STRICT INEQUALITIES FOR CONNECTIVE CONSTANTS OF TRANSITIVE GRAPHS

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ABSTRACT. The *connective constant* of a graph is the exponential growth rate of the number of self-avoiding walks starting at a given vertex. Strict inequalities are proved for connective constants of vertex-transitive graphs. Firstly, the connective constant *decreases* strictly when the graph is replaced by a non-trivial quotient graph. Secondly, the connective constant *increases* strictly when a quasi-transitive family of new edges is added. These results have the following implications for Cayley graphs. The connective constant of a Cayley graph decreases strictly when a new relator is added to the group, and increases strictly when a non-trivial group element is declared to be a generator.

1. INTRODUCTION

A self-avoiding walk (abbreviated to SAW) is a path that revisits no vertex. Self-avoiding walks were first introduced in the context of long-chain polymers in chemistry (see [10]), and they have been studied intensively since by mathematicians and physicists (see [22]). If the underlying graph G has some periodicity, the number of n-step SAWs with a given starting point grows (asymptotically) exponentially as $n \to \infty$, with some growth rate $\mu(G)$ called the *connective constant* of the graph. There are only few graphs G for which $\mu(G)$ is known exactly, and a substantial part of the associated literature is devoted to inequalities for such constants. The purpose of the current work is to establish conditions under which a systematic change to G results in a *strict* change to $\mu(G)$.

We have two main results for an infinite, vertex-transitive graph G, as follows. The automorphism group of G is denoted by $\mathrm{Aut}(G)$. Precise conditions are given in the formal statements of the theorems.

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1. (Theorem 3.8) Let the subgroup $\Gamma \subseteq \mathrm{Aut}(G)$ act transitively on G, and let $\mathcal{A} \subseteq \Gamma$ be a non-trivial, normal subgroup of Γ (satisfying a minor condition). The (directed) quotient graph $\vec{G} = G/\mathcal{A}$ satisfies $\mu(\vec{G}) < \mu(G)$.
2. (Theorem 3.2) Suppose new edges are added in such a way that the resulting graph is quasi-transitive (subject to a certain algebraic condition). The connective constant of the new graph $\overline{G}$ satisfies $\mu(G) < \mu(\overline{G})$.

These inequalities have the following implications for Cayley graphs. Let G be the Cayley graph of an infinite, finitely generated group $\mathcal{G}$ with generator set S and relator set R.

3. (Corollary 4.1) Let G_ρ be the Cayley graph obtained by adding to $\mathcal{G}$ a further non-trivial relator ρ. Then $\mu(G_\rho) < \mu(G)$.
4. (Corollary 4.3) Let w be a non-trivial element of $\mathcal{G}$ that is not a generator, and let $\overline{G}_w$ be the Cayley graph obtained by declaring w to be a further generator. Then $\mu(G) < \mu(\overline{G}_w)$.

The proofs follow partly the general approach of Kesten's proof of the pattern theorem, see [19] and [22, Sect. 7.2]. Any n-step SAW π in the smaller graph G lifts to a SAW π' in the larger graph G'. The idea is to show that 'most' such π contain at least an sub-walks for which the corresponding sections of π' may be replaced by SAWs on G'. Different subsets of these sub-walks give rise to different SAWs on G'. The number of such subsets grows exponentially in n, and this introduces an exponential 'entropic' factor in the count of SAWs.

Whereas Kesten's proof and subsequent elaborations were directed at certain specific lattices, our results apply in the general setting of vertex-transitive graphs, and they require new algebraic and combinatorial techniques. Indeed, the work reported here may be the first systematic study of SAWs on general vertex-transitive graphs.

Related questions have been considered in the contexts of percolation and disordered systems. Consider a given model on a graph G, such as a percolation or an Ising/Potts model. There is generally a singularity at some parameter-value called the 'critical point'. For percolation, the parameter in question is the *density* of open sites or bonds, and for the models of statistical physics it is the *temperature*. It is important and useful to understand something of how the critical point varies with the choice of graph. In particular, under what conditions does a systematic change in the graph cause a *strict* change in the value of the critical point? A general approach to this issue was presented in [1] and developed further in [9, 12] and [13, Chap. 3].

Turning back to SAWs, the SAW generating function has radius of convergence $1/\mu$, and is believed (for lattice-graphs at least) to have power-law behaviour near its critical point, see [22]. The above theorems amount to strict inequalities for the critical point as the underlying graph G varies. Despite a similarity of the problem with that of disordered systems, the required techniques for SAWs are substantially different. We concentrate here on vertex-transitive graphs, and the required conditions are expressed in the language of algebra. There is another feasible approach to proving strict inequalities, namely the bridge-decomposition method introduced by Hammersley and Welsh in [17], and used more recently in various works including [6] and [22, Thm 8.2.1].

Basic notation and facts about SAWs and connective constants are presented in Section 2. There is a large literature concerning SAWs, of which we mention [2, 5, 18, 22].

This paper has two companion papers, [14, 15]. In [14], we prove bounds on connective constants of vertex-transitive graphs, in particular $\mu \geq \sqrt{\Delta - 1}$ when G is an infinite, connected, Δ-regular, vertex-transitive, simple graph. In [15], we explore the effect on SAWs of the Fisher transformation, applied to a cubic or partially cubic graph.

2. Notation and definitions

All graphs considered here are connected and infinite. Subject to a minor exception in Section 3, they may not contain *loops* (that is, edges both of whose endpoints are the same vertex) but, in certain circumstances, they are permitted to have *multiple edges* (that is, two or more edges with the same pair of endpoints). A graph $G = (V, E)$ is called *simple* if it has neither loops nor multiple edges. An edge e with endpoints u, v is written $e = \langle u, v\rangle$, and two edges with the same endpoints are called *parallel*. If $\langle u, v\rangle \in E$, we call u and v *adjacent* and write $u \sim v$. Let $\partial v = \{u : u \sim v\}$ denote the set of neighbours of $v \in V$.

The *degree* of vertex v is the number of edges incident to v. We assume that the vertex-degrees of a given graph G are finite with supremum $\Delta < \infty$. The *graph-distance* between two vertices u, v is the number of edges in the shortest path from u to v, denoted $d_G(u, v)$.

The automorphism group of the graph $G = (V, E)$ is denoted $\mathrm{Aut}(G)$. A subgroup $\Gamma \subseteq \mathrm{Aut}(G)$ is said to *act transitively* on G (or on the vertex-set V) if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma v = w$. It is said to *act quasi-transitively* if there is a finite set W of vertices (called a *fundamental domain*) such that, for $v \in V$, there exist $w \in W$

and $\gamma \in \Gamma$ with $\gamma v = w$. The graph is called *vertex-transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively). The identity element of any group is denoted by ι.

A *walk* w on G is an alternating sequence $w_0 e_0 w_1 e_1 \cdots e_{n-1} w_n$ of vertices w_i and edges $e_i = \langle v_i, v_{i+1} \rangle$. We write $|w| = n$ for the *length* of w, that is, the number of edges in w. The walk w is called *closed* if $w_0 = w_n$. A *cycle* (or n-*cycle*) is a closed walk w with distinct edges and $w_i \neq w_j$ for $1 \leq i < j \leq n$. Thus, two parallel edges form a 2-cycle.

Let $n \in \{1, 2, \dots\} \cup \{\infty\}$. An n-*step self-avoiding walk* (SAW) on G is a walk containing n edges no vertex of which appears more than once. Let $\sigma_n(v)$ be the number of n-step SAWs starting at $v \in V$. We are interested here in the exponential growth rate of $\sigma_n(v)$. Note that, in the presence of parallel edges, two SAWs with identical vertex-sets but different edge-sets are considered as distinct SAWs.

Theorem 2.1. [16] *Let $G = (V, E)$ be an infinite, connected, quasi-transitive graph with finite vertex-degrees. There exists $\mu = \mu(G) \in [1, \infty)$, called the* connective constant*, such that*

$$\lim_{n \to \infty} \sigma_n(v)^{1/n} = \mu, \qquad v \in V. \tag{2.1}$$

Subadditivity plays a key part in the proof of this theorem. It yields the inequality

$$\sup_{v \in V} \sigma_n(v) \geq \mu^n, \qquad n \geq 0, \tag{2.2}$$

which will be useful later in this paper.

See [15, Thm 3.1] (and also [20, Prop. 1.1]) for an elaboration of Theorem 2.1 in the absence of quasi-transitivity. We note for use in Section 3 that the above notation may be extended in a natural way to *directed* graphs, and we omit the details here. In particular, one may define the connective constant $\vec{\mu} = \mu(\vec{G})$ of a *directed*, quasi-transitive graph $\vec{G}$ by (2.1) with $\sigma_n(v)$ replaced by the number of *directed* n-step SAWs (whenever the relevant limits exist).

3. Strict inequalities for vertex-transitive graphs

In this section, we present and discuss two strict inequalities for connective constants of vertex-transitive graphs. The first (Theorem 3.2) deals with the effect of adding a quasi-transitive family of new edges, and the second (Theorem 3.8) deals with quotient graphs. The implications of these inequalities for Cayley graphs are presented in Section 4, see Corollaries 4.1 and 4.3.

Let $G = (V, E)$ be an infinite, connected, vertex-transitive, simple graph. We assume throughout that the vertex-degree Δ of G satisfies

$\Delta < \infty$. We state first our result for quasi-transitive augmentations of G. This is followed in Section 3.2 by a discussion of quotient graphs. Proofs of theorems in this section are found in Sections 5–6.

3.1. **Quasi-transitive augmentation.** Let $G = (V, E)$ be as above, and let $\overline{G} = (V, \overline{E})$ be obtained from G by adding further edges, possibly in parallel to existing edges. We assume that E is a *proper* subset of $\overline{E}$, and introduce next a certain property.

Definition 3.1. *A subgroup* $\mathcal{A} \subseteq \mathrm{Aut}(G)$ *is said to* have the finite coset property *with* root $\rho \in V$ *if there exist* $\nu_0, \nu_1, \dots, \nu_s \in \Gamma$*, with* $\nu_0 = \iota$ *and* $s < \infty$*, such that* V *is partitioned as* $\bigcup_{i=0}^{s} \nu_i \mathcal{A}\rho$*. It is said simply to have the finite coset property if it has this property with some root.*

Theorem 3.2. *Let* Γ *act transitively on* G*, and let* $\mathcal{A}$ *be a subgroup of* Γ *with the finite coset property. If* $\mathcal{A} \subseteq \mathrm{Aut}(\overline{G})$*, then* $\mu(G) < \mu(\overline{G})$*.*

Two categories with the finite coset property are given next.

Theorem 3.3. *Let* Γ *act transitively on* G*, and* $\rho \in V$*. The subgroup* $\mathcal{A}$ *of* Γ *has the finite coset property with root* ρ *if either of the following holds.*

(a) $\mathcal{A}$ *is a normal subgroup of* Γ *which acts quasi-transitively on* G*.*
(b) *The index* $[\Gamma : \mathcal{A}]$ *is finite.*

We ask whether the condition of Theorem 3.2 may be relaxed. More generally, is it the case that $\mu(G) < \mu(\overline{G})$ whenever $G = (V, E)$ is transitive and $\overline{G} = (V, \overline{E})$ is quasi-transitive, with E a proper subset of $\overline{E}$.

The conclusion of Theorem 3.2 is generally invalid if G is assumed only quasi-transitive. Consider, for example, the pair G, $\overline{G}$ of Figure 3.1, each of which has connective constant 1.

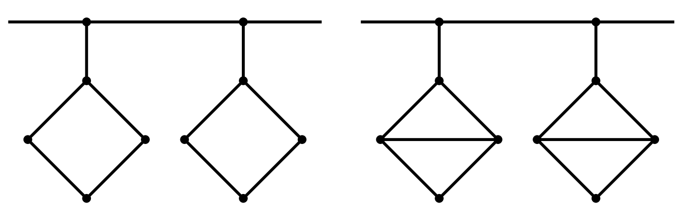

FIGURE 3.1. The pair G, $\overline{G}$. The patterns are extended infinitely in both directions. Each graph is quasi-transitive with connective constant 1, and the second is obtained from the first by the addition of edges.

The assumption of normality will recur in this paper, and we recall the following 'standard fact'.

Remark 3.4. *Let Γ act transitively on the infinite graph $G = (V, E)$. A partition $\mathcal{P}$ of V is called* Γ-invariant *if, for $u, v \in V$ belonging to the same set of the partition, and for $\gamma \in \Gamma$, γu and γv belong to the same set of the partition.*

Let $\mathcal{A}$ be a subgroup of Γ. The orbits of $\mathcal{A}$ form a partition $\mathcal{P}(\mathcal{A})$ of V. If $\mathcal{A}$ is a normal subgroup, the partition $\mathcal{P}(\mathcal{A})$ is Γ-invariant. If $\mathcal{P}(\mathcal{A})$ is Γ-invariant, there exists a normal subgroup $\mathcal{N}$ of Γ such that $\mathcal{P}(\mathcal{N}) = \mathcal{P}(\mathcal{A})$. In the latter case, $\mathcal{N}$ may be taken as the normal closure of $\mathcal{A}$, that is, the intersection of all normal subgroups of Γ containing $\mathcal{A}$.

