BOUNDED ENTANGLEMENT ENTROPY IN THE QUANTUM ISING MODEL

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Abstract. We give a rigorous proof of the boundedness of the entanglement of a block of spins for the ground state of the one-dimensional quantum Ising model with sufficiently strong transverse field. This is proved by a refinement of the arguments in the earlier work by the same authors (J. Statist. Phys. 131 (2008) 305–339). The proof is geometrical, and utilises a transformation to a model of classical probability called the continuum random-cluster model. The same conclusion has been announced by M. Campanino and M. Gianfelice using the different method of cluster expansions. Our method of proof is fairly robust, and applies also to certain disordered systems.

1. The quantum Ising model and entanglement

The purpose of the current note is to explain how the geometrical approach of [11] may be elaborated to obtain the boundedness of the entanglement entropy of a block of spins in the ground state of the one-dimensional quantum Ising model with sufficiently strong transverse field. The current paper is presented as a elaboration of the earlier work [11] by the same authors, to which the reader is referred for details of the background and basic theory.

We shall consider a block of L spins in a line of length $2m+L$. Let $L \geq 0$. For $m \geq 0$, let

$$\Delta_m = \{-m, -m+1, \dots, m+L\}$$

be a subset of the one-dimensional lattice $\mathbb{Z}$, and attach to each vertex $x \in \Delta_m$ a quantum spin-$\frac{1}{2}$ with local Hilbert space $\mathbb{C}^2$. The Hilbert space $\mathcal{H}$ for the system is $\mathcal{H} = \bigotimes_{x=-m}^{m+L} \mathbb{C}^2$. A convenient basis for each spin is provided by the two eigenstates $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, of the Pauli operator

$$\sigma_x^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

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at the site x, corresponding to the eigenvalues ± 1. The other two Pauli operators with respect to this basis are represented by the matrices

$$\sigma_x^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_x^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.1}$$

A complete basis for $\mathcal{H}$ is given by the tensor products (over x) of the eigenstates of $\sigma_x^{(3)}$. In the following, $|\phi\rangle$ denotes a vector and $\langle\phi|$ its adjoint. As a notational convenience, we shall represent sub-intervals of $\mathbb{Z}$ as real intervals, writing for example $\Delta_m = [-m, m+L]$.

The spins in Δ_m interact via the quantum Ising Hamiltonian

$$H_m = -\tfrac{1}{2} \sum_{\langle x,y\rangle} \lambda \sigma_x^{(3)} \sigma_y^{(3)} - \sum_x \delta \sigma_x^{(1)}, \tag{1.2}$$

generating the operator $e^{-\beta H_m}$ where β denotes inverse temperature. Here, $\lambda \geq 0$ and $\delta \geq 0$ are the spin-coupling and external-field intensities, respectively, and $\sum_{\langle x,y\rangle}$ denotes the sum over all (distinct) unordered pairs of neighbouring spins. While we phrase our results for the translation-invariant case, our approach can be extended to certain random couplings and field intensities, much as in [11, Sect. 8]. See Section 4.

The Hamiltonian H_m has a unique pure ground state $|\psi_m\rangle$ defined at zero temperature (as $\beta \to \infty$) as the eigenvector corresponding to the lowest eigenvalue of H_m. This ground state $|\psi_m\rangle$ depends only on the ratio $\theta = \lambda/\delta$. We work here with a free boundary condition on Δ_m, but we note that the same methods are valid with a periodic (or wired) boundary condition, in which Δ_m is embedded on a circle.

Write $\rho_m(\beta) = e^{-\beta H_m} / \operatorname{tr}(e^{-\beta H_m})$, and

$$\rho_m = \lim_{\beta\to\infty} \rho_m(\beta) = |\psi_m\rangle\langle\psi_m|$$

for the density operator corresponding to the ground state of the system. The ground-state entanglement of $|\psi_m\rangle$ is quantified by partitioning the spin chain Δ_m into two disjoint sets $[0, L]$ and $\Delta_m \setminus [0, L]$ and by considering the entropy of the *reduced density operator*

$$\rho_m^L = \operatorname{tr}_{\Delta_m \setminus [0,L]}(|\psi_m\rangle\langle\psi_m|). \tag{1.3}$$

One may similarly define, for finite β, the reduced operator $\rho_m^L(\beta)$. In both cases, the trace is performed over the Hilbert space of spins belonging to $\Delta_m \setminus [0, L]$. Note that ρ_m^L is a positive semi-definite operator on the Hilbert space $\mathcal{H}_L$ of dimension $d = 2^{L+1}$ of spins indexed by the interval $[0, L]$. By the spectral theorem for normal matrices [6], this operator may be diagonalised and has real, non-negative eigenvalues, which we denote in decreasing order by $\lambda_j^{\downarrow}(\rho_m^L)$.

Definition 1.4. *The* entanglement (entropy) *of the interval* $[0, L]$ *relative to its complement* $\Delta_m \setminus [0, L]$ *is given by*

$$S(\rho_m^L) = -\operatorname{tr}(\rho_m^L \log_2 \rho_m^L) = -\sum_{j=1}^{2^{L+1}} \lambda_j^{\downarrow}(\rho_m^L) \log_2 \lambda_j^{\downarrow}(\rho_m^L), \tag{1.5}$$

where $0 \log_2 0$ *is interpreted as* 0.

Here are our two main theorems.

Theorem 1.6. *Let* $\lambda, \delta \in (0, \infty)$ *and* $\theta = \lambda/\delta$. *There exists* $C = C(\theta) \in (0, \infty)$, *and a constant* $\gamma = \gamma(\theta)$ *satisfying* $0 < \gamma < \infty$ *if* $\theta < 1$, *such that, for all* $L \geq 1$,

$$\|\rho_m^L - \rho_n^L\| \leq \min\{2, Ce^{-\gamma m}\}, \qquad 2 \leq m \leq n. \tag{1.7}$$

Furthermore, we may find such γ *satisfying* $\gamma \to \infty$ *as* $\theta \downarrow 0$.

Equation (1.7) is in terms of the operator norm:

$$\|\rho_m^L - \rho_n^L\| \equiv \sup_{\|\psi\|=1} \Big| \langle \psi | \rho_m^L - \rho_n^L | \psi \rangle \Big|, \tag{1.8}$$

where the supremum is taken over all vectors $|\psi\rangle \in \mathcal{H}_L$ with unit L^2-norm.

Theorem 1.9. *Consider the quantum Ising model* (1.2) *on* $n = 2m + L + 1$ *spins, with parameters* λ, δ, *and let* γ *and* C *be as in Theorem 1.6. If* $\gamma > 4 \ln 2$, *there exists* $c_1 = c_1(\theta, \gamma)$ *such that*

$$S(\rho_m^L) \leq c_1, \qquad m, L \geq 0. \tag{1.10}$$

Weaker versions of these two theorems were proved in [11, Thms 2.2, 2.8], namely that (1.7) holds subject to a power factor of the form L^α, and (1.10) holds with c_1 replaced by $C_1 + C_2 \log L$.

