THE WORK OF LUCIO RUSSO ON PERCOLATION

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Dedicated in friendship to Lucio Russo

ABSTRACT. The contributions of Lucio Russo to the mathematics of percolation and disordered systems are outlined. The context of his work is explained, and its ongoing impact on current work is described and amplified.

1. A PERSONAL APPRECIATION

Prior to his mid-career move to the history of science in the early 1990s, Lucio Russo enjoyed a very successful and influential career in the theory of probability and disordered systems, in particular of percolation and the Ising model. His ideas have shaped these significant fields of science, and his name will always be associated with a number of fundamental techniques of enduring importance.

The author of this memoir is proud to have known Lucio in those days, and to have profited from his work, ideas, and company. He hopes that this brief account of some of Lucio's results will stand as testament to the beauty and impact of his ideas.

2. SCIENTIFIC SUMMARY

Lucio Russo has worked principally on the mathematics of percolation, that is, of the existence (or not) of infinite connected clusters within a disordered spatial network. The principal model in this field is the so called *percolation model*, introduced to mathematicians by Broadbent and Hammersley in 1957, [12]. Consider, for definiteness, the hypercubic lattice $\mathbb{Z}^d$ with $d \geq 2$, and let $p \in [0, 1]$. We declare each edge to be *open* with probability p and *closed* otherwise, and different edges receive independent states. The main questions are centred around the existence (or not) of an infinite open component in $\mathbb{Z}^d$. It turns out that there exists a critical probability $p_c = p_c(d)$ such that no infinite open cluster exists when $p < p_c$, and there exists a

unique such cluster when $p > p_c$. (It is not still known which of these two occurs when $p = p_c$ for general d, specifically when $3 \leq d \leq 10$. See [18].)

Let C be the open cluster of $\mathbb{Z}^d$ containing the origin. Two functions that play important roles in the theory are the *percolation probability* θ and the *mean cluster size* χ given by

$$\theta(p) = P_p(|C| = \infty), \qquad \chi(p) = E_p|C|,$$

where P_p and E_p are the appropriate product measure and expectation. The above model is the *bond* percolation model; the *site* percolation model is defined similarly, with sites being open/closed. A fairly recent account of percolation may be found in [23].

The question was raised in 1960 (by Harris, [28]) of whether or not $p_c(2) = \frac{1}{2}$, and the search for a rigorous proof attracted a number of fine mathematicians into the field, including Lucio. Several important partial results were proved, culminating in 1980 with Kesten's complete proof that $p_c(2) = \frac{1}{2}$, [34]. The interest of the community then migrated towards the case $d \geq 3$, before returning firmly to $d = 2$ with the 2001 proof by Smirnov, [41, 42], of Cardy's formula.

Lucio contributed a number of fundamental techniques to percolation theory during the period 1978–1988, and the main purpose of the current paper is to describe these and to explore their significance. We mention Russo's formula, the Russo–Seymour–Welsh (RSW) inequalities, his study of percolation surfaces in three dimensions, and of the uniqueness of the infinite open cluster, and finally Russo's approximate zero–one law. Russo's formula and RSW theory have proved of especially lasting value in, for example, recent developments concerning conformal invariance for critical percolation.

In Section 8, we mention some of Lucio's results concerning percolation of $+/-$ spins in the two-dimensional Ising model.

This short account is confined to Lucio's contributions to percolation, and does not touch on his work lying closer to ergodic theory and dynamical systems, namely [R2, R6, R8, R11], and neither does it refer to the paper [R7]. A comprehensive list of Lucio's mathematical publications from MathSciNet may be found at the end of this paper.

Results from Lucio's work will be described here using 'modern' notation. No serious attempt is made to include comprehensive citations of the related work of others.

3. Russo's formula

Let $\Omega = \{0, 1\}^E$ where E is finite, and let P_p be product measure on the partially ordered set Ω with density $p \in [0, 1]$. An event $A \subseteq \Omega$ is called *increasing* if:

$$\omega \in A, \ \omega \leq \omega' \quad \Rightarrow \quad \omega' \in A.$$

Let $\omega \in \Omega$. An element $e \in E$ is called *pivotal* for an increasing event A if $\omega_e \notin A$ and $\omega^e \in A$, where ω^e and ω_e are obtained from ω by varying the state of the edge e thus:

$$\omega_e(f) = \begin{cases} 0 & \text{if } f = e, \\ \omega(f) & \text{if } f \neq e, \end{cases} \qquad \omega^e(f) = \begin{cases} 1 & \text{if } f = e, \\ \omega(f) & \text{if } f \neq e. \end{cases}$$

In other words, e is said to be pivotal for A if the occurrence of A depends on the state of e.

