1}$ to v, where $\mathbf{1}$ is the identity of the group.

(d) The *square/octagon lattice* of Figure 3.2 is the Cayley graph of the group with generators s_1, s_2, s_3 and relators $\{s_1^2, s_2^2, s_3^2, s_1s_2s_1s_2, s_1s_3s_2s_3s_1s_3s_2s_3\}$. It has no graph height function with $\mathcal{H}$ acting *transitively*. There are numerous ways to define a height function with quasi-transitive $\mathcal{H}$, of which we mention one. Let $\mathcal{H}$ be the automorphism subgroup generated by the shifts that map $\mathbf{1}$ to $\mathbf{1}'$ and $\mathbf{1}''$, respectively, and let the graph height function be as in the figure. We have $d = 1$ and $r = 5$.

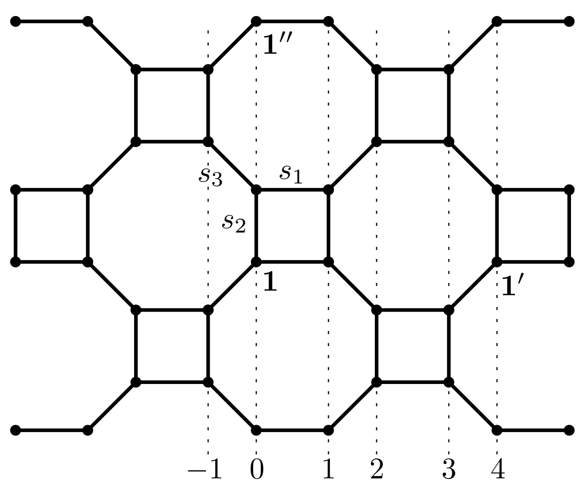

FIGURE 3.2. The square/octagon lattice. The subgroup $\mathcal{H}$ is generated by the shifts τ, τ'' that map $\mathbf{1}$ to $\mathbf{1}'$ and $\mathbf{1}''$, respectively, and $h(v)$ measures the horizontal displacement of v as marked.

(e) The *hexagonal lattice* of Figure 3.3 is the Cayley graph of a finitely presented group. It possesses a graph height function h with $\mathcal{H}$ acting transitively, as follows. Let $\mathcal{T}$ be the set of automorphisms of the lattice that act by translation of the figure, and let ρ be reflection in the vertical line as indicated there. With the heights as given in the figure, we take $\mathcal{H}$ to be the group generated by $\mathcal{T}$ and ρ. This is an example for which $\mathcal{H}$ is not torsion-free, and we have $d = 1$ and $r = 0$. Since $\mathcal{T}$ acts quasi-transitively, one may also work with $\mathcal{H} := \mathcal{T}$, in which case $r = 1$.

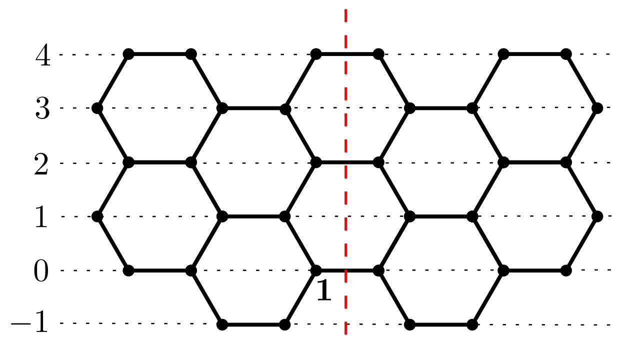

FIGURE 3.3. The hexagonal lattice. The heights of vertices are as marked, and ρ denotes reflection in the dashed line.

(f) The *Diestel–Leader graphs* $\mathrm{DL}(m, n)$ with $m \neq n$ were proposed in [10] as candidates for transitive graphs that are quasi-isometric to no Cayley graph, and this conjecture was proved in [12] (see [11] for a further example). They arise through a certain combination (details of which are omitted here) of a $(m+1)$-regular tree and a $(n+1)$-regular tree, and the horocyclic graph height function of either tree provides a graph height function for the combination. We have $d = 1$ and $r = 0$.

Proof of Proposition 3.2. Let $(h, \mathcal{H})$ be as in Definition 3.1, and assume that $\mathcal{H}$ acts quasi-transitively but not transitively. For $v, w \in V$ in distinct orbits of $\mathcal{H}$, we write $v \to w$ if there exist $v' \in \mathcal{H}v$, $w' \in \mathcal{H}w$ such that (i) $h(v') < h(w')$, and (ii) there is a SAW $\nu = (\nu_0, \nu_1, \ldots, \nu_m)$ with $\nu_0 = v'$, $\nu_m = w'$ and $h(v') < h(\nu_j) < h(w')$ for $1 \leq j < m$. We prove next that $v \to w$ for all such $v, w \in V$. Since $\mathcal{H}$ has only finitely many orbits, this will imply that $r(h, \mathcal{H}) < \infty$.

Let $\overline{G} = (\overline{V}, \overline{E})$ be the simple, undirected quotient graph obtained from $\vec{G}$ by placing a single undirected edge between $\overline{a}, \overline{b} \in \overline{V}$ if and only if $\partial a \cap \overline{b} \neq \varnothing$. For given $u \in V$, $\overline{G}$ lifts to a connected subgraph G_u of G which contains u as well as exactly one member of each orbit of $\mathcal{H}$. Since $\overline{G}$ is connected, it has a spanning tree $\overline{T}$ that lifts to a spanning tree T_u of G_u. With $N = |\overline{V}| = |G/\mathcal{H}|$, the tree T_u has $N - 1$ edges. Let

$$\Delta_u = \max\{|h(a) - h(b)| : a, b \in G_u\}.$$

By (3.1),

$$|\Delta_u| \le (N-1)d, \qquad u \in V. \tag{3.2}$$

For $v \in V$, there exists by Definition 3.1(c) a doubly infinite SAW $\pi(v) = (\pi_j(v) : j \in \mathbb{Z})$ with $\pi_0(v) = v$ and such that $h(\pi_j(v))$ is strictly increasing in j. Since h takes integer values,

$$h(\pi_{j+1}(v)) - h(\pi_j(v)) \ge 1, \qquad j \in \mathbb{Z},\ v \in V. \tag{3.3}$$

Let $v, w \in V$ be in distinct orbits of $\mathcal{H}$. Let $v' = \pi_R(v)$ and $w' = \pi_{-R}(w)$ where $R \ge 1$ will be chosen soon. Find $\alpha \in \mathcal{H}$ such that $\alpha v' \in G_{w'}$. Let ν be the walk obtained by following the sub-SAW of $\alpha\pi(v)$ from αv to $\alpha v'$, followed by the sub-path of $T_{w'}$ from $\alpha v'$ to w', followed by the sub-SAW of $\pi(w)$ from w' to w. The length of ν is at most $2R + N - 1$.