There is a considerable and growing literature in the physics journals concerning entanglement entropy in one and more dimensions. For example, paper [8] is an extensive review of area laws. The relationship between entanglement entropy and the spectral gap has been explored in [2, 3], and polynomial-time algorithms for simulating the ground state are studied in [4].

We make next some remarks about the proofs of the above two theorems. These follow the proofs of [11, Thms 2.2, 2.8] subject to certain improvements in the probabilistic estimates. The general approach and many details are the same as in the earlier paper. We make frequent reference here to [11], and will highlight where the current proofs differ, while omitting arguments that may be taken directly from [11]. In particular, the reader is referred to [11, Sects. 4, 5] for details of the percolation representation of the ground state, and of the associated continuum random-cluster model. In Section 2, we review the relationship between the reduced density operator and the random-cluster model, and we state the fundamental inequalities of Theorem 2.6 and Lemma

2.8. Once the last two results have been proved, Theorems 1.6 and 1.9 follow as in [11]: the first as in the proof of [11, Thm 2.2], and the second as in that of [11, Thm 2.8].

We reflect in Section 4 on the extension of our methods and conclusions when the edge-couplings λ and field strengths δ are permitted to vary, either deterministically or randomly, about the line. In this disordered case, the Hamiltonian (1.2) is replaced by

$$H_m = -\tfrac{1}{2}\sum_{\langle x,y\rangle} \lambda_{x,y}\sigma_x^{(3)}\sigma_y^{(3)} - \sum_x \delta_x \sigma_x^{(1)}, \tag{1.11}$$

where the sum is over neighbouring pairs $\langle x,y\rangle$ of Δ_m. We write $\boldsymbol{\lambda} = (\lambda_{x,x+1} : x \in \mathbb{Z})$ and $\boldsymbol{\delta} = (\delta_x : x \in \mathbb{Z})$.

Theorem 1.12. *Consider the quantum Ising model on $\mathbb{Z}$ with Hamiltonian* (1.11)*, such that, for some $\lambda, \delta > 0$, $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$ satisfy*

$$\lambda_{x,y}/\delta_x \le \lambda/\delta, \qquad y = x-1, x+1,\ x \in \mathbb{Z}. \tag{1.13}$$

(a) *If $\lambda/\delta < 1$, then* (1.8) *holds with C and γ as given there.*
(b) *If, further, $\gamma > 4\ln 2$, then* (1.10) *holds with c_1 as given there.*

If $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$ are random sequences satisfying (1.13) *with probability one, then parts* (a) *and* (b) *are valid a.s.*

The situation is more complicated when $\boldsymbol{\lambda}$, $\boldsymbol{\delta}$ are random but do not a.s. satisfy (1.13). See Section 4 for a short discussion of this case.

Remark 1. *The authors acknowledge Massimo Campanino's announcement in a lecture on 12 June 2019 of his proof with Michele Gianfelice of a version of Theorem 1.9 using cluster expansions. That announcement provoked the current work.*

2. Estimates via the continuum random-cluster model

The *continuum percolation model* is constructed as in [10, 11]. For $x \in \mathbb{Z}$, let D_x be a Poisson process of points in $\{x\}\times\mathbb{R}$ with intensity δ; the processes $\{D_x : x \in \mathbb{Z}\}$ are independent, and the points in the D_x are termed 'deaths'. The lines $\{x\}\times\mathbb{R}$ are called 'time lines'.

For $x \in \mathbb{Z}$, let B_x be a Poisson process of points in $\{x+\frac{1}{2}\}\times\mathbb{R}$ with intensity λ; the processes $\{B_x : x \in \mathbb{Z}\}$ are independent of each other and of the D_y. For $x \in \mathbb{Z}$ and each $(x+\frac{1}{2}, t) \in B_x$, we draw a unit line-segment in $\mathbb{R}^2$ with endpoints (x,t) and $(x+1,t)$, and we refer to this as a 'bridge' joining its two endpoints. For $(x,s),(y,t) \in \mathbb{Z}\times\mathbb{R}$, we write $(x,s) \leftrightarrow (y,t)$ if there exists a path π in $\mathbb{R}^2$ with endpoints (x,s), (y,t) such that: π comprises sub-intervals of $\mathbb{Z}\times\mathbb{R}$ containing no deaths, together possibly with bridges. For $\Lambda, \Delta \subseteq \mathbb{Z}\times\mathbb{R}$, we write $\Lambda \leftrightarrow \Delta$ if there exist $a \in \Lambda$ and $b \in \Delta$ such that

$a \leftrightarrow b$. Let $\mathbb{P}_{\Lambda,\lambda,\delta}$ denote the associated probability measure when restricted to the set Λ.

We make a note concerning exponential decay which will be important later. The critical point of continuum percolation is given by $\theta = 1$ where $\theta = \lambda/\delta$ (see [5, Thm 1.12]). In particular (as in [11, Thm 6.7]) there is exponential decay when $\theta < 1$. Let $\Lambda_m = [-m, m]^2 \subseteq \mathbb{Z} \times \mathbb{R}$, with boundary $\partial\Lambda_m$.

Theorem 2.1 ([5, Thm 1.7])**.** *Let $\lambda, \delta \in (0, \infty)$, and $I = \{0\} \times [-\frac{1}{2}, \frac{1}{2}] \subseteq \mathbb{Z} \times \mathbb{R}$. There exist $C = C(\lambda, \delta) \in (0, \infty)$ and $\gamma = \gamma(\lambda, \delta)$ satisfying $\gamma > 0$ when $\theta = \lambda/\delta < 1$, such that*

$$\mathbb{P}_{\lambda,\delta}\big(I \leftrightarrow \partial\Lambda_m\big) \leq Ce^{-\gamma m}, \qquad m \geq 0.$$

The function $\gamma(\lambda, \delta)$ may be chosen to satisfy $\gamma \to \infty$ as $\delta \to \infty$ for fixed λ.

Henceforth the function γ denotes that of Theorem 2.1.

The *continuum random-cluster model* on $\mathbb{Z} \times \mathbb{R}$ is defined as follows. Let $a, b \in \mathbb{Z}$, $s, t \in \mathbb{R}$ satisfy $a \leq b$, $s \leq t$, and write $\Lambda = [a, b] \times [s, t]$ for the box $\{a, a + 1, \ldots, b\} \times [s, t]$. Its boundary $\partial\Lambda$ is the set of all points $(x, y) \in \Lambda$ such that: either $x \in \{a, b\}$, or $y \in \{s, t\}$, or both.