The proof of Theorem 3.3 makes use of the relationship between the index of $\mathcal{A}$ and the number of its orbits when acting on V. Consider Γ as a group acting on V with orbit-set denoted by V/Γ. It is said to act *freely* if every stabilizer is trivial, that is, if

$$\mathrm{Stab}_v := \{\gamma \in \Gamma : \gamma v = v\}$$

satisfies

$$\mathrm{Stab}_v = \{\iota\}, \qquad v \in V. \tag{3.1}$$

We abuse notation by saying that $\Gamma \setminus \mathcal{A}$ acts freely on V if

$$\mathrm{Stab}_v \subseteq \mathcal{A}, \qquad v \in V. \tag{3.2}$$

The proof of the next proposition is at the beginning of Section 6.

Proposition 3.5. *Let Γ act transitively on the countable set V, and let $\mathcal{A}$ be a subgroup of Γ. Then*

$$|V/\mathcal{A}| \leq [\Gamma : \mathcal{A}]. \tag{3.3}$$

If $|V/\mathcal{A}| < \infty$, equality holds in (3.3) if and only if $\Gamma \setminus \mathcal{A}$ acts freely on the set V.

We make three comments concerning Theorem 3.3.

1. Let $\mathcal{A} \subseteq \mathrm{Aut}(\overline{G})$ act quasi-transitively on G, and suppose there exists $\Gamma \subseteq \mathrm{Aut}(G)$ such that Γ acts transitively on G and $\Gamma \setminus \mathcal{A}$ acts freely on V. We have by Proposition 3.5 that $[\Gamma : \mathcal{A}] = |V/\mathcal{A}| < \infty$, whence $\mathcal{A}$ has the finite coset property (with arbitrary root) by Theorem 3.3(b).
2. By Remark 3.4, condition (a) of Theorem 3.3 may be replaced by the apparently weaker assumption that $\mathcal{A}$ is a subgroup of Γ acting quasi-transitively on G, whose orbits form a Γ-invariant partition of V.

3. The *normal core* of $\mathcal{A}$ is the intersection of the conjugate subgroups of $\mathcal{A}$. If $[\Gamma : \mathcal{A}] < \infty$, the normal core $\mathcal{N}$ of $\mathcal{A}$ satisfies $[\Gamma : \mathcal{N}] < \infty$ (see [24, 1.6.9]). By Proposition 3.5, $\mathcal{N}$ acts quasi-transitively on V.

3.2. **Quotient graphs.** Let Γ be a subgroup of the automorphism group $\mathrm{Aut}(G)$ that acts transitively, and let $\mathcal{A}$ be a subgroup of Γ. We denote by $\vec{G} = (\overline{V}, \vec{E})$ the (directed) quotient graph $G/\mathcal{A}$ constructed as follows. Let $\approx$ be the equivalence relation on V given by $v_1 \approx v_2$ if and only if there exists $\alpha \in \mathcal{A}$ with $\alpha v_1 = v_2$. The vertex-set $\overline{V}$ comprises the equivalence classes of $(V, \approx)$, that is, the orbits $\overline{v} := \mathcal{A}v$ as v ranges over V. For $v, w \in V$, we place $|\partial v \cap \overline{w}|$ directed edges from $\overline{v}$ to $\overline{w}$ (if $\overline{v} = \overline{w}$, these edges are directed loops), and we write $\overline{v} \sim \overline{w}$ if $|\partial v \cap \overline{w}| \geq 1$ and $\overline{v} \neq \overline{w}$. By the next lemma, the number $|\partial v \cap \overline{w}|$ is independent of the choice of $v \in \overline{v}$.

Lemma 3.6. *Let $\mathcal{A}$ be a subgroup of Γ, and let $\overline{v}, \overline{w} \in \overline{V}$. The number $|\partial v \cap \overline{w}|$ is independent of the representative $v \in \overline{v}$.*

Proof. Let $v, v' \in \overline{v}$ and $v' \neq v$. Choose $\alpha \in \mathcal{A}$ such that $\alpha v = v'$. Then $x \in \partial v \cap \overline{w}$ is mapped to $\alpha x \in \partial v' \cap \overline{w}$, whence α acts as an injection from $\partial v \cap \overline{w}$ to $\partial v' \cap \overline{w}$. Therefore, $|\partial v \cap \overline{w}| \leq |\partial v' \cap \overline{w}|$, and the claim follows by symmetry. $\square$

We recall that the orbits of $\mathcal{A}$ are invariant under Γ if and only if they are the orbits of some normal subgroup of Γ (see Remark 3.4). *Assume henceforth that $\mathcal{A}$ is a normal subgroup of* Γ. It is standard that $\alpha \in \Gamma$ acts on $\vec{G}$ by $\alpha(\mathcal{A}v) = \mathcal{A}(\alpha v)$, and that Γ acts transitively on $\vec{G}$. Furthermore, for $v, w \in V$,

$$\overline{v} = \overline{w} \quad \Leftrightarrow \quad \forall \gamma \in \Gamma,\ \overline{\gamma v} = \overline{\gamma w}. \tag{3.4}$$

Any walk π on G induces a (directed) walk $\vec{\pi}$ on $\vec{G}$, and we say that π *projects* onto $\vec{\pi}$. For a walk $\vec{\pi}$ on $\vec{G}$, there exists a walk π on G that projects onto $\vec{\pi}$, and we say that $\vec{\pi}$ *lifts* to π. There are generally many choices for such π, and we fix such a choice as follows. For $\overline{v}_1, \overline{v}_2 \in \overline{V}$, we label the $N(v_1, v_2) := |\partial v_1 \cap \overline{v}_2|$ directed edges from $\overline{v}_1$ to $\overline{v}_2$ in a fixed but arbitrary way with the integers $1, 2, \ldots, N(v_1, v_2)$. For $v \in \overline{v}_1$, we label similarly the edges from v to vertices in the set $\overline{v}_2$. For $v \in \overline{v}$, any $\vec{\pi}$ from $\overline{v}$ lifts to a unique π from v that conserves edge-labellings, and thus walks from a given v on G are in one–one correspondence with walks from $\overline{v}$ on $\vec{G}$. Since a SAW on $\vec{G}$ lifts to a SAW on G, $\vec{\mu} := \mu(\vec{G})$ satisfies $\vec{\mu} \leq \mu(G)$.

It is sometimes convenient to work with an undirected graph derived from $\vec{G}$. There are two such graphs, depending on whether or not the multiplicities of edges are retained. The first is the simple graph, denoted $\overline{G}_0$, derived from $\vec{G}$ by declaring two distinct vertices $\overline{v}$, $\overline{w}$ to be adjacent if and only if there is a directed edge between $\overline{v}$ to $\overline{w}$ (this property is symmetric in $\overline{v}$, $\overline{w}$).

The second such graph, denoted $\overline{G} = (\overline{V}, \overline{E})$, is a multigraph derived from $\overline{G}_0$ by retaining the multiplicities of parallel edges of $\vec{G}$. We make this precise as follows. We call $\mathcal{A}$ *symmetric* if

$$|\partial v \cap \overline{w}| = |\partial w \cap \overline{v}|, \qquad v, w \in V. \tag{3.5}$$

For symmetric $\mathcal{A}$, we obtain $\overline{G}$ by placing $|\partial v \cap \overline{w}|$ (undirected) parallel edges between each distinct pair $\overline{v}, \overline{w} \in \overline{V}$, and adding $|\partial v \cap \overline{v}|$ (undirected) loops at each $\overline{v} \in \overline{V}$.

We introduce next two possible features of the pair $(G, \mathcal{A})$, the first of which is *unimodularity*. We say that $\gamma \in \Gamma$ *fixes* vertex v if $\gamma \in \mathrm{Stab}_v$. Note that $|\mathrm{Stab}_v w| < \infty$ for $v, w \in V$, since G is locally finite and all elements of $\mathrm{Stab}_v w$ are at the same distance from v.

Let $\mathrm{Stab}_v^0 = \mathrm{Stab}_v \cap \mathcal{A}$. As shown in [27] (see also [7, 25]), when viewed as a topological group with the usual topology, $\mathcal{A}$ is unimodular if and only if

$$|\mathrm{Stab}_u^0 v| = |\mathrm{Stab}_v^0 u|, \qquad u, v \in V. \tag{3.6}$$

Since all groups considered here are subgroups of $\mathrm{Aut}(G)$, we may follow [21, Chap. 8] by *defining* $\mathcal{A}$ to be *unimodular* if (3.6) holds.

The second feature of $(G, \mathcal{A})$ concerns the length of the shortest SAW of G with endpoints in the same orbit. Let $v \in V$, and let $w \neq v$ satisfy: $\overline{w} = \overline{v}$ and $d_G(v, w)$ is minimal with this property. By the transitive action of Γ,

$$d_G(x, y) \geq d_G(v, w), \qquad x \neq y, \ \overline{x} = \overline{y}. \tag{3.7}$$

We say that v is of:

$$\begin{aligned} &\text{type 1} \quad \text{if } d_G(v, w) = 1, \\ &\text{type 2} \quad \text{if } d_G(v, w) = 2, \\ &\text{type 3} \quad \text{if } d_G(v, w) \geq 3. \end{aligned}$$

By (3.4), every vertex has the same type, and thus we shall speak of the *type* of $\mathcal{A}$. Note that $d_G(v, w)$ is the length of the shortest cycle of $\vec{G}$.

Lemma 3.7.

(a) *If $\mathcal{A}$ is unimodular, then it is symmetric (in that (3.5) holds).*
(b) *If $\mathcal{A}$ has type 3, then $|\partial v \cap \overline{v}| = 0$ for $v \in V$, and $|\partial v \cap \overline{w}| = |\partial w \cap \overline{v}| = 1$ whenever $\overline{v} \sim \overline{w}$. In particular, $\mathcal{A}$ is symmetric.*

Proof. (a) Suppose $\mathcal{A}$ is unimodular, and let $\overline{v}, \overline{w} \in \overline{V}$ with $\overline{v} \neq \overline{w}$. For $x, y \in V$, let

$$f(x,y) = \begin{cases} 1 & \text{if } x \in \overline{v},\ y \in \overline{w}, \text{ and } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

The function f is invariant under the diagonal action of $\mathcal{A}$, in that $f(\alpha x, \alpha y) = f(x, y)$ for $\alpha \in \mathcal{A}$. By the mass-transport principle as enunciated in, for example, [21, Thm 8.7],

$$\sum_{w' \in \overline{w}} f(v, w') = \sum_{v' \in \overline{v}} f(v', w) \frac{|\mathrm{Stab}^0_{v'} w|}{|\mathrm{Stab}^0_{w} v'|}, \qquad v, w \in V.$$

Equation (3.5) follows by (3.6).
(b) When $\mathcal{A}$ has type 3, the claim is a consequence of (3.7). □

There follows the main theorem of this section. A group is called *trivial* if it comprises the identity ι only.

Theorem 3.8. *Let $\mathcal{A}$ be a non-trivial, normal subgroup of Γ. The connective constant $\vec{\mu} = \mu(\vec{G})$ satisfies $\vec{\mu} < \mu(G)$ if: either*
- (a) *the type of $\mathcal{A}$ is 1 or 3, or*
- (b) *$\mathcal{A}$ has type 2 and either of the following holds.*
  - (i) *G contains a SAW v_0, w, v' satisfying $\overline{v}_0 = \overline{v}'$ and $|\partial v_0 \cap \overline{w}| \geq 2$,*
  - (ii) *G contains a SAW $v_0\ (= w_0), w_1, w_2, \dots, w_l\ (= v')$ satisfying $\overline{v}_0 = \overline{v}'$, $\overline{w}_i \neq \overline{w}_j$ for $0 \leq i < j < l$, and furthermore $v' = \beta v_0$ for some $\beta \in \mathcal{A}$ which fixes no w_i.*

In the special case when $\mu(G) = 1$, by [14, Thm 1.1] G has degree 2 and is therefore the line $\mathbb{Z}$. It is easily seen that $\overline{V}$ is finite, so that $\vec{\mu} = 0$.

Suppose $\mathcal{A}$ has type 2. Condition (i) of Theorem 3.8(b) holds if $\mathcal{A}$ is symmetric, since $|\partial w \cap \overline{v}| \geq 2$. While symmetry is sufficient for Theorem 3.8, we shall see below that it is not necessary.

Conditions (i)–(ii) are necessary in the type-2 case, in the sense illustrated by the following example. Let G be the infinite 3-regular tree with a distinguished end ω. Let Γ be the set of automorphisms that preserve ω, and let $\mathcal{A}$ be the normal subgroup generated by the interchange of two children of a given vertex v (and the associated relabelling of their descendants). The graph $\vec{G}$ is obtained from $\mathbb{Z}$ by placing two directed edges between consecutive integers in one direction, and one directed edge in the reverse direction. Thus, $\mathcal{A}$ has type 2. It is easily seen that neither (i) nor (ii) holds, and indeed $\mu(\vec{G}) = \mu(G) = 2$. We develop this example as follows.

Let $k \geq 0$, and let $\mathcal{A}_k$ be the normal subgroup generated by $\mathcal{A}$ together with the map that shifts v to its ancestor k generations earlier. Note that $\mathcal{A}_k$ has type 2 for $k \neq 1$. The case of $\mathcal{A}_1$ is trivial since $\vec{G}$ has a unique vertex.