Theorem 3.1 (Russo's formula, [R14]). *Let A be an increasing event. We have that*

$$\frac{d}{dp} P_p(A) = \sum_{e \in E} P_p(e \text{ is pivotal for } A).$$

Similar techniques are encountered independently in related fields. For example, Russo's formula is essentially equation (4.4) of Barlow and Proschan's book [5, p. 210] on reliability theory. Such a formula appeared also in the work of Margulis, [36], in the Russian literature. A characteristic of Russo's work is the geometric context of the formula when applied in situations such as percolation.

Russo's formula is key to the study of geometrical probability governed by a product measure. It has so many applications that is a challenge to single out any one. We mention here its use in the derivation of exact values for critical exponents in two dimensions, [35, 43].

Similarly, extensions of Russo's formula have been central in several related fields, including but not limited to the contact model [8, Thm 2.13], continuum percolation [19, 31], and the random-cluster model [9, Prop. 4].

4. RUSSO–SEYMOUR–WELSH INEQUALITIES

For twenty years from about 1960 to 1980, mathematicians attempted to prove that the critical probability p_c of bond percolation on the square lattice satisfies $p_c = \frac{1}{2}$. This prominent open problem was in the spirit of that of the critical temperature of the Ising model, resolved in 1944 by Onsager, [39]. Harris [28] showed how to use duality to obtain $p_c \geq \frac{1}{2}$, but the corresponding upper bound was elusive. Then, in 1978, a powerful technique emerged in independent and contemporaneous work of Lucio, [R12], and Seymour and Welsh, [40]. It has come to be known simply as 'RSW'.

Consider bond percolation with density p on the square lattice $\mathbb{Z}^2$. A *left–right crossing* of a rectangle B is an open path in B which joins some vertex on its left side to some vertex on its right side. For positive integers m and n, we define the rectangle

$$B(m, n) = [0, 2m] \times [0, 2n],$$

and let $\text{LR}(m, n)$ be the event that there exists a left–right crossing of $B(m, n)$.

Lemma 4.1 (Russo–Seymour–Welsh (RSW), [R12, 40]). *Let $p \in (0, 1)$. We have that*

$$P_p\big(\text{LR}(\tfrac{3}{2}n, n)\big) \geq \big(1 - \sqrt{1 - \tau}\big)^3,$$

where $\tau = P_p(\text{LR}(n, n))$.

This fundamental but superficially innocuous lemma implies that, if the chance of crossing a square is bounded from 0 uniformly in its size, then so is the chance of crossing a rectangle with aspect ratio $\frac{3}{2}$. Using the self-duality of $\mathbb{Z}^2$, we have as input to the RSW lemma that

$$(4.1) \qquad\qquad P_{\frac{1}{2}}(\text{LR}(n, n)) \geq \tfrac{1}{2}.$$

Let A_n be the event that the annulus $[-3n, 3n]^2 \setminus [-n, n]^2$ contains an open cycle with the origin in the bounded component of its complement in $\mathbb{R}^2$. Using elementary geometrical arguments and the FKG inequality, it follows by the RSW lemma and (4.1) that there exists $\sigma > 0$ such that $P_p(A_n) \geq \sigma$ for $n \geq 1$ and $p \geq \frac{1}{2}$.

The RSW lemma and the ensuing annulus inequality have proved to be key to the study of percolation in two dimensions. In common with other useful methods of mathematics, there is now a cluster of related inequalities, see for example [10], [24, Sect. 5.5], and [47, Chap. 5].

RSW methods were used by their discoverers to make useful but incomplete progress towards proving that $p_c = \frac{1}{2}$, and they played a role in Kesten's full proof, [34]. (The principle novelty of Kesten's paper was a bespoke theory of sharp threshold, see Section 5.) More precisely, they led to the following result, which is presented in terms of *site* percolation on the square lattice $\mathbb{Z}^2$ and its *matching lattice* $\mathbb{Z}_*^2$, derived by adding the two diagonals to each face of $\mathbb{Z}^2$.