As sample space we take the set Ω_Λ comprising all finite subsets (of Λ) of deaths and bridges, and we assume that no death is the endpoint of any bridge. For $\omega \in \Omega_\Lambda$, we write $B(\omega)$ and $D(\omega)$ for the sets of bridges and deaths, respectively, of ω.

The top/bottom periodic boundary condition is imposed on Λ: for $x \in [a, b]$, we identify the two points (x, s) and (x, t). The remaining boundary of Λ, denoted $\partial^{\mathrm{h}}\Lambda$, is the set of points of the form $(x, u) \in \Lambda$ with $x \in \{a, b\}$ and $u \in [s, t]$.

For $\omega \in \Omega_\Lambda$, let $k(\omega)$ be the number of its clusters (subject to the above boundary condition). Let $q \in (0, \infty)$, and define the 'continuum random-cluster' probability measure $\mathbb{P}_{\Lambda,\lambda,\delta,q}$ by

$$d\mathbb{P}_{\Lambda,\lambda,\delta,q}(\omega) = \frac{1}{Z} q^{k(\omega)} d\mathbb{P}_{\Lambda,\lambda,\delta}(\omega), \qquad \omega \in \Omega_\Lambda, \tag{2.2}$$

where Z is the appropriate partition function. As at [11, eqn (5.3)],

$$\mathbb{P}_{\Lambda,\lambda,\delta,q} \leq_{\mathrm{st}} \mathbb{P}_{\Lambda,\lambda,\delta}, \qquad q \geq 1, \tag{2.3}$$

in the sense of stochastic ordering.

We introduce next a variant in which the box Λ possesses a 'slit' at its centre. Let $L \geq 0$ and $S_L = [0, L] \times \{0\}$. We think of S_L as a collection of $L + 1$ vertices labelled in the obvious way as $x = 0, 1, 2, \ldots, L$. For $m \geq 2$, $\beta > 0$, let $\Lambda_{m,\beta}$ be the box

$$\Lambda_{m,\beta} = [-m, m + L] \times [-\tfrac{1}{2}\beta, \tfrac{1}{2}\beta]$$

subject to a 'slit' along S_L. That is, $\Lambda_{m,\beta}$ is the usual box except that each vertex $x \in S_L$ is replaced by two distinct vertices x^+ and x^-. The vertex x^+ (respectively, x^-) is attached to the half-line $\{x\} \times (0, \infty)$ (respectively, the half-line $\{x\} \times (-\infty, 0)$); there is no direct connection between x^+ and x^-. Write $S_L^\pm = \{x^\pm : x \in S_L\}$ for the upper and lower sections of the slit S_L. Let $\phi_{m,\beta}$ be the continuum random-cluster measure on $\Lambda_{m,\beta}$ with top/bottom periodic boundary condition and parameters λ, δ, $q = 2$.

It is explained in [11] that a random-cluster configuration ω gives rise to an Ising configuration on Λ, which serves (see [1]) as a two-dimensional representation of the quantum Ising model of (1.2). We shall use $\phi_{m,\beta}$ to denote the coupling of the continuum random-cluster measure and the corresponding (Ising) spin-configuration.

Let $\Omega_{m,\beta}$ be the sample space of the continuum random-cluster model on $\Lambda_{m,\beta}$, and $\Sigma_{m,\beta}$ the set of admissible allocations of spins to the clusters of configurations in $\Omega_{m,\beta}$. For $\sigma \in \Sigma_{m,\beta}$ and $x \in S_L$, write $\sigma_x^\pm$ for the spin-state of $x^\pm$. Let $\Sigma_L = \{-1, +1\}^{L+1}$ be the set of spin-configurations of the vectors $\{x^+ : x \in S_L\}$ and $\{x^- : x \in S_L\}$, and write $\sigma_L^+ = (\sigma_x^+ : x \in S_L)$ and $\sigma_L^- = (\sigma_x^- : x \in S_L)$.

Let
$$a_{m,\beta} = \phi_{m,\beta}(\sigma_L^+ = \sigma_L^-). \tag{2.4}$$
Then,
$$a_{m,\beta} \to a_m = \phi_m(\sigma_L^+ = \sigma_L^-) \qquad \text{as } \beta \to \infty, \tag{2.5}$$
where $\phi_m = \lim_{\beta\to\infty} \phi_{m,\beta}$.

Here is the main estimate of this section, of which Theorem 1.6 is an immediate corollary with adapted values of the constants. It differs from [12, Thm 6.5] in the removal of a factor of order L^α.

Theorem 2.6. *Let $\lambda, \delta \in (0, \infty)$ and write $\theta = \lambda/\delta$. If $\theta < 1$, there exist $C, M \in (0, \infty)$, depending on θ only, such that the following holds. For $L \geq 1$ and $M \leq m \leq n < \infty$,*
$$\sup_{\|c\|=1} \left| \frac{\phi_m(c(\sigma_L^+)c(\sigma_L^-))}{a_m} - \frac{\phi_n(c(\sigma_L^+)c(\sigma_L^-))}{a_n} \right| \leq Ce^{-\frac{1}{3}\gamma m}, \tag{2.7}$$
where γ is as in Theorem 2.1, and the supremum is over all functions $c : \Sigma_L \to \mathbb{R}$ with L^2-norm satisfying $\|c\| = 1$.

The theorem should presumably be valid subject to the weaker condition $\theta < 2$, since it is now known that $\theta = 2$ is the critical value of the associated continuum random-cluster model on $\mathbb{Z} \times \mathbb{R}$ (see [7, Thm 7.1]). In constrast, the value $\theta = 1$ is the critical point of the continuum percolation model (see [5, Thm 1.12]).

In the proof of Theorem 2.6, we make use of the following two lemmas (corresponding, respectively, to [11, Lemmas 6.8, 6.9]), which are proved in Section 3 using the method of ratio weak-mixing.

Lemma 2.8. *Let $\lambda, \delta \in (0, \infty)$ satisfy $\theta = \lambda/\delta < 1$, and let γ be as in Theorem 2.1. There exist constants $A(\lambda, \delta), C_1(\lambda, \delta) \in (0, \infty)$ such that the following holds. Let*

$$R_\xi = R(K, L, \xi) = C_1\big(e^{-\frac{1}{2}\gamma K} + Le^{-\frac{1}{4}\gamma\xi}\big). \tag{2.9}$$

For all $K, L, m \geq 1$, $\beta \geq 1$, and all $\epsilon^+, \epsilon^- \in \Sigma_L$, we have that

$$A^{2K}(1 - R_\beta) \leq \frac{\phi_{m,\beta}(\sigma_L^+ = \epsilon^+,\ \sigma_L^- = \epsilon^-)}{\phi_{m,\beta}(\sigma_L^+ = \epsilon^+)\phi_{m,\beta}(\sigma_L^- = \epsilon^-)} \leq A^{-2K}(1 + R_\beta),$$

whenever K, L, β are such that $R_\beta \leq \frac{1}{2}$.

