LOCALITY OF CONNECTIVE CONSTANTS, I. TRANSITIVE GRAPHS

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Abstract. The connective constant $\mu(G)$ of a quasi-transitive graph G is the exponential growth rate of the number of self-avoiding walks from a given origin. We prove a locality theorem for connective constants, namely, that the connective constants of two graphs are close in value whenever the graphs agree on a large ball around the origin (and a further condition is satisfied). The proof exploits a generalized bridge decomposition of self-avoiding walks, which is valid subject to the assumption that the underlying graph is quasi-transitive and possesses a so-called graph height function.

1. Introduction, and summary of results

There is a rich theory of interacting systems on infinite graphs. The probability measure governing a process has, typically, a continuously varying parameter, z say, and there is a singularity at some 'critical point' z_c. The numerical value of z_c depends in general on the choice of underlying graph G, and a significant part of the associated literature is directed towards estimates of z_c for different graphs. In most cases of interest, the value of z_c depends on more than the geometry of some bounded domain only. This observation provokes the question of 'locality': to what degree is the value of z_c determined by knowledge of a bounded domain of G?

The purpose of the current paper is to present a locality theorem (namely, Theorem 6.1) for the connective constant $\mu(G)$ of the graph G. A self-avoiding walk (SAW) is a path that visits no vertex more than once. SAWs were introduced in the study of long-chain polymers in chemistry (see, for example, the 1953 volume of Flory, [19]), and their theory has been much developed since (see the book of Madras and Slade, [38], and the recent review [2]). If the underlying graph G has some periodicity, the number of n-step SAWs from a given origin grows exponentially with some growth rate $\mu(G)$ called the *connective constant* of the graph G.

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There are only few graphs G for which the numerical value of $\mu(G)$ is known exactly (some examples are given in [25]), and a substantial part of the literature on SAWs is devoted to inequalities for $\mu(G)$. The current paper may be viewed in this light, as a continuation of the series of papers on the broad topic of connective constants of transitive graphs by the same authors, see [22, 23, 24, 25].

The main result (Theorem 6.1) of this paper is as follows. Let G, G' be infinite, vertex-transitive graphs, and write $S_K(v,G)$ for the K-ball around the vertex v in G. If $S_K(v,G)$ and $S_K(v',G')$ are isomorphic as rooted graphs, then

$$|\mu(G)-\mu(G')| \le \epsilon_K(G), \tag{1.1}$$

where $\epsilon_K(G)\to 0$ as $K\to\infty$. (A related result holds for quasi-transitive graphs.)

This is proved subject to certain conditions on the graphs G, G', of which the primary condition is that they support so-called 'graph height functions' (see Section 3 for the definition of a graph height function). The existence of graph height functions permits the use of a 'bridge decomposition' of SAWs (in the style of the work of Hammersley and Welsh [32]), and this leads in turn to computable sequences that converge to $\mu(G)$ from above and below, respectively. The locality result of (1.1) may be viewed as a partial answer to a question of Benjamini, [4, Conj. 2.3], made independently of the work reported here.

A class of vertex-transitive graphs of special interest is provided by the Cayley graphs of finitely generated groups. Cayley graphs have algebraic as well as geometric structure, and this allows a deeper investigation of locality and of graph height functions. The corresponding investigation is reported in the companion paper [26] where, in particular, we present a method for the construction of a graph height function via a suitable harmonic function on the graph.

The locality question for percolation was approached by Benjamini, Nachmias, and Peres [5] for tree-like graphs. Let G be vertex-transitive with degree $d+1$. It is elementary that the percolation critical point satisfies $p_{\mathrm{c}}\ge 1/d$ (see [9, Thm 7]), and an asymptotically equivalent upper bound for p_{c} was developed in [5] for graphs with degree $d+1$ and large girth. In recent work of Martineau and Tassion [39], a locality result has been proved for percolation on abelian graphs. The proof extends the methods and conclusions of [28], where it is proved that the slab critical points converge to $p_{\mathrm{c}}(\mathbb{Z}^d)$, in the limit as the slabs become infinitely 'fat'. (A related result for connective constants is included here at Example 6.3.)

We are unaware of a locality theorem for the critical temperature T_{c} of the Ising model. Of the potentially relevant work on the Ising model to date, we mention [8, 10, 14, 15, 35, 41].

This paper is organized as follows. Relevant background and notation is described in Section 2. The concept of a graph height function is presented in Section 3, where examples are included of infinite graphs with graph height functions. A sufficient condition is presented in Section 4 for the existence of a graph height function (see Theorem 4.1), with an application to the Cayley graphs of residually finite groups (see Theorem 4.2). Bridges and the bridge constant are defined in Section 5, and it is proved in Theorem 5.2 that the bridge constant equals the connective constant whenever there exists a graph height function. The main 'locality theorem' is given at Theorem 6.1. Theorem 6.2 is an application of the locality theorem in the context of a sequence of quotient graphs; this parallels the Grimmett–Marstrand theorem [28] for percolation on slabs, but with the underlying lattice replaced by a transitive graph with a graph height function. Sections 7, 8, and 9 contain the proofs of Theorems 4.1, 4.2, and 5.2.

2. Notation and background

The graphs $G = (V, E)$ considered here are generally assumed to be infinite, connected, locally finite, and also simple, in that they have neither loops nor multiple edges. An undirected edge e with endpoints u, v is written as $e = \langle u, v\rangle$, and if directed from u to v as $[u, v\rangle$. Graphs are assumed by default to be undirected, unless otherwise stated. (When used in the context of directed graphs, the words 'directed' and 'oriented' are synonyms.) If $\langle u, v\rangle \in E$, we call u and v *adjacent*, and we write $u \sim v$, and $\partial v = \{u \in V : \langle u, v\rangle \in E\}$ denotes the set of neighbours of v.

Loops and multiple edges have been excluded for cosmetic reasons only. A SAW can traverse no loop, and thus loops may be removed without changing the connective constant. The same proofs are valid in the presence of multiple edges. When there are multiple edges, we are effectively considering SAWs on a weighted simple graph, and indeed our results are valid for edge-weighted graphs with strictly positive weights, and for counts of SAWs in which the contribution of a given SAW is the product of the weights of the edges therein.

The *degree* of vertex v is the number of edges incident to v, denoted $\deg_G(v)$, and G is called *locally finite* if every vertex-degree is finite. The *graph-distance* between two vertices u, v is the number of edges in the shortest path from u to v, denoted $d_G(u, v)$.

The automorphism group of the graph $G = (V, E)$ is denoted $\mathrm{Aut}(G)$. A subgroup $\Gamma \le \mathrm{Aut}(G)$ is said to *act transitively* on G (or on its vertex-set V) if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma v = w$. It is said to *act quasi-transitively* if there is a finite set W of vertices such that, for $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $\gamma v = w$. The graph is called *(vertex-)transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively). For a subgroup $\mathcal{H} \le \mathrm{Aut}(G)$ and a vertex $v \in V$, the orbit of v under $\mathcal{H}$ is written $\mathcal{H}v$. The number of such orbits is written as $|G/\mathcal{H}|$.

A *walk* w on G is an (ordered) alternating sequence $(w_0, e_0, w_1, e_1, \dots, e_{n-1}, w_n)$ of vertices w_i and edges $e_i = \langle w_i, w_{i+1}\rangle$. We write $|w| = n$ for the *length* of w, that is, the number of edges in w. The walk w is called *closed* if $w_0 = w_n$. We note that w is directed from w_0 to w_n. When, as is normally the case, G is simple, we abbreviate w to the sequence $(w_0, w_1, \dots, w_n)$ of vertices visited.

A *cycle* is a closed walk w traversing distinct edges, and satisfying $w_i \neq w_j$ for $1 \le i < j \le n$. Strictly speaking cycles (thus defined) have orientations derived from the underlying walk, and for this reason we may refer to them sometimes as *directed cycles* of G. When factoring out this direction (and also the privileged role of w_0), we shall use the usual term *cycle*, or *undirected cycle*.

In the definition of 'quotient' graphs in Section 4, we shall encounter multigraphs possibly containing loops and parallel edges. Any further notation for multigraphs will be introduced as needed.

An *n-step self-avoiding walk* (SAW) on G is a walk containing n edges no vertex of which appears more than once. Let $\Sigma_n(v)$ be the set of n-step SAWs starting at v, with cardinality $\sigma_n(v) := |\Sigma_n(v)|$, and let

$$\sigma_n = \sigma_n(G) := \sup\{\sigma_n(v) : v \in V\}. \tag{2.1}$$

We have in the usual way (see [31, 38]) that

$$\sigma_{m+n} \le \sigma_m \sigma_n, \tag{2.2}$$

whence the *connective constant*

$$\mu = \mu(G) := \lim_{n\to\infty} \sigma_n^{1/n}$$

exists, and furthermore

$$\sigma_n \ge \mu^n, \qquad n \ge 0. \tag{2.3}$$

Hammersley [30] proved that, if G is quasi-transitive,

$$\lim_{n\to\infty} \sigma_n(v)^{1/n} = \mu, \qquad v \in V. \tag{2.4}$$

We select a vertex of G and call it the *identity* or *origin*, denoted $\mathbf{1}$. Further notation concerning SAWs will be introduced when needed. The concept of a 'graph height function' is explained in the next section, and that leads in Section 5 to the definition of a 'bridge'.

The set of integers is written as $\mathbb{Z} = \{\dots, -1, 0, -1, \dots\}$, the natural numbers as $\mathbb{N} = \{1, 2, 3 \dots\}$, and the rationals as $\mathbb{Q}$.

3. Quasi-transitive graphs and graph height functions

We consider quasi-transitive graphs with certain properties, and begin with some notation. Let $G = (V, E)$ be an infinite, connected, quasi-transitive, locally finite, simple graph.

Definition 3.1. *A* graph height function *on G is a function $h : V \to \mathbb{Z}$ such that:*

(a) $h(\mathbf{1}) = 0$,
(b) *there exists a subgroup $\mathcal{H} \le \mathrm{Aut}(G)$ acting quasi-transitively on G such that h is $\mathcal{H}$-*difference-invariant *in the sense that*
$$h(\alpha v) - h(\alpha u) = h(v) - h(u), \qquad \alpha \in \mathcal{H},\ u, v \in V,$$
(c) *for $v \in V$, there exist $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$.*

The expression 'graph height function' is chosen in contrast to the 'group height function' of [26]. It is explained in [26] that a group height function of a finitely generated group is a graph height function of its Cayley graph, but not necessarily *vice versa*.

For a given graph height function h, we may select a corresponding group of automorphisms denoted $\mathcal{H} = \mathcal{H}(h)$, and we shall sometimes refer to the pair $(h, \mathcal{H})$ as the graph height function. Associated with the pair $(h, \mathcal{H})$ are two integers d, r which will play roles in the following sections and which we define next. Let

$$d = d(h) = \max\big\{|h(u) - h(v)| : u, v \in V,\ u \sim v\big\}. \tag{3.1}$$

If $\mathcal{H}$ acts transitively, we set $r = 0$. Assume $\mathcal{H}$ does not act transitively, and let $r = r(h, \mathcal{H})$ be the least integer r such that the following holds. For $u, v \in V$ in different orbits of $\mathcal{H}$, there exists $v' \in \mathcal{H}v$ such that $h(u) < h(v')$, and a SAW $\nu(u, v')$ from u to v', with length r or less, all of whose vertices x, other than its endvertices, satisfy $h(u) < h(x) < h(v')$. The following proposition is proved at the end of this section.