Theorem 4.2 (Russo, [R12]). *Consider site percolation on the square lattice $\mathbb{Z}^2$. The critical points*

$$p_c = \sup\{p : \theta(p) = 0\}, \qquad \pi_c = \sup\{p : \chi(p) < \infty\},$$

satisfy

$$(4.2) \qquad\qquad p_c + \pi_c^* = 1, \qquad p_c^* + \pi_c = 1,$$

where an asterisk denotes the corresponding values on the matching lattice.

The parallel work of Seymour and Welsh, [40], was directed at the bond model on $\mathbb{Z}^2$, of which the dual model lies on a translate of $\mathbb{Z}^2$. Following Kesten's proof of $p_c(\mathbb{Z}^2) = \frac{1}{2}$ for bond percolation, Lucio revisited Theorem 4.2 in [R14] with a proof that $\pi_c^* = p_c^*$, and the consequent improvement of (4.2), namely $p_c + p_c^* = 1$. He proved also that θ is a continuous function on $[0, 1]$.

RSW theory is now recognised as fundamental to rigorous proofs of conformal invariance of critical two-dimensional percolation and all that comes with that. The proof of Cardy's formula, [41, 42], provides a major illustration. It was observed by Aizenman and Burchard, [3], that certain connection probabilities belong to a space of uniformly Hölder functions. Since this space is compact, such functions have subsequential limits as the mesh of the lattice approaches 0. The above Hölder property is proved using annulus inequalities.

Indeed the power of RSW arguments extends beyond percolation to a host of problems involving two-dimensional stochastic geometry, such as the FK-Ising model [16] and Voronoi percolation [46]. In addition, RSW theory provides one of the main techniques for the proof by Beffara and Duminil-Copin, [7], that the random-cluster model on $\mathbb{Z}^2$ with cluster-weighting parameter $q \geq 1$ has critical value $p_c(q) = \sqrt{q}/(1 + \sqrt{q})$. We retrieve Kesten's theorem by setting $q = 1$.

5. Approximate zero–one law

Kolmogorov's zero–one law may be stated as follows. Consider the infinite product space $\Omega = \{0,1\}^{\mathbb{N}}$ endowed with the product σ-algebra and the product measure P_p. If A is an event that is independent of any finite subcollection $\{\omega(e) : e \in E\}$, $E \subseteq \mathbb{N}$, $|E| < \infty$, then $P_p(A)$ equals either 0 or 1. It follows that, for an increasing event A, there exists $p_0 \in [0,1]$ such that

$$P_p(A) = \begin{cases} 0 & \text{if } p < p_0, \\ 1 & \text{if } p > p_0. \end{cases}$$

This law is intrinsically an *infinite-volume* effect, in that the index set is the infinite set $\mathbb{N}$. Lucio posed the farsighted question in [R15] of whether there exists a *finite volume* version of this result, and this led him to his 'approximate zero–one law', following.

Let $\Omega = \{0,1\}^E$ where E is finite, and let P_p be product measure on the partially ordered set Ω with density $p \in [0,1]$. The *influence* $I_{A,p}(e)$ of $e \in E$ on the event $A \subseteq \Omega$ is defined by

$$I_{A,p}(e) = P_p\big(1_A(\omega_e) \neq 1_A(\omega^e)\big),$$

where 1_A denotes the indicator function of A. When A is increasing, this may be written

(5.1) $$I_{A,p}(e) = P_p(\omega_e \notin A, \ \omega^e \in A) = P_p(e \text{ is pivotal for } A).$$

Theorem 5.1 (Russo's approximate zero–one law, [R15])**.** *For $\epsilon > 0$, there exists $\eta > 0$ such that, if A is an increasing event and*

(5.2) $$I_{A,p}(e) < \eta, \qquad e \in E, \ p \in [0,1],$$

then there exists $p_0 \in [0,1]$ such that

$$(5.3) \qquad P_p(A) \begin{cases} \leq \epsilon & \text{if } p < p_0 - \epsilon, \\ \geq 1 - \epsilon & \text{if } p > p_0 + \epsilon. \end{cases}$$