We have that $\mathcal{A}_2$ is symmetric, and condition (i) of Theorem 3.8(b) applies. In contrast, $\mathcal{A}_3$ is asymmetric (and therefore non-unimodular, see [27] and also [23, 25]), and condition (ii) applies. In either case, $\vec{\mu} < \mu(G)$. The situation is in fact trivial since $\vec{G}$ is a directed k-cycle with two directed edges clockwise and one anticlockwise. Thus $\vec{\mu} = 0$ and $\mu(G) = 2$. The same argument shows $\vec{\mu} < \mu(G)$ in the less trivial case with G the direct product of $\mathbb{Z}^d$ and the tree.

The relationship between $\vec{\mu}$ and the connective constants of the undirected graphs derived from $\vec{G}$ is as follows. It is easily seen that $\mu(\overline{G}_0) \leq \vec{\mu}$. The graph $\overline{G}$ is defined whenever $\mathcal{A}$ is symmetric, and in this case $\mu(\overline{G}) = \vec{\mu}$.

The proof of Theorem 3.8 does not appear to yield an explicit non-trivial upper bound for the ratio $\vec{\mu}/\mu(G)$. One may, however, show the following theorem.

Theorem 3.9. *Suppose there exists a real sequence $(b_n : n = 1, 2, \dots)$, each term of which may be calculated in finite time, such that $b_n \leq \mu(G)$ and $\lim_{n\to\infty} b_n = \mu(G)$. There exists an algorithm which terminates in finite time and, on termination, yields a number $R = R(G, \mathcal{A}) < 1$ such that $\vec{\mu}/\mu(G) \leq R$.*

If μ is known, we may set $b_n \equiv \mu(G)$. In certain other cases, such a sequence (b_n) may be found; for example, when $G = \mathbb{Z}^d$, we may take b_n to be the nth root of the number of n-step bridges from the origin (see [17, 19]). Theorem 3.9 is unlikely to be useful in practice since it relies on successive enumerations of the numbers of n-step SAWs on G and $\vec{G}$.

The conclusion of Theorem 3.8 is generally invalid for *quasi-transitive* graphs. Consider, for example, the graph G of Figure 3.2, with $\mathcal{A} = \{\iota, \rho\}$ where ρ is reflection in the horizontal axis. Both G and its quotient graph have connective constant 1.

For accounts of algebraic graph theory, see [4, 11, 21, 25]. Theorem 3.8 is related to but distinct from the strict inequalities of [6, 18, 26] and [22, Thm 8.2.1], where specific examples are considered of graphs that are not vertex-transitive.

The relationship between the *percolation* critical points of a graph G and a version of the quotient graph $\overline{G}$ is the topic of a conjecture of Benjamini and Schramm [8, Qn 1]. As observed above (see also [8]), it is not necessary for the definition of quotient graph to assume that $\mathcal{A}$ is

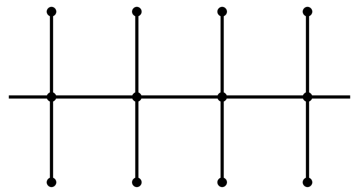

FIGURE 3.2. The pattern is extended infinitely in both directions.

a *normal* subgroup of Γ. However, [8] appears to make the additional assumption that $\mathcal{A}$ acts freely on V. This is a stronger assumption than unimodularity.

Finally, we discuss the assumption of normality in Theorem 3.8. By Remark 3.4, this assumption may be replaced by the following: there exists an automorphism group Γ acting transitively on G, and a subgroup $\mathcal{A}$ whose partition of V is Γ-invariant.

Following [8], we ask whether $\mu(\vec{G}) < \mu(G)$ where $\vec{G} = G/\mathcal{A}$, under the weaker assumption that $\mathcal{A}$ is a non-trivial (not necessarily normal) subgroup of Γ acting freely on V, such that $\vec{G}$ is vertex-transitive. The proof of Theorem 3.8 may be adapted to give an affirmative answer to this question under an extra condition, as follows. An outline proof is included at the end of Section 5.

Theorem 3.10. *Let G be an infinite, locally finite graph on which the automorphism group Γ acts transitively. Let $\mathcal{A}$ be a non-trivial subgroup of Γ acting freely on V, such that the quotient graph $\vec{G} := G/\mathcal{A}$ is vertex-transitive. Assume there exists $l \geq 1$ such that $\vec{G}$ possesses an l-cycle but G does not. Then $\mu(\vec{G}) < \mu(G)$.*

4. Strict inequalities for Cayley graphs

We turn to the special case of Cayley graphs. Let $\mathcal{G}$ be an infinite group with a finite set S of generators, where S is assumed symmetric in that $S = S^{-1}$, and the identity ι satisfies $\iota \notin S$. Thus $\mathcal{G}$ has a presentation as $\mathcal{G} = \langle S \mid R \rangle$ where R is a set of relators. The Cayley graph $G = G(\mathcal{G}, S)$ is the simple graph defined as follows. The vertex-set V of G is the set of elements of $\mathcal{G}$. Distinct elements $g, h \in V$ are connected by an edge if and only if there exists $s \in S$ such that $h = gs$. It is easily seen that G is connected and vertex-transitive.

The group $\mathcal{G}$ may be viewed as a subgroup of the automorphism group of $G = (V, E)$ by: $\gamma \in \mathcal{G}$ acts on V by $v \mapsto \gamma v$. Thus, $\mathcal{G}$ acts transitively.

The reader is reminded of an elementary property of Cayley graphs, namely that $\mathcal{G}$ acts freely on $v \in V$. This is seen as follows. Suppose

$\gamma \in \mathrm{Stab}_v$. For $w \in V$, there exists $h \in \mathcal{G}$ with $w = vh$, so that γ fixes every $w \in V$, whence $\gamma = \iota$. See [4] for a general account of the theory of Cayley graphs, and [21, Sect. 3.4] for a brief account.

Suppose a product $s_1 s_2 \cdots s_l$ of generators is a relator. The relation $s_1 s_2 \cdots s_l = \iota$ corresponds to the closed walk $(\iota, s_1, s_1 s_2, \dots, s_1 s_2 \cdots s_l = \iota)$ of G passing through the identity ι. Consider now the effect of adding a further relator. Let $s_1, s_2, \dots, s_l \in S$ be such that $\rho := s_1 s_2 \cdots s_l \neq \iota$, and write $\mathcal{G}_\rho = \langle S \mid R \cup \{\rho\} \rangle$. Then $\mathcal{G}_\rho$ is isomorphic to the quotient group $\mathcal{G}/\mathcal{N}$ where $\mathcal{N}$ is the normal subgroup of $\mathcal{G}$ generated by ρ. Let $G(\mathcal{G}_\rho, S)$ be the Cayley graph of $\mathcal{G}_\rho$.

Corollary 4.1. *Let $G = G(\mathcal{G}, S)$ be the Cayley graph of the infinite, finitely-presented group $\mathcal{G} = \langle S \mid R \rangle$. Let $\rho \in \mathcal{G}$, $\rho \notin R \cup \{\iota\}$, and let $G_\rho = G(\mathcal{G}_\rho, S)$. The connective constants of G and G_ρ satisfy $\mu(G_\rho) < \mu(G)$.*

Proof. The Cayley graph G_ρ is obtained from the quotient graph $G/\mathcal{G}_\rho$ by replacing any set of parallel edges by a single edge. Since $\mathcal{G}_\rho$ acts freely on V, it is unimodular, and the claim follows by Lemma 3.7 and Theorem 3.8. □

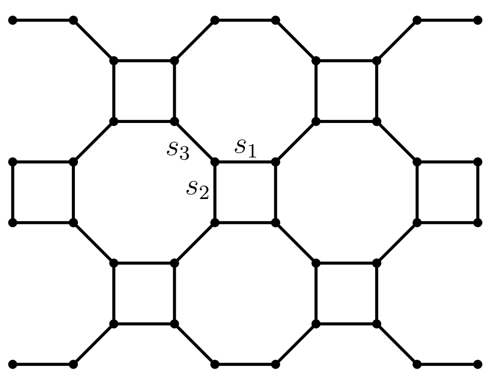

FIGURE 4.1. The square/octagon lattice, also denoted $(4, 8^2)$.

Example 4.2. *The* square/octagon *lattice, otherwise known as the Archimedean lattice $(4, 8^2)$, is illustrated in Figure 4.1. It is the Cayley graph of the group with generator set $S = \{s_1, s_2, s_3\}$ and relators $\{s_1^2, s_2^2, s_3^2, s_1 s_2 s_1 s_2, s_1 s_3 s_2 s_3 s_1 s_3 s_2 s_3\}$. The horizontal edges correspond to s_1, the vertical edges to s_2, and the other edges to s_3. Adding the further relator $s_2 s_3 s_2 s_3$, we obtain a graph isomorphic to the ladder graph of Figure 4.2, whose connective constant is the golden mean $\phi := \frac{1}{2}(\sqrt{5} + 1)$.*

By Corollary 4.1, the connective constant μ of the square/octagon lattice is strictly greater than $\phi = 1.618\dots$. The best lower bound currently known appears to be $\mu > 1.804\dots$, see [18].

We ask whether $\mu(G) \ge \phi$ for all infinite, vertex-transitive, simple, cubic graphs G, see [14, 15].

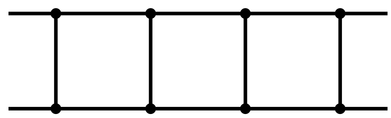

FIGURE 4.2. The (doubly-infinite) ladder graph has connective constant the golden mean $\phi := \frac{1}{2}(\sqrt{5}+1)$; see [3, p. 184] and [15].

Our second inequality for Cayley graphs concerns the addition of a generator. Let $\mathcal{G} = \langle S \mid R\rangle$ be a finitely-generated group as above, and let w be a group element satisfying $w \notin S$ and $w \ne \iota$. Let $\overline{\mathcal{G}}_w = \langle S \cup \{w, w^{-1}\} \mid R\rangle$.

Corollary 4.3. *The connective constants of the Cayley graphs G, $\overline{G}_w$ of the above presentations of the groups $\mathcal{G}$, $\overline{\mathcal{G}}_w$ satisfy $\mu(G) < \mu(\overline{G}_w)$.*

Proof. This is an immediate corollary of Theorem 3.2, on noting that $\mathcal{G}$ acts transitively on both G and $\overline{G}_w$. □

In the special case when $\mu(G) = 1$, we have that $G = \mathbb{Z}$ (as noted after Theorem 3.8). Therefore, $\overline{G}_w$ has degree either 3 or 4, whence $\mu(\overline{G}_w) \ge \sqrt{2}$ by [14, Thm 1.1].

As a simple example of the above construction, consider $\mathbb{Z}^2$ as the Cayley graph of the abelian group with $S = \{a, b\}$ and $R = \{aba^{-1}b^{-1}\}$. Adding the generator ab (and its inverse) amounts to adding a diagonal to each square of $\mathbb{Z}^2$. One may easily construct more interesting examples based on, for example, the square/octagon lattice of Figure 4.1.

For both Corollary 4.1 and Corollary 4.3, there exists an algorithm which, under a certain condition, terminates in finite time and, on termination, yields an explicit non-trivial bound for the ratio of the connective constants under consideration. This holds just as in Theorem 3.9, and the details are omitted.

5. Proofs of Theorems 3.8 and 3.9

The proof of Theorem 3.8 is inspired in part by Kesten's proof of the pattern theorem in [19] (see also [22, Sect. 7.2]). The overall shape of the argument from [19] recurs more than once in this paper; at later occurrences we shall outline any necessary adaptation rather than attempt to systematize the method. We begin with an elementary lemma.

Lemma 5.1. *Let $v \in V$, and let $\mathcal{A}$ be a normal subgroup of Γ. We have that $\mathcal{A} \subseteq \mathrm{Stab}_v$ if and only if $\mathcal{A}$ is trivial.*

Proof. If $\mathcal{A}$ is trivial then $\mathcal{A} = \{\iota\} \subseteq \mathrm{Stab}_v$. Conversely, suppose $\mathcal{A} \subseteq \mathrm{Stab}_v$ and let $w \in V$. Since G is vertex-transitive, there exists $\gamma \in \Gamma$ such that $\gamma v = w$. Since $\mathcal{A}$ is normal,

$$\mathcal{A}w = \mathcal{A}\gamma v = \gamma \mathcal{A} v = \{\gamma v\} = \{w\}.$$

Therefore, $\mathcal{A} \subseteq \mathrm{Stab}_w$ for all $w \in V$, and hence $\mathcal{A}$ is trivial. $\square$

Let G, $\mathcal{A} \subseteq \Gamma$, $\vec{G} = G/\mathcal{A}$, etc, be given as for Theorem 3.8, and fix $v_0 \in V$. Let Σ_n (respectively, $\vec{\Sigma}_n$) be the set of n-step SAWs of G (respectively, $\vec{G}$) starting from v_0 (respectively, $\overline{v}_0$), and write $\sigma_n = |\Sigma_n|$ (respectively, $\vec{\sigma}_n = |\vec{\Sigma}_n|$).