Proposition 3.2. *Let $(h, \mathcal{H})$ be a graph height function on the graph G. Then*
$$0 \le r(h, \mathcal{H}) \le (M - 1)(2d + 1) + 2,$$
where $M = |G/\mathcal{H}|$ and d is given by (3.1).

Not every quasi-transitive graph has a graph height function. For example, (c), above, fails if G has a cut-vertex whose removal breaks G into an infinite and a finite part. We do not have a useful necessary and sufficient condition for the existence of a graph height function.

Remark 3.3. *There exist infinite Cayley graphs that support no graph height function. Examples are provided in the paper [27], which postdates the current work.*

Here are several examples of transitive graphs with graph height functions.

(a) The *hypercubic lattice* $\mathbb{Z}^n$ with, say, $h(x_1, x_2, \dots, x_n) = x_1$. With $\mathcal{H}$ the set of translations of $\mathbb{Z}^n$, we have $d(h) = 1$ and $r(h) = 0$.

(b) The *d-regular tree* T may be drawn as in Figure 3.1. Let ω be a ray of T, and 'suspend' T from ω as in the figure. A given vertex on ω is designated as identity **1** and is given height 0, and other vertices have heights as indicated. The set $\mathcal{H}$ is the subgroup of automorphisms that fix the end of T determined by ω, and $\mathcal{H}$ acts transitively. We have $d = 1$ and $r = 0$.

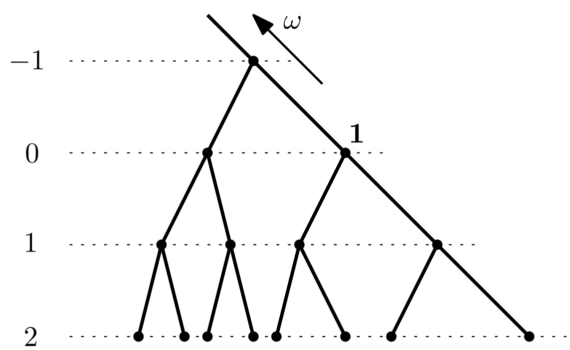

FIGURE 3.1. The 3-regular tree with the 'horocyclic' height function.

(c) There follow three examples of Cayley graphs of finitely presented groups (readers are referred to [26] for further information on Cayley graphs). The *discrete Heisenberg group*

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

has generator set $S = \{s_1, s_2, s_3, s_1', s_2', s_3'\}$ where

$$s_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and relator set

$$R = \{s_1 s_1', s_2 s_2', s_3 s_3'\} \cup \{s_1 s_2 s_1' s_2' s_3', s_1 s_3 s_1' s_3', s_2 s_3 s_2' s_3'\}.$$

Consider its Cayley graph. To a directed edge of the form $[v, vs_1\rangle$ (respectively, $[v, vs_1'\rangle$) we associate the height difference 1 (respectively, -1), and to all other edges height difference 0. The height $h(v)$ of vertex v is given by adding the height differences along any directed path from the identity $\mathbf{1}$ to v, which is to say that

$$h\left[\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right] = x.$$

We have $d = 1$ and $r = 0$.

(d) The *square/octagon lattice* of Figure 3.2 is the Cayley graph of the group with generators s_1, s_2, s_3 and relators $\{s_1^2, s_2^2, s_3^2, s_1s_2s_1s_2, s_1s_3s_2s_3s_1s_3s_2s_3\}$. It has no graph height function with $\mathcal{H}$ acting *transitively*. There are numerous ways to define a height function with quasi-transitive $\mathcal{H}$, of which we mention one. Let $\mathcal{H}$ be the automorphism subgroup generated by the shifts that map $\mathbf{1}$ to $\mathbf{1}'$ and $\mathbf{1}''$, respectively, and let the graph height function be as in the figure. We have $d = 1$ and $r = 6$.

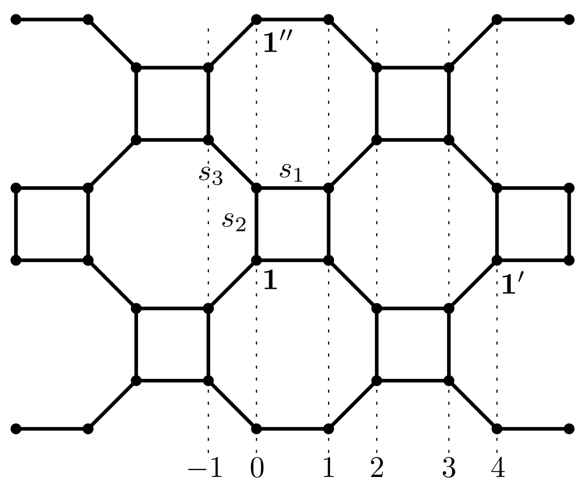

FIGURE 3.2. The square/octagon lattice. The subgroup $\mathcal{H}$ is generated by the shifts τ, τ'' that map $\mathbf{1}$ to $\mathbf{1}'$ and $\mathbf{1}''$, respectively, and the heights of vertices are as marked.

(e) The *hexagonal lattice* of Figure 3.3 is the Cayley graph of a finitely presented group. It possesses a graph height function h with $\mathcal{H}$ acting transitively, as follows. Let $\mathcal{T}$ be the set of automorphisms of the lattice that act by translation of the figure, and let ρ be reflection in the vertical line as indicated there. With the heights as given in the figure, we take $\mathcal{H}$ to be the group generated by $\mathcal{T}$ and ρ. This is an example for which $\mathcal{H}$ is not torsion-free, and we have $d = 1$ and $r = 0$. Since $\mathcal{T}$ acts quasi-transitively, one may also work with $\mathcal{H} := \mathcal{T}$, in which case $r = 1$.

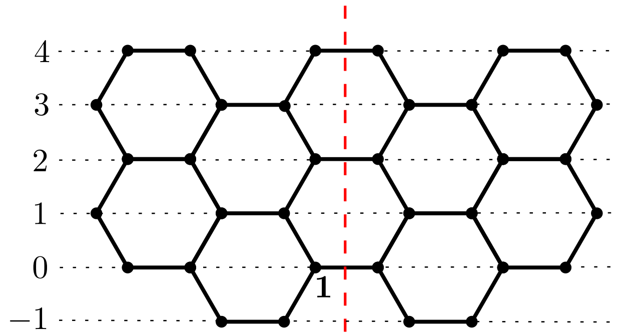

FIGURE 3.3. The hexagonal lattice. The heights of vertices are as marked, and ρ denotes reflection in the vertical dashed line.

(f) The *Diestel–Leader graphs* $\mathrm{DL}(m,n)$ with $m \neq n$ were proposed in [13] as candidates for transitive graphs that are quasi-isometric to no Cayley graph, and this conjecture was proved in [18] (see [17] for a further example). They arise through a certain combination (details of which are omitted here) of a $(m+1)$-regular tree and a $(n+1)$-regular tree, and the horocyclic graph height function of either tree provides a graph height function for the combination. We have $d = 1$ and $r = 0$.

Proof of Proposition 3.2. Let $(h, \mathcal{H})$ be as in Definition 3.1, and assume that $\mathcal{H}$ acts quasi-transitively but not transitively. For $v, w \in V$, we write $v \to w$ if there exist $v' \in \mathcal{H}v$, $w' \in \mathcal{H}w$ such that (i) $h(v') < h(w')$, and (ii) there is a SAW $\nu = (\nu_0, \nu_1, \dots, \nu_m)$ with $\nu_0 = v'$, $\nu_m = w'$, and $h(v') < h(\nu_j) < h(w')$ for $1 \le j < m$. We prove next that $v \to w$ for all pairs v, w lying in distinct orbits of V. Since $\mathcal{H}$ has only finitely many orbits, this will imply that $r(h, \mathcal{H}) < \infty$.

Let $u \in V$, and let T_u be a sub-tree of G containing u and exactly one representative of each orbit of $\mathcal{H}$. (The tree T_u may be obtained as a lift of a spanning tree of the quotient graph $G/\mathcal{H}$ of Section 4.) With $M = |G/\mathcal{H}|$, the tree T_u has $M - 1$ edges. Let

$$\Delta_u = \max\bigl\{|h(a) - h(b)| : a, b \in V(T_u)\bigr\},$$

where $V(T_u)$ is the vertex-set of T_u. By (3.1),

$$|\Delta_u| \le (M-1)d, \qquad u \in V. \tag{3.2}$$

By Definition 3.1(c), for $v \in V$, we may pick a doubly infinite SAW $\pi(v) = (\pi_j(v) : j \in \mathbb{Z})$ with $\pi_0(v) = v$, such that $h(\pi_j(v))$ is strictly increasing in j. Since h takes integer values,

$$h(\pi_{j+1}(v)) - h(\pi_j(v)) \ge 1, \qquad j \in \mathbb{Z},\ v \in V. \tag{3.3}$$

Let $v, w \in V$ be in distinct orbits of $\mathcal{H}$. Let $v' = \pi_R(v)$ and $w' = \pi_{-R}(w)$ where $R \geq 1$ will be chosen soon. Find $\alpha \in \mathcal{H}$ such that $\alpha v' \in V(T_{w'})$. Let ν be the walk obtained by following the sub-SAW of $\alpha\pi(v)$ from αv to $\alpha v'$, followed by the sub-path of $T_{w'}$ from $\alpha v'$ to w', followed by the sub-SAW of $\pi(w)$ from w' to w. The length of ν is at most $2R + M - 1$.

By (3.3), we can pick R sufficiently large that

$$\begin{aligned} h(\alpha v) &< \min\{h(a) : a \in V(T_{w'})\} \\ &\leq h(\alpha v'), h(w') \\ &\leq \max\{h(a) : a \in V(T_{w'})\} < h(w), \end{aligned}$$

and indeed, by (3.2), it suffices that $R = (M-1)d + 1$. By loop-erasure of ν, we obtain a SAW $\nu' = (\nu'_0, \nu'_1, \dots, \nu'_m)$ with $\nu'_0 = \alpha v$, $\nu'_m = w$,

$$m \leq 2R + M - 1 \leq 2(M-1)d + 2 + (M-1), \tag{3.4}$$

and $h(\nu'_0) < h(\nu'_j) < h(\nu'_m)$ for $1 \leq j < m$. Therefore, $v \to w$ as required. The upper bound for r follows from (3.4). □

4. A sufficient condition for a graph height function

This section is devoted to a sufficient condition for the existence of a graph height function for an undirected, transitive graph G. The cycle space $\mathcal{C} = \mathcal{C}(G)$ of G is the vector subspace of $\mathbb{Z}_2^E$ generated by the cycles (see [12]). Let $\mathcal{H} \leq \mathrm{Aut}(G)$ act quasi-transitively on G, and let $\mathcal{B}$ be a finite set of cycles of G. The cycle space is said to be *generated by* $\mathcal{B}$ (*with respect to* $\mathcal{H}$) if the set of images of $\mathcal{B}$ under $\mathcal{H}$ is a generating family of $\mathcal{C}$. If this holds, $\mathcal{C}$ may be viewed as a finitely generated $\mathcal{H}$-module (see [29]). It is elementary that the Cayley graph of any finitely presented group Γ is finitely generated by the cycles corresponding to its relators (with respect to Γ). The Cayley graphs of groups that are finitely generated but not finitely presented are potentially relevant in the context of graphs whose cycle spaces are not finitely generated, and we mention the interesting examples of Baumslag [3], Dunwoody [16], and Grigorchuk [21].