By (3.3), we can pick R sufficiently large that

$$\begin{aligned} h(\alpha v) &< \min\{h(a) : a \in G_{w'}\} \\ &\le h(\alpha v'), h(w') \\ &\le \max\{h(a) : a \in G_{w'}\} < h(w), \end{aligned}$$

and indeed, by (3.2), it suffices that $R = (N-1)d + 1$. By loop-erasure of ν, we obtain a SAW $\nu' = (\nu'_0, \nu'_1, \dots, \nu'_m)$ with $\nu'_0 = \alpha v$, $\nu'_m = w$,

$$m \le 2R + N - 1 \le 2(N-1)d + 2 + (N-1), \tag{3.4}$$

and $h(\nu'_0) < h(\nu'_j) < h(\nu'_m)$ for $1 \le j < m$. Therefore, $v \to w$ as required. The upper bound for r follows from (3.4). □

4. Bridges and the bridge constant

Assume that G is quasi-transitive with graph height function $(h, \mathcal{H})$. The forthcoming definitions depend on the choice of pair $(h, \mathcal{H})$.

Let $v \in V$ and $\pi = (\pi_0, \pi_1, \dots, \pi_n) \in \Sigma_n(v)$. We call π a *half-space SAW* if

$$h(\pi_0) < h(\pi_i), \qquad 1 \le i \le n,$$

and we write $c_n(v)$ for the number of half-space walks with initial vertex v. We call π a *bridge* if

$$h(\pi_0) < h(\pi_i) \le h(\pi_n), \qquad 1 \le i \le n, \tag{4.1}$$

and a *reversed bridge* if (4.1) is replaced by

$$h(\pi_n) \le h(\pi_i) < h(\pi_0), \qquad 1 \le i \le n.$$

The *span* of a SAW π is defined as

$$\operatorname{span}(\pi) = \max_{0 \le i \le n} h(\pi_i) - \min_{0 \le i \le n} h(\pi_i).$$

The number of n-step bridges from v with span s is denoted $b_{n,s}(v)$, and in addition

$$b_n(v) = \sum_{s=0}^{\infty} b_{n,s}(v)$$

is the total number of n-step bridges from v. Let

$$b_n = b_n(G) := \inf\{b_n(v) : v \in V\}. \tag{4.2}$$

It is easily seen (as in [24]) that

$$b_{m+n} \geq b_m b_n, \tag{4.3}$$

from which we deduce the existence of the *bridge constant*

$$\beta = \beta(G) = \lim_{n\to\infty} b_n^{1/n} \tag{4.4}$$

satisfying

$$b_n \leq \beta^n, \qquad n \geq 0. \tag{4.5}$$

Proposition 4.1. *Let $G = (V, E)$ be an infinite, connected, quasi-transitive, locally finite, simple graph possessing a height function $(h, \mathcal{H})$. Then*

$$b_n(v)^{1/n} \to \beta, \qquad v \in V, \tag{4.6}$$

and furthermore

$$b_n(v) \leq \beta^{n+r}, \qquad n \geq 1,\ v \in V, \tag{4.7}$$

where $r = r(h, \mathcal{H})$ is given after (3.1).

Theorem 4.2. *Let $G = (V, E)$ be an infinite, connected, quasi-transitive, locally finite, simple graph possessing a height function h. Then $\beta = \mu$.*

This theorem extends that of Hammersley and Welsh [24] for lattices, and has as corollary that the value of the bridge constant is independent of the choice of pair $(h, \mathcal{H})$. The proof of the theorem is deferred to Section 7.

It is proved in [19] that systematic changes to a graph G lead to a strict change in the value of its connective constant, and the question is posed there of whether one can establish a concrete numerical bound on the magnitude of the change of value. It is proved in [19, Thm 3.7] that this can be done whenever there exists a real sequence (a_n) satisfying $a_n \uparrow \mu(G)$, and which can be calculated in finite time. For any graph G satisfying the hypothesis of Theorem 4.2, we may take $a_n = b_n^{1/n}$.

Proof of Proposition 4.1. Assume G has graph height function $(h, \mathcal{H})$. If G is transitive, the claim is trivial, so we assume G is quasi-transitive but not transitive. For $v, w \in V$ with $w \notin \mathcal{H}v$, let $\nu(v, w)$ be a SAW from v to some $w' \in \mathcal{H}w$ with $h(v) < h(w')$, every vertex x of which, other than its endvertices, satisfies

$h(v) < h(x) < h(w')$. We may assume that the length $l(v,w)$ of $\nu(v,w)$ satisfies $l(v,w) \le r$ for all such pairs v, w.

By counting bridges from the endvertices of $\nu(v,x)$ and $\nu(x,v)$, we obtain that

$$b_n(v) \ge b_{n-l}(x) \ge b_{n-l-l'}(v) \ge b_{n-2r}(v),$$

where $l = l(v,x)$ and $l' = l(x,v)$, and we have used Definition 3.1(c) for the last inequality. The limit (4.6) follows by (4.4). Now choose $x \in V$ such that $b_{n+r}(x) = b_{n+r}$, and assume $x \notin \mathcal{H}v$. Since $b_{n+l}(x) \le b_{n+r}(x)$,

$$b_n(v) \le b_{n+l}(x) \le b_{n+r}(x) = b_{n+r},$$

and the final claim follows by (4.5). □

5. Locality of connective constants

Let $\mathcal{G}$ be the set of all infinite, connected, quasi-transitive, locally finite, simple graphs. For $G \in \mathcal{G}$, we choose a vertex which we label $\mathbf{1} = \mathbf{1}_G$ and call the *identity* or *origin* of G. The *sphere* $S_k(v) = S_k(v,G)$, with centre v and radius k, is the subgraph of G induced by the set of its vertices within graph-distance k of v. For $G, G' \in \mathcal{G}$, we write $S_k(v,G) \simeq S_k(v',G')$ if there exists a graph-isomorphism from $S_k(v,G)$ to $S_k(v',G')$ that maps v to v'. We define the *similarity function* K on $\mathcal{G} \times \mathcal{G}$ by

$$K(G,G') = \max\bigl\{k : S_k(\mathbf{1}_G, G) \simeq S_k(\mathbf{1}_{G'}, G')\bigr\}, \qquad G, G' \in \mathcal{G},$$

and the corresponding distance-function $d(G,G') = 2^{-K(G,G')}$. The corresponding metric space was introduced by Babai [1]; see also [5, 10].