This result was motivated by a desire to generalise certain results for box-crossing probabilities in percolation. Its impact extends far beyond percolation, and it is a precursor of a more recent theory, pioneered by Kahn, Kalai, Linial, [32] and Talagrand, [44, 45], of influence and sharp threshold. It is proved at [44, Thm 1.1] that there exists an absolute constant $c > 0$ such that, for $p \in (0,1)$ and an increasing event A,

$$(5.4) \qquad \sum_{e \in E} I_{A,p}(e) \geq \left(\frac{c}{p(1-p)\log[2/(p(1-p))]} \right) P_p(A)(1 - P_p(A))\log(1/m_p),$$

where

$$m_p = \max\{I_{A,p}(e) : e \in E\}.$$

It follows that, when $p \in (0,1)$ and A is increasing,

$$(5.5) \qquad \sum_{e \in E} I_{A,p}(e) \geq c' P_p(A)(1 - P_p(A))\log(1/m_p),$$

where $c' > 0$ is an absolute constant.

Amongst the implications of (5.5) is a quantification of the relationship between ϵ and η in Theorem 5.1. Suppose (5.2) holds with $\eta \in (0,1)$, so that $m_p \leq \eta$. By (5.5) and Russo's formula,

$$(5.6) \qquad \frac{d}{dp} P_p(A) \geq c' P_p(A)(1 - P_p(A))\log(1/\eta).$$

Choose p_0 such that $P_{p_0}(A) = \frac{1}{2}$, and integrate (5.6) to obtain

$$(5.7) \qquad P_p(A) \begin{cases} \leq \epsilon_p & \text{if } p < p_0, \\ \geq 1 - \epsilon_p & \text{if } p > p_0, \end{cases}$$

with

$$\epsilon_p = \frac{1}{1 + (1/\eta)^{c'|p - p_0|}}.$$

Such inequalities have found numerous applications in percolation and related topics, see for example [10, 17, 22]. They have been extended to general product measures, [11, 25], and to probability measures satisfying the FKG lattice condition, [21]. Recent overviews include [24, Chap. 4] and [33].

6. Percolation in dimension $d \geq 3$

In 1983, Lucio spent a sabbatical at Princeton University. His work during that period led to two significant publications [R1, R3] on aspects of percolation in three dimensions. The first of these caused quite a stir in the community at the time of its appearance, largely since most work until then had been for models in only two dimensions. Whereas the dual of a bond model in two dimensions is another bond model, the dual model in three or more dimensions is a 'plaquette' model. Since the topology of surfaces of plaquettes is much more complicated that that of paths, the ensuing percolation duality poses a number of challenging topological questions.

The authors of [R1] consider bond percolation on $\mathbb{Z}^3$ with density p, together with its dual 'plaquette' model on $\mathbb{Z}^3_* := \mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. A *plaquette* is a unit square with vertices in $\mathbb{Z}^3_*$, and its bounding lines are *edges* of $\mathbb{Z}^3_*$. Each edge e of $\mathbb{Z}^3$ intersects a unique plaquette Π_e, and Π_e is termed *occupied* if and only if e is closed (and *unoccupied* otherwise). Thus, a plaquette is occupied with probability $1 - p$. For any collection F of plaquettes, the boundary ∂F is defined to be set of edges of $\mathbb{Z}^3_*$ belonging to an *odd* number of members of F.

Let γ be a cycle of $\mathbb{Z}^3_*$. The main results of [R1] concern the probability there exists a set F of occupied plaquettes which spans γ in the sense that $\partial F = \gamma$. For simplicity, we shall suppose here that γ is a $m \times n$ rectangle of the x/y plane, and we denote the above event as W_γ. Note that γ has area mn and perimeter $2(m+n)$.

Theorem 6.1 (Aizenman, Chayes, Chayes, Fröhlich, Russo, [R1])**.** *There exist constants* $\pi_c, \rho_c \in (0, 1)$ *such that*

$$-\log P_p(W_\gamma) \sim \begin{cases} \alpha mn & \text{if } 1 - p < \pi_c, \\ \beta(m+n) & \text{if } 1 - p > \rho_c, \end{cases}$$

where $\alpha, \beta > 0$ *depend on* p, *and the asympotic relation is as* $m, n \to \infty$.

The constants π_c, ρ_c are the critical densities of the bond percolation model on $\mathbb{Z}^3$ given by

$$\pi_c = \sup\{p : \chi(p) < \infty\}, \qquad \rho_c = \lim_{k \to \infty} \widehat{p}_c(k),$$

where $\widehat{p}_c(k)$ is the *slab critical point*

$$\widehat{p}_c(k) = \sup\{p : P_p(0 \leftrightarrow \infty \text{ in } [0, \infty)^2 \times [0, k]) = 0\}.$$

It was conjectured in [R1] that $\pi_c = p_c = \rho_c$. The first equality was proved later in [2, 38], and the second in [6, 27].