We introduce next the definition of a quotient graph, as adapted to the current context. Let $\mathcal{H}$ be a subgroup of $\mathrm{Aut}(G)$. We denote by $\overline{G} = (\overline{V}, \overline{E})$ the undirected quotient graph $G/\mathcal{H}$ constructed as follows. The vertex-set $\overline{V}$ comprises the orbits $\overline{v} := \mathcal{H}v$ as v ranges over V. For distinct $\overline{v}, \overline{w} \in \overline{V}$, let $E_{\overline{v},\overline{w}} \subseteq E$ be the set of edges with endpoints v, w satisfying $v \in \overline{v}$, $w \in \overline{w}$. We define an equivalence relation on $E_{\overline{v},\overline{w}}$ as follows. Two edges $e_1 = \langle v_1, w_1 \rangle$, $e_2 = \langle v_2, w_2 \rangle$ are declared *equivalent* if and only if there exists $\alpha \in \mathcal{H}$ such that $v_2 = \alpha v_1$ and $w_2 = \alpha w_1$. Let $C^1_{\overline{v},\overline{w}}, C^2_{\overline{v},\overline{w}}, \dots, C^N_{\overline{v},\overline{w}}$ be the equivalence classes, with $N = N(\overline{v}, \overline{w})$. We place N edges between $\overline{v}$ and $\overline{w}$ in $\overline{G}$, and label these edges by the N equivalence classes.

The quotient graph may also contain loops, and slightly greater care is required with these. Let $\overline{v} \in \overline{V}$, and let $E_{\overline{v}}$ be the set of *ordered* pairs (v_1, v_2) such that $v_1, v_2 \in \overline{v}$ and $\langle v_1, v_2 \rangle \in E$. Two such pairs (v_1, v_2), (v_3, v_4) are declared equivalent if there exists $\alpha \in \mathcal{H}$ with $v_3 = \alpha v_1$ and $v_4 = \alpha v_2$. We place N loops at $\overline{v} \in \overline{V}$, where N is the number of equivalence classes, and we label these loops with the classes. This completes the definition of the (multi)graph $\overline{G} = (\overline{V}, \overline{E})$, which we note to be locally finite since the degree of $\overline{v} \in \overline{V}$ is bounded above by the degree of $v \in V$, for $v \in \overline{V}$.

We deal next with the *projection* of vertices and edges.

(a) A vertex $v \in V$ projects onto $\overline{v} \in \overline{V}$.
(b) An edge $e = \langle v, w \rangle \in E$ with $\overline{v} \neq \overline{w}$ projects onto the edge of $\overline{G}$, denoted $\pi(e)$, that has endpoints $\overline{v}$, $\overline{w}$ and is labelled with the equivalence class containing e.
(c) An edge of the form $e = \langle v, v' \rangle$ with $v' \in \overline{v}$ needs to be considered in conjunction with a orientation. Let $\vec{e}$ be obtained from e by adding an orientation, say $\vec{e} = [v, v'\rangle$. Then $\vec{e}$ projects to the loop at $\overline{v}$, denoted $\pi(\vec{e})$, labelled by the equivalence class of the ordered pair (v, v'). In particular, $[v, v'\rangle$ and $[v', v\rangle$ do not necessarily project onto the same edge of $\overline{G}$.

A walk $\pi = (w_0, e_0, w_1, \dots, w_n)$ on G *projects* onto the walk $\overline{\pi} = (\overline{w}_0, \overline{e}_0, \overline{w}_1, \dots, \overline{w}_n)$, where

$$\overline{e} = \begin{cases} \pi(e) & \text{if } e = \langle w_i, w_{i+1} \rangle \text{ with } \overline{w}_i \neq \overline{w}_{i+1}, \\ \pi(\vec{e}) & \text{if } e = \langle w_i, w_{i+1} \rangle \text{ with } w_{i+1} \in \overline{w}_i, \text{ where } \vec{e} = [w_i, w_{i+1}\rangle. \end{cases}$$

We say that $\overline{\pi}$ *lifts* to π, and indeed $\overline{\pi}$ has many lifts. Note that a cycle of G projects onto a closed walk of $\overline{G}$, which may or may not be a cycle. Furthermore, the action of $\mathcal{H}$ on walks w of G satisfies: for $\gamma \in \mathcal{H}$, the projections of w and $\gamma(w)$ are the same.

Theorem 4.1. *Let $G = (V, E)$ be an infinite, connected, transitive, locally finite, simple graph. Suppose there exist a subgroup $\Gamma \leq \mathrm{Aut}(G)$ acting transitively on V, a normal subgroup $\mathcal{H} \trianglelefteq \Gamma$ with index satisfying $[\Gamma : \mathcal{H}] < \infty$, and a non-empty, finite set $\mathcal{B}$ of cycles of G such that the following hold:*

(a) *$\mathcal{C}(G)$ is generated by $\mathcal{B}$ (with respect to $\mathcal{H}$),*
(b) *every $B \in \mathcal{B}$ projects onto a cycle $\overline{B}$ of $\overline{G}$,*
(c) *there exists some cycle of $\overline{G}$ of which every lift is a SAW of G.*

Then G has a graph height function of the form $(h, \mathcal{H})$.

Furthermore, such h may be found satisfying

$$d(h) \leq D_N, \quad r(h, \mathcal{H}) \leq R_N, \tag{4.1}$$

for constants D_N, R_N depending only on the number N of edges of $\overline{G}$.

The idea of the proof of Theorem 4.1, which is found in Section 7 and illustrated in Example 7.5, is to construct a suitable function on the quotient graph $\overline{G}$, and to export this to a graph height function h on G via the action of $\mathcal{H}$.

In [26, Thm 3.3] is presented an alternative version of Theorem 4.1 which, on the one hand, is more restrictive, but, on the other hand, has an interesting and informative proof using the language of harmonic functions and random walk. The restriction is satisfied automatically in the context of Cayley graphs, which are the objects of study of the associated article [26].

The following application of Theorem 4.1 is included here as partial motivation for the conditions of the theorem. A group Γ is called *residually finite* if, for any non-identity element $\gamma \in \Gamma$, there exists a homomorphism ξ from Γ to a finite group H such that $\xi(\gamma) \neq \mathbf{1}_H$, where $\mathbf{1}_H$ is the identity element of H. It is standard (see [40, Sect. 2.2]) that Γ is residually finite if and only if, for each non-identity element $\gamma \in \Gamma$, there exists a normal subgroup with finite index not containing γ.

Theorem 4.2. *A locally finite Cayley graph of an infinite, finitely presented, residually finite group has a graph height function.*

The proof of Theorem 4.2 is given in Section 8, where it is explained that we consider simple Cayley graphs obtained by removal of loops and merging of parallel edges.

5. Bridges and the bridge constant

Assume that G is quasi-transitive with graph height function $(h, \mathcal{H})$. The forthcoming definitions depend on the choice of pair $(h, \mathcal{H})$.

Let $v \in V$ and $\pi = (\pi_0, \pi_1, \dots, \pi_n) \in \Sigma_n(v)$. We call π a *half-space SAW* if

$$h(\pi_0) < h(\pi_i), \qquad 1 \le i \le n,$$

and we write $c_n(v)$ for the number of half-space walks with initial vertex v. We call π a *bridge* if

$$h(\pi_0) < h(\pi_i) \le h(\pi_n), \qquad 1 \le i \le n, \tag{5.1}$$

and a *reversed bridge* if (5.1) is replaced by

$$h(\pi_n) \le h(\pi_i) < h(\pi_0), \qquad 1 \le i \le n.$$

The *span* of a SAW π is defined as

$$\mathrm{span}(\pi) = \max_{0 \le i \le n} h(\pi_i) - \min_{0 \le i \le n} h(\pi_i).$$

The number of n-step bridges from v with span s is denoted $b_{n,s}(v)$, and in addition

$$b_n(v) = \sum_{s=0}^{\infty} b_{n,s}(v)$$

is the total number of n-step bridges from v. Let

$$b_n = b_n(G) := \min\{b_n(v) : v \in V\}. \tag{5.2}$$

It is easily seen (as in [32]) that

$$b_{m+n} \geq b_m b_n, \tag{5.3}$$

from which we deduce the existence of the *bridge constant*

$$\beta = \beta(G) = \lim_{n\to\infty} b_n^{1/n} \tag{5.4}$$

satisfying

$$b_n \leq \beta^n, \qquad n \geq 0. \tag{5.5}$$

Proposition 5.1. *Let $G = (V, E)$ be an infinite, connected, quasi-transitive, locally finite, simple graph possessing a height function $(h, \mathcal{H})$. Then*

$$b_n(v)^{1/n} \to \beta, \qquad v \in V, \tag{5.6}$$

and furthermore

$$b_n(v) \leq \beta^{n+r}, \qquad n \geq 1,\ v \in V, \tag{5.7}$$

where $r = r(h, \mathcal{H})$ is given after (3.1).

Theorem 5.2. *Let $G = (V, E)$ be an infinite, connected, quasi-transitive, locally finite, simple graph possessing a height function $(h, \mathcal{H})$. Then $\beta = \mu$.*

This theorem extends that of Hammersley and Welsh [32] for certain lattices, and has as corollary that the value of the bridge constant is independent of the choice of pair $(h, \mathcal{H})$. The proof of the theorem is deferred to Section 9.

Remark 5.3. *It is proved in [24] that non-trivial quotienting of a graph G leads to strict reduction in the value of its connective constant, and the question is posed there of whether one can establish a concrete lower bound on the magnitude of the change in value. It is proved in* [24, Thm 3.11] *that this can be done whenever there exists a real sequence (a_n) satisfying $a_n \uparrow \mu(G)$, and which can be calculated in finite time. For any transitive graph G satisfying the hypothesis of Theorem 5.2, we may take $a_n = b_n^{1/n}$.*

Proof of Proposition 5.1. Assume G has graph height function $(h, \mathcal{H})$. If G is transitive, the claim is trivial, so we assume G is quasi-transitive but not transitive. For $v, w \in V$ with $w \notin \mathcal{H}v$, let $\nu(v, w)$ be a SAW from v to some $w' \in \mathcal{H}w$ with $h(v) < h(w')$, every vertex x of which, other than its endvertices, satisfies $h(v) < h(x) < h(w')$. We may assume that the length $l(v, w)$ of $\nu(v, w)$ satisfies $l(v, w) \leq r$ for all such pairs v, w.

Choose $x \in V$ such that $b_{n+r}(x) = b_{n+r}$, and assume $x \notin \mathcal{H}v$. Let $l = l(x,v)$. Since $b_m(x)$ is non-decreasing in m, and $l \le r$,

$$b_n(v) \le b_{n+l}(x) \le b_{n+r}(x) = b_{n+r},$$

and (5.7) follows by (5.5). The limit (5.6) follows by (5.2) and (5.4). □

6. Locality of connective constants

Let $\mathcal{G}$ be the class of infinite, connected, quasi-transitive, locally finite, simple, rooted graphs. For $G \in \mathcal{G}$, we label the root as $\mathbf{1} = \mathbf{1}_G$ and call it the *identity* or *origin* of G. The *ball* $S_k(v) = S_k(v,G)$, with centre v and radius k, is the subgraph of G induced by the set of its vertices within graph-distance k of v. For $G, G' \in \mathcal{G}$, we write $S_k(v,G) \simeq S_k(v',G')$ if there exists a graph-isomorphism from $S_k(v,G)$ to $S_k(v',G')$ that maps v to v'. We define the *similarity function* K on $\mathcal{G} \times \mathcal{G}$ by

$$K(G,G') = \max\bigl\{k : S_k(\mathbf{1}_G, G) \simeq S_k(\mathbf{1}_{G'}, G')\bigr\}, \qquad G, G' \in \mathcal{G},$$

and the distance-function $d(G,G') = 2^{-K(G,G')}$. Thus d defines a metric on $\mathcal{G}$ quotiented by graph-isomorphism, and this metric space was introduced by Babai [1]; see also [6, 13].