For $D, R \in \mathbb{N}$, let $\mathcal{G}_{D,R}$ be the set of all $G \in \mathcal{G}$ which possess a graph height function h satisfying $d(h) \le D$ and $r(h) \le R$. For a quasi-transitive graph G, we write $M = M(G) = |G/\mathrm{Aut}(G)|$ for the number of orbits under its automorphism group. The locality theorem for quasi-transitive graphs follows, with proof at the end of the section. The theorem may be regarded as a partial resolution of a question of Benjamini, [3, Conj. 2.3], which was posed independently of the work reported here.

Theorem 5.1 (Locality theorem for connective constants)**.** *Let $G, G' \in \mathcal{G}$ with $K = K(G,G')$. Write $M = M(G)$, $M' = M(G')$, and $S = \max\{M, M'\} - 1$.*

(a) *There exist $\epsilon_k = \epsilon_k(G, M')$, satisfying $0 < \epsilon_k \downarrow 0$ as $k \to \infty$, such that*

$$\mu(G') \le \mu(G) + \epsilon_K. \tag{5.1}$$

(b) *Let $D, R \ge 1$ and $G' \in \mathcal{G}_{D,R}$. There exists a constant $B = B(D,R) \in (0,\infty)$ such that, for $K > S$,*

$$\frac{\mu(G)}{f(K-S)} \le \beta(G') = \mu(G'), \tag{5.2}$$

where $f(x) = [Bx^3 e^{B\sqrt{x}}]^{1/x}$.

(c) *Let* $D, R \geq 1$. *Let* $G \in \mathcal{G}$ *and* $G_m \in \mathcal{G}_{D,R}$ *for* $m \geq 1$ *be such that* $K(G, G_m) - M(G_m) \to \infty$ *as* $m \to \infty$. *Then* $\mu(G_m) \to \mu(G)$.

When G is transitive, $M(G) = 1$ and one may take $R = 0$. The statement of the theorem is thus simpler when restricted to transitive graphs.

The following application of Theorem 5.1 is prompted in part by a result in percolation theory. Let $p_c(G)$ be the critical probability of either bond or site percolation on an infinite graph G, and let $\mathbb{Z}^d$ be the d-dimensional hypercubic lattice with $d \geq 3$, and $S_k = \mathbb{Z}^2 \times \{0, 1, \dots, k\}^{d-2}$. It was proved by Grimmett and Marstrand [20] that

$$p_c(S_k) \to p_c(\mathbb{Z}^d) \qquad \text{as } k \to \infty. \tag{5.3}$$

Does a corresponding limit hold for connective constants? For simplicity, we consider this question for *transitive* graphs only.

Let $G \in \mathcal{G}$ and let Γ be a subgroup of $\text{Aut}(G)$ that acts transitively. For $m \geq 1$, let $\alpha_m \in \Gamma$ and let $\mathcal{A}_m$ be the normal subgroup of Γ generated by α_m. The group $\mathcal{A}_m$ gives rise to a quotient graph $G_m = G/\mathcal{A}_m$, which we regard as an undirected, simple graph. Since $\mathcal{A}_m$ is a normal subgroup of Γ, Γ acts transitively on $G/\mathcal{A}_m$ (by [19, Remark 3.5]), whence G_m is transitive.

Theorem 5.2. *Let* $D \geq 1$, *and assume* $G_m = G/\mathcal{A}_m \in \mathcal{G}_{D,0}$ *for all* m, *and further that* $d_G(\mathbf{1}, \alpha_m \mathbf{1}) \to \infty$ *as* $m \to \infty$. *Then* $\mu(G_m) \to \mu(G)$ *as* $m \to \infty$.

Proof. The quotient graph is obtained from G by identifying any two vertices $v \neq w$ with $w = \alpha v$ and $\alpha \in \mathcal{A}_m$. For such v, w, we have $d_G(v, w) \geq d_G(\mathbf{1}, \alpha_m \mathbf{1})$. Therefore, $K(G, G_m) \geq \frac{1}{2} d_G(\mathbf{1}, \alpha_m \mathbf{1}) - 1$, and the result follows by Theorem 5.1. □

Example 5.3. *Let* G *be the hypercubic lattice* $\mathbb{Z}^n$ *with* $n \geq 2$, *and let* Γ *be the set of its translations. Choose* $v = (v_1, v_2, \dots, v_n) \in \mathbb{Z}^n$ *with* $v \neq \mathbf{0}$, *and let* $\alpha_v \in \Gamma$ *be the translation* $w \mapsto w + v$. *Let* p_v *be a unit vector perpendicular to* v, *and let* k_v *be the smallest positive integer such that* $k_v p_v$ *is a vector of integers. For* $z \in \mathbb{Z}^n$, *let* $h_v(z) = z \cdot (k_v p_v)$, *so that* (h_v, Γ) *is a graph height function with* $d(h_v) \leq k_v$ *and* $r(h_v) = 0$.

Let $\mathcal{A}_v$ *be the normal subgroup of* Γ *generated by* α_v. *In the notation of Theorem 5.2, we have that* $\mu(G/\mathcal{A}_{v_m}) \to \mu(G)$ *as* $m \to \infty$, *so long as the sequence* (v_m) *satisfies* $|v_m| \to \infty$ *and* $\limsup_{m \to \infty} d(h_{v_m}) < \infty$. *This may be regarded as a version of the limit* (5.3) *with connective constants in place of critical percolation probabilities.*