Assumption 5.2. *We assume henceforth that either condition (a) of Theorem 3.8 holds, or condition (b)(i). An explanation of the sufficiency of condition (b)(ii) for type 2 is given in Remark 5.5 at the end of this proof.*

Any walk π on G induces a walk $\vec{\pi}$ on $\vec{G}$, and we say that π *projects* onto $\vec{\pi}$. For $\vec{\pi} \in \vec{\Sigma}_n$, there exists a SAW $\pi \in \Sigma_n$ that projects onto $\vec{\pi}$, and we say that $\vec{\pi}$ *lifts* to π. There are generally many choices for such π, and we fix such a choice as explained after Lemma 3.6. Let $\mu = \mu(G)$ and $\vec{\mu} = \mu(\vec{G})$. By the above remarks, $\vec{\mu} \le \mu$.

The idea of the proof is to replace certain sub-walks of $\vec{\pi} \in \vec{\Sigma}_n$ by new walks that lift to SAWs on G. Such replacements are given in terms of a certain SAW on G that we introduce next.

Let ℓ_{v_0} be a shortest SAW from v_0 to some $w \ne v_0$ satisfying $\overline{v}_0 = \overline{w}$. Such a walk exists by Lemma 5.1 and the non-triviality of $\mathcal{A}$. We consider ℓ_{v_0} as a directed walk from v_0 to w. Let $\vec{\ell}_{v_0}$ be the projection of ℓ_{v_0} in $\vec{G}$. Thus: if G has type 1, $\vec{\ell}_{v_0}$ is a loop; if type 2, $\vec{\ell}_{v_0}$ traverses a directed edge from $\overline{v}_0$ to $\overline{w}$, and then returns along a directed edge from $\overline{w}$ to $\overline{v}_0$; if type 3, $\vec{\ell}_{v_0}$ is a (directed) cycle of $\vec{G}$ of length 3 or more.

When $\mathcal{A}$ has type 2, we take ℓ_{v_0} to be the path given in condition (b)(i) of the theorem.

For $w \in V$, let L_w (or $L(w)$) denote the set of all walks in G starting from w which are images of ℓ_{v_0} under elements of Γ, and let $\vec{L}_{\overline{w}}$ (or

$\vec{L}(\overline{w})$) be the corresponding collection of walks in $\vec{G}$. Let $L = \bigcup\{L_w : w \in V\}$ and $\vec{L} = \bigcup\{\vec{L}_{\overline{w}} : \overline{w} \in \overline{V}\}$. All walks in L have equal length, written Λ. Thus, Λ is the number of distinct vertices in $\vec{\ell}$. We shall refer to elements of $\vec{L}$ as *cycles*, and we shall use later the fact that the graph G, 'decorated' with members of L, is preserved under the action of Γ.

For $u \in V$ and a positive integer r, we define the *ball* $B_r(u) = \{v \in V : d_G(u, v) \le r\}$. Since G is Δ-regular,

$$|B_r(u)| \le 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{r-1} < \Delta^{r+1}. \tag{5.1}$$

Let $\vec{\pi}$ be an r-step SAW on $\vec{G}$, and write $\vec{\pi}_0, \vec{\pi}_1, \dots, \vec{\pi}_r$ for the vertices traversed by $\vec{\pi}$ in order. For $k \in \mathbb{N}$, we say that $E_k = E_k(\vec{\pi})$ occurs at the jth step of $\vec{\pi}$ if there exists $\vec{\ell} \in \vec{L}_{\vec{\pi}_j}$, denoted $\vec{\ell} = \vec{\ell}(\vec{\pi}_j)$, such that at least k vertices of $\vec{\ell}$ are visited by $\vec{\pi}$.

Now let $\vec{\pi}$ be an n-step SAW on $\vec{G}$. For $m \in \mathbb{N}$, we say that E_k^m occurs at the jth step of $\vec{\pi}$ if $E_k(\vec{\pi}_{j-m}, \dots, \vec{\pi}_{j+m})$ occurs at the mth step. (If $j - m < 0$ or $j + m > n$, an obvious modification must be made in this definition: if $j - m < 0$, we require that $E_k(\vec{\pi}_0, \dots, \vec{\pi}_{j+m})$ occurs at the jth step, and similarly if $j + m > n$.) In particular, if E_k^m occurs at the jth step of $\vec{\pi}$, then E_k occurs at the jth step of $\vec{\pi}$.

For $r \ge 0$, let $\vec{\sigma}_n(r, E_k)$ (respectively, $\vec{\sigma}_n(r, E_k^m)$) be the number of SAWs in $\vec{\Sigma}_n$ for which E_k (respectively, E_k^m) occurs at no more than r different steps. Observe that, for given n, r, the count $\vec{\sigma}_n(r, E_k^m)$ is non-increasing in m.

It is easily seen that $\vec{\sigma}_{m+n}(0, E_k) \le \vec{\sigma}_m(0, E_k)\vec{\sigma}_n(0, E_k)$. By the subadditive limit theorem, the limit

$$\lambda_k := \lim_{n \to \infty} \vec{\sigma}_n(0, E_k)^{1/n} \tag{5.2}$$

exists and satisfies

$$\lambda_k \le \vec{\sigma}_n(0, E_k)^{1/n}, \qquad n \ge 1. \tag{5.3}$$

Thus, $\lambda_k < \mu$ if and only if

$$\exists \epsilon > 0,\ M \in \mathbb{N} \text{ such that: } \vec{\sigma}_m(0, E_k) < [\mu(1 - \epsilon)]^m \text{ for } m \ge M. \tag{5.4}$$

Lemma 5.3. *Let k satisfy $1 \le k \le \Lambda$ and*

$$\lambda_k < \mu. \tag{5.5}$$

Let ϵ, M satisfy (5.4), and let $m \ge M$ satisfy

$$\vec{\sigma}_m \le [\mu(1 + \epsilon)]^m. \tag{5.6}$$

There exist $a = a(\epsilon, m) > 0$ and $R = R(\epsilon, m) \in (0,1)$ such that

$$\limsup_{n\to\infty} \vec{\sigma}_n(an, E_k^m)^{1/n} < R\mu. \tag{5.7}$$

Proof. Assume k is such that (5.5) holds, and let ϵ, M, m satisfy (5.4) and (5.6). Since $\vec{\sigma}_m(0, E_k) = \vec{\sigma}_m(0, E_k^m)$,

$$\vec{\sigma}_m(0, E_k^m) < [\mu(1-\epsilon)]^m. \tag{5.8}$$

Let $\vec{\pi} \in \vec{\Sigma}_n$ and $N = \lfloor n/m \rfloor$. If E_k^m occurs at no more than r steps in $\vec{\pi}$, then E_k^m occurs at no more than r of the N m-step subwalks

$$(\vec{\pi}_{(j-1)m}, \vec{\pi}_{(j-1)m+1}, \dots, \vec{\pi}_{jm}), \qquad 1 \le j \le N.$$

Counting the number of ways in which k or fewer of these subwalks can contain an occurrence of E_k^m, we have by (5.6) and (5.8) that

$$\begin{aligned} \vec{\sigma}_n(r, E_k^m) &\le \sum_{i=0}^{r} \binom{N}{i} (\vec{\sigma}_m)^i \vec{\sigma}_m(0, E_k^m)^{N-i} \vec{\sigma}_{n-Nm} \\ &\le \mu^{Nm} \vec{\sigma}_{n-Nm} \sum_{i=0}^{r} \binom{N}{i} (1+\epsilon)^{im} (1-\epsilon)^{(N-i)m}. \end{aligned} \tag{5.9}$$

It suffices to show that there exist $\zeta > 0$ and $t < 1$, depending on ϵ, m only, such that

$$\vec{\sigma}_n(\zeta N, E_k^m)^{1/N} < t\mu^m, \tag{5.10}$$

for all sufficiently large n, since this yields (5.7) with $0 < a < \zeta/m$ and $R = t^{1/m}$. For ζ small and positive,

$$\begin{aligned} &\sum_{i=0}^{\zeta N} \binom{N}{i} (1+\epsilon)^{im} (1-\epsilon)^{(N-i)m} \\ &\qquad \le (\zeta N + 1) \binom{N}{\zeta N} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\zeta N m} (1-\epsilon)^{Nm}. \end{aligned} \tag{5.11}$$

The Nth root of the right side converges as $N \to \infty$ to

$$\frac{1}{\zeta^\zeta (1-\zeta)^{1-\zeta}} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\zeta m} (1-\epsilon)^m,$$

which is strictly less than 1 for $0 < \zeta < \zeta_0$, and some $\zeta_0 = \zeta_0(\epsilon, m) > 0$. Combining this with (5.9), we obtain (5.10) for $0 < \zeta < \zeta_0$, suitable $t = t(\epsilon, m)$, and n sufficiently large. $\square$

Lemma 5.4. *We have that $\lambda_\Lambda < \mu$.*

Proof. Since each vertex π_j is visited by π,

$$\vec{\sigma}_n(0, E_1) = 0, \qquad n \geq 1. \tag{5.12}$$

Case 1. If G has type 1, then $\Lambda = 1$, and $\lambda_\Lambda = 0$ by (5.12).

Cases 2/3. Assume G has type 2 or 3, and that the lemma is false in that

$$\lambda_\Lambda = \mu. \tag{5.13}$$

Now, $\vec{\sigma}_n(0, E_k)$ (and hence λ_k also) is non-decreasing in k. By (5.12), we may choose k with $1 \leq k < \Lambda$ such that

$$\lambda_k < \mu, \quad \lambda_{k+1} = \mu. \tag{5.14}$$

Let ϵ, M satisfy (5.4), and let $m \geq M$ satisfy (5.6). By Lemma 5.3, there exists $a = a(\epsilon, m) > 0$ such that

$$\limsup_{n\to\infty} \vec{\sigma}_n(an, E_k^m)^{1/n} < \mu. \tag{5.15}$$

Let T_n be the subset of $\vec{\Sigma}_n$ comprising SAWs for which E_{k+1} never occurs but E_k^m occurs at least an times. We have that

$$|T_n| \geq \vec{\sigma}_n(0, E_{k+1}) - \vec{\sigma}_n(an, E_k^m),$$

whence, by (5.14) and (5.15),

$$\lim_{n\to\infty} |T_n|^{1/n} = \mu. \tag{5.16}$$

Thus, under (5.13), for 'most' SAWs $\vec{\pi}$, there exist many cycles in $\overline{L}$ having exactly k $(< \Lambda)$ vertices visited by $\vec{\pi}$ and none with more than k such vertices. The rest of the proof is devoted to showing the existence of $S = S(\epsilon, m, \Delta, \Lambda) < 1$ such that

$$\limsup_{n\to\infty} |T_n|^{1/n} \leq S\mu. \tag{5.17}$$

This contradicts (5.16), and the claim follows.

The idea is as follows. Let $\vec{\pi} \in T_n$, and consider the set of cycles $\vec{\ell}(\vec{\pi}_j)$ with exactly k vertices visited by $\vec{\pi}$. Where such a cycle is met by $\vec{\pi}$, we may augment $\vec{\pi}$ with an entire copy of it. The ensuing transformation is not one–one, and the length of the resulting walk differs from that of $\vec{\pi}$. By selecting the places where the new elements of $\vec{L}$ are added, we shall show that the number of resulting walks exceeds $|T_n|$ by an exponential factor. It is key that such augmented walks lift to SAWs on G while traversing cycles in $\vec{G}$, and thus we shall contradict (5.16).

Let $\vec{\pi} \in T_n$, so that $\vec{\pi}$ contains at least an occurrences of E_k^m. We can find $j_1 < \cdots < j_u$ with $u = \lfloor \kappa n \rfloor - 2$ where

$$\kappa = \frac{a}{(2m+2)\Delta^{2\Lambda+1}}, \tag{5.18}$$

such that

$$E_k^m \text{ occurs at the } j_1\text{th, } j_2\text{th, } \dots, \ j_u\text{th steps of } \overline{\pi}, \tag{5.19}$$

(and perhaps at other steps as well), and in addition

$$0 < j_1 - m, \quad j_u + m < n, \tag{5.20}$$

$$j_t + m < j_{t+1} - m, \qquad 1 \le t < u, \tag{5.21}$$

$$\forall \vec{\ell}_t \in \overline{L}(\vec{\pi}_t), \text{ the } \vec{\ell}_1, \vec{\ell}_2, \dots, \vec{\ell}_u \text{ are pairwise vertex-disjoint.} \tag{5.22}$$

Such j_t may be found by the following iterative construction. First, j_1 is the smallest $j > m$ such that E_k^m occurs at the jth step of $\vec{\pi}$. Having found $j_1, j_2, \dots, j_r$, let j_{r+1} be the smallest j such that

1. $j_r + m < j - m$,
2. every element of $\vec{L}(\vec{\pi}_j)$ is disjoint from every element of $\vec{L}(\vec{\pi}_{j_1})$, $\vec{L}(\vec{\pi}_{j_2}), \dots, \vec{L}(\vec{\pi}_{j_r})$,
3. E_k^m occurs at the jth step of $\vec{\pi}$.

Condition 1 gives rise to the factor $2m+2$ in the denominator of (5.18), and, by (5.1), condition 2 gives rise to the factor Δ^{2k+1} ($\le \Delta^{2\Lambda+1}$).