For integers $D \ge 1$ and $R \ge 0$, let $\mathcal{G}_{D,R}$ be the set of all $G \in \mathcal{G}$ which possess a graph height function h satisfying $d(h) \le D$ and $r(h, \mathcal{H}) \le R$. For a quasi-transitive graph G, we write $M = M(G) = |G/\mathrm{Aut}(G)|$ for the number of orbits under its automorphism group. The locality theorem for quasi-transitive graphs follows, with proof at the end of the section. The theorem may be regarded as a partial resolution of a question of Benjamini, [4, Conj. 2.3], which was posed independently of the work reported here.

Theorem 6.1 (Locality theorem for connective constants)**.** *Let $G \in \mathcal{G}$.*

(a) *Let $m \ge 1$. There exists a non-increasing real sequence $(\epsilon_k : k \ge 1)$, depending on G and m only, and satisfying $0 < \epsilon_k \downarrow 0$ as $k \to \infty$, such that, for $G' \in \mathcal{G}$ with $M(G') \le m$,*

$$\mu(G') \le \mu(G) + \epsilon_{K(G,G')}, \qquad \text{if } K(G,G') \ge \max\{M(G), m\}. \tag{6.1}$$

(b) *Let $D \ge 1$, $R \ge 0$, $b, m \ge 1$, and let $G' \in \mathcal{G}_{D,R}$ satisfy $\beta(G') \le b$ and $M(G') \le m$. There exists $B = B(D,R,b) \in (0,\infty)$ such that,*

$$\frac{\mu(G)}{f(K-L)} \le \beta(G') = \mu(G'), \qquad \text{if } K > L, \tag{6.2}$$

where $L = \max\{M(G), m\} - 1$ and $f(x) = \bigl(Bx^4 e^{B\sqrt{x}}\bigr)^{1/x}$.

(c) *Let $D \ge 1$ and $R \ge 0$, and let $G_n \in \mathcal{G}_{D,R}$ for $n \ge 1$. If $K(G, G_n) \to \infty$ as $n \to \infty$, then $\mu(G_n) \to \mu(G)$.*

When G is transitive, $M(G) = 1$ and one may take $R = 0$. The statement of the theorem is thus simpler when restricted to transitive graphs.

The following application of Theorem 6.1 is prompted in part by a result in percolation theory. Let $p_c(G)$ be the critical probability of either bond or site percolation on an infinite graph G, and let $\mathbb{Z}^d$ be the d-dimensional hypercubic lattice with $d \geq 3$, and $S_k = \mathbb{Z}^2 \times \{0, 1, \dots, k\}^{d-2}$. It was proved by Grimmett and Marstrand [28] that

$$p_c(S_k) \to p_c(\mathbb{Z}^d) \qquad \text{as } k \to \infty. \tag{6.3}$$

By Theorem 6.1(c) and the bridge construction of Hammersley and Welsh [32], the connective constants satisfy

$$\mu(\widehat{S}_k) \to \mu(\mathbb{Z}^d) \qquad \text{as } k \to \infty, \tag{6.4}$$

where $\widehat{S}_k$ is obtained from S_k by imposing periodic boundary conditions in its $d-2$ bounded dimensions. Such a limit may be extended as follows to more general situations. For simplicity, we consider the case of *transitive* graphs only.

Let $G \in \mathcal{G}$ and let Γ be a subgroup of $\mathrm{Aut}(G)$ that acts transitively. For $n \geq 1$, let $\alpha_n \in \Gamma$ and let $\mathcal{A}_n$ be the normal subgroup of Γ generated by α_n. The group $\mathcal{A}_n$ gives rise to a quotient graph $G_n := G/\mathcal{A}_n$ (see Section 3). Since $\mathcal{A}_n$ is a normal subgroup of Γ, Γ acts on $G/\mathcal{A}_n$ (see [24, Remark 3.5]), whence G_n is transitive.

Theorem 6.2. *Let $D \geq 1$. Assume $G_n = G/\mathcal{A}_n \in \mathcal{G}_{D,0}$ for all n, and further that $\delta_n := d_G\big(\mathbf{1}, \mathcal{A}_n\mathbf{1} \setminus \{\mathbf{1}\}\big)$ satisfies $\delta_n \to \infty$ as $n \to \infty$. Then $\mu(G_n) \to \mu(G)$ as $n \to \infty$.*

Proof. The quotient graph G_n is obtained from G by identifying any two vertices $v \neq w$ with $w = \alpha v$ and $\alpha \in \mathcal{A}_n$. For such v, w, we have $d_G(v, w) \geq \delta_n$. Therefore, $K(G, G_n) \geq \frac{1}{2}\delta_n - 1$, and the result follows by Theorem 6.1(c). □

Example 6.3. *Let G be the hypercubic lattice $\mathbb{Z}^d$ with $d \geq 2$, and let Γ be the group of its translations. Choose $v = (v_1, v_2, \dots, v_d) \in \mathbb{Z}^d$ with $v \neq \mathbf{0}$, and let $\alpha_v \in \Gamma$ be the translation $w \mapsto w + v$. Let $\mathcal{P}_v$ be the set of integer vectors perpendicular to v, and choose $p_v = (p_1, p_2, \dots, p_d) \in \mathcal{P}_v$ in such a way that $\|p\| := \max_i |p_i|$ is a minimum. For $z \in \mathbb{Z}^d$, let $h_v(z) = z \cdot p_v$, so that (h_v, Γ) is a graph height function with $d(h_v) \leq \|p\|$ and $r(h_v, \Gamma) = 0$.*

Let $\mathcal{A}_v$ be the subgroup of Γ generated by α_v (which is invariably normal). It may be seen that $(h_v, \Gamma/\mathcal{A}_v)$ is a height function for $G/\mathcal{A}_v$ with d and r as above. In the notation of Theorem 6.2, we have that $\mu(G/\mathcal{A}_{v_m}) \to \mu(G)$ as $m \to \infty$, so long as the sequence (v_m) satisfies $|v_m| \to \infty$ and $\limsup_{m\to\infty} d(h_{v_m}) < \infty$. This may be regarded as a general version of the limit (6.4).

Proof of Theorem 6.1. Let $G \in \mathcal{G}$. Since the quotient graph $\overline{G} := G/\mathrm{Aut}(G)$ is connected, G has some subtree T containing $\mathbf{1}$ and comprising exactly one member of each orbit under $\mathrm{Aut}(G)$. Therefore,

$$\sigma_n = \sigma_n(v) \text{ for some } v \in V_T, \tag{6.5}$$

where V_T is the vertex-set of T.

(a) Let $G \in \mathcal{G}$ and $m \geq 1$, and write

$$L = \max\{M(G), m\} - 1. \tag{6.6}$$

By (2.3), there exist $\eta_k = \eta_k(G)$ such that $0 < \eta_k \downarrow 0$ as $k \to \infty$ and

$$\mu(G)^n \leq \sigma_n(G) \leq (\mu(G) + \eta_k)^n, \qquad n \geq k. \tag{6.7}$$

Let $G' \in \mathcal{G}$ be such that $M(G') \leq m$, and write $K = K(G, G')$. Since $V_T \subseteq S_L(\mathbf{1}_G, G)$ and $S_K(\mathbf{1}_G, G) \simeq S_K(\mathbf{1}_{G'}, G')$,

$$\sigma_{K-L}(G') = \sigma_{K-L}(G) \qquad \text{if } K \geq L. \tag{6.8}$$

Assume $K > L$. By (6.7)–(6.8) and (2.2),

$$\begin{aligned}\sigma_{(K-L)r}(G') &\leq \sigma_{K-L}(G')^r \\ &= \sigma_{K-L}(G)^r \leq (\mu(G) + \eta_{K-L})^{(K-L)r}, \qquad r \geq 1.\end{aligned}$$

Take $(K-L)r^{\text{th}}$ roots and let $r \to \infty$, to obtain that $\mu(G') \leq \mu(G) + \eta_{K-L}$, and the claim follows with $\epsilon_k = \eta_{k-L}$.

(b) Let D, R, b, m, G' satisfy the given conditions, and let L be given by (6.6). By (6.7)–(6.8) and the forthcoming Proposition 9.3, there exists $B = B(D, R, b) > 0$ such that, for $K > L$,

$$\begin{aligned}\beta(G')^{K-L} &\geq \frac{\sigma_{K-L}(G')}{B(K-L)^4 e^{B\sqrt{K-L}}} \\ &= \frac{\sigma_{K-L}(G)}{f(K-L)^{K-L}} \geq \frac{\mu(G)^{K-L}}{f(K-L)^{K-L}}.\end{aligned}$$

Therefore,

$$\beta(G') \geq \frac{\mu(G)}{f(K-L)}.$$

By Theorem 5.2, $\mu(G') = \beta(G')$, and (6.2) is proved.

(c) Since $G_n \in \mathcal{G}_{D,R}$ and $K(G, G_n) \to \infty$, $M(G_n)$ and $\beta(G_n)$ are uniformly bounded. The claim is now immediate by parts (a) and (b). $\square$

7. Proof of Theorem 4.1

Recall the quotient graph $\overline{G} = (\overline{V}, \overline{E})$ of Section 4. Let $\vec{G} = (\overline{V}, \vec{E})$ be the directed graph obtained from $\overline{G}$ by replacing each edge e by two edges with opposite orientations, denoted $\vec{e}$ and $-\vec{e}$. A cycle D of $\overline{G}$ is a closed walk $D = (\overline{w}_0, \overline{e}_0, \overline{w}_1, \dots, \overline{w}_n = \overline{w}_0)$ such that $\overline{e}_i$ has endpoints $\overline{w}_i$, $\overline{w}_{i+1}$, and the $\overline{w}_i$ and $\overline{e}_i$ are distinct. Such D gives rise to a *(directed) cycle* $\vec{D}$ of $\vec{G}$ obtained by replacing each edge by the corresponding directed edge of $\vec{G}$, having the same endpoints and oriented in the direction of the cycle.

We propose to construct a graph height function h by (i) finding a 'difference' function $\overline{\delta} : \vec{E} \to \mathbb{Q}$ which sums to 0 around any cycle of $\vec{G}$ that lifts to a cycle of G, (ii) exporting $\overline{\delta}$ via the action of $\mathcal{H}$ to a function δ on the edges of G, endowed with orientations, and (iii) summing δ to obtain a height function h.

Assume the conditions of Theorem 4.1. Let $\overline{\mathcal{B}}$ be the set of projections of $\mathcal{B}$ onto $\overline{G}$, and let $\mathcal{C}(\overline{\mathcal{B}})$ be the subspace of $\mathcal{C}(\overline{G})$ generated by $\overline{\mathcal{B}}$. Standard facts about undirected and directed cycle bases may be found in [20, 34, 36, 37].