In the second lemma we allow a general boundary condition on $\Lambda_{m,\beta}$. As in the discussion at the end of [11, Sect. 5], there are two types of legitimate boundary conditions, depending on whether they indicate random-cluster connectivity, or spin-values on the clusters thereof.

Lemma 2.10. *Let $\lambda, \delta \in (0, \infty)$, and let γ be as in Theorem 2.1. There exists a constant $C_1 \in (0, \infty)$ such that: for all $L, m \geq 1$, $\beta \geq 1$, all events $A \subseteq \Sigma_L \times \Sigma_L$, and all admissible random-cluster boundary conditions τ and spin boundary conditions η of $\Lambda_{m,\beta}$,*

$$\left| \frac{\phi^\alpha_{m,\beta}((\sigma_L^+, \sigma_L^-) \in A)}{\phi_{m,\beta}((\sigma_L^+, \sigma_L^-) \in A)} - 1 \right| \leq C_1 e^{-\frac{2}{7}\gamma m}, \qquad \text{for } \alpha = \tau, \eta,$$

whenever the right side of the inequality is less than or equal to 1.

The above two lemmas are stated in terms of the box $\Lambda_{m,\beta}$ with top/bottom periodic boundary conditions. Their proofs are valid under other boundary conditions also, including free boundary conditions.

Proof of Theorem 2.6. Let $0 < \lambda < \delta$, and let γ be as in Theorem 2.1. Let A, C_1, R_β be as in Lemma 2.8, and let $L \geq 1$ and $1 \leq K < \frac{1}{2}L$ be such that $C_1 e^{-\frac{1}{2}\gamma K} < \frac{1}{4}$. (Other pairs K, L are covered in (2.7) by adjusting C.) By (2.9),

$$\lim_{\beta\to\infty} R_\beta = C_1 e^{-\frac{1}{2}\gamma K} \leq \tfrac{1}{4}, \tag{2.11}$$

and we choose $l = l(\lambda, \delta, L) \geq 1$ such that $R_l < \frac{1}{2}$.

Let $2l \leq m \leq n < \infty$ and take $\beta > m$. Later we shall let $\beta \to \infty$. Since $\phi_{m,\beta} \leq_{\text{st}} \phi_{n,\beta}$, we may couple $\phi_{m,\beta}$ and $\phi_{n,\beta}$ via a probability measure ν on pairs (ω_1, ω_2) of configurations on $\Lambda_{n,\beta}$ in such a way that $\nu(\omega_1 \leq \omega_2) = 1$. It is standard (as in [9, 15]) that we may find ν such that ω_1 and ω_2 are identical configurations within the region of $\Lambda_{m,\beta}$ that is not connected to $\partial^{\text{h}}\Lambda_{m,\beta}$ in the upper configuration ω_2. Let D be the set of all pairs $(\omega_1, \omega_2) \in \Omega_{n,\beta} \times \Omega_{n,\beta}$ such that: ω_2 contains no path joining ∂B to $\partial^{\mathrm{h}} \Lambda_{m,\beta}$, where

$$B = [-r, r + L] \times [-r, r]$$

and r will be chosen later to satisfy

$$l \le r < \tfrac{1}{2} m, \tag{2.12}$$

implying in particular that

$$R_{2r} \le R_\beta < \tfrac{1}{2}. \tag{2.13}$$

We take free boundary conditions on B. The relevant regions are illustrated in Figure 1.

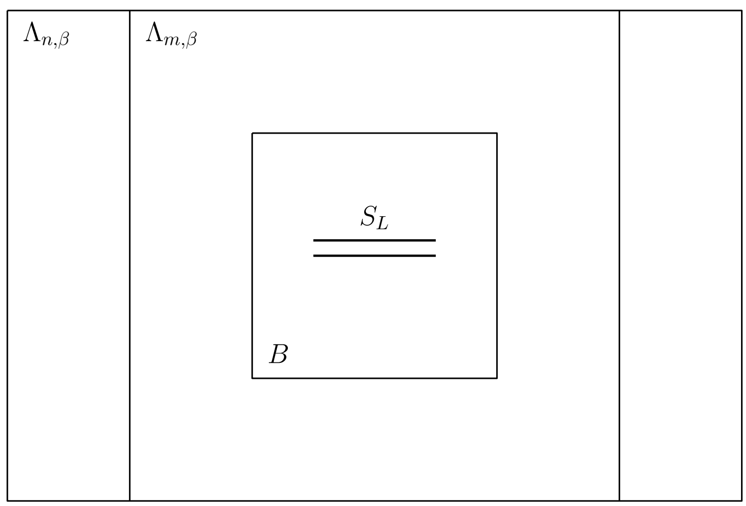

FIGURE 1. The boxes $\Lambda_{n,\beta}$, $\Lambda_{m,\beta}$, and B.

Having constructed the measure ν accordingly, we may now allocate spins to the clusters of ω_1 and ω_2 in the manner described in [11, Sect. 5]. This may be done in such a way that, on the event D, the spin-configurations associated with ω_1 and ω_2 within B are identical. We write σ_1 (respectively, σ_2) for the spin-configuration on the clusters of ω_1 (respectively, ω_2), and $\sigma^{\pm}_{i,L}$ for the spins of σ_i on the slit S_L.

By the remark following [11, eqn (6.4)], it suffices to consider non-negative functions $c : \Sigma_L \to \mathbb{R}$, and thus we let $c : \Sigma_L \to [0, \infty)$ with $\|c\| = 1$. Let

$$S_c = \frac{c(\sigma^+_{1,L})c(\sigma^-_{1,L})}{a_{m,\beta}} - \frac{c(\sigma^+_{2,L})c(\sigma^-_{2,L})}{a_{n,\beta}}, \tag{2.14}$$

so that

$$\frac{\phi_{m,\beta}(c(\sigma_L^+)c(\sigma_L^-))}{a_{m,\beta}} - \frac{\phi_{n,\beta}(c(\sigma_L^+)c(\sigma_L^-))}{a_{n,\beta}} = \nu(S_c 1_D) + \nu(S_c 1_{\overline{D}}), \tag{2.15}$$

where $\overline{D}$ is the complement of D, and 1_E is the indicator function of E.