Let $t \in \{1, 2, \dots, u\}$. Since E_k^m but not E_{k+1} occurs at the j_tth step, $\vec{\pi}$ visits at most k vertices in each cycle of $\vec{L}(\vec{\pi}_{j_t})$. Let $\vec{L}(\vec{\pi}_{j_t}, k)$ be the subset of $\vec{L}(\vec{\pi}_{j_t})$ containing all such cycles with exactly k vertices visited by $\vec{\pi}$, and such that these k vertices lie between $\vec{\pi}_{j_t - m}$ and $\vec{\pi}_{j_t + m}$ on $\vec{\pi}$. Choose a specific cycle $\vec{\ell}(\vec{\pi}_{j_t}) \in \vec{L}(\vec{\pi}_{j_t}, k)$. For $t = 1, 2, \dots, u$, let

$$\alpha_t = \min\{i : \vec{\pi}_i \in \vec{\ell}(\vec{\pi}_{j_t})\}, \quad \omega_t = \max\{i : \vec{\pi}_i \in \vec{\ell}(\vec{\pi}_{j_t})\}, \tag{5.23}$$

so that

$$j_t - m \le \alpha_t \le j_t \le \omega_t \le j_t + m, \qquad 1 \le t \le u.$$

We describe next the strategy for replacement of the subwalk $(\vec{\pi}_{\alpha_t}, \vec{\pi}_{\alpha_t+1}, \dots, \vec{\pi}_{\omega_t})$. Starting from $\vec{\pi}_{\alpha_t}$, the new walk winds once around the cycle $\vec{\ell} := \vec{\ell}(\vec{\pi}_{j_t})$; having returned to the vertex $\vec{\pi}_{\alpha_t}$, it continues around the same cycle $\vec{\ell}$ until it reaches the vertex $\vec{\pi}_{\omega_t}$. This new subwalk is inserted into $\vec{\pi}$ at the appropriate place. The resulting walk $\widetilde{\pi}^{(t)}$ is evidently not self-avoiding in $\vec{G}$ (it includes a unique cycle, namely $\vec{\ell}$ in some order), but we shall see that it lifts to a SAW $\pi_*^{(t)}$ on G. The precise definition and properties of $\widetilde{\pi}^{(t)}$ and $\pi_*^{(t)}$ are described next.

Suppose $\vec{\pi} \in T_n$ lifts to $\pi \in \Sigma_n$. The initial segment $(\vec{\pi}_0, \dots, \vec{\pi}_{j_t})$ of $\vec{\pi}$ lifts to a SAW of G that traverses the vertices $\pi_0, \pi_1, \dots, \pi_{j_t}$. We write $v := \pi_{j_t}$, and we shall consider graphs of type 2 and 3 separately.

Case 3. Assume that G has type 3, and think of $\vec{\ell}$ as a rooted, directed cycle of $\vec{G}$ with root $\overline{v}$. The cycle $\vec{\ell}$ lifts to the SAW $\ell := \ell_v$ of G traversing the vertices $v\,(= w_0), w_1, w_2, \dots, w_\Lambda$. We have that

$$v\,(= w_0), w_1, \dots, w_{\Lambda-1} \text{ belong to different equivalence classes} \tag{5.24}$$

of $(V, \approx)$, and we may choose $\beta \in \mathcal{A}$ such that $w_\Lambda = \beta v$.

We prove next that

$$w_r \neq \beta w_r, \qquad 1 \leq r < \Lambda. \tag{5.25}$$

Suppose first that $w_1 = \beta w_1$, and consider the walk

$$v\,(= w_0), w_1, \dots, w_\Lambda\,(= \beta v), \beta w_1\,(= w_1).$$

By the triangle inequality,

$$\Lambda = d_G(v, \beta v) \leq d_G(v, w_1) + d_G(\beta w_1, \beta v) = 2, \tag{5.26}$$

a contradiction since $\Lambda \geq 3$. Applying the same argument to the walk

$$w_1, w_2, \dots, w_\Lambda\,(= \beta v), \beta w_1, \beta w_2,$$

we obtain by (3.7) that $w_2 \neq \beta w_2$, and (5.25) follows by iteration.

Suppose $\vec{\pi}_{\alpha_t}$ lifts to vertex $x \in V$ with $\overline{x} = \overline{w}_i$ and $i \geq 1$; similarly, suppose $\vec{\pi}_{\omega_t} = \overline{w}_j$. We show next that the replacement of the subwalk $(\vec{\pi}_{\alpha_t}, \dots, \vec{\pi}_{\omega_t})$ lifts to some SAW of G. Find $\gamma \in \mathcal{A}$ such that $x = \gamma w_i$.

Case 3.1. Suppose $i < j$. Consider the walk

$$x\,(= \gamma w_i), \gamma w_{i+1}, \dots, \gamma w_\Lambda\,(= \gamma\beta v), \gamma\beta w_1, \dots, \gamma\beta w_i, \tag{5.27}$$

followed by $\gamma\beta w_{i+1}, \gamma\beta w_{i+2}, \dots, \gamma\beta w_j$. By (5.24), two vertices of this walk are equal if and only if there exists r such that $1 \leq r < \Lambda$ and $w_r = \beta w_r$. By (5.25), this does not occur. We have proved that $(\vec{\pi}_0, \dots, \vec{\pi}_{\alpha_t})$, followed by the above walk, lifts to a SAW ν on G. Thus, $\widetilde{\pi}^{(t)}$ lifts to the SAW ν followed by the image of $(\pi_{\omega_t}, \dots, \pi_n)$ under the map that sends π_{ω_t} to $\gamma\beta w_j$. Since $\vec{\pi} \in T_n$, $\widetilde{\pi}^{(t)}$ lifts to a SAW $\pi_*^{(t)}$ of G.

Case 3.2. Suppose $i > j$. Consider the walk

$$x\,(= \gamma w_i), \gamma w_{i-1}, \dots, \gamma w_0\,(= \gamma v), \gamma\beta^{-1} w_{\Lambda-1}, \dots, \gamma\beta^{-1} w_i, \tag{5.28}$$

followed by $\gamma\beta^{-1} w_{i-1}, \gamma\beta^{-1} w_{i-2}, \dots, \gamma\beta^{-1} w_j$. By (5.25), this is a SAW on G, and the step is completed as above.

Case 2. Assume that G has type 2, so that $\Lambda = 2$ and $k = 1$. The required argument is slightly different since (5.26) is no longer a contradiction. Let j_t be as above, and let $\vec{\ell} = \vec{\ell}(\vec{\pi}_{j_t}) \in \vec{L}(\vec{\pi}_{j_t})$ (with corresponding $\ell \in L$) be a witness to the occurrence of E_1^m at $\overline{\pi}_{j_t}$. As in (5.23),

$$\begin{aligned} \alpha_t &= \alpha_t(\vec{\pi}) = \min\{i : \vec{\pi}_i \in \vec{\ell}\}, \\ \omega_t &= \omega_t(\vec{\pi}) = \max\{i : \vec{\pi}_i \in \vec{\ell}\}. \end{aligned} \tag{5.29}$$

Let π be the lift of $\vec{\pi}$ to a SAW in G from v_0, and write $v = \pi_{j_t}$. Recall that ℓ is a SAW visiting $v, w, \beta v$ in G for some $\beta = \beta_t \in \mathcal{A}$ with $\beta v \neq v$, and the pair v, w are visited (in some order) at the α_tth and ω_tth steps of π. We may assume that $\beta w \neq v$, since otherwise w is adjacent to βw, and G is of type 1. We describe next the required substitution.

Case 2.1. Suppose that $\beta w \neq w$. As in Cases 3.1 and 3.2 above, both $v, w, \beta v, \beta w$ and $w, v, \beta^{-1} w, \beta^{-1} v$ are SAWs on G. We write $\pi_*^{(t)}$ for the SAW on G obtained by replacing the segment of π between π_{α_t} and π_{ω_t} by

$$\begin{aligned} v, w, \beta v, \beta w &\quad \text{if } \pi \text{ visits } v \text{ before } w, \\ w, v, \beta^{-1} w, \beta^{-1} v &\quad \text{if } \pi \text{ visits } w \text{ before } v, \end{aligned}$$

and the walk after βw (respectively, $\beta^{-1} v$) by the image under β (respectively, β^{-1}) of π after w (respectively, v).

Case 2.2.1. Suppose that $\beta w = w$, and π visits w before v. Only in the following shall we use the assumed condition (b)(i) of Theorem 3.8. Since $|\partial v \cap \overline{w}|, |\partial w \cap \overline{v}| \geq 2$, there exist $w' \in \overline{w}$ and $v' \in \overline{v}$ such that

$$w, v, w', v' \tag{5.30}$$

is a SAW of G. Let $\gamma \in \mathcal{A}$ be such that $v' = \gamma v$. We write $\pi_*^{(t)}$ for the SAW on G obtained by replacing the segment of π between π_{α_t} and π_{ω_t} by (5.30), and the walk after v' ($= \gamma v$) by the image under γ of π after v.

Case 2.2.2. Suppose that $\beta w = w$, and π visits v before w. Since $|\partial v \cap \overline{w}| \geq 2$, there exists $w' \in \overline{w}$ such that

$$v, w, \beta v, w' \tag{5.31}$$

is a SAW on G. We find $\gamma \in \mathcal{A}$ such that $w' = \gamma w$, and we write $\pi_*^{(t)}$ for the SAW on G obtained by replacing the segment of π between π_{α_t} and π_{ω_t} by (5.31), and the walk after w' ($= \gamma w$) by the image under γ of π after w.

This ends the definitions of the required substitutions, and we consider next their enactment. Let $\delta > 0$, to be chosen later, and set $s = \delta n$. (Here and later, for simplicity of notation, we omit the integer-part symbols.) Let $H = (h_1, h_2, \dots, h_s)$ be an ordered subset of $\{j_1, j_2, \dots, j_u\}$. We shall make an appropriate substitution in the neighbourhood of each $\vec{\pi}_{h_t}$, by an iterative construction.

We consider the cases $t = 1, 2, \dots, u$ in order. First, let $t = 1$. If $j_1 \notin H$, we do nothing, and we set $\pi^{(1)} = \pi$. If $j_1 \in H$, we make the appropriate substitution around the point $\vec{\pi}_{j_1}$, and write $\pi^{(1)}$ for the resulting SAW on G. Now let $t = 2$. Once again, if $j_2 \notin H$, we do nothing, and set $\pi^{(2)} = \pi^{(1)}$. Otherwise, let $\alpha_2' = \alpha_2(\vec{\pi}^{(1)})$, $\omega_2' = \omega_2(\vec{\pi}^{(1)})$. The sub-SAW of $\pi^{(1)}$ between steps α_2' and ω_2' is the image of π between steps α_2 and ω_2 under some $\gamma' \in \mathcal{A}$. Therefore, we may perform operations on $\pi^{(1)}$ after the α_2'th step as discussed above for the α_2th step of π, obtaining thus a SAW $\pi^{(2)}$. This process may be iterated to obtain a sequence $\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(u)}$ of SAWs on G, and we set $\pi_* = \pi^{(u)}$. The role of H is emphasized by writing $\pi_* = \pi_*(\vec{\pi}, H)$. It follows from the construction that $\pi_*(\vec{\pi}, H) \neq \pi_*(\vec{\pi}, H')$ if $H \neq H'$.

One small issue arises during the iteration, namely that the projections of the SAWs $\pi^{(t)}$ are not generally self-avoiding. However, by (5.20)–(5.22), the subwalk of $\vec{\pi}$ under inspection at any given step is disjoint from all previously inspected subwalks, and thus the current substitution is unaffected by the past.

By (5.22) and the discussion above, π_* is self-avoiding on G. By inspection of such π_* and its projection, one may reconstruct the places at which cycles have been added to $\vec{\pi}$. The length of π_* does not exceed $n + 2\Lambda s$.