Lemma 7.1. *Let $\rho := \dim(\mathcal{C}(\overline{\mathcal{B}}))$ and $\Delta := \dim(\mathcal{C}(\overline{G}))$.*

(a) *The cycle space $\mathcal{C}(\overline{G})$ has a basis $\{C_1, \dots, C_\rho, C_{\rho+1}, \dots, C_\Delta\}$ of cycles such that $\{C_1, C_2, \dots, C_\rho\}$ is a basis of the subspace $\mathcal{C}(\overline{\mathcal{B}})$.*

(b) *We have that $\rho < \Delta$.*

Proof. (a) First, a reminder. Let $H = (W, F)$ be an undirected graph, and let T be a spanning forest of H with exactly one component in each component of H. The set of cycles formed by adding one further edge to T forms a basis for the cycle space $\mathcal{C}(H)$, denoted $B(H, T)$. In particular, the dimension of the space is the number of edges not belonging to T, so that $\dim(\mathcal{C}(H)) = |F| - |T|$, where $|T|$ is the number of edges of T.

Let $\overline{E}'$ ($\subseteq \overline{E}$) be the union of the edges of $\overline{\mathcal{B}}$. Let T' (respectively, T) be a spanning forest of $\overline{G}' := (\overline{V}, \overline{E}')$ (respectively, $\overline{G}$) such that (i) T' (respectively, T) has exactly one component in each component of $\overline{G}'$ (respectively, $\overline{G}$), and (ii) T' is a subgraph of T. The cycle space $\mathcal{C}(\overline{G})$ has dimension $\Delta := |\overline{E}| - |T|$, and $\mathcal{C}(\overline{\mathcal{B}})$ has dimension $\rho := |\overline{E}'| - |T'|$. Note that $B(\overline{G}', T') \subseteq B(\overline{G}, T)$, and (a) follows.

(b) Write $\mathcal{B} = \{B_1, B_2, \dots, B_I\}$, and let $\overline{\sigma} \in \mathcal{C}(\overline{\mathcal{B}})$. Then $\overline{\sigma}$ may be expressed in the form $\overline{\sigma} = \sum_i \mu_i \overline{B}_i$, where $\mu_i \in \{0, 1\}$ and addition is modulo 2. There are many lifts of $\overline{\sigma}$, including $\sigma := \sum_i \mu_i \gamma_i(B_i)$ with $\gamma_i \in \mathcal{H}$. We choose the γ_i such that $\sigma \in \mathcal{C}(G)$ is non-empty. In particular, every $\overline{\sigma} \in \mathcal{C}(\overline{\mathcal{B}})$ has a lift that is not a SAW.

By assumption (c), there exists a cycle C_Δ in $\mathcal{C}(\overline{G})$ all of whose lifts are SAWs. It follows that $\mathcal{C}(\overline{\mathcal{B}})$ is a strict subspace of $\mathcal{C}(\vec{G})$, whence $\rho < \Delta$. $\square$

We remind the reader of the directed cycle space $\mathcal{C}(\vec{G})$ over the rationals $\mathbb{Q}$. For a directed cycle $\vec{D}$ of $\vec{G}$, we define the *incidence vector* $\delta_{\vec{D}} = (\delta_{\vec{e}} : \vec{e} \in \vec{E})$ where $\delta_{\vec{e}} = 1$ if $\vec{e} \in \vec{D}$, and $\delta_{\vec{e}} = 0$ otherwise. The *directed cycle space* $\mathcal{C}(\vec{G})$ is the subspace of $\mathbb{Q}^{\vec{E}}$ spanned by the vectors $\delta_{\vec{D}}$ as $\vec{D}$ ranges over the directed cycles of $\vec{G}$.

Remark 7.2. *The incidence vectors of the basis $\{C_1, C_2, \dots, C_\Delta\}$ of $\mathcal{C}(\overline{G})$ together with the set of length-2 cycles $\{C_e := (\vec{e}, -\vec{e}) : e \in \overline{E}\}$ of $\vec{G}$ form a basis of the directed cycle space $\mathcal{C}(\vec{G})$, denoted $\mathbb{B} = \{C_x\}$. This basis is weakly fundamental and is, therefore, an integral cycle basis also. See, for example,* [7, Thm 3.4], [36, Thm 4], [37, Lemma 4], *and* [20, Lemma 7]. *The incidence vector of a directed closed walk W of $\vec{G}$ is expressed in terms of the basis $\mathbb{B}$ in the form*

$$W = \sum_x \lambda_x C_x, \qquad \lambda_x \in \mathbb{Z}. \tag{7.1}$$

Corresponding facts hold for the undirected cycle space $\mathcal{C}(\overline{\mathcal{B}})$ over $\mathbb{Z}_2$, and the corresponding directed space space $\mathcal{C}(\vec{\mathcal{B}})$ over $\mathbb{Q}$.

Let $\overline{\delta} : \overline{E} \to \mathbb{Q}$, and extend the domain of $\overline{\delta}$ to $\vec{E}$ in the following manner. To each $e \in \overline{E}$, we assign an arbitrary orientation. For $\vec{e} \in \vec{E}$, we set

$$\overline{\delta}(\vec{e}) := \begin{cases} \overline{\delta}(e) & \text{if } e \text{ is oriented in the direction of } \vec{e}, \\ -\overline{\delta}(e) & \text{otherwise.} \end{cases} \tag{7.2}$$

Thus, $\overline{\delta}$ sums to 0 around each 2-cycle of $\vec{G}$ of the form $(\vec{e}, -\vec{e})$ with $e \in \overline{E}$.

We shall require $\overline{\delta}$ to have certain properties as follows. Consider the following system of linear equations over $\mathbb{Q}$ in the values $\overline{\delta}(e)$, $e \in \overline{E}$:

$$\sum_{\vec{e} \in C_i} \overline{\delta}(\vec{e}) = 0, \qquad 1 \le i \le \Delta - 1, \tag{7.3}$$

$$\sum_{\vec{e} \in C_\Delta} \overline{\delta}(\vec{e}) = 1. \tag{7.4}$$

By Remark 7.2, the incidence vectors of the C_i, regarded as directed cycles, are independent over $\mathbb{Q}$. Therefore, the coefficient matrix and augmented matrix of the above system have equal rank Δ, whence there exists a rational-valued solution. Let $\overline{\delta}$ be such a solution.

We now lift $\overline{\delta}$ from $\overline{G}$ to G through the action of $\mathcal{H}$. Let $\overleftrightarrow{G} = (V, \overleftrightarrow{E})$ be the directed graph derived from G by replacing each $e \in E$ by two edges oriented in opposite directions. For $\vec{f} = [u, v\rangle \in \overleftrightarrow{E}$, let $\delta(\vec{f}) = \delta(\vec{e})$ where $\vec{e} = [\overline{u}, \overline{v}\rangle \in \vec{E}$ is the projection onto $\vec{G}$ of $\vec{f}$. Note that $\mathcal{H}$ acts on $\overleftrightarrow{E}$, and δ is invariant under this action.

Lemma 7.3. *For a directed closed walk W on G,*

$$\sum_{\vec{e}\in W} \delta(\vec{e}) = 0. \tag{7.5}$$

Proof. We make use of Remark 7.2 applied to $\mathcal{C}(\overline{\mathcal{B}})$. Write $\mathcal{B} = \{B_1, B_2, \dots, B_I\}$ and let D be a cycle of G. Since $D \in \mathcal{C}(G)$, D may be expressed in the form $D = \sum_i \mu_i \gamma_i(B_i)$, for $\mu_i \in \{0,1\}$ and $\gamma_i \in \mathcal{H}$, where addition is modulo 2. Thus the projection of D is $\overline{D} = \sum_i \mu_i \overline{B}_i$, which satisfies $\overline{D} \in \mathcal{C}(\overline{\mathcal{B}})$. When considered as a directed cycle, $\overline{D}$ lies in the directed cycle space $\mathcal{C}(\vec{\overline{\mathcal{B}}})$, which has basis

$$\mathbb{B}' := \{C_1, C_2, \dots, C_\rho\} \cup \Big\{ C_e : e \in \bigcup_i \overline{B}_i \Big\}.$$

Since $\mathbb{B}'$ is weakly fundamental, it is an integral cycle basis, whence $\overline{D}$ (considered as a directed closed walk of $\vec{G}$) can be written in the form

$$\overline{D} = \sum_{C_x \in \mathbb{B}'} \lambda_x C_x, \qquad \lambda_x \in \mathbb{Z}.$$

By (7.2)–(7.3),

$$\sum_{\vec{e}\in D} \delta(\vec{e}) = \sum_{\vec{e}\in \overline{D}} \overline{\delta}(\vec{e}) = 0. \tag{7.6}$$

Let W be a directed closed walk of G. By progressive cycle removal, the sum of the $\delta(\vec{e})$ around W may be expressed in the form

$$\sum_{\vec{e}\in W} \delta(\vec{e}) = \sum_{j=1}^{J} \sum_{\vec{e}\in \gamma_j D_j} \delta(\vec{e}),$$

for suitable directed cycles D_j of G and elements $\gamma_j \in \mathcal{H}$. By the $\mathcal{H}$-invariance of δ and (7.6), this equals 0 as required. □

Let $h' : V \to \mathbb{Q}$ be given as follows. Let $h'(\mathbf{1}) = 0$. For $v \in V$, find a directed path l_v from $\mathbf{1}$ and v, and define

$$h'(v) = \sum_{\vec{e}\in l_v} \delta(\vec{e}). \tag{7.7}$$

By Lemma 7.3, h' is well defined in the sense that $h'(v)$ is independent of the choice of l_v.

The function h' takes rational values. Since δ takes only finitely many values, there exists $m \in \mathbb{N}$ such that $h := mh'$ takes integer values. The resulting h may fail to be a graph height function only in that it may fail to satisfy condition (c) of

Definition 3.1. We shall explain in the following construction how to find a solution $\overline{\delta}$ of (7.3)–(7.4) such that the ensuing δ satisfies (c).

We show next how such $\overline{\delta} : \overline{E} \to \mathbb{Q}$ may be found via an explicit iterative construction, illustrated in Example 7.5. The values $\overline{\delta}(e)$ will be revealed one by one. At each stage, edges $\vec{e}, -\vec{e} \in \vec{E}$ (or, equivalently, the edge $e \in \overline{E}$) are said to be *explored* if $\overline{\delta}(e)$ is known. The set of explored edges will increase as the stages progress, until it becomes the entire edge-set of $\vec{G}$.

The cycle C_Δ plays a special role, and we abbreviate it to C. Let $\overline{c}$ be a vertex of C. Since $\mathcal{H}$ is a normal subgroup of Γ, and Γ acts transitively on G, Γ acts transitively on $\overline{G}$ also. For $\overline{v} \in \overline{V}$, we pick $\gamma \in \Gamma$ such that $\gamma(\overline{c}) = \overline{v}$. Let $C(\overline{v}) = \gamma C$, so that $C(\overline{v})$ may be viewed as a directed cycle of $\vec{G}$ through $\overline{v}$.

Let U be the union of the edge-sets of cycles $\{C(\overline{v}) : \overline{v} \in \overline{V}\}$ viewed as cycles of $\overline{G}$, and let $S_1, S_2, \dots, S_k$ be the vertex-sets of the connected components of $(\overline{V}, U)$. If $k = 1$, we may proceed directly to the forthcoming Stage 1. Otherwise, since U touches every member of $\overline{V}$, for each i there exists $j \neq i$ and $\overline{e} = \langle \overline{v}_i, \overline{v}_j \rangle \in \overline{E}$ such that $\overline{v}_i \in S_i$, $\overline{v}_j \in S_j$. Starting from S_1, we find an edge $\overline{e}_1 \notin U$ that connects S_1 and some S_{i_1} with $i_1 \neq 1$, then an edge $\overline{e}_2 \notin U$ that connects $S_1 \cup S_{i_1}$ and some S_{i_2}, and so on. This results in a set $F = \{\overline{e}_1, \overline{e}_2, \dots, \overline{e}_{k-1}\}$ such that $(\overline{V}, U \cup F)$ is a connected subgraph of $\overline{G}$.