Consider first the term $\nu(S_c 1_D)$ in (2.15). On the event D, we have that $\sigma_{1,L}^\pm = \sigma_{2,L}^\pm$, so that

$$|\nu(S_c 1_D)| \le \left|1 - \frac{a_{m,\beta}}{a_{n,\beta}}\right| \frac{\phi_{m,\beta}(c(\sigma_L^+)c(\sigma_L^-))}{a_{m,\beta}}. \tag{2.16}$$

By Lemma 2.8, (2.13), and [11, Lemma 6.10],

$$\begin{aligned}
\phi_{m,\beta}(c(\sigma_L^+)c(\sigma_L^-)) &= \sum_{\epsilon^\pm \in \Sigma_L} c(\epsilon^+)c(\epsilon^-)\phi_{m,\beta}(\sigma_L^+ = \epsilon^+,\, \sigma_L^- = \epsilon^-) \\
&\le A^{-2K}(1+R_\beta)\phi_{m,\beta}(c(\sigma_L^+))\phi_{m,\beta}(c(\sigma_L^-)) \\
&= A^{-2K}(1+R_\beta)\left(\sum_{\epsilon\in\Sigma_L} c(\epsilon)\phi_{m,\beta}(\sigma_L^+ = \epsilon)\right)^2 \\
&\le A^{-2K}(1+R_\beta)\sum_{\epsilon\in\Sigma_L}\phi_{m,\beta}(\sigma_L^+ = \epsilon)^2, \qquad (2.17)
\end{aligned}$$

where we have used reflection-symmetry in the horizontal axis at the intermediate step. By Lemma 2.8 and reflection-symmetry again,

$$\begin{aligned}
a_{m,\beta} &= \sum_{\epsilon\in\Sigma_L}\phi_{m,\beta}(\sigma_L^+ = \sigma_L^- = \epsilon) \\
&\ge A^{2K}(1-R_\beta)\sum_{\epsilon\in\Sigma_L}\phi_{m,\beta}(\sigma_L^+ = \epsilon)^2.
\end{aligned}$$

Therefore,

$$\frac{\phi_{m,\beta}(c(\sigma_L^+)c(\sigma_L^-))}{a_{m,\beta}} \le A^{-4K}\frac{1+R_\beta}{1-R_\beta}. \tag{2.18}$$

We set $A = \{\sigma_L^+ = \sigma_L^-\}$ in Lemma 2.10 to find that, for sufficiently large $m \ge M_1(\lambda,\delta)$,

$$\left|\frac{\phi_{m,\beta}^\eta(\sigma_L^+ = \sigma_L^-)}{\phi_{m,\beta}(\sigma_L^+ = \sigma_L^-)} - 1\right| \le Ce^{-\frac{2}{7}\gamma m} < \frac{1}{2}.$$

By averaging over η, sampled according to $\phi_{n,\beta}$, we deduce that

$$\left|\frac{\phi_{n,\beta}(\sigma_L^+ = \sigma_L^-)}{\phi_{m,\beta}(\sigma_L^+ = \sigma_L^-)} - 1\right| \le Ce^{-\frac{2}{7}\gamma m} < \frac{1}{2},$$

which is to say that

$$\left|\frac{a_{n,\beta}}{a_{m,\beta}} - 1\right| \le Ce^{-\frac{2}{7}\gamma m} < \frac{1}{2}. \tag{2.19}$$

We make a note for later use. By the remark after Lemma 2.10, a version of inequality (2.18) holds with $\phi_{m,\beta}$ replaced by the continuum random-cluster measure ϕ_B on the box B with free boundary conditions, namely,

$$\frac{\phi_B(c(\sigma_L^+)c(\sigma_L^-))}{a_B} \le A^{-4K}\frac{1+R_{2r}}{1-R_{2r}}, \tag{2.20}$$

where $R_{2r} < \frac{1}{2}$ by (2.13). By (2.19), we may take C and M_1 above such that

$$\left|\frac{a_{n,\beta}}{a_B} - 1\right| \le Ce^{-\frac{2}{7}\gamma r} < \frac{1}{2}, \qquad r \ge M'(\lambda, \delta), \tag{2.21}$$

where $a_B = \phi_B(\sigma_L^+ = \sigma_L^-)$.

Inequalities (2.18) and (2.19) may be combined as in (2.16) to obtain

$$|\nu(S_c 1_D)| \le C_1 A^{-4K}\frac{1+R_\beta}{1-R_\beta}e^{-\frac{2}{7}\gamma m} \tag{2.22}$$

for an appropriate constant $C_1 = C_1(\lambda, \delta)$ and all $m \ge M_1$.

We turn to the term $\nu(S_c 1_{\overline{D}})$ in (2.15). Evidently,

$$|\nu(S_c 1_{\overline{D}})| \le A_m + B_n, \tag{2.23}$$

where

$$A_m = \frac{\nu(c(\sigma_{1,L}^+)c(\sigma_{1,L}^-)1_{\overline{D}})}{a_{m,\beta}}, \quad B_n = \frac{\nu(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-)1_{\overline{D}})}{a_{n,\beta}}.$$

There exist constants C_2, M'' depending on λ, δ, such that, for $m > r \ge M_2$,

$$\begin{aligned} B_n &= \frac{\nu(\overline{D})}{a_{n,\beta}}\nu(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-) \mid \overline{D}) \\ &= \frac{\nu(\overline{D})}{a_{n,\beta}}\phi_{n,\beta}\big(\phi_B^\tau(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-)) \mid \overline{D}\big) \\ &\le \frac{\nu(\overline{D})}{a_B}C_2\phi_B(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-)) \end{aligned} \tag{2.24}$$

by Lemma 2.10 with $\phi_{m,\beta}$ replaced by ϕ_B, and (2.21). At the middle step, we have used conditional expectation given the configuration τ on $\Lambda_{m,\beta} \setminus B$. By (2.20), there exists $C_3 = C_3(\lambda, \delta)$ such that

$$\frac{1}{a_B}\phi_B(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-)) \le C_3 A^{-4K}\frac{1+R_{2r}}{1-R_{2r}}. \tag{2.25}$$

Inequalities (2.24)–(2.25) imply an upper bound for B_n in terms of $\nu(\overline{D})$.

A similar upper bound is valid for A_m, on noting that the conditioning on $\overline{D}$ imparts certain information about the configuration ω_1 outside B but nothing further about ω_1 within B. Combining this with (2.23)–(2.25), we find that, for $r \geq M_3(\lambda, \delta)$ and some $C_4 = C_4(\lambda, \delta)$,

$$|\nu(S_c 1_{\overline{D}})| \leq \nu(\overline{D}) C_4 A^{-4K} \frac{1 + R_{2r}}{1 - R_{2r}}. \tag{2.26}$$

Let $r = \max\{2l, M_3\}$ to obtain by (2.3), (2.12), and Theorem 2.1 that

$$\nu(\overline{D}) \leq C_5 r e^{-\frac{1}{2}\gamma m} \leq C_6 e^{-\frac{1}{3}\gamma m}, \qquad m \geq M_4, \tag{2.27}$$

for some C_5, C_6, $M_4 \geq 2M_3$. We combine (2.22), (2.26), (2.27) as in (2.15). Letting $\beta \to \infty$ and recalling (2.11), we obtain (2.7) from (2.5), for $m \geq M := \max\{M_1, M_2, M_4\}$.