Proof of Theorem 5.1. Let G be quasi-transitive with $M = M(G)$. Since the quotient graph $G/\text{Aut}(G)$ is connected, G has some connected subgraph H containing $\mathbf{1}$ and comprising exactly one member of each orbit under $\text{Aut}(G)$. Since $|H| = M$, H has a spanning tree containing $M - 1$ edges. Therefore, for $n \geq M - 1$,

$$\sigma_n = \sigma_n(v) \text{ for some } v \in H. \tag{5.4}$$

(a) Let $G, G' \in \mathcal{G}$, and recall (2.3). Pick $\eta_k = \eta_k(G)$ such that $0 < \eta_k \downarrow 0$ as $k \to \infty$ and

$$\mu(G)^n \le \sigma_n(G) \le (\mu(G) + \eta_k)^n, \qquad n \ge k. \tag{5.5}$$

Since $K(G, G') = K$, we have that $S_K(\mathbf{1}_G, G) \simeq S_K(\mathbf{1}_{G'}, G')$, and by (5.4)

$$\sigma_{K-S}(G') = \sigma_{K-S}(G). \tag{5.6}$$

By (5.5)–(5.6) and (2.2), for $r \ge 1$,

$$\begin{aligned}\sigma_{(K-S)r}(G') &\le \sigma_{K-S}(G')^r \\ &= \sigma_{K-S}(G)^r \le (\mu(G) + \eta_{K-S})^{(K-S)r}.\end{aligned}$$

We take $(K-S)r^{\text{th}}$ roots and let $r \to \infty$, to obtain that $\mu(G') \le \mu(G) + \epsilon_K$, where $\epsilon_K = \eta_{K-S}$.

(b) Let $D, R \ge 1$ and $G' \in \mathcal{G}_{D,R}$. By the forthcoming Proposition 7.3 and (5.5)–(5.6), there exists $B = B(D, R)$ such that

$$\begin{aligned}\beta(G')^{K-S} &\ge \frac{\sigma_{K-S}(G')}{B(K-S)^3 e^{B\sqrt{K-S}}} \\ &= \frac{\sigma_{K-S}(G)}{f(K-S)^{K-S}} \ge \frac{\mu(G)^{K-S}}{f(K-S)^{K-S}},\end{aligned}$$

whence

$$\beta(G') \ge \frac{\mu(G)}{f(K-S)}.$$

By Theorem 4.2, $\mu(G') = \beta(G')$, and (5.2) is proved.

(c) This is immediate from part (b). □

6. Proof of Theorem 3.4

Edges, walks, and cycles may sometimes be directed and sometimes undirected in the following proof. We use notation and words to distinguish between these two situations, and we hope our presentation is clear to the reader.

The strategy of the proof is to contruct a function δ on the edges of $\vec{G} = (\overline{V}, \vec{E})$ that, when lifted to G, sums to 0 around directed cycles. The graph height function is obtained by summing δ along suitable paths of G. The function δ will be defined in a sequential manner.

For $v \in V$, let l_v be the shortest directed path of G from v to some $v' \in \overline{v}$ with $v' \ne v$. We write $\vec{l}_v$ for the projection of l_v onto $\vec{G}$. By assumption (c) of Theorem 3.4, $\vec{l}_v$ is a directed cycle of $\vec{G}$, and $l \ge 3$. The key property of the $\vec{l}_v$ is that they are cycles of $\vec{G}$ that lift to SAWs of G.

As discussed in [19], when $\mathcal{H}$ is a normal subgroup of some transitive Γ, then $\vec{G}$ is finite and transitive, and the length $l := |l_v|$ is independent of the choice of $v \in V$.

Let $\mathcal{D}$ be the set of directed cycles of $\vec{G}$. Each $\sigma \in \mathcal{D}$ lifts to a directed walk of G which may or may not be a cycle. (In fact, there are many lifts of a given σ, either all or none of which are directed cycles in G.) We write $\mathcal{D}'$ for the subset of $\mathcal{D}$ containing members that lift to cycles of G.

We aim to construct a function $\delta : \vec{E} \to \mathbb{Q}$ such that:

1. for any directed edge $\vec{e} \in \vec{l}_{\mathbf{1}}$, we have $\delta(\vec{e}) = 1$,
2. for any two edges $\vec{e}, \vec{f}$ of $\vec{G}$ with the same endpoints, we have $\delta(\vec{e}) = \delta(\vec{f})$ if they have the same orientations, and $\delta(\vec{e}) = -\delta(\vec{f})$ otherwise,
3. for any $\sigma \in \mathcal{D}'$, the sum of the $\delta(\vec{e})$ over $\vec{e} \in \sigma$ is 0,
4. for $\overline{v} \in \overline{V}$, there exist $\overline{u}, \overline{w} \in \overline{V}$ such that

$$\delta([\overline{u}, \overline{v}\rangle), \delta([\overline{v}, \overline{w}\rangle) > 0. \tag{6.1}$$

These properties are referred to below as Properties 1–4.

Let such δ be given, and let $G^{\pm} = (V, E^{\pm})$ be the graph obtained from G by replacing each edge by two directed edges with opposite orientations. Let $\Delta h : E^{\pm} \to \mathbb{Q}$ be given by $\Delta h(e^{\pm}) = \delta(\vec{e})$, where $\vec{e}$ is the projection of $e^{\pm}$ onto $\vec{G}$. Every $e^{\pm} \in E^{\pm}$ receives a δ-value, and furthermore Δh is invariant under the action of $\mathcal{H}$.

For $\gamma \in \mathrm{Aut}(G)$ and a directed $B \in \mathcal{B}$, there exist $\sigma \in \mathcal{D}'$ and $\alpha \in \mathcal{H}$ such that $\alpha\sigma = \gamma B$. By Property 3 above, the sum of $\Delta h(e^{\pm})$ over $e^{\pm}$ lying in any directed $\gamma B \in \gamma\mathcal{B}$ is zero. Any directed element $C \in \mathcal{C}$ is the sum over $\mathbb{Z}_2$ of directed elements of $\mathcal{H}\mathcal{B}$, and thus the sum of $\Delta h(e^{\pm})$ over $e^{\pm}$ lying in any directed $C \in \mathcal{C}$ is zero.

The height $h(v)$ of $v \in V$ is obtained by summing $\Delta h(e^{\pm})$ over a directed path of $G^{\pm}$ from $\mathbf{1}$ to v. By the above, the height of v does not depend on the choice of path. By Property 4, each $v \in V$ has neighbours $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$. The function h does not generally take integer values. However, δ takes only finitely many rational values, so h may be replaced by mh where m is the lowest common multiple of their denominators. The ensuing h is the required height function.

We show next how such δ may be found via an iterative construction, as illustrated in Example 6.3. The values $\delta(\vec{e})$ will be revealed one by one. At each stage, edge $\vec{e}$ is said to be *explored* if $\delta(\vec{e})$ is known. The set of explored edges will increase as the stages progress, until it becomes the entire edge-set of $\vec{G}$. When $\delta(\vec{e})$ is revealed, then so is $\delta(\vec{f})$ for any $\vec{f}$ with the same endpoints as $\vec{e}$, by Property 2.