Let S_1 comprise the cycles $C(\overline{v}_i)$ for $1 \leq i \leq d$. We may assume without loss of generality that $\overline{v}_1 = \overline{c}$ and, for $2 \leq i \leq d$, $C(\overline{v}_i)$ and $\bigcup_{1 \leq j < i} C(\overline{v}_j)$ have at least one vertex in common. We explain next how to define $\overline{\delta}$ on the edges of S_1.

Stage 1. We consider first the edges in C. For each directed edge $\vec{e} \in C$, we set $\delta(\vec{e}) = -\delta(-\vec{e}) = 1/|C|$.

Stage 2. We describe how the domain of $\overline{\delta}$ may be extended in stages. At each stage, we shall consider a path of unexplored edges, and assign $\overline{\delta}$-values according to a given set of rules. The argument proceeds by an induction of which the hypothesis is as follows.

For any set H of currently explored edges of $\vec{G}$, let

$$\Sigma(H) = \sum_{\vec{e} \in H} \overline{\delta}(\vec{e}).$$

By (7.1), for any directed closed walk W of $\vec{G}$,

$$\Sigma(W) = \sum_x \lambda_x \Sigma(C_x). \tag{7.8}$$

Hypothesis 7.4. *Let F be the set of currently explored edges of $\vec{G}$, and let P be the set of their endpoints.*

(a) *For every distinct pair* $\overline{a}, \overline{b} \in P$*, there exists a directed SAW* R *of* (P, F) *from* $\overline{a}$ *to* $\overline{b}$ *such that* $\Sigma(R) \notin \mathbb{Z}$.

(b) *For any directed, closed walk* W *of the graph* (P, F),

$$\Sigma(W) = \lambda_\Delta, \tag{7.9}$$

where $W = \sum_x \lambda_x C_x$ *is the unique representation of* W *in terms of the basis* $\mathbb{B} = \{C_x\}$ *of the cycle space* $\mathcal{C}(\vec{G})$*, and* $\lambda_\Delta = \lambda_\Delta(W)$ *is the coefficient of the special cycle* C_Δ.

Note that the hypothesis is valid at the end of Stage 1, at which point we may write $F = \{e \in \overline{E} : \vec{e} \in C\}$ for the updated set of explored edges. Next consider $C(\overline{v}_2)$, which, by construction, intersects C. The cycle $C(\overline{v}_2)$ is cut by C into edge-disjoint segments, endpoints of which lie in C. Let $(\overline{a} \leftrightarrow \overline{b})$ denote an undirected sub-path of $C(\overline{v}_2)$ containing one or more edges, and with endvertices $\overline{a}, \overline{b} \in C$ such that no vertex other than these endvertices lies in C. Let $\pi = (\overline{a} \to \overline{b})$ denote a *directed* segment along $C(\overline{v}_2)$ such that there are no vertices of C between $\overline{a}$ and $\overline{b}$. There are two cases depending on whether or not $\overline{a} = \overline{b}$.

Case I. Suppose first that $\overline{a} \neq \overline{b}$. By the induction hypothesis, we may find a subset R of F that forms a directed SAW from $\overline{b}$ and $\overline{a}$ and satisfies

$$\Sigma(R) \notin \mathbb{Z}. \tag{7.10}$$

Then R, combined with π, forms a directed closed walk of $\vec{G}$, denoted by W. By (7.1), W has a unique representation in the form

$$W = \sum_x \lambda_x C_x, \tag{7.11}$$

with $\lambda_x = \lambda_x(W) \in \mathbb{Z}$. We assign $\overline{\delta}$-values to the edges of π in such a way that $\Sigma(W) = \lambda_\Delta$, that is, such that, by (7.10),

$$\Sigma(\pi) = \Sigma(W) - \Sigma(R) = \lambda_\Delta - \Sigma(R) \neq 0. \tag{7.12}$$

If π contains a unique edge $\vec{e}$, (7.12) determines the value $\overline{\delta}(\vec{e})$. Otherwise, we distribute the sum $\Sigma(\pi)$ between the edges of π in such a way that the $\overline{\delta}(\vec{e})$, $\vec{e} \in \pi$, are non-zero and have the same sign. There is an additional condition, as follows. Let F' be the updated set of explored edges, and P' their endvertices. We require that, for any distinct pair $\overline{x}, \overline{y} \in P'$, there exists a directed SAW R of F' from $\overline{x}$ to $\overline{y}$ with $\Sigma(R) \notin \mathbb{Z}$. This may be achieved by small variations in an equidistribution of $\Sigma(\pi)$ around π.

It may be checked (see the general induction step below) that Hypothesis 7.4(b) remains true after completion of the above stage.

Case II. Assume now that $\overline{a} = \overline{b}$. This can happen only if $\overline{a}$ is the unique common vertex of C and $C(\overline{v}_2)$. In this case, we set $\Sigma(C(\overline{v}_2)) = 1$, and we distribute this sum between the edges of $C(\overline{v}_2)$ in such a way that the values are non-zero with the same sign, and furthermore Hypothesis 7.4(a) is preserved. By (7.8), Hypothesis (b) is satisfied by the ensuing set of explored edges.

We iterate the above process, at each stage exploring another segment of $C(\overline{v}_2)$, until no such segment remains.

The general induction step is as follows. Assume Hypothesis 7.4 is valid so far, and we are required to assign $\overline{\delta}$-values to the edges belonging to a (currently unexplored) directed SAW $\pi = (\overline{a} \to \overline{b})$ satisfying the condition that $\overline{a}$ and $\overline{b}$ are incident to one or more currently explored edges, but no other vertex of π has this property.

Case I′. Assume first that $\overline{a} \neq \overline{b}$. Let R be an explored SAW on $\vec{G}$ which is directed from $\overline{b}$ to $\overline{a}$ and satisfies $\Sigma(R) \notin \mathbb{Z}$. Then R, combined with π, forms a directed closed walk W', which has a unique representation in the form (7.11) with $\lambda_i \in \mathbb{Z}$. We assign values to the $\overline{\delta}(\vec{e})$, $\vec{e} \in \pi$, in the same manner as above, such that they are non-zero with the same sign, and such that Hypothesis 7.4(a) is preserved.

We show next that Hypothesis 7.4(b) remains true after this has been done. Let W be some explored, directed, closed walk. If W uses no edge of $\pi := (\overline{a} \to \overline{b})$ in either direction, then (7.9) holds as before. Assume W uses one or more edges of π in one or the other direction. Since no internal vertex of π is incident to a previously explored edge, each excursion of W into π enters at either $\overline{a}$ or $\overline{b}$, and leaves at either $\overline{a}$ or $\overline{b}$. If it leaves at the same place at it enters, then the excursion contributes 0 to $\Sigma(W)$, since each of its edges inside π is traversed equally often in the two directions. In the other situation, it contributes $\pm\Sigma(\pi)$. Suppose W contains f 'forward' excursions that enter at $\overline{a}$ and leave at $\overline{b}$, and g 'backward' excursions that enter at $\overline{b}$ and leave at $\overline{a}$, and let $q = f - g$.

Let W'' be the walk that follows W but subject to the differences that: (i) whenever W wishes to traverse π forwards (respectively, backwards) it is redirected along R backwards (respectively, forwards), and (ii) excursions of W into π that enter and leave at the same vertex are removed. Then W'' is a closed walk of previously explored edges which uses no edges of π. By the hypothesis, $\Sigma(W'') = \lambda_\Delta(W'')$, so that, by (7.8),

$$\begin{aligned}\Sigma(W) &= \Sigma(W'') + q\Sigma(W') \\ &= \lambda_\Delta(W'') + q\lambda_\Delta(W') = \lambda_\Delta(W),\end{aligned}$$

as required. It follows that the values assigned to the $\vec{e} \in \pi$ do not depend on the choice of R, above, and furthermore that (7.9) holds for all currently explored walks.

Case II′. When $\overline{a} = \overline{b}$, we proceed as in Case II above.

Stage 3. Having assigned $\overline{\delta}$-values to edges in C and $C(\overline{v}_2)$, we continue by induction to the remaining cycles in S_1.

Stage 4. We set $\overline{\delta}(\vec{e}_1) = 0$ where $\vec{e}_1 = [\overline{v}_1, \overline{v}_2\rangle$, and we consider the exploration of S_2, beginning with $C(\overline{v}_2)$. We now assign $\overline{\delta}$-values to the edges of $C(\overline{v}_2)$ by the recipe of Stage 1 above, subject to the further condition that

(C) for $\overline{x} \in S_1$ and $\overline{y} \in P \cap S_2$ with $\overline{y} \neq \overline{v}_2$, there exists a directed SAW R of currently explored edges from $\overline{x}$ to $\overline{y}$ such that $\Sigma(R) \notin \mathbb{Z}$,

where, as usual, P denotes the set of endpoints of currently explored edges. The cycles $C(\overline{v}_k)$ of S_2 are then explored in sequence, following the previous process of Stage 2, and subject to condition (C). When S_2 is complete, we declare $\overline{\delta}(\vec{e}_2) = 0$ and continue with S_3, and so on.

Stage 5. Once Stage 4 is complete, every vertex of $\vec{G}$ is incident to some explored edge. There may however remain unexplored edges, and any such $\vec{e} = [\overline{a}, \overline{b}\rangle \in \vec{E}$ is allocated a $\overline{\delta}$-value by the recipe of Stage 2. Let R be an explored SAW from $\overline{b}$ to $\overline{a}$, and let W be the walk obtained by combining $\vec{e}$ with R. We set $\overline{\delta}(\vec{e}) = \Sigma(W) - \Sigma(R)$, where $\Sigma(W)$ satisfies (7.9). It may happen that $\overline{\delta}(\vec{e}) = 0$, and it may be checked as above that Hypothesis 7.4(b) is preserved (condition (a) is unimportant at this stage).

The construction of $\overline{\delta}$ is complete. A final note. By inspection of the construction, for $\overline{v} \in \overline{V}$, there exist $\overline{u}, \overline{w} \in \partial\overline{v}$ such that $\overline{\delta}([\overline{u}, \overline{v}\rangle), \overline{\delta}([\overline{v}, \overline{w}\rangle) > 0$. Therefore, the ensuing h' of (7.7) satisfies Definition 3.1(c).

Example 7.5. *The above construction of a height function is illustrated by a simple example. Take $G = \mathbb{Z}^2$, let Γ be the group of translations, and let $\mathcal{H} \leq \Gamma$ be the subgroup of translations generated by the horizontal and vertical shift by three units. The undirected quotient graph $\overline{G}$ has vertex-set $\{0,1,2\}^2$ with toroidal edges, as in Figure 7.1. We take each $C(\overline{v})$ to be the upwards oriented 3-cycle through $\overline{v}$.*

Finally, (4.1) holds for suitable R_N, D_N since in the above construction, there are (up to isomorphism) only finitely many connected graphs $\overline{G}$ with N edges.

8. Proof of Theorem 4.2

The reader is referred to [26] for a more extensive account of graph height functions and Cayley graphs. The identity element of a group is denoted $\mathbf{1}$.

Let $\Gamma = (S_0, R_0)$ be an infinite, residually finite group with finite generator set S_0 satisfying $\mathbf{1} \notin S_0$ and $S_0 = S_0^{-1}$, and finite relator set R_0. Let G_0 be the Cayley graph of Γ with respect to S_0. We adapt G_0, if necessary, as follows:

(a) remove all loops,
(b) replace any collection of parallel edges by a single edge.