Finally, we remark that C and M depend on λ and δ. The left side of (2.7) is invariant under re-scalings of the time-axes, that is, under the transformations $(\lambda, \delta) \mapsto (\lambda\eta, \delta\eta)$ for $\eta \in (0, \infty)$. We may therefore work with the new values $\lambda' = \theta$, $\delta' = 1$, with appropriate constants $\alpha(\theta, 1)$, $C(\theta, 1)$, $M(\theta, 1)$. □

3. Proofs of Lemmas 2.8 and 2.10

Let Λ be a box in $\mathbb{Z} \times \mathbb{R}$ (we shall later consider a box Λ with a slit S_L, for which the same definitions and results are valid). A *path* π of Λ is an alternating sequence of disjoint intervals (contained in Λ) and unit line-segments of the form $[z_0, z_1]$, b_{12}, $[z_2, z_3]$, b_{34}, $\dots$, $b_{2k-1,2k}$, $[z_{2k}, z_{2k+1}]$, where: each pair z_{2i}, z_{2i+1} is on the same time line of Λ, and $b_{2i-1,2i}$ is a unit line-segment with endpoints z_{2i-1} and z_{2i}, perpendicular to the time-lines. The path π is said to join z_0 and z_{2k+1}. The *length* of π is its one-dimensional Lebesgue measure. A *circuit* D of Λ is a path except inasmuch as $z_0 = z_{2k+1}$. A set D is called *linear* if it is a disjoint union of paths and/or circuits. Let Δ, Γ be disjoint subsets of Λ. The linear set D is said to *separate* Δ and Γ if every path of Λ from Δ to Γ passes through D, and D is minimal with this property in that no strict subset of D has the property.

Let $\omega \in \Omega_\Lambda$. An *open path* π of ω is a path of Λ such that, in the notation above, the intervals $[z_{2i}, z_{2i+1}]$ contain no death of ω, and the line-segments $b_{2i-1,2i}$ are bridges of ω.

The (one-dimensional) Lebesgue measure of a measurable subset S of $\mathbb{Z} \times \mathbb{R}$ is denoted $|S|$. Let S and T be measurable subsets of Λ. The distance $d(S, T)$ from S to T is defined to be the infimum of the lengths of paths having one endpoint in S and one in T.

Let ϕ_Λ denote the random-cluster measure on Ω_Λ with parameters λ, δ, and $q = 2$ (with top/bottom periodic boundary condition). Let Γ be a measurable subset and Δ a finite subset of Λ such that $\Delta \cap \Gamma = \varnothing$. We shall make use of the 'ratio weak-mixing property' of the spin-configurations in Δ and Γ that is stated and proved in [11, Thm 7.1]. Let $\overline{\phi}$ denote the continuum random-cluster measure on Λ with parameters λ, δ, $q = 2$, but subject to the difference that the set of clusters that intersect $\Delta \cup \Gamma$ count only 1 in all towards the cluster count $k(\omega)$ in (2.2). We call $\overline{\phi}$ a 'wired random-cluster measure'.

We now prove Lemmas 2.8 and 2.10. Consider the box $\Lambda_{m,\beta}$ with slit S_L. Let K be an integer satisfying $1 \le K < \frac{1}{2}L$, and let

$$\begin{aligned} \Delta &= \{x^+ : x \in S_L,\ K \le x \le L-K\}, \\ \Gamma &= \{x^- : x \in S_L,\ K \le x \le L-K\}. \end{aligned} \tag{3.1}$$

The following replaces [11, Lemma 7.24].

Lemma 3.2. *Let $\lambda, \delta \in (0,\infty)$ satisfy $\theta = \lambda/\delta < 1$. There exists $C_1 = C_1(\lambda,\delta) \in (0,\infty)$ such that the following holds. Let*

$$R = R(K,L,\beta) = C_1(e^{-\frac{1}{2}\gamma K} + Le^{-\frac{1}{4}\gamma\beta}),$$

and let $\gamma > 0$ be as in Theorem 2.1. For $\epsilon_K^+ \in \Sigma_\Delta$, $\epsilon_K^- \in \Sigma_\Gamma$, we have that

$$\left| \frac{\phi_{m,\beta}(\sigma_\Delta = \epsilon_K^+,\ \sigma_\Gamma = \epsilon_K^-)}{\phi_{m,\beta}(\sigma_\Delta = \epsilon_K^+)\phi_{m,\beta}(\sigma_\Gamma = \epsilon_K^-)} - 1 \right| \le R,$$

whenever $R \le \frac{1}{2}$.

Proof. Take

$$D = \Big([-m,0) \times \{0\}\Big) \cup \Big((L, L+m]) \times \{0\}\Big) \cup \Big([-m, m+L] \times \{\tfrac{1}{2}\beta\}\Big),$$

the union of the two horizontal line-segments that, when taken with the slit S_L, complete the 'equator' of $\Lambda_{m,\beta}$, together with the top/bottom of $\Lambda_{m,\beta}$. Recalling that we are dealing with the top/bottom periodic boundary condition, D is a linear subset of $\Lambda_{m,\beta}$ that separates Δ and Γ. Let t_1, t_2, t be as in [11, Thm 7.1], namely,

$$t_1 = \overline{\phi}(\Delta \leftrightarrow D), \quad t_2 = \sqrt{\overline{\phi}(D \leftrightarrow \Gamma)}, \quad t = t_1 + 2t_2 + \frac{t_1 + t_2}{1 - t_1 - 2t_2}. \tag{3.3}$$

Since $\overline{\phi} \le_{\text{st}} \mathbb{P}_{\Lambda,\lambda,\delta}$, there exist constants C_2, C_3, depending on λ and δ alone, such that

$$\begin{aligned} t_1 &\le 2 \sum_{i=K}^{\lfloor L/2 \rfloor} C_2 e^{-\gamma i} + 2Le^{-\frac{1}{2}\gamma\beta} \\ &\le C_3 e^{-\gamma K} + 2Le^{-\frac{1}{2}\gamma\beta}, \end{aligned}$$

and furthermore $t_2^2 = t_1$. The claim now follows by [11, Thm 7.1]. □

Proof of Lemma 2.8. Let γ be as in Theorem 2.1. With $1 \le K < \frac{1}{2}L$, write $\sigma_{L,K}^\pm = (\sigma_x^\pm : K \le x \le L-K)$. First, let $x = (L,0)$, and let $\epsilon^+, \epsilon^- \in$

$\{-1,+1\}^{L+1}$ be possible spin-vectors of the sets S_L^+ and S_L^-, respectively. By [11, Lemma 7.25] with $S = S_L^+ \cup S_L^- \setminus \{x^+\}$,

$$\phi_{m,\beta}(\sigma_L^+ = \epsilon^+,\, \sigma_L^- = \epsilon^-) \\ \geq \tfrac{1}{2}\phi_{m,\beta}(\sigma_y^+ = \epsilon_y^+ \text{ for } y \in S_L^+ \setminus \{x^+\},\, \sigma_L^- = \epsilon^-)\mathbb{P}_{\Lambda_{m,\beta},\lambda,\delta}(x^+ \nleftrightarrow S).$$