Let U be the union of the edge-sets of the $\{\vec{l}_v : \overline{v} \in \overline{V}\}$ viewed as undirected cycles, and let $S_1, S_2, \dots, S_k$ be the vertex-sets of the connected components of $(\overline{V}, U)$. Since U touches every member of $\overline{V}$, for each i there exists $j \neq i$ and $\vec{e} = [v_i, v_j\rangle \in \vec{E}$ such that $\overline{v}_i \in S_i$, $\overline{v}_j \in S_j$. Starting from S_1, we find an edge $\vec{e}_1 \notin U$ that connects S_1 and some S_{i_1} with $i_1 \neq 1$, then an edge $\vec{e}_2 \notin U$ that connects $S_1 \cup S_{i_1}$ and some S_{i_2}, and so on. This results in a set $F = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{k-1}\}$ such that $(\overline{V}, U \cup F)$ is connected.

Let S_1 comprise the cycles $\vec{l}_{v_i}$ for $1 \leq i \leq d$. We may assume without loss of generality that $v_1 = \mathbf{1}$ and, for $2 \leq i \leq d$, $\vec{l}_{v_i}$ and $\bigcup_{1 \leq j < i} \vec{l}_{v_j}$ have at least one vertex in common. We explain next how to define δ on the edges of S_1.

Stage 1. For each $\vec{e} \in \vec{l}_{v_1}$, we set $\delta(\vec{e}) = 1$.

Stage 2. Next consider $\vec{l}_{v_2}$, which, by construction, intersects $\vec{l}_{v_1}$. The cycle $\vec{l}_{v_2}$ is cut by $\vec{l}_{v_1}$ into edge-disjoint segments, endpoints of which lie in $\vec{l}_{v_1}$. Let $(\overline{z}_a \leftrightarrow \overline{z}_b)$ denote an undirected sub-path of $\vec{l}_{v_2}$ with one or more edges, and with endvertices $\overline{z}_a, \overline{z}_b \in \vec{l}_{v_1}$ such that no vertex other than these endvertices lies in $\vec{l}_{v_1}$. It can be the case that $\overline{z}_a = \overline{z}_b$.

The walk $(\overline{z}_a \leftrightarrow \overline{z}_b)$ lifts to a SAW in G. (In fact, there are many lifts of $(\overline{z}_a \leftrightarrow \overline{z}_b)$ to G, all of which are SAWs.) Consider one lift of $(\overline{z}_a \leftrightarrow \overline{z}_b)$, which is a SAW connecting z_1 and z_2, where $z_1 \in \overline{z}_a$ and $z_2 \in \overline{z}_b$. Let $U(z_1, z_2)$ be the set of all directed explored walks of $\vec{G}$ from $\overline{z}_a$ to $\overline{z}_b$ which lift to SAWs of G from z_1 to z_2. Note that, even when $\overline{z}_a = \overline{z}_b$, it may be that $U(z_1, z_2) \neq \varnothing$. For any directed walk π on $\vec{G}$, let $\Delta(\pi)$ be the sum of the $\delta(\vec{e})$ over $\vec{e} \in \pi$.

Lemma 6.1. *For $\pi_1, \pi_2 \in U(z_1, z_2)$, we have that $\Delta(\pi_1) = \Delta(\pi_2) \neq 0$.*

Proof. Each π_i, lifts to an SAW of G from z_1 to z_2. Therefore, the directed walk π_1, followed by the reversal of π_2, lifts to a closed walk of G that includes z_1 and z_2. Since this composite walk uses only explored edges, it is restricted to $\vec{l}_{\mathbf{1}}$. Using the properties of the lifts, one sees that the walk must follow $\vec{l}_{\mathbf{1}}$ from $\overline{z}_a$ to $\overline{z}_b$ (in one of the two available directions), and then back. The sum of the $\delta(\vec{e})$ along this closed walk is zero. That is, $\Delta(\pi_1 - \pi_2) = 0$ for $\pi_1, \pi_2 \in U(z_1, z_2)$. By the rule for assigning the $\delta(\vec{e})$ for $\vec{e} \in \vec{l}_{\mathbf{1}}$, we have $\Delta(\pi_1) = \Delta(\pi_2) \neq 0$. □

(i) If $U(z_1, z_2) \neq \varnothing$, we assign rational values to the $\delta(\vec{e})$, for $\vec{e}$ lying in the directed path $(\overline{z}_a \leftrightarrow \overline{z}_b)$, in such a way that: (i) they are non-zero and have the same sign, and (ii) the sum of the $\delta(\vec{e})$, for $\vec{e}$ in $(\overline{z}_a \leftrightarrow \overline{z}_b)$, equals $\Delta(\pi)$, for any given $\pi \in U(z_1, z_2)$.

(ii) If $U(z_1, z_2) = \varnothing$, we assign rational values to such $\delta(\vec{e})$ which are non-zero with the same sign.

We impose a further constraint on the values of δ which is not burdensome, and which implies that future values of the $\delta(\vec{e})$ along the $\vec{l}_v$ may be chosen to be non-zero. For $\overline{v} \in \overline{V}$, let $\Pi_{\overline{v}}$ be the set of directed walks on $\vec{G}$ from $\overline{\mathbf{1}}$ to $\overline{v}$ that use currently explored edges such that the following hold.

1. Each walk in $\Pi_{\overline{v}}$ lifts to a SAW on G.

2. Let $\pi \in \Pi_{\overline{v}}$ and let z_1, z_2 be the endpoints of an arbitrary lift of π. Then $d_G(z_1, z_2) \le 2|\overline{V}|$.

For $\pi_{\overline{v}} \in \Pi_{\overline{v}}$, let $h_{\overline{v}}(\pi_{\overline{v}})$ be the sum of the $\delta(\vec{e})$ for $\vec{e} \in \pi_{\overline{v}}$, and let $H_{\overline{v}} = \{h_{\overline{v}}(\pi_{\overline{v}}) : \pi_{\overline{v}} \in \Pi_{\overline{v}}\}$ be the set of such sums for given $\overline{v}$. Note that H_v may have two or more possibly different entries, but we list only the distinct values. By condition 2, we have $|H_{\overline{v}}| < \infty$ for $\overline{v} \in \overline{V}$.

We require that

$$\text{the elements of the collection } (H_{\overline{v}} : \overline{v} \in \overline{V}), \text{ are distinct.} \tag{6.2}$$

We assume that the new δ-values are chosen in such a way that (6.2) remains true after their assignation.