We estimate the number of pairs $(\vec{\pi}, H)$ as follows. First, the number $|(\vec{\pi}, H)|$ is at least the cardinality of T_n multiplied by the minimum number of possible choices of H as $\vec{\pi}$ ranges over T_n. Any subset of $\{j_1, j_2, \dots, j_u\}$ with cardinality $s = \delta n$ may be chosen for H, whence

$$|(\vec{\pi}, H)| \geq |T_n| \binom{\kappa n - 2}{\delta n}. \tag{5.32}$$

We bound $|(\vec{\pi}, H)|$ above by counting the number of SAWs π_* of G with length not exceeding $n + 2\Lambda\delta n$, and multiplying by an upper bound for the number of pairs $(\vec{\pi}, H)$ giving rise to a particular π_*. The number of possible choices for π_* is no greater than $\sum_{i=0}^{n+2\Lambda\delta n} \sigma_i$. A given π_* contains $|H| = \delta n$ elements of L. At the tth such occurrence, $\vec{\pi}_{h_t}$ is a point on the corresponding cycle, and there are no more than 2Λ different choices for $\vec{\pi}_{h_t}$. For given π and $(\vec{\pi}_{h_t} : t = 1, 2, \dots, s)$, there are at most $\left(\sum_{i=1}^{2m} \vec{\sigma}_i\right)^{\delta n}$ corresponding SAWs $\vec{\pi}$ of $\vec{G}$. Therefore,

$$|(\vec{\pi}, H)| \le \left(2\Lambda \sum_{i=1}^{2m} \vec{\sigma}_i\right)^{\delta n} \left(\sum_{i=0}^{n+2\Lambda\delta n} \sigma_i\right). \tag{5.33}$$

Let $\tau := \limsup |T_n|^{1/n}$. We combine (5.32)–(5.33), take nth roots and the limit as $n \to \infty$, to obtain, by the fact that $\sigma_N^{1/N} \to \mu$,

$$\tau \frac{\kappa^\kappa}{\delta^\delta (\kappa-\delta)^{\kappa-\delta}} \le \left(2\Lambda \sum_{i=1}^{2m} \vec{\sigma}_i\right)^{\delta} \mu^{1+2\Lambda\delta}.$$

There exists $Z = Z(\epsilon, m, \Delta, \Lambda) < \infty$ such that

$$2\Lambda\mu^{2\Lambda} \sum_{i=1}^{2m} \vec{\sigma}_i \le Z.$$

Therefore,

$$\tau \le f(\eta)^\kappa \mu,$$

where $f(\eta) = Z^\eta \eta^\eta (1-\eta)^{1-\eta}$ and $\eta = \delta/\kappa$. Since

$$\lim_{\eta \downarrow 0} f(\eta) = 1, \quad \lim_{\eta \downarrow 0} f'(\eta) = -\infty,$$

we have that $f(\eta) < 1$ for sufficiently small $\eta = \eta(Z) > 0$, and (5.17) follows for suitable $S < 1$. The proof is complete. $\square$

Proof of Theorem 3.8 under Assumption 5.2. By Lemma (5.4), $\lambda_\Lambda < \mu$. Let ϵ, M satisfy (5.4) and let $m \ge M$ satisfy (5.6), with $k = \Lambda$. By Lemma 5.3, there exist $a = a(\epsilon, m) > 0$ and $R = R(\epsilon, m) \in (0, 1)$ such that

$$\limsup_{n \to \infty} \vec{\sigma}_n(an, E_\Lambda^m)^{1/n} < R\mu. \tag{5.34}$$

Let T_n be the subset of $\vec{\Sigma}_n$ comprising SAWs for which E_Λ^m occurs at least an times. Thus

$$|T_n| \ge \vec{\sigma}_n - \vec{\sigma}_n(an, E_\Lambda^m). \tag{5.35}$$

We follow the route of the previous proof. For $\vec{\pi} \in T_n$ and κ as in (5.18), we may find $j_1 < \cdots < j_u$ with $u = \kappa n - 2$ such that (5.19)–(5.22) hold with $k = \Lambda$.

Let $\delta > 0$, set $s = \delta n$, and let $\vec{\pi} \in T_n$. We choose a subset H of $\{j_1, j_2, \dots, j_u\}$ with cardinality s, and we construct a SAW $\pi_* = \pi_*(\vec{\pi}, H)$ on G accordingly. This may be done exactly as in the previous proof *if G has type 2 or 3* to obtain as in (5.17) that there exists $S = S(\epsilon, m, \Delta, \Lambda) < 1$ such that

$$\limsup_{n\to\infty} |T_n|^{1/n} \le S\mu. \tag{5.36}$$

By (5.34)–(5.35),

$$\vec{\mu} = \lim_{n\to\infty} \vec{\sigma}_n^{1/n} \le \max\{R, S\}\mu,$$

and the claim of the theorem is proved in this case.

Assume G has type 1. This case is easier than the others. Suppose $\vec{\pi}_{j_t} = \overline{v}$, and $\vec{\pi}$ lifts to π with $\pi_{j_t} = v$. We obtain $\pi_*^{(t)}$ by replacing v by the pair $v, \gamma v$ for some $\gamma \in \mathcal{A}$ with $\gamma \ne \iota$, and translating by γ the subwalk of π starting at v. The above counting argument yields the result. □

Remark 5.5. *Theorem 3.8 has been proved subject to Assumption 5.2, and it remains to prove it when $\mathcal{A}$ has type 2 and condition (b)(ii) holds. Suppose the latter holds, so that $l \ge 2$. If $l = 2$, the argument in the proof of Lemma 5.4 is valid (this is Case 2.1 of the proof, which does not use condition (i)), and the theorem follows similarly. Assume $l \ge 3$. By vertex-transitivity, we may set $v_0 = v$, and replace ℓ_{v_0} by the path of (ii), so that $\Lambda = l$. Case 3 of the proof of Lemma 5.4 may be followed with one difference, namely that (5.25) holds by the assumption that β fixes no w_i.*

Proof of Theorem 3.9. We use the notation of the previous proofs. The constants R, S of the last proof may be calculated explicitly in terms of ϵ, m, Δ, Λ, and so it suffices to describe how to choose ϵ and m.

Let $a_n = \sigma_n^{1/n}$, noting by (2.2) that $a_n \ge \mu$ and $a_n \to \mu$. By changing (b_n) if necessary, we may assume for convenience that (b_n) is a non-decreasing sequence, so that $b_n \uparrow \mu$.

By Lemma 5.4, $\lambda_\Lambda < \mu$. Since $b_r(1 - r^{-1}) \uparrow \mu$, we may find the earliest r such that

$$\vec{\sigma}_r(0, E_\Lambda)^{1/r} < b_r(1 - r^{-1}).$$

Let $\epsilon = r^{-1}$, and let $s \ge r$ be such that $b_s(1+\epsilon) \ge a_s(1 + \frac{1}{2}\epsilon)$. Since $b_s \ge b_r$,

$$\vec{\sigma}_r(0, E_\Lambda)^{1/r} < b_s(1 - \epsilon).$$

By (5.3), $\lambda_\Lambda < b_s(1-\epsilon)$, so that

$$\vec{\sigma}_m(0, E_\Lambda)^{1/m} < b_s(1 - \epsilon) \tag{5.37}$$

for infinitely many values of m.

Since $a_s \ge \mu$, we may find the earliest m such that (5.37) holds and in addition $\vec{\sigma}_m^{1/m} \le b_s(1+\epsilon)$. With ϵ and m defined thus, (5.6) and (5.8) are valid as required.

Each of the above computations requires a finite enumeration of a suitable family of SAWs. $\square$

Outline Proof of Theorem 3.10. Let $v_0 \in V$ with orbit $\overline{v}_0 = \mathcal{A}v_0$. With $l \ge 1$ as given, let $\vec{\ell}$ be a (directed) cycle of $\vec{G}$ of length l. Since $\vec{G}$ is vertex-transitive, we may assume $\vec{\ell}$ goes through $\overline{v}_0$, and thus we write $\vec{\ell}_{\overline{v}_0}$ for $\vec{\ell}$, considered as a (directed) walk from $\overline{v}_0$ to $\overline{v}_0$. Now, $\vec{\ell}_{\overline{v}_0}$ lifts to a walk ℓ_{v_0} from v_0 on G. Since every vertex of $\vec{\ell}_{\overline{v}_0}$ other than its endpoints are distinct, and G possesses no cycle of length l, ℓ_{v_0} is a SAW from v_0 on G.

Let $\mathcal{B}$ be a subgroup of $\mathrm{Aut}(\vec{G})$ acting transitively on $\vec{G}$. For $w \in V$, let $\vec{L}(\vec{w})$ be the set of images of $\vec{\ell}_{\overline{v}_0}$ under $\mathcal{B}$, and let $L(w)$ be the set of lifts of such images. Since G possesses no cycle of length l, every member of $L(w)$ is a SAW on G.

We now follow the above proof of Theorem 3.8. No assumption is made on the relationship between the groups Γ and $\mathcal{B}$. If $l \ne 2$, the argument is as presented previously except insomuch as that (5.25) holds (since Γ acts freely). If $l = 2$, we follow Case 2.1 as in Remark 5.5. $\square$

6. Proof of Theorems 3.2–3.3

Proof of Proposition 3.5. Fix $v_0 \in V$, and choose $v_1, v_2 \dots$ such that $\mathcal{A}v_0, \mathcal{A}v_1, \dots$ are the (distinct) orbits of $\mathcal{A}$. Let $\gamma_0 = \iota$ and, for $i \ge 1$, find $\gamma_i \in \Gamma$ such that $\gamma_i v_0 = v_i$. Let $\Gamma/\mathcal{A}$ denote the set of right cosets of $\mathcal{A}$, and let $\phi : V/\mathcal{A} \to \Gamma/\mathcal{A}$ be given by $\phi(\mathcal{A}v_i) = \mathcal{A}\gamma_i$. We check next that ϕ is an injection.

It suffices to show that $i = j$ whenever $\mathcal{A}\gamma_i = \mathcal{A}\gamma_j$. Suppose $\mathcal{A}\gamma_i = \mathcal{A}\gamma_j$. There exists $\alpha \in \mathcal{A}$ such that $\gamma_i = \alpha\gamma_j$. Therefore,

$$v_i = \gamma_i v_0 = \alpha\gamma_j v_0 = \alpha v_j,$$

whence $\mathcal{A}v_i = \mathcal{A}v_j$ and $i = j$. Thus, ϕ is an injection, and (3.3) follows.

Suppose $|V/\mathcal{A}| < \infty$, and write $U := \bigcup_{i=0}^{\infty} \mathcal{A}\gamma_i$. Equality holds in (3.3) if and only if ϕ is a surjection, which is to say that $\Gamma = U$. Assume there exists $\rho \in \Gamma \setminus U$. Find j such that $\rho v_0 \in \mathcal{A}v_j$, say $\rho v_0 = \alpha v_j$ with $\alpha \in \mathcal{A}$. Then $\alpha^{-1}\rho\gamma_j^{-1} \in \mathrm{Stab}_{v_j}$. If $\alpha^{-1}\rho\gamma_j^{-1} \in \mathcal{A}$, then $\rho \in \mathcal{A}\gamma_j$ in contradiction of the assumption $\rho \notin U$. Therefore, $\alpha^{-1}\rho\gamma_j^{-1} \in \Gamma \setminus \mathcal{A}$, and $\Gamma \setminus \mathcal{A}$ does not act freely.

Suppose conversely that there exist $\rho \in \Gamma \setminus \mathcal{A}$ and $v \in V$ with $\rho v = v$. Set $w_0 = v$ and find $w_i \in V$ such that the (distinct) orbits of $\mathcal{A}$ are $\mathcal{A}w_0, \mathcal{A}w_1, \dots$. Let $\gamma_0 = \iota$ and, for $i \geq 1$, find $\gamma_i \in \Gamma$ such that $\gamma_i w_0 = w_i$. If $\rho \in \mathcal{A}\gamma_i$, then $\rho = \alpha\gamma_i$ for some $\alpha \in \mathcal{A}$, so that $w_0 = \rho w_0 = \alpha\gamma_i w_0 = \alpha w_i$. This implies that $i = 0$, and hence $\rho \in \mathcal{A}$, a contradiction. Therefore, $\rho \in \Gamma \setminus U$ where $U = \bigcup_i \mathcal{A}\gamma_i$. It follows as above that $|V/\mathcal{A}| < [\Gamma : \mathcal{A}]$. □

The graphs G, $\overline{G}$ have the same vertex-set V, but $\overline{G}$ possesses cycles not present in G. Let $v_0, w_0 \in V$ be such that there are strictly more edges of the form $\langle v_0, w_0 \rangle$ in $\overline{E}$ than in E. Since G is connected, it has a shortest path from v_0 to w_0, written

$$v_0 f_1 v_1 f_2 v_2 \cdots f_\Lambda v_\Lambda \, (= w_0) \quad \text{where} \quad f_t = \langle v_{t-1}, v_t \rangle. \tag{6.1}$$

The path (6.1), followed by the new edge $f := \langle w_0, v_0 \rangle$, forms a cycle in $\overline{G}$ but not in G. Let C_1 denote this cycle of $\overline{G}$, and note that C_1 has length $\Lambda + 1$. As usual, Δ denotes the vertex-degree of G.

Let C be a finite set of vertices of G with $v_0 \in C$, and denote by $\gamma C = \{\gamma z : z \in C\}$ the image of C under the automorphism $\gamma \in \Gamma$. Let $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ be an n-step SAW on G from v_0. For $k \in \mathbb{N}$, we say that $E_k(C)$ occurs at the jth step of π if there exists $\gamma \in \Gamma$ with $\gamma v_0 = \pi_j$ such that at least k vertices of γC are visited by π. Let $\sigma_n(r, E_k(C))$ (respectively, $\sigma_n(r, E_*(C))$) be the number of n-step SAWs on G from v_0 in which $E_k(C)$ (respectively, $E_{|C|}(C)$) occurs at no more than r steps. When C is the vertex-set of C_1, we write $E_k^1 = E_k(C_1)$ for $E_k(C)$, and so on.

The general idea of the proof of Theorem 3.2 is similar to that of Theorem 3.8, but with some differences. A key element is the following lemma.