Now, $\mathbb{P}_{\Lambda_{m,\beta},\lambda,\delta}(x \nleftrightarrow S)$ is at least as large as the probability that the first event (death or bridge) encountered on moving northwards from x is a death, so that

$$\mathbb{P}_{\Lambda_{m,\beta},\lambda,\delta}(x \nleftrightarrow S) \geq \frac{\delta}{2\lambda+\delta}.$$

On iterating the above, we obtain that

$$\phi_{m,\beta}(\sigma_L^+ = \epsilon^+,\, \sigma_L^- = \epsilon^-) \geq A^{2K}\phi_{m,\beta}(\sigma_{L,K}^+ = \epsilon_K^+,\, \sigma_{L,K}^- = \epsilon_K^-), \tag{3.4}$$

where $\epsilon_K^\pm$ is the vector obtained from $\epsilon^\pm$ by removing the entries labelled by vertices x satisfying $0 \leq x < K$ and $L-K < x \leq L$, and

$$A = \left(\frac{\delta}{2\lambda+\delta}\right)^2. \tag{3.5}$$

In summary, for $\epsilon^\pm \in \Sigma_L$,

$$A^{2K}\phi_{m,\beta}(\sigma_{L,K}^+ = \epsilon_K^+,\, \sigma_{L,K}^- = \epsilon_K^-) \leq \phi_{m,\beta}(\sigma_L^+ = \epsilon^+,\ \sigma_L^- = \epsilon^-) \\ \leq \phi_{m,\beta}(\sigma_{L,K}^+ = \epsilon_K^+,\, \sigma_{L,K}^- = \epsilon_K^-). \tag{3.6}$$

With Δ, Γ as in (3.1), we apply Lemma 3.2 to obtain that there exists $C_1 = C_1(\lambda,\delta) < \infty$ such that

$$\left|\frac{\phi_{m,\beta}(\sigma_{L,K}^+ = \epsilon_K^+,\, \sigma_{L,K}^- = \epsilon_K^-)}{\phi_{m,\beta}(\sigma_{L,K}^+ = \epsilon_K^+)\phi_{m,\beta}(\sigma_{L,K}^- = \epsilon_K^-)} - 1\right| \leq C_1(e^{-\frac{1}{2}\gamma K} + Le^{-\frac{1}{4}\gamma\beta}),$$

whenever the right side is less than or equal to $\frac{1}{2}$.

By a similar argument to (3.6),

$$A^K\phi_{m,\beta}(\sigma_{L,K}^\pm = \epsilon_K^\pm) \leq \phi_{m,\beta}(\sigma_L^\pm = \epsilon^\pm) \leq \phi_{m,\beta}(\sigma_{L,K}^\pm = \epsilon_K^\pm).$$

The claim follows. □

Proof of Lemma 2.10. Let $\Delta = S_L^+ \cup S_L^-$ and $\Gamma = \partial^{\mathrm{h}}\Lambda_{m,\beta}$. Let $k = \frac{3}{7}m$ and assume for simplicity that k is an integer. (If either m is small or k is non-integral, the constant C may be adjusted accordingly.) Let D_0 be the circuit illustrated in Figure 2, comprising a path in the upper half-plane from $(-k,0)$ to $(L+k,0)$ together with its reflection in the x-axis. Let $D = D_0 \cap \Lambda_{m,\beta}$. Thus, $D = D_0$ in the case $\beta = \beta_2$ of the figure. In the case $\beta = \beta_1$, D comprises two disjoint circuits of $\Lambda_{m,\beta}$ (with top/bottom periodic boundary condition). In each case, D separates Δ and Σ.

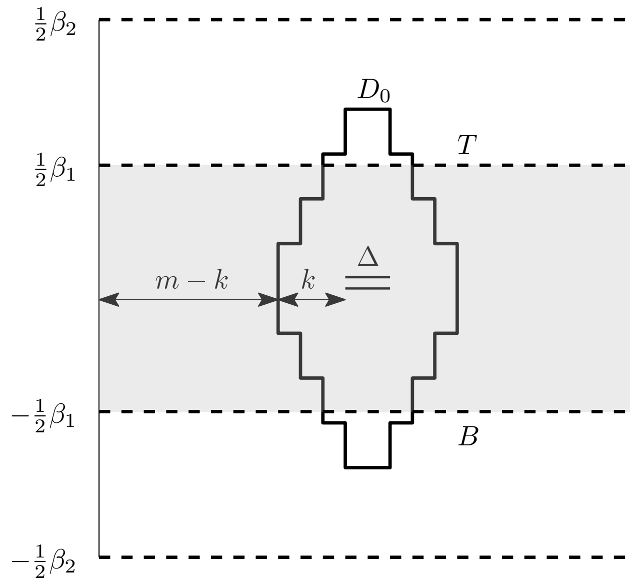

FIGURE 2. The circuit D_0 is approximately a parallelogram with Δ at its centre. The sides comprise vertical steps of height 2 followed by horizontal steps of length 1. The horizontal and vertical diagonals of D_0 have lengths $2k+L$ and (approximately) $4k+2L$ respectively, where $k = \frac{3}{7}m$. Two values of β are indicated. When $\beta = \beta_2$, D_0 is contained in $\Lambda_{m,\beta}$ and we take $D = D_0$. When $\beta = \beta_1$, $\Lambda_{m,\beta}$ is the shaded area only, and we work with $D = D_0 \cap \Lambda_{m,\beta}$ considered as the union of two disjoint circuits that separates D and Σ.

Let t_1, t_2, t be as in (3.3). By the ratio weak-mixing theorem [11, Thm 7.1],

$$\left| \frac{\phi^{\alpha}_{m,\beta}((\sigma^+_L, \sigma^-_L) = (\epsilon^+, \epsilon^-))}{\phi_{m,\beta}((\sigma^+_L, \sigma^-_L) = (\epsilon^+, \epsilon^-))} - 1 \right| \le 2t, \qquad \alpha = \eta, \tau, \ \epsilon^{\pm} \in \Sigma_L,$$

whenever $t \le \frac{1}{2}$. We ‘multiply up’ and sum over $(\epsilon^+, \epsilon^-) \in A$ to obtain

$$\left| \frac{\phi^{\alpha}_{m,\beta}(\sigma_\Delta \in A)}{\phi_{m,\beta}(\sigma_\Delta \in A)} - 1 \right| \le 2t, \tag{3.7}$$

whenever $t \le \frac{1}{2}$.