Stage 3. Having assigned δ-values to edges in $\vec{l}_{v_1}$ and $\vec{l}_{v_2}$, we continue by induction to the remaining paths in S_1. The induction hypothesis is as follows.

(a) The currently explored edges have rational δ-values satisfying Property 2.
(b) Let $u, v \in V$ be such that there exists a SAW from u to v which projects onto a SAW or a cycle of $\vec{G}$. For any directed closed walk W of G which is decomposable into two SAWs between u and v, whose projections use only explored edges of $\vec{G}$, the sum of the $\delta(\vec{e})$ along the projection of W is 0.
(c) Assumption (6.2) holds.
(d) For every vertex $\overline{v}$ incident to some explored edge, there exist $\overline{u}, \overline{w} \in \overline{V}$ such that $[\overline{u}, \overline{v}\rangle$ and $[\overline{v}, \overline{w}\rangle$ are explored and satisfy (6.1).

Here is the general step of the induction. Assume (a)–(d) hold for the explored edges so far, and let h_v, H_v be as prior to (6.2). Suppose we meet a new segment $(\overline{z}_a \leftrightarrow \overline{z}_b)$ when considering a directed cycle $\vec{l}_{v_i}$, and assume this segment intersects no explored path except at its endvertices $\overline{z}_a$, $\overline{z}_b$. (Note that $\overline{z}_a = \overline{z}_b$ is possible.) Let $(\overline{z}_a \leftrightarrow \overline{z}_b)$ lift to an SAW of G connecting z_1 and z_2, where $z_1 \in \overline{z}_a$ and $z_2 \in \overline{z}_b$. Let $Q(z_1, z_2)$ be the set of all directed explored walks of $\vec{G}$ which lift to SAWs of G from z_1 to z_2.

Lemma 6.2. *For $\pi_1, \pi_2 \in Q(z_1, z_2)$, we have that $\Delta(\pi_1) = \Delta(\pi_2) \ne 0$.*

Proof. As in the proof of Lemma 6.1, the directed walk π_1, followed by the reversal of π_2, lifts to a closed walk of G that includes z_1 and z_2. The conclusion follows from hypothesis (b). That $\Delta(\pi) \ne 0$ follows from assumption (c). □

(i) If $Q(z_1, z_2) \ne \varnothing$, let $\pi \in Q(z_1, z_2)$, so that $\Delta(\pi) \ne 0$. We assign rational values to the $\delta(\vec{e})$, with $\vec{e}$ in the directed path $(\overline{z}_a \leftrightarrow \overline{z}_b)$, in such a way that: (i) they are non-zero and have the same sign, and (ii) the sum of the $\delta(\vec{e})$, for $\vec{e}$ in $(\overline{z}_a \leftrightarrow \overline{z}_b)$, equals $\Delta(\pi)$. By Lemma 6.2, this may be done in a way that does not depend on the choice of π.

(ii) If $Q(z_1, z_2) = \varnothing$, we assign rational values to such $\delta(\vec{e})$ which are non-zero with the same sign.

In either case, the values may be assigned in such a way that the induction hypothesis remains valid.

We continue this process until all edges in S_1 have been explored, and we turn to the sets $S_2, S_3, \dots, S_k$.

Stage 4. We set $\delta(\vec{e}_1) = 0$, and we consider the exploration of S_2, beginning with $\vec{l}_v$ where v is the endpoint of e_1 lying in S_2. Assume $\vec{l}_v$ lifts to a SAW in G connecting z_1 and z_2 (so that, in particular, $\overline{v} = \overline{z}_1 = \overline{z}_2$). Let $Q(z_1, z_2)$ be the set of all explored closed walks of $\vec{G}$ which lift to SAWs of G from z_1 to z_2. We now assign δ-values to the edges of $\vec{l}_v$ by the recipe of Stage 3 above.

After the edges of S_2 have been explored in sequence, following the process described above, we declare $\delta(\vec{e}_2) = 0$ and continue with S_3, and so on.

Stage 5. Once Stage 4 is complete, every vertex of $\vec{G}$ is incident to some explored edge. There may however remain unexplored edges, and any such $\vec{e} \in \vec{E}$ is allocated a δ-value by the process of Stage 3. The proof of the existence of the height function h is complete.

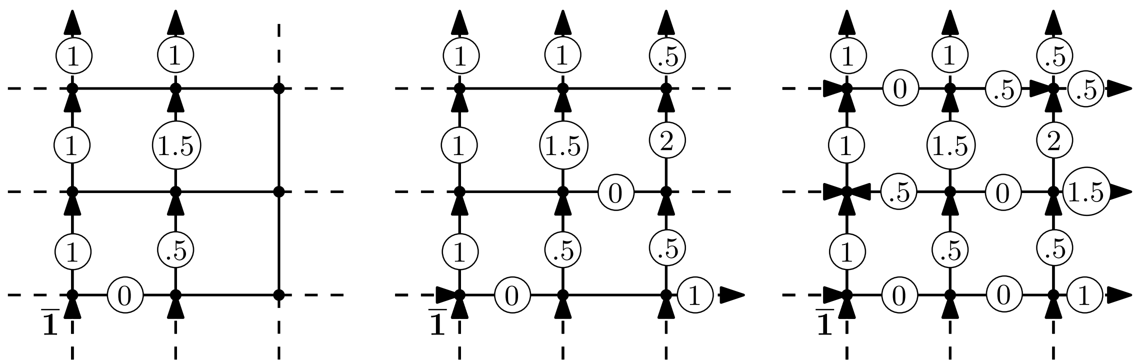

FIGURE 6.1. An illustration of the construction of the function δ on the quotient graph $\vec{G}$ of Example 6.3. Directions and magnitudes are assigned to the edges in an iterative manner, in such a way that the sum of the values around any cycle that lifts to a cycle of G is 0.