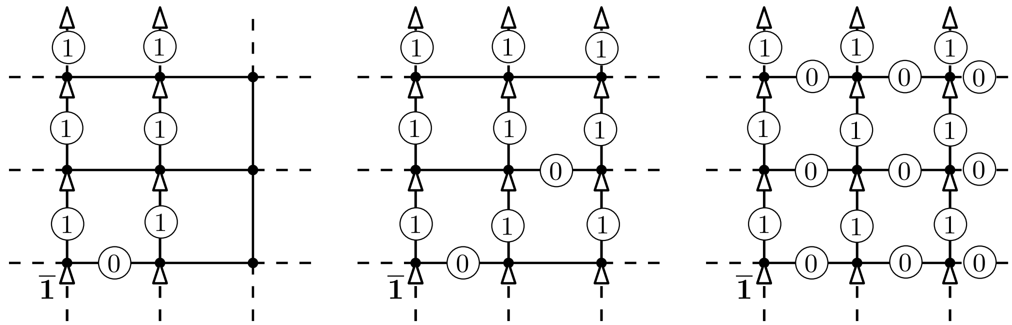

FIGURE 7.1. An illustration of the construction of the function $3\overline{\delta}$ on the quotient graph $\vec{G}$ of Example 7.5. Directions and magnitudes are assigned to the edges in an iterative manner, in such a way that the sum of the values around any cycle that lifts to a cycle of G is 0. In the leftmost figure, the values have been determined on the cycles $C(\overline{\mathbf{1}})$ and $C(1,0)$, where $\overline{\mathbf{1}} = (0,0)$. In the central figure, the values on $C(2,1)$ have been added, and the missing values are provided in the rightmost figure.

Let G be the resulting *simple* graph. We change the set of generators, accordingly, from S_0 to S, such that G is a Cayley graph of Γ, with respect to S. More precisely, S is obtained from S_0 as follows:

(c) if $s \in S_0$ corresponds to a loop of G, we remove s from S,
(d) for any subset of generators $T = \{s_1, \dots, s_k\} \subseteq S$, such that each s_i corresponds to an edge with the same two given endpoints, we remove $T \setminus \{s_1, s_1^{-1}\}$ from S.

Let R be a set of relators for Γ with respect to the new generator set S. We shall think of R as words in the alphabet S of letters. We further assume that

(e) R is minimal in that no relation follows from the others, and each $r \in R$ corresponds to a cycle of G through the identity $\mathbf{1}$ with length at least 3.

It may be seen that assumptions (a)–(e) do not affect the occurrence (or not) of a graph height function.

If $R = \varnothing$, then Γ is a free group. Thus, G is a regular tree, and regular trees have graph height functions (see Section 3). Suppose henceforth that $R \neq \varnothing$. Let $r = s_1 s_2 \cdots s_k \in R$ be a word with $s_i \in S$ and $k = k(r) \geq 3$. For $1 \leq i \leq k$, let $C_{r,i}$ be the cycle of G through $\mathbf{1}$ obtained by following the edges corresponding to the generators $s_i, s_{i+1}, \dots, s_k, s_1, s_2, \dots, s_{i-1}$. Let $C_R = \{C_{r,i} : 1 \leq i \leq k,\ r \in R\}$, and let V_R be the (finite) union of the vertex-sets of cycles in C_R.

Since Γ is assumed residually finite, for $v \in V_R$, $v \neq \mathbf{1}$, there exists a finite-index normal subgroup $\mathcal{N}_v \trianglelefteq \Gamma$ such that $v \notin \mathcal{N}_v$. Then $\mathcal{N} = \bigcap_{v \in V_R} \mathcal{N}_v$ is a finite-index normal subgroup of Γ, and $V_R \cap \mathcal{N} = \{\mathbf{1}\}$. Let $\overline{G} = G/\mathcal{N}$ be the quotient graph obtained by the procedure described before Theorem 4.1.

Let $v \in \Gamma$ and $r = s_1 s_2 \cdots s_k \in R$ with $s_i \in S$ and $k \geq 3$, and let $\sigma_{v,r}$ be the cycle $\sigma_{v,r} = vC_{r,1}$. We claim that

$$\text{for } v \in \Gamma \text{ and } r \in R,\ \sigma_{v,r} \text{ projects to a cycle in } \overline{G}. \tag{8.1}$$

Suppose on the contrary that $\sigma_{v,r}$ projects to a closed walk but not a cycle of $\overline{G}$. There exist $0 \leq p < q < k$, such that $\left[v \prod_{i=1}^{p} s_i\right]^{-1} \left[v \prod_{j=1}^{q} s_j\right] \in \mathcal{N}$, whence $t := \prod_{i=p+1}^{q} s_i$ satisfies $t \in \mathcal{N}$. This contradicts the facts that $t \in V_R$ and $t \neq \mathbf{1}$. Statement (8.1) follows.

Since $\mathcal{N} \trianglelefteq \Gamma$ has finite index, there exists $W \subseteq \Gamma$ satisfying $|W| < \infty$ such that: for $v \in \Gamma$ there exists $w \in W$ and $\alpha \in \mathcal{N}$ such that $\alpha v = w$. The cycle space $\mathcal{C}(G)$ is generated by the finite family $\{\sigma_{v,r} : v \in W,\ r \in R\}$ under the action of $\mathcal{N}$.

Let $u \in \mathcal{N}$, $u \neq \mathbf{1}$, be such that $d_G(\mathbf{1}, u)$ is minimal among all non-identity vertices in $\mathcal{N}$. Let l_u be a shortest path of G from $\mathbf{1}$ and u, and write $|l_u|$ for the number of edges of l_u. As in [24, Sect. 3.4],

$$d_G(x, y) \geq |l_u| \quad \text{if} \quad x\mathcal{N} = y\mathcal{N},\ x \neq y. \tag{8.2}$$

We claim that the projection $\bar{l}_u$ satisfies

$$\bar{l}_u \text{ is a cycle of } \overline{G} \text{ of which every lift is a SAW of } G. \tag{8.3}$$

There are three cases to consider in proving (8.3).

(a) If $|l_u| = 1$, then $\bar{l}_u$ is a loop and every lift is a 1-step SAW of G.

(b) Suppose $|l_u| = 2$, so that $l_u = (\mathbf{1}, s, st = u)$ for some $s, t \in S$. Then l_u traverses the directed edges $e_1 = [\mathbf{1}, s\rangle$, $e_2 = [s, st\rangle$. Its projection is $\bar{l}_u = (\overline{\mathbf{1}}, \bar{s}, \overline{\mathbf{1}})$, where the two edges $\bar{e}_1 = [\overline{\mathbf{1}}, \bar{s}\rangle$, $\bar{e}_2 = [\bar{s}, \overline{\mathbf{1}}\rangle$ are determined by the endvertices of the e_i. We claim that $\bar{e}_1$, $\bar{e}_2$, when undirected, are different edges of the multigraph $\overline{G}$. By the definition of $\overline{G}$ in Section 4, they are the same edges if and only if there exists $\alpha \in \mathcal{N}$ such that $\alpha(\mathbf{1}) = st$ and $\alpha(s) = s$. Since α acts by left-multiplication, this implies $st = \mathbf{1}$. This is a contradiction, whence the claim follows. We have proved that $\bar{l}_u$ is a 2-cycle of $\overline{G}$, and moreover that every lift of $\bar{l}_u$ is a SAW of G.

(c) Assume $|l_u| \geq 3$, and $\bar{l}_u$ lifts to the path l_u and also to some closed walk W through $\mathbf{1}$. By following l_u until the first vertex at which W diverges from l_u, we obtain distinct vertices $x \in W$, $y \in l_u$ lying in the same coset of $\mathcal{N}$ and such that $d_G(x, y) = 2$. This contradicts (8.2), and therefore every lift of $\bar{l}_u$ is a SAW of G.

By (8.2)–(8.3) and Theorem 4.1, G admits a graph height function.

Example 8.1. *The group* $\mathrm{SL}_3(\mathbb{Z})$ *is an infinite, finitely-presented, residually finite, Property (T), and hence non-amenable group. By Theorem 4.2, any locally finite Cayley graph of* $\mathrm{SL}_3(\mathbb{Z})$ *admits a graph height function.*

A graph height function may also be constructed by the recipe at the end of the proof of Theorem 4.1. Following [11], for $1 \le i \ne j \le 3$*, let* T_{ij} *be the matrix with* 1 *on the diagonal and in the* (i,j)*th position, and with* 0 *elsewhere. Then* $\mathrm{SL}_3(\mathbb{Z})$ *is generated by the* T_{ij} *and has a presentation in terms of these generators, with relations as follows:*

(a) $[T_{ij}, T_{kl}] = \mathbf{1}$ *whenever* $j \ne k$ *and* $i \ne l$,
(b) $[T_{ij}, T_{jk}] = T_{ik}$ *whenever* i, j, k *are distinct,*
(c) $(T_{12}T_{21}^{-1}T_{12})^4 = \mathbf{1}$.

Here, $[A, B] = ABA^{-1}B^{-1}$ *is the commutator of* A *and* B.

Now, $\mathrm{SL}_3(\mathbb{Z}_m)$ *is a quotient group of* $\mathrm{SL}_3(\mathbb{Z})$ *with the same set of generators, and a presentation is obtained by adding the single relation* $T_{12}^m = 1$ *to (a)–(c) above. The relators for* $\mathrm{SL}_3(\mathbb{Z})$ *in (a)–(c) have length at most 12. We take* $m \ge 13$*, whence each cycle in* G*, corresponding to a relator in (a)–(c), projects to a cycle in* G_m*. Consider a Cayley graph* G *(respectively,* G_m*) with respect to the above presentation of* $\mathrm{SL}_3(\mathbb{Z})$ *(respectively,* $\mathrm{SL}_3(\mathbb{Z}_m)$*). By following the procedure at the end of the proof of Theorem 4.1, with* G_m *playing the role of the quotient graph* $\overline{G}$*, we may obtain a graph height function for* G*.*

9. Proof of Theorem 5.2

We adapt and extend the 'bridge decomposition' approach of Hammersley and Welsh [32], which was originally specific to the hypercubic lattice. A *distinct partition* Π of the integer $n \ge 1$ is an expression of the form $n = a_1 + a_2 + \cdots + a_k$ with integers a_i satisfying $a_1 > a_2 > \cdots > a_k > 0$ and some $k = k(\Pi) \ge 1$. The number $k(\Pi)$ is the *order* of the partition Π, and the number of distinct partitions of n is denoted $P(n)$. We recall two facts about such distinct partitions.

Lemma 9.1. *The order* $k = k(\Pi)$ *and the number* $P(n)$ *satisfy*

$$k(k+1) \le 2n \qquad \text{for all distinct partitions } \Pi \text{ of } n, \tag{9.1}$$

$$\log P(n) \sim \pi\sqrt{n/3} \qquad \text{as } n \to \infty. \tag{9.2}$$

Proof. The sum of the first r natural numbers is $\frac{1}{2}r(r+1)$. Therefore, if r satisfies $\frac{1}{2}r(r+1) > n$, the order of Π is at most $r-1$. See [33] for a proof of (9.2). □

Let G be a graph with the given properties, and let $(h, \mathcal{H})$ be a graph height function on G. For the given $(h, \mathcal{H})$, and $v \in V$, we let $b_n(v)$ and $c_n(v)$ be the counts of bridges and half-space SAWs starting at v, respectively, as in Section 5. Recall the constants $d = d(h)$, $r = r(h, \mathcal{H})$ given after Definition 3.1, and note by Proposition 3.2 that $r < \infty$.