There are only few percolation models on finite-dimensional lattices for which the numerical values of the critical probabilities are known exactly, and all such exact results are in two dimensions only (see, for example, [26]). In contrast, quite a lot

of work has been devoted to obtaining rigorous upper and lower bounds for critical probabilities, and there is a host of numerical estimates.

Consider site percolation on the simple cubic lattice $\mathbb{Z}^3$. By a comparison with the site model on the triangular lattice, Lucio has shown (with Campanino, in [R3]) that $p_c \leq \frac{1}{2}$. (See also [37].) They obtained also the strict inequality, with a distinctly more complicated argument.

Theorem 6.2 (Campanino, Russo, [R3]). *The critical probability of site percolation on $\mathbb{Z}^3$ satisfies $p_c < \frac{1}{2}$.*

By examining the proof, one obtains a small $\epsilon > 0$ such that $p_c < \frac{1}{2} - \epsilon$, but the current author is unaware of a further rigorous improvement. It is expected that $p_c \approx 0.31$.

7. Uniqueness of the infinite open cluster

Let I be the number of infinite open clusters of a percolation model in a finite-dimensional space. For a period in the 1980s, the 'next' problem was to prove that $P_p(I = 1) = 1$ in the supercritical phase (when $p > p_c$). This problem was solved by Aizenman, Kesten, and Newman [4] in 1987. Their proof seemed slightly mysterious at the time, and it was simplified by Lucio in the jointly written paper [R9]. The key step was to show, using a large-deviation estimate present already in [4], that there is density 0 of sites that are adjacent to two distinct infinite clusters.

This useful argument was soon overshadowed by the beautiful proof of uniqueness by Burton and Keane, [13]. The last proof uses translation-invariance of the underlying measure together with a property of so called 'finite energy', and may thus be extended to more general measures than product measures. On the other hand, since the proof uses no quantitative estimate, it yields no 'rate'. The methods of [4, R9] provide a missing rate, and this has been useful in the later work [14, 15].

The question of uniqueness for dependent models is potentially harder, since the large-deviation estimate of [4, R9] is not available. In joint work [R10] with Gandolfi and Keane, Lucio used path-intersection arguments to show uniqueness for ergodic, positively associated measures in two dimensions, satisfying certain translation and reflection symmetries. Unlike the Burton–Keane proof, they needed no finite-energy assumption.

8. Ising model

Lucio has written three papers on the geometry of the d-dimensional Ising model, [R4, R5, R13]. In this work, he (and his coauthors) studied the relationship between properties of the infinite-volume Gibbs measures and the existence or not of an infinite cluster of either $+$ or $-$ spins (that is, of percolation in the Ising model).

The first two of these papers [R4, R5] explore a relationship between the Ising magnetization and the above percolation probability, and yield the non-existence of percolation in the high-temperature phase. This is complemented when $d = 2$ with the proof that percolation (of the corresponding spin) exists in the low-temperature phase for either of the pure infinite-volume limits μ_+, μ_-, obtained respectively as the weak limits with $+/-$ boundary conditions. These methods were developed further in [R5], where a phase diagram was proposed for the existence of infinite clusters in the two-dimensional ferromagnetic Ising model, as a function of external field h and temperature T. The principal features of this diagram were later proved by Higuchi, [30].

One of the central problem in two dimensions of the late 1970s was to prove or disprove the statement that every infinite-volume Gibbs measure is a convex combination of the two extremal measures μ_+, μ_-. Lucio obtained the following important result for this problem.

Theorem 8.1 (Russo, [R13])**.** *Any infinite-volume Gibbs measure μ, which is translation-invariant in one or both of the axial directions, is a convex combination of μ_+ and μ_-.*

Lucio proved this by considering the existence (or not) of infinite $+/-$ clusters on $\mathbb{Z}^2$ and its matching lattice. The full conclusion, without an assumption of partial invariance, was obtained later in independent work of Aizenman, [1], and Higuchi, [29] (see also [20]). Therefore, in two dimensions (unlike three dimensions) there exists no non-translation-invariant Gibbs measure.

Acknowledgement

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