Example 6.3. *The above proof is illustrated by an example. Take $G = \mathbb{Z}^2$, and let $\mathcal{H}$ be the group of translations of G generated by the shift by three units rightwards and the shift by three units upwards. The quotient graph $\vec{G}$, when undirected, has vertex-set $\{0, 1, 2\}^2$ with toroidal edges, as in Figure 6.1. We take each $\vec{l}_v$ to be the upwards oriented 3-cycle through $\overline{v}$. In the leftmost figure, the values have been determined* *on the cycles $\vec{l}_{\mathbf{1}}$ and $\vec{l}_{(1,0)}$, where $\mathbf{1}=(0,0)$. In the central figure, the values on $\vec{l}_{(2,1)}$ have been added, and the missing values are provided in the rightmost figure.*

7. Proof of Theorem 4.2

We adapt and extend the 'bridge decomposition' approach of Hammersley and Welsh [24], which was originally specific to the hypercubic lattice. A *partition* Π of the integer $n \ge 1$ into distinct integers is an expression of the form $n = a_1 + a_2 + \cdots + a_k$ with integers a_i satisfying $a_1 > a_2 > \cdots > a_k > 0$ and some $k = k(\Pi) \ge 1$. The number $k(\Pi)$ is the *order* of the partition Π, and the number of partitions of n is denoted $P(n)$. We recall two facts about such partitions.

Lemma 7.1. *The order $k = k(\Pi)$ and the number $P(n)$ satisfy*

$$k(k+1) \le 2n \qquad \text{for all partitions } \Pi \text{ of } n, \tag{7.1}$$

$$\log P(n) \sim \pi\sqrt{n/3} \qquad \text{as } n \to \infty. \tag{7.2}$$

Proof. The sum of the first r natural numbers is $\frac{1}{2}r(r+1)$. Therefore, if r satisfies $\frac{1}{2}r(r+1) > n$, the order of Π is at most $r-1$. See [25] for a proof of (7.2). □

Let G be a graph with the given properties, and let $(h, \mathcal{H})$ be a graph height function on G. For the given $(h, \mathcal{H})$, and $v \in V$, we let $b_n(v)$ and $c_n = c_n(\mathbf{1})$ be the counts of bridges and half-space SAWs starting at v and $\mathbf{1}$, respectively, as in Section 4. Recall the constants $d = d(h)$, $r = r(h, \mathcal{H})$ given after Definition 3.1.

Proposition 7.2. *There exists $A = A(r)$ such that $c_n \le dne^{A\sqrt{n}}\beta^n$.*

Proof. Let $n \ge 1$, and let $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ be an n-step half-space SAW starting at $\pi_0 = \mathbf{1}$. Let $n_0 = 0$, and for $j \ge 1$, define $S_j = S_j(\pi)$ and $n_j = n_j(\pi)$ recursively as follows:

$$S_j = \max_{n_{j-1} \le m \le n} (-1)^j \big[h(\pi_{n_{j-1}}) - h(\pi_m)\big],$$

and n_j is the largest value of m at which the maximum is attained. The recursion is stopped at the smallest integer $k = k(\pi)$ such that $n_k = n$, so that S_{k+1} and n_{k+1} are undefined. Note that S_1 is the span of π and, more generally, S_{j+1} is the span of the SAW $\overline{\pi}^{j+1} := (\pi_{n_j}, \pi_{n_j+1}, \dots, \pi_{n_{j+1}})$. Moreover, each of the subwalks $\overline{\pi}^{j+1}$ is either a bridge or a reversed bridge. We observe that $S_1 > S_2 > \cdots > S_k > 0$.

For a decreasing sequence of $k \ge 2$ positive integers $a_1 > a_2 > \cdots > a_k > 0$, let $B_n(a_1, a_2, \dots, a_k)$ be the set of (n-step) half-space walks from $\mathbf{1}$ such that $k(\pi) = k$, $S_1(\pi) = a_1$, …, $S_k(\pi) = a_k$ and $n_k(\pi) = n$ (and hence S_{k+1} is undefined). In particular, $B_n(a)$ is the set of n-step bridges from $\mathbf{1}$ with span a.

Let $\pi \in B_n(a_1, a_2, \dots, a_k)$. We describe next how to perform surgery on π in order to obtain a SAW π' satisfying

$$\pi' \in \begin{cases} B_{n+\sigma}(a_1 + a_2 + a_3 + \delta, a_4, \dots, a_k) & \text{if } k \geq 3, \\ B_{n+\sigma}(a_1 + a_2 + \delta) & \text{if } k = 2, \end{cases} \tag{7.3}$$

for some $\sigma = \sigma(\pi)$ and $\delta = \delta(\pi)$ satisfying $0 \leq \sigma \leq 2r$ and $\delta \geq 0$. The argument is different from that of the corresponding step of [24] since G may not be invariant under reflections, and the conclusion (7.3) differs from that of [24] through the inclusion of the terms σ, δ. The proof of (7.3) is easier when $\mathcal{H}$ acts transitively, and thus we assume that $\mathcal{H}$ acts quasi-transitively but not transitively.

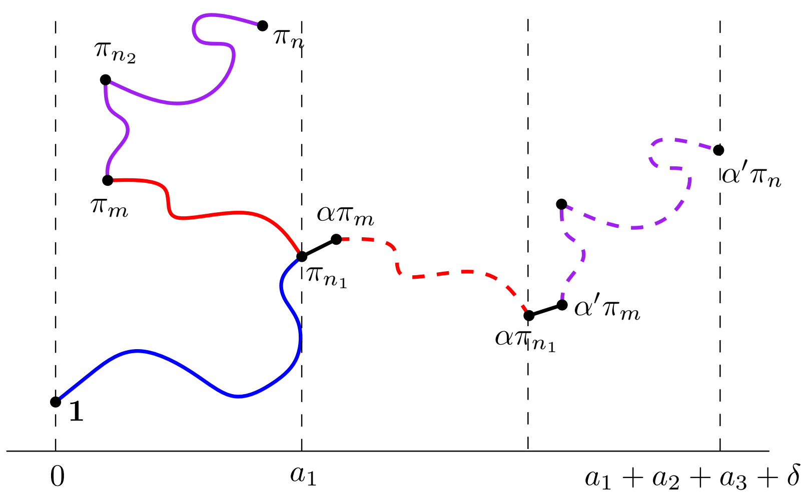

FIGURE 7.1. The solid SAW lies in $B_n(a_1, a_2, a_3)$. We translate the blue path connecting π_{n_1} to π_m, and also the third sub-SAW of π, thereby obtaining a SAW in $B_{n+\sigma}(a_1 + a_2 + a_3 + \delta)$. After translation, the paths are dashed.

The new SAW π' is constructed in the following way, as illustrated in Figure 7.1. Suppose first that $k \geq 3$.