Lemma 6.1. *With $\mu = \mu(G)$, we have that*

$$\liminf_{n\to\infty} \sigma_n(0, E_2^1)^{1/n} < \mu. \tag{6.2}$$

As in (5.2), the lim inf is in fact a limit (here and later). An automorphism $\gamma \in \Gamma$ acts on the edge $e = \langle x, y \rangle$ by $\gamma e = \langle \gamma x, \gamma y \rangle$. Let $\asymp$ be the equivalence relation on the edge-set of $\overline{G}$ given by $e \asymp e'$ if there exists $\gamma \in \Gamma$ such that $\gamma e = e'$. We write $\overline{e}$ for the equivalence class containing edge e, and $\{\overline{e} : e \in E\}$ for the set of such classes. Since Γ acts transitively on $\overline{G}$, for every edge e and vertex v, there exists a member of $\overline{e}$ that is incident to v.

The proof of Lemma 6.1 will make use of the intermediate Lemmas 6.2–6.3.

Lemma 6.2. *Assume that*

$$\lim_{n\to\infty} \sigma_n(0, E_2^1)^{1/n} = \mu. \tag{6.3}$$

There exist $h \geq 2$ and a sequence $(\overline{e}_r : r = 1, 2, \dots, h-1)$ of distinct equivalence classes of edges of G, with $\overline{e}_1 = \overline{f}_1$, such that the following holds.

For $r \geq 1$, let

$$F_r = \overline{e}_1 \cup \overline{e}_2 \cup \cdots \cup \overline{e}_r,$$

and let G_r be the graph obtained from G by the deletion of F_r.

There exists a sequence $C_2, C_3, \dots, C_h$ of cycles in G such that:

$$\text{for } 2 \leq r \leq h, \text{ there exists } e_{r-1} \in \overline{e}_{r-1} \text{ that is incident to } v_0 \text{ and belongs to both } C_{r-1} \text{ and } C_r, \tag{6.4}$$

$$\lim_{n\to\infty} \sigma_{n,r-1}(0, E_2(C_r))^{1/n} = \mu, \qquad 2 \leq r < h, \tag{6.5}$$

$$\liminf_{n\to\infty} \sigma_{n,h-1}(0, E_2(C_h))^{1/n} < \mu, \tag{6.6}$$

where $\sigma_{n,r-1}(0, E_2(C_r))$ is the number of n-step SAWs on G_{r-1} from v_0 for which $E_2(C_r)$ never occurs.

Proof of Lemma 6.2. We construct the $\overline{e}_r$ and C_r by iteration. Let G_r' be obtained from G_{r-1} (where $G_0 := G$) by the deletion of all edges in $\overline{e}_r$ incident to v_0. Assume (6.3) holds, and let $\overline{e}_1 = \overline{f}_1$.

The case $r = 2$. By (6.3), for 'most' n-step SAWs π on G from v_0, and for all j and all γC_1 with $\gamma \in \Gamma$ and $\gamma v_0 = \pi_j$, π_j is the unique vertex of γC_1 visited by π. Since no edge in F_1 is traversed by any path contributing to $\sigma_n(0, E_2^1)$, by (6.3),

$$\mu(G_1) = \mu. \tag{6.7}$$

Let $v_0^{(1)}$ be the neighbour of v_0 such that $f_1 = \langle v_0, v_0^{(1)} \rangle$. We claim that, subject to (6.3),

$$v_0 \text{ and } v_0^{(1)} \text{ are connected in } G_1', \tag{6.8}$$

and shall prove this by contradiction. Assume (6.3) holds, and that v_0 and $v_0^{(1)}$ are not connected in G_1'.

Let $\pi^{(1)}$, $\pi^{(2)}$ be two SAWs from v_0 on G_1, and consider a walk ν on G given as follows: ν follows $\pi^{(1)}$ from v_0 to its other endvertex z, then traverses an edge of the form $f = \langle z, w \rangle \in \overline{e}_1$, and then follows $\gamma\pi^2$ for some $\gamma \in \Gamma$ satisfying $\gamma v_0 = w$. By the above assumption and vertex-transitivity, the removal of all edges of $\overline{e}_1$ incident to z disconnects z and w, and thus any such ν is a SAW on G.

This process of concatenation may be iterated as follows. Let $\pi^{(1)}$, $\pi^{(2)}, \dots, \pi^{(N)}$ be SAWs on G_1 from v_0. We aim to construct a SAW ν on G as follows. Suppose we have concatenated $\pi^{(1)}, \dots, \pi^{(r)}$ to obtain a SAW $\nu^{(r)}$ on G from v_0 to some vertex z. We now traverse an edge of the form $f = \langle z, w\rangle \in \overline{e}_1$, followed by the SAW $\gamma\pi^{(2)}$ for some $\gamma \in \Gamma$ with $\gamma v_0 = w$. Unless (i) there is a unique edge of $\overline{e}_1$ incident to z and (ii) $\pi^{(r)}$ is the SAW with zero length, then there exists a choice of w such that the resulting walk $\nu^{(r+1)}$ is a SAW. Therefore, if every $\pi^{(r)}$ has length 1 or more, the sequence $(\pi^{(r)})$ may be concatenated to obtain a SAW $\nu = \nu^{(N)}$ on G.

Let $\sigma_{n,1}$ be the number of n-step SAWs from v_0 on G_1, and let $\mu_1 = \mu(G_1)$. By (2.2),

$$\sigma_{2n,1} \ge \mu_1^{2n}, \qquad n \ge 1. \tag{6.9}$$

Let Σ_{2n}^{o} be the set of $2n$-step SAWs from v_0 on G that traverse edges in F_1 (in either direction) at the odd-numbered steps only, and write $\sigma_{2n}^{\mathrm{o}} = |\Sigma_{2n}^{\mathrm{o}}|$. Our purpose in considering only the odd steps is to avoid component walks of zero length.

Let $0 \le j \le n$, and consider the set of all $\pi \in \Sigma_{2n}^{\mathrm{o}}$ that traverse exactly j edges of F_1. There are $\binom{n}{j}$ ways of choosing the indices of these edges within π. By the construction described above, and (6.9), there are at least μ_1^{2n-j} choices for the sequence of SAWs on G_1 whose concatenation makes π. Therefore,

$$\sigma_{2n}^{\mathrm{o}} \ge \sum_{j=0}^{n} \binom{n}{j} \mu_1^{2n-j} = \mu_1^{2n}(1+\mu_1^{-1})^n. \tag{6.10}$$

This implies

$$\sigma_{2n} \ge \sigma_{2n}^{\mathrm{o}} \ge \mu_1^{2n}(1+\mu_1^{-1})^n, \tag{6.11}$$

whence $\mu \ge \mu_1\sqrt{1+\mu_1^{-1}} > \mu_1$, in contradiction of (6.7). Statement (6.8) is proved.

By (6.8), G has a shortest (directed) walk W_2 from v_0 to $v_0^{(1)}$ using no edge of $\overline{e}_1$ incident to v_0. The walk W_2, followed by the (directed) edge $\langle v_0^{(1)}, v_0\rangle$, forms the required (directed) cycle C_2 of G, having e_1 in common with C_1. Let $e_2 = \langle v_0, v_0^{(2)}\rangle$ be the first edge of W_2, and write $E_k^2 = E_k(C_2)$ and so on.

If

$$\liminf_{n\to\infty} \sigma_{n,1}(0, E_2^2)^{1/n} < \mu, \tag{6.12}$$

we have proved the claim with $h = 2$. Suppose conversely that

$$\lim_{n\to\infty} \sigma_{n,1}(0, E_2^2)^{1/n} = \mu, \tag{6.13}$$

whence $\mu(G_2) = \mu$ as in (6.7).

The general case. Suppose $u \ge 2$, and cycles $C_2, C_3, \dots, C_u$ from v_0 have been found, with corresponding classes $\overline{e}_2, \overline{e}_3, \dots, \overline{e}_u$ and edges $e_i = \langle v_0, v_0^{(i)}\rangle \in \overline{e}_i$, such that (6.4) holds with h replaced by u, and

$$\lim_{n\to\infty} \sigma_{n,i-1}(0, E_2^i)^{1/n} = \mu, \qquad 1 \le i \le u, \tag{6.14}$$

where $E_2^i = E_2(C_i)$. By (6.14) with $r = u$, we have as above that $\mu(G_u) = \mu$.

Let $\sigma^{\circ}_{2n,u-1}$ be the number of $2n$-step SAWs on G_{u-1} from v_0 that traverse edges in F_u only at odd steps. If v_0 and $v_0^{(u)}$ are disconnected in G'_u, we have

$$\sigma_{2n} \ge \sigma^{\circ}_{2n,u-1} \ge \sum_{j=0}^{n} \binom{n}{j} \mu_u^{2n-j} = \mu_u^{2n}(1 + \mu_u^{-1})^n, \tag{6.15}$$

where $\mu_u = \mu(G_u)$. This contradicts $\mu_u = \mu$, whence v_0 and $v_0^{(u)}$ are connected in G'_u.

Therefore, v_0 and $v_0^{(u)}$ are connected by a shortest (directed) walk W_{u+1} of G_{u-1} using no edge of $\overline{e}_u$ incident to v_0, and we write $e_{u+1} = \langle v_0, v_0^{(u+1)}\rangle$ for the first edge of W_{u+1}, and C_{u+1} for the cycle in G_{u-1} formed by W_{u+1} followed by $\langle v_0^{(u)}, v_0\rangle$. If

$$\liminf_{n\to\infty} \sigma_{n,u}(0, E_2^{u+1})^{1/n} < \mu, \tag{6.16}$$

we stop and set $h = u + 1$, and otherwise we continue as above.

Either this iterative process terminates at some earliest u satisfying (6.16), or not. In the step from G_u to G_{u+1}, the degree of v_0 is reduced strictly. If the process does not terminate in the manner of (6.16), all edges incident to v_0 in the last non-trivial graph encountered, G_t say, lie in the same equivalence class. Furthermore, the equation of (6.14) holds with i replaced by $t + 1$, and hence $\mu_{t+1} = \mu$. This is impossible since v_0 is isolated in G_{t+1}. The proof is complete. □

If (6.3) fails, we set $h = 1$ and $G_0 = G$. If (6.3) holds, we adopt the notation of Lemma 6.3, and write $E_k^i = E_k(C_i)$ and so on. For an n-step SAW $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ on G, we say that $E_k^i(m)$ occurs at the jth step of π if E_k^i occurs at the mth step of the $2m$-step SAW $(\pi_{j-m}, \dots, \pi_{j+m})$ (with a modified definition if either $j - m < 0$ or $j + m > n$, as in the second paragraph after (5.1)).

Lemma 6.3. *Let $1 \le i \le h$ and $1 \le k \le |C_i|$. If*
$$\liminf_{n\to\infty} \sigma_{n,i-1}(0, E_k^i) < \mu,$$
then there exist $a > 0$ and $m \in \mathbb{N}$ such that
$$\limsup_{n\to\infty} \sigma_{n,i-1}(an, E_k^i(m))^{1/n} < \mu.$$

Proof. This is proved in the same manner as was Lemma 5.3. □

Proof of Lemma 6.1. Assume the converse of (6.2), namely (6.3), and adopt the notation of Lemma 6.2. Exactly one of the following holds:
$$\liminf_{n\to\infty} \sigma_{n,h-1}(0, E_*^h)^{1/n} < \mu, \tag{6.17}$$
$$\lim_{n\to\infty} \sigma_{n,h-1}(0, E_*^h)^{1/n} = \mu, \tag{6.18}$$
and we shall derive a contradiction in each case.

Assume first that (6.17) holds. No SAW counted in $\sigma_{n,h-2}(0, E_2^{h-1})$ traverses any edge of $\overline{e}_{h-1}$, whence $\sigma_{n,h-2}(0, E_2^{h-1}) = \sigma_{n,h-1}(0, E_2^{h-1})$. Since C_h and C_{h-1} have at least one common edge, we deduce that
$$\sigma_{n,h-2}(0, E_2^{h-1}) \le \sigma_{n,h-1}(0, E_*^h),$$
and hence
$$\liminf_{n\to\infty} \sigma_{n,h-2}(0, E_2^{h-1})^{1/n} < \mu,$$
in contradiction of the minimality of h.

Assume now that (6.18) holds. By (6.6) and the monotonicity in k of $\sigma_{n,h-1}(0, E_k^h)$, there exists $k \in \mathbb{N}$ satisfying $2 \le k < \ell := |C_h|$ such that
$$\liminf_{n\to\infty} \sigma_{n,h-1}(0, E_k^h)^{1/n} < \mu, \tag{6.19}$$
$$\lim_{n\to\infty} \sigma_{n,h-1}(0, E_{k+1}^h)^{1/n} = \mu. \tag{6.20}$$
By (6.19) and Lemma 6.3, there exist $a > 0$ and $m \in \mathbb{N}$ such that
$$\limsup_{n\to\infty} \sigma_{n,h-1}(an, E_k^h(m))^{1/n} < \mu. \tag{6.21}$$
We shall derive a contradiction from (6.20)–(6.21) in a similar manner to the proof of (5.17).