Proposition 9.2. *There exists $A = A(r, \beta)$, that is non-decreasing in r and β, such that $c_n(v) \le dn^{3/2}e^{A\sqrt{n}}\beta^n$ for $n \ge 1$ and $v \in V$.*

Proof. We assume for simplicity that $v = \mathbf{1}$, and the same proof is valid for general v. Let $n \ge 1$, and let $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ be an n-step half-space SAW starting at $\pi_0 = \mathbf{1}$. Let $n_0 = 0$, and for $j \ge 1$, define $S_j = S_j(\pi)$ and $n_j = n_j(\pi)$ recursively as follows:

$$S_j = \max_{n_{j-1} \le m \le n} (-1)^j \big[h(\pi_{n_{j-1}}) - h(\pi_m)\big],$$

and n_j is the largest value of m at which the maximum is attained. The recursion is stopped at the smallest integer $k = k(\pi)$ such that $n_k = n$, so that S_{k+1} and n_{k+1} are undefined. Note that S_1 is the span of π and, more generally, S_{j+1} is the span of the SAW $\overline{\pi}^{j+1} := (\pi_{n_j}, \pi_{n_j+1}, \dots, \pi_{n_{j+1}})$. Moreover, each of the subwalks $\overline{\pi}^{j+1}$ is either a bridge or a reversed bridge. We observe that $S_1 > S_2 > \cdots > S_k > 0$.

For a decreasing sequence of $k \ge 2$ positive integers $a_1 > a_2 > \cdots > a_k > 0$, let $B_n(a_1, a_2, \dots, a_k)$ be the set of (n-step) half-space walks from $\mathbf{1}$ such that $k(\pi) = k$, $S_1(\pi) = a_1$, ..., $S_k(\pi) = a_k$ and $n_k(\pi) = n$ (and hence S_{k+1} is undefined). In particular, $B_n(a)$ is the set of n-step bridges from $\mathbf{1}$ with span a.

Let $\pi \in B_n(a_1, a_2, \dots, a_k)$. We describe next how to perform surgery on π in order to obtain a SAW π' satisfying

$$\pi' \in \begin{cases} B_{n+\sigma}(a_1 + a_2 + a_3 + \delta, a_4, \dots, a_k) & \text{if } k \ge 3, \\ B_{n+\sigma}(a_1 + a_2 + \delta) & \text{if } k = 2, \end{cases} \tag{9.3}$$

for some $\sigma = \sigma(\pi)$ and $\delta = \delta(\pi)$ satisfying $0 \le \sigma \le 2r$ and $\delta \ge 0$. The argument is different from that of the corresponding step of [32] since G may not be invariant under reflections, and the conclusion (9.3) differs from that of [32] through the inclusion of the terms σ, δ. The proof of (9.3) is easier when $\mathcal{H}$ acts transitively, and thus we assume that $\mathcal{H}$ acts quasi-transitively but not transitively.

The new SAW π' is constructed in the following way, as illustrated in Figure 9.1. Suppose first that $k \ge 3$.

1. Let π'_1 be the sub-SAW $\overline{\pi}^1$ from $\pi_0 = \mathbf{1}$ to the vertex π_{n_1}.
2. Let $m = \min\{N > n_1 : h(\pi_N) = S_1 - S_2\}$, and let $\nu_1 := (\pi_{n_1}, \dots, \pi_m)$ and $\nu_2 = (\pi_m, \dots, \pi_n)$ be the two sub-SAWs of π with the given endvertices.
   (a) If $\pi_m \notin \mathcal{H}\pi_{n_1}$, we find $\alpha \in \mathcal{H}$ such that: (i) $a_1 < h(\alpha\pi_m)$ and (ii) there is a SAW $\nu(\pi_{n_1}, \alpha\pi_m)$ with the given endvertices, of length not exceeding r, and of which every vertex x, other than its endvertices, satisfies $a_1 < h(x) < h(\alpha\pi_m)$.

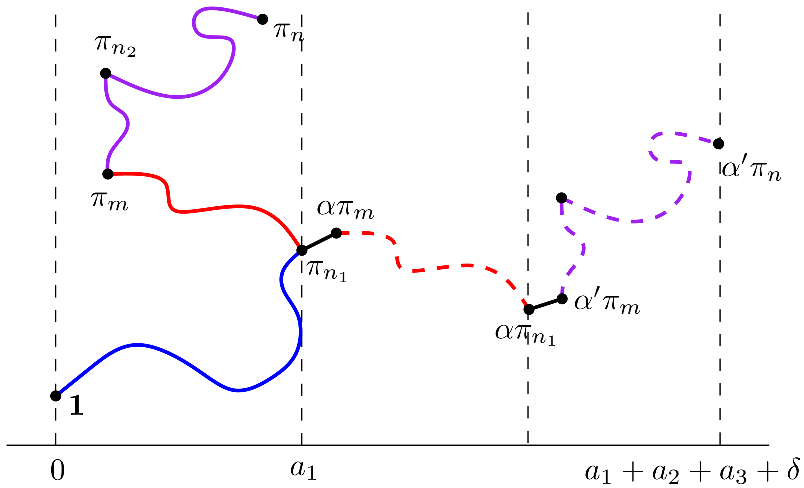

FIGURE 9.1. The solid SAW lies in $B_n(a_1, a_2, a_3)$. We translate the blue path connecting π_{n_1} to π_m, and also the third sub-SAW of π, thereby obtaining a SAW in $B_{n+\sigma}(a_1 + a_2 + a_3 + \delta)$. After translation, the paths are dashed.

(b) If $\pi_m \in \mathcal{H}\pi_{n_1}$, we find $\alpha \in \mathcal{H}$ such that $\alpha\pi_m = \pi_{n_1}$, and write $\nu(\pi_{n_1}, \alpha\pi_m)$ for the 0-step SAW at π_{n_1}.

The union of the three (undirected) SAWs π_1', $\nu(\pi_{n_1}, \alpha\pi_m)$, and $\alpha\nu_1$ is a SAW, denoted π_2', from $\mathbf{1}$ to $\alpha\pi_{n_1}$. In concluding that π_2' is a SAW, we have made use of Definition 3.1(b). Note that $a_1 + a_2 \le h(\alpha\pi_{n_1})$.

3. As in Step 2, we next find $\alpha' \in \mathcal{H}$ such that $h(\alpha\pi_{n_1}) \le h(\alpha'\pi_m)$ and a SAW $\nu(\alpha\pi_{n_1}, \alpha'\pi_m)$ with the given endvertices and the previous type. The union of the three (undirected) SAWs π_2', $\nu(\alpha\pi_{n_1}, \alpha'\pi_m)$, and $\alpha'\nu_2$, is a SAW, denoted π_3', from $\mathbf{1}$ to $\alpha'\pi_n$.

We note the repeated use of Definition 3.1(b). It follows that $\pi' \in B_{n+\sigma}(a_1 + a_2 + a_3 + \delta, a_4, \dots, a_k)$ for some $0 \le \sigma \le 2r$ and $\delta \ge 0$. The mapping $\pi \mapsto \pi'$ is not one-to-one since π may not be reconstructible from knowledge of π' without identification of the intermediate SAWs $\nu(\cdot)$ in steps 2 and 3. However, since the intermediate SAWs have length no greater than r, the mapping $\pi \mapsto \pi'$ is at most $(r+1)^2$-to-one.

Suppose now that $k = 2$. At step 2 above, we have that $h(\pi_n) = S_1 - S_2$, so that $\pi' \in B_{n+\sigma}(a_1 + a_2 + \delta)$ for some $0 \le \sigma \le r$ and $\delta \ge 0$. As above, the map $\pi \mapsto \pi'$ is at most $(r+1)$-to-one.

Let $T = a_1 + a_2 + \dots + a_k$, and write $\sum_a^{(k,T)}$ for the summation over all finite integer sequences $a_1 > \dots > a_k > 0$ with given length k and sum T. By iteration of (9.3), for $\pi \in B_n(a_1, a_2, \dots, a_k)$, there exists $0 \le s \le kr$ and a $(n+s)$-step bridge π'

from **1**. The map $\pi \mapsto \pi'$ is at most $(r+1)^k$-to-one. Therefore,

$$\begin{aligned} c_n(\mathbf{1}) &\le \sum_{T=1}^{dn} \sum_{k=1}^{n} \sum_{a}^{(k,T)} \bigl|B_n(a_1,\dots,a_k)\bigr| \\ &\le \sum_{T=1}^{dn} \sum_{k=1}^{n} \sum_{a}^{(k,T)} (r+1)^k \sum_{s=0}^{kr} b_{n+s}(\mathbf{1}). \end{aligned}$$

By (9.1)–(9.2), and (5.7), there exists a constant $A = A(r,\beta)$ with the required properties such that

$$\begin{aligned} c_n(\mathbf{1}) &\le \sum_{T=1}^{dn} \sum_{k=1}^{n} \sum_{a}^{(k,T)} (r+1)^k (kr+1)\beta^{n+kr+r} \\ &\le \sum_{T=1}^{dn} \sum_{k=1}^{n} \sum_{a}^{(k,T)} \beta^{n+r}\{\beta^r(r+1)\}^{\sqrt{2n}}\bigl(r\sqrt{2n}+1\bigr) \\ &\le dn^{3/2} e^{A\sqrt{n}} \beta^n, \end{aligned}$$

as required. $\square$

Proposition 9.3. *There exists $B = B(r,\beta) > 0$, that is non-decreasing in r and β, such that $\sigma_n(v) \le d^2 n^4 e^{B\sqrt{n}}\beta^n$ for $n \ge 1$ and $v \in V$.*

Proof. As before, we take $v = \mathbf{1}$, and let $\pi = (\pi_0, \pi_1, \dots, \pi_n) \in \Sigma_n$. Let $H = \min_{0\le i\le n} h(\pi_i)$ and $m = \max\{i : h(\pi_i) = H\}$. We construct two half-space SAWs as follows.

By Definition 3.1(c), we may choose $w \in \partial\mathbf{1}$ with $h(w) < h(\mathbf{1})$. The prolongation of $(\pi_0, \pi_1, \dots, \pi_m)$ by the edge $\langle \mathbf{1}, w\rangle$, when reversed, yields a half-space SAW from w with length $m+1$. Similarly, the addition of $\langle w, \mathbf{1}\rangle$ at the start of $(\pi_m, \pi_{m+1}, \dots, \pi_n)$ yields a half-space SAW from w with length $n-m+1$.

By Proposition 9.2,

$$\sigma_n(\mathbf{1}) \le \sum_{m=0}^{n} U_{m+1} U_{n-m+1},$$

where $U_s = ds^{3/2} e^{A\sqrt{s}}\beta^s$. Therefore,

$$\begin{aligned} \sigma_n(\mathbf{1}) &\le d^2\beta^{n+2} \sum_{m=0}^{n} \bigl\{(m+1)(n-m+1)\bigr\}^{3/2} \exp\Bigl(A\sqrt{m+1} + A\sqrt{n-m+1}\Bigr) \\ &\le d^2 \beta^{n+2} (n+1)^4 e^{A\sqrt{2(n+2)}}, \end{aligned}$$

by the arithmetic/geometric mean inequality and the fact that $\sqrt{x}+\sqrt{y} \le \sqrt{2x+2y}$. The claim follows. $\square$

It is trivial that $b_n \le \sigma_n$, whence $\beta \le \mu$. The reverse inequality follows by Proposition 9.3, and the theorem is proved.

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