1. Let π'_1 be the sub-SAW $\overline{\pi}^1$ from $\pi_0 = \mathbf{1}$ to the vertex π_{n_1}.
2. Let $m = \min\{N > n_1 : h(\pi_N) = S_1 - S_2\}$, and let $\nu_1 := (\pi_{n_1}, \dots, \pi_m)$ and $\nu_2 = (\pi_m, \dots, \pi_n)$ be the two sub-SAWs of π with the given endvertices.
   (a) If $\pi_m \notin \mathcal{H}\pi_{n_1}$, we find $\alpha \in \mathcal{H}$ such that: (i) $a_1 < h(\alpha\pi_m)$ and (ii) there is a SAW $\nu(\pi_{n_1}, \alpha\pi_m)$ with the given endvertices, of length not exceeding r, and of which every vertex x, other than its endvertices, satisfies $a_1 < h(x) < h(\alpha\pi_m)$.

(b) If $\pi_m \in \mathcal{H}\pi_{n_1}$, we find $\alpha \in \mathcal{H}$ such that $\alpha\pi_m = \pi_{n_1}$, and write $\nu(\pi_{n_1}, \alpha\pi_m)$ for the 0-step SAW at π_{n_1}.

The union of the three (undirected) SAWs π'_1, $\nu(\pi_{n_1}, \alpha\pi_m)$, and $\alpha\nu_1$ is a SAW, denoted π'_2, from $\mathbf{1}$ to $\alpha\pi_{n_1}$. In concluding that π'_2 is a SAW, we have made use of Definition 3.1(b). Note that $a_1 + a_2 \le h(\alpha\pi_{n_1})$.

3. As in Step 2, we next find $\alpha' \in \mathcal{H}$ such that $h(\alpha\pi_{n_1}) \le h(\alpha'\pi_m)$ and a SAW $\nu(\alpha\pi_{n_1}, \alpha'\pi_m)$ with the given endvertices and the previous type. The union of the three (undirected) SAWs π'_2, $\nu(\alpha\pi_{n_1}, \alpha'\pi_m)$, and $\alpha'\nu_2$, is a SAW, denoted π'_3, from $\mathbf{1}$ to $\alpha'\pi_n$.

We note the repeated use of Proposition 3.2, and also Definition 3.1(b). It follows that $\pi' \in B_{n+\sigma}(a_1 + a_2 + a_3 + \delta, a_4, \dots, a_k)$ for some $0 \le \sigma \le 2r$ and $\delta \ge 0$. The mapping $\pi \mapsto \pi'$ is not one-to-one since π may not be reconstructable from knowledge of π' without identification of the intermediate SAWs $\nu(\cdot)$ in steps 2 and 3. However, since the intermediate SAWs have length no greater than r, the mapping is at most r^2-to-one.

Suppose now that $k = 2$. At step 2 above, we have that $h(\pi_n) = S_1 - S_2$, so that $\pi' \in B_{n+\sigma}(a_1 + a_2 + \delta)$ for some $0 \le \sigma \le r$ and $\delta \ge 0$. As above, the map $\pi \mapsto \pi'$ is at most r-to-one.

Let $T = a_1 + a_2 + \cdots + a_k$, and write $\sum_a^{(k,T)}$ for the summation over all finite integer sequences $a_1 > \cdots > a_k > 0$ with given length k and sum T. By iteration of (7.3),

$$\begin{aligned} c_n &\le \sum_{T=1}^{dn} \sum_{k=1}^{n} \sum_a {}^{(k,T)} \big| B_n(a_1, \dots, a_k) \big| \\ &\le \sum_{T=1}^{dn} \sum_{k=1}^{n} \sum_a {}^{(k,T)} r^k \big| B_{n+s}(T + \Delta) \big|, \end{aligned}$$

for some $0 \le s \le kr$ and $\Delta \ge 0$. By (7.1)–(7.2), and (4.7), there exists a constant $A = A(r)$ such that

$$\begin{aligned} c_n &\le \sum_{T=1}^{dn} \sum_{k=1}^{n} \sum_a {}^{(k,T)} r^k \beta^{n+kr+r} \\ &\le \sum_{T=1}^{dn} \sum_{k=1}^{n} \sum_a {}^{(k,T)} \beta^{n+r} (\beta^r r)^{\sqrt{2n}} \le dn e^{A\sqrt{n}} \beta^n, \end{aligned}$$

as required. □

Proposition 7.3. *There exists $B = B(d, r) > 0$ such that*

$$\sigma_n \le B n^3 e^{B\sqrt{n}} \beta^n, \qquad n \ge 1.$$

Proof. Let $\pi = (\pi_0, \pi_1, \dots, \pi_n) \in \Sigma_n$. Let $\mu = \min_{0\le i\le n} h(\pi_i)$ and $m = \max\{i : h(\pi_i) = \mu\}$. We construct two half-space SAWs as follows.

Find $\mathbf{1}' \in \mathcal{H}\mathbf{1}$ such that, by Proposition 3.2, there exists a SAW $\nu(\mathbf{1}', \pi_m)$ with the given endvertices, of length not exceeding r, and of which every vertex x, other than its endvertices, satisfies $h(\mathbf{1}') < h(x) < h(\pi_m)$. If $\pi_m \in \mathcal{H}\mathbf{1}$, we take $\mathbf{1}' = \pi_m$ and $\nu(\mathbf{1}', \pi_m)$ the 0-step SAW at π_m. The union of the two SAWs $\nu(\mathbf{1}', \pi_m)$ and $(\pi_0, \dots, \pi_m)$ contains a half-space SAW from $\mathbf{1}'$ with length $m+\sigma$ for some $0 \le \sigma \le r$. Similarly, the union of $\nu(\mathbf{1}', \pi_m)$ and $(\pi_m, \dots, \pi_n)$ contains a half-space SAW from $\mathbf{1}'$ with length $n - m + \sigma$.

By Proposition 7.2,

$$\sigma_n \le \sum_{m=0}^{n} U_{m+r} U_{n-m+r},$$

where $U_s = dse^{A\sqrt{s}}\beta^s$. Therefore,

$$\begin{aligned}\sigma_n &\le d^2\beta^{n+2r} \sum_{m=0}^{n} (m+r)(n-m+r) \exp\Big(A\sqrt{m+r} + A\sqrt{n-m+r}\Big)\\ &\le d^2\beta^{n+2r} n(n+2r)^2 e^{A\sqrt{2(n+2r)}},\end{aligned}$$

by the inequality $\sqrt{x} + \sqrt{y} \le \sqrt{2x+2y}$. The claim follows. □

It is trivial that $b_n \le \sigma_n$, whence $\beta \le \mu$. The reverse inequality follows by Proposition 7.3, and the theorem is proved.

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