Let T_n be the set of n-step SAWs on G_{h-1} from v_0 such that E_{k+1}^h never occurs, and $E_k^h(m)$ occurs at least an times. By (6.20)–(6.21),
$$\lim_{n\to\infty} |T_n|^{1/n} = \mu. \tag{6.22}$$

Let $\pi = (\pi_0, \pi_1, \dots, \pi_n) \in T_n$ and $u = \lfloor \kappa n \rfloor - 2$ where
$$\kappa = \frac{a}{(2m+2)\ell}.$$

As after (5.18), we may find $j_1 < j_2 < \cdots < j_u$ and $\gamma_1, \dots, \gamma_u \in \Gamma$ such that

$$\gamma_t(v_0) = \pi_{j_t}, \qquad 1 \le t \le u, \tag{6.23}$$

$$\text{each } 2m\text{-step SAW } (\pi_{j_t - m}, \dots, \pi_{j_t+m}) \text{ visits at least } k \text{ vertices of } \gamma_t C_h, \tag{6.24}$$

$$0 < j_1 - m, \quad j_u + m < n, \tag{6.25}$$

$$j_t + m < j_{t+1} - m, \qquad 1 \le t < u, \tag{6.26}$$

$$\gamma_1 C_h, \gamma_2 C_h, \dots, \gamma_u C_h \text{ are pairwise vertex-disjoint.} \tag{6.27}$$

For $t = 1, 2, \dots, u$, let

$$\alpha_t = \min\{i : \pi_i \in \gamma_t C_h\}, \quad \omega_t = \max\{i : \pi_i \in \gamma_t C_h\}. \tag{6.28}$$

Since $E_k^h(m)$ occurs at the j_tth step but not E_{k+1}^h, there are exactly k points of $\gamma_t C_h$ that are visited by π, and these points lie on π between positions $j_t - m$ and $j_t + m$. Therefore,

$$j_t - m \le \alpha_t < \omega_t \le j_t + m, \qquad 1 \le t \le u. \tag{6.29}$$

We propose to replace the subwalk $(\pi_{\alpha_t}, \dots, \pi_{\omega_t})$ by the part of the cycle $\gamma_t C_h$ with the same endpoints and using at least one edge in $\gamma_t C_{h-1}$; this may be done since $\gamma_t C_h$ and $\gamma_t C_{h-1}$ have at one edge in common, namely $\gamma_t e_{h-1}$. By (6.27), such a replacement may be performed simultaneously for all t. The resulting walk ψ is a SAW on G with length n' satisfying

$$n' < n + u\ell.$$

Furthermore, since $\pi \in T_n$, the only edges of $\overline{e}_{h-1}$ in ψ are those introduced during a substitution.

Let $\delta > 0$ and $s = \delta n$, where δ will be chosen later (and we omit integer-part symbols as before). Consider the set of pairs (π, H) where $\pi \in T_n$, and $H = (h_1, h_2, \dots, h_s)$ is an ordered subset of $\{j_1, j_2, \dots, j_u\}$. We may make the above replacement around each π_{h_i} to obtain a SAW $\psi = \psi(\pi, H)$ on G

As in (5.32),

$$|(\pi, H)| \ge |T_n| \binom{\kappa n - 2}{\delta n}. \tag{6.30}$$

For an upper bound, consider a given pair (π, H). Edges in $\overline{e}_{h-1}$ are traversed between $|H| = \delta n$ and $\ell \delta n$ times on ψ. Therefore, given ψ, there are at most $2m\binom{\ell\delta n}{\delta n}$ $(\le m 2^{\ell \delta n + 1})$ possibilities for the location of the earliest point of ψ in $\gamma_t C_h$. Given ψ and the locations of these earliest points, there are at most Δ^ℓ different choices for each $\gamma_t C_h$.

Once these are determined, each such $\gamma_t C_h$ determines a subwalk of ψ that replaces some subwalk of π. Since each of the replaced subwalks of π has length not exceeding $2m$, there are at most $Y^{\delta N}$ possibilities for π, where $Y = \sum_{i=0}^{2m} \sigma_i$. Therefore,

$$|(\pi, H)| \le m2^{\ell\delta n+1}\Delta^\ell Y^{\delta n} \sum_{i=0}^{n+\ell\delta n} \sigma_i. \tag{6.31}$$

We combine (6.30) and (6.31), take the nth root and let $n \to \infty$, obtaining by (6.22) that

$$\mu \frac{\kappa^\kappa}{\delta^\delta(\kappa-\delta)^{\kappa-\delta}} \le 2^{\ell\delta} Y^\delta \mu^{1+\ell\delta}.$$

Setting $Z = (2\mu)^\ell Y$ and $\eta = \delta/\kappa$, we deduce that

$$1 \le [Z^\eta \eta^\eta (1-\eta)^{1-\eta}]^\kappa.$$

As at the end of the proof of Lemma 5.4, this is a contradiction for small $\eta > 0$.

In conclusion, if (6.3) and (6.18) hold, we have a contradiction, and the lemma is proved. □

Proof of Theorem 3.2. Write $\overline{\mu} := \mu(\overline{G})$, and let F be the set of edges of $\overline{G}$ not in G. Let f and C_1 be given as around (6.1), and let $\ell = \Lambda + 1$ be the number of vertices in C_1 (or, equivalently, the number of edges in C_1 viewed as a cycle of $\overline{G}$). It suffices to make the following assumption.

Assumption 6.4. *We have that $F = \mathcal{A}f$.*

For $v, w \in V$ and a graph H with vertex-set V, let $N(v, w; H)$ denote the number of edges between v and w in H. For $\gamma \in \Gamma$, we construct the graph denoted $\gamma\overline{G}$ as follows. First, $\gamma\overline{G}$ has vertex-set V. For $v, w \in V$, we place $N(v, w; \overline{G})$ edges between γv and γw in $\gamma\overline{G}$. Thus, $\gamma\overline{G}$ is obtained from γG by adding the edges of γF. Two elementary properties of $\gamma\overline{G}$ are as follows.

Lemma 6.5. *Let $\gamma \in \Gamma$.*

(a) *$\gamma\overline{G}$ is isomorphic to $\overline{G}$.*
(b) *$\gamma\mathcal{A}\gamma^{-1} \subseteq \mathrm{Aut}(\gamma\overline{G})$.*

Proof. (a) This is an immediate consequence of the definition of $\gamma\overline{G}$.

(b) By the definition of $\gamma\overline{G}$, for $\alpha \in \mathcal{A}$,

$$\begin{aligned} N\big(\gamma\alpha\gamma^{-1}(\gamma v), \gamma\alpha\gamma^{-1}(\gamma w); \gamma\overline{G}\big) &= N(\gamma\alpha v, \gamma\alpha w; \gamma\overline{G}) \\ &= N(\alpha v, \alpha w; \overline{G}) \\ &= N(v, w; \overline{G}) \\ &= N(\gamma v, \gamma w; \gamma\overline{G}), \end{aligned}$$

and the claim follows. □

Assume $\overline{\mu} = \mu$. By Lemma 6.1 with $E_k := E_k^1$,

$$\liminf_{n\to\infty} \sigma_n(0, E_2)^{1/n} < \mu \ (= \overline{\mu}). \tag{6.32}$$

Exactly one of the following holds:

$$\liminf_{n\to\infty} \sigma_n(0, E_\ell)^{1/n} < \mu, \tag{6.33}$$

$$\lim_{n\to\infty} \sigma_n(0, E_\ell)^{1/n} = \mu. \tag{6.34}$$

Assume first that (6.33) holds. By Lemma 6.3, there exist $a > 0$ and $m \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \sigma_n(an, E_\ell(m))^{1/n} < \mu. \tag{6.35}$$

Let R_n be the set of n-step SAWs from v_0 on G on which $E_\ell(m)$ occurs at least an times. By (6.35),

$$\lim_{n\to\infty} |R_n|^{1/n} = \mu. \tag{6.36}$$

Let $\rho \in V$ be such that $\mathcal{A}$ has the finite coset property with root ρ, and let $\nu_0, \nu_1, \dots, \nu_s$ be given accordingly as in Definition 3.1. Let $\mathcal{O} = \Gamma C_1$ be the orbit under Γ of C_1 viewed as a set of labelled vertices, and let $\mathcal{O}_i = \nu_i \mathcal{A} C_1$. Let $0 \le i \le s$. We say that $E_\ell^i(m)$ occurs at the jth step of a SAW π on G if there exists $\gamma \in \Gamma$ with $\gamma\pi_0 = \pi_j$ and $\gamma C_1 \in \mathcal{O}_i$, such that all the vertices of γC_1 are visited by the $2m$-step SAW $(\pi_{j-m}, \dots, \pi_{j+m})$ (subject to the usual amendment if $j - m < 0$ or $j + m > n$).

Let $R_n^{(i)}$ be the set of n-step SAWs from v_0 on G for which $E_\ell^i(m)$ occurs at least $an/(s+1)$ times. Thus,

$$R_n \subseteq \bigcup_{i=0}^{s} R_n^{(i)}. \tag{6.37}$$

By (6.36)–(6.37), there exists i satisfying $0 \le i \le s$ such that

$$\limsup_{n\to\infty} |R_n^{(i)}|^{1/n} = \mu, \tag{6.38}$$

and we choose i accordingly.

We now apply the argument in the proof of Lemma 6.1 with the constant a replaced by $a/(s+1)$. Let $\pi \in R_n^{(i)}$ be such that $E_\ell^i(m)$ occurs at steps $j_1 < j_2 < \cdots < j_u$, and in addition (6.25)–(6.27) hold with h replaced by 1, and with each $\gamma_t C_1 \in \mathcal{O}_i$. Since $\gamma_t C_1 \in \mathcal{O}_i = \nu_i \mathcal{A} C_1$, there exists $\alpha_t \in \mathcal{A}$ such that $\gamma_t C_1 = [\nu_i \alpha_t \nu_i^{-1}]\nu_i C_1$. By Lemma 6.5(b), $[\nu_i \alpha_t \nu_i^{-1}]\nu_i \overline{G}$ is isomorphic to $\nu_i \overline{G}$. Therefore, $\gamma_t f$ is an edge of $\nu_i \overline{G}$ but not of $\nu_i G$. We think of π as a SAW on the graph $\nu_i G$.

Let α_t, ω_t be as in (6.28), and note that (6.29) holds as before. Consider the replacement of the subwalk $(\pi_{\alpha_t}, \dots, \pi_{\omega_t})$ by a walk that goes along that part of $\gamma_t C_1$ $(= \gamma_t' \nu_i C_1$ where $\gamma_t' = \nu_i \alpha_t \nu_i^{-1})$, viewed as a cycle, that includes an edge of $\gamma_t F$ $(= \gamma_t' \nu_i F)$. By Lemma 6.5(b), the new walk is a SAW on $\nu_i \overline{G}$. Furthermore it uses edges of $\nu_i \overline{G}$ not belonging to $\nu_i G$. Lower and upper bounds may be derived as in the proof of Lemma 6.1, and these lead to a contradiction when working along the subsequence implicit in (6.38). This implies $\mu(G) < \mu(\nu_i \overline{G})$ subject to (6.33). By Lemma 6.5(a), $\mu(\nu_i \overline{G}) = \overline{\mu}$ and hence $\mu(G) < \overline{\mu}$.

Assume finally that (6.34) holds. By (6.32)–(6.34), there exists $k \in \mathbb{N}$ satisfying $2 \le k < \ell$ such that

$$\begin{aligned} &\liminf_{n\to\infty} \sigma_n(0, E_k)^{1/n} < \mu, \\ &\lim_{n\to\infty} \sigma_n(0, E_{k+1})^{1/n} = \mu. \end{aligned} \tag{6.39}$$

By Lemma 6.3, there exist $a > 0$ and $m \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \sigma_n(an, E_k(m))^{1/n} < \mu. \tag{6.40}$$

Let S_n be the set of n-step SAWs from v_0 on G such that E_{k+1} never occurs and $E_k(m)$ occurs at least an times. By (6.39)–(6.40),

$$\lim_{n\to\infty} |S_n|^{1/n} = \mu. \tag{6.41}$$

For $0 \le i \le s$, let $S_n^{(i)}$ be the set of n-step SAWs from v_0 on G such that E_{k+1} never occurs, and $E_k^i(m)$ occurs at least an/s times. Thus,

$$S_n \subseteq \bigcup_{i=0}^{s} S_n^{(i)}. \tag{6.42}$$

By (6.41)–(6.42), there exists i satisfying $0 \le i \le s$ such that

$$\limsup_{n\to\infty} |S_n^{(i)}|^{1/n} = \mu,$$

and we choose i accordingly.

We apply the argument of the proof of Lemma 6.1 once again. Let $\pi \in S_n^{(i)}$ be such that $E_k^i(m)$ occurs at steps $j_1 < j_2 < \cdots < j_u$, and in addition (6.25)–(6.27) hold with $\gamma_t C_1 \in \mathcal{O}_i$. The argument is as above, and we do not repeat the details. This implies $\mu(G) < \mu(\overline{G})$ when (6.34) holds, and the proof is complete. □

Proof of Theorem 3.3. (a) Suppose $\mathcal{A}$ is a normal subgroup of Γ acting quasi-transitively on G. With $\rho \in V$, the orbits of $\mathcal{A}$ may be written as $\mathcal{A}\nu_i\rho$ for suitable $\nu_i \in \mathcal{A}$ and $0 \le i \le s < \infty$. Since $\mathcal{A}$ is normal, $\mathcal{A}\nu_i = \nu_i\mathcal{A}$, and the claim follows.

(b) Suppose $[\Gamma : \mathcal{A}] < \infty$. The finite coset property holds with the ν_i chosen so that the $\nu_i\mathcal{A}$ are the left cosets of $\mathcal{A}$ in Γ. □

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