Since $\overline{\phi} \le_{\mathrm{st}} \mathbb{P}_{\Lambda,\lambda,\delta}$, there exist $C_2, C_3, c_4 > 0$, depending on λ, δ, such that

$$t_1 \le 4 \sum_{i=0}^{\lfloor L/2 \rfloor} \mathbb{P}_{\lambda,\delta}((i,0) \leftrightarrow D_0) \le 4 \sum_{i=0}^{\lfloor L/2 \rfloor} C_2 e^{-\gamma\frac{2}{3}(k+i)} \le C_3 e^{-\frac{2}{7}\gamma m}, \tag{3.8}$$

and similarly,

$$t_2^2 \le 8 \sum_{i=0}^{\lceil k+L/2 \rceil} C_2 e^{-\gamma(\frac{4}{7}m + c_4 i)} \le C_3 e^{-\frac{4}{7}\gamma m}. \tag{3.9}$$

The claim of the lemma follows by [11, Thm 7.1]. □

4. Quenched disorder

The parameters λ and δ have so far been assumed constant. The situation is more complicated in the disordered case, when either they vary deterministically, or they are random. The arguments of this paper may be applied in both cases, and the outcomes are summarised in this section. Let the Hamiltonian (1.2) be replaced by (1.11), and write $\boldsymbol{\lambda} = (\lambda_{x,x+1} : x \in \mathbb{Z})$ and $\boldsymbol{\delta} = (\delta_x : x \in \mathbb{Z})$.

The fundamental bound of Theorem 2.6 depends only on the ratio $\theta = \lambda/\delta$. In the disordered setting, the connection probabilities of the continuum random-cluster model are increasing in $\boldsymbol{\lambda}$ and decreasing in $\boldsymbol{\delta}$, and the function $A(\lambda, \delta)$ of (3.5) is replaced by functions of the form

$$A'_{x,k} = \prod_{i=1}^{k} \left(\frac{\delta_{x+i}}{\delta_{x+i} + \lambda_{x+i,x+i-1} + \lambda_{x+i,x+i+1}} \right), \tag{4.1}$$

which are decreasing in $\boldsymbol{\lambda}$ and increasing in $\boldsymbol{\delta}$. By examination of the earlier lemmas and proofs, the conclusions of the paper are found to be valid with $\gamma = \gamma(\lambda, \delta)$ whenever (1.13) holds with some $\lambda, \delta > 0$. Hence, in the disordered case where (1.13) holds with probability one, the corresponding conclusions are valid a.s. (subject to appropriate bounds on the ratio λ/δ). This proves Theorem 1.12.

Consider now the situation in which (1.13) does not hold with probability one. Suppose that the $\lambda_{x,x+1}$, $x \in \mathbb{Z}$, are independent, identically distributed random variables, and similarly the δ_x, $x \in \mathbb{Z}$, and assume that the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$ are independent. (One may work with weaker assumptions on the dependence structure of $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$, but for convenience we assume the above independence.) We write P for the corresponding probability measure, viewed as the measure governing the 'random environment', and Λ, Δ for random variables with the same distributions as $\lambda_{x,y}$ and δ_z, respectively. The mean of a random variable Z under P is denoted $P(D)$. Let $\mathbb{P}_{\boldsymbol{\lambda},\boldsymbol{\delta}}$ be the probability measure of the quenched continuum percolation process, condiitonal on $\boldsymbol{\lambda}$, $\boldsymbol{\delta}$. In applying the methods of this paper within the random environment, one needs to deal with sub-domains of $\mathbb{Z}$ where the environment is not propitious for the bound of Theorem 2.6.

As before, we perform a comparison of the continuum random-cluster and percolation models in a random environment, and we shall appeal to [11, Thm

8.2] (see [14] and [1, Thm 1.6]). For $(x,s),(y,t) \in \mathbb{Z} \times \mathbb{R}$, let

$$d(x,s;y,t) = \max\big\{|x-y|, \ln^+|s-t|\big\}, \tag{4.2}$$

where $\ln^+ x = \max\{\ln x, 0\}$.

For the remainder of this section we assume that the conditions of [11, Thm 8.2] are valid, and we shall work with $\gamma > 1$, and the identically distributed random variables D_x given in the theorem. We let $L \geq 8$, $1 \leq K < \frac{1}{2}(L-1)$, and consider the event

$$A_{K,L} = \bigcap_{x=K}^{L-K} \big\{D_x < \min\{x, L-x\}\big\},$$

noting that

$$P(A_{K,L}) \geq 1 - 2\sum_{x=K}^{\infty} P(D \geq x),$$

where D has the distribution of the D_x. By [11, Thm 8.2], there exists $\eta > 1$ such that $P(D^\eta) < \infty$, whence

$$P(A_{K,L}) \geq 1 - CK^{1-\eta} \to 1 \qquad \text{as } K, L \to \infty. \tag{4.3}$$

The conclusion of Lemma 3.2 is valid whenever the event $A_{K,L}$ occurs. The conclusion of Lemma 2.8 holds on $A_{K,L}$ with $A(\lambda,\delta)$ replaced by $X_{K,L}$, where

$$X_{K,L} = \left(\prod_{x \in \Theta} \frac{\delta_x}{\delta_x + \lambda_{x,x-1} + \lambda_{x,x+1}}\right)^2,$$

where, in the notation of the proof of Lemma 2.8, $\Theta = (S_L^+ \setminus \Delta) \cup (S_L^- \setminus \Gamma)$; cf. (4.1). Now,

$$\ln X_{K,L} = -2\sum_{x=0}^{K-1} Z_x - 2\sum_{x=L-K+1}^{L} Z_x \tag{4.4}$$

where

$$Z_x = \ln\left(1 + \frac{\lambda_{x,x-1} + \lambda_{x,x+1}}{\delta_x}\right).$$

The two summations in (4.4) are independent of one another, and each is the sum of a 1-dependent sequence of random variables. Also,

$$Z_x \leq \ln\left(1 + \frac{\lambda_{x,x-1}}{\delta_x}\right) + \ln\left(1 + \frac{\lambda_{x,x+1}}{\delta_x}\right),$$

so that, by [11, ass. (8.4)] and the Minkowski inequality,

$$\sqrt{P(Z_x^2)} \leq 2\sqrt{P\Big(\big[\ln(1 + (\Lambda/\Delta))\big]^2\Big)} < \infty.$$

By the central limit theorem for 1-dependent sequences (see, for example, [13, Thm 19.2.1]),

$$P\left(X_{K,L} \geq e^{-a_K\sqrt{K}}\right) \to 1 \qquad \text{as } K, L \to \infty, \tag{4.5}$$

for any a_K satisfying $a_k \to \infty$ as $K \to \infty$. Let $B_{K,L}(a) = \{X_{K,L} \geq e^{-a\sqrt{K}}\}$.

Some changes are necessary to the proof of Lemma 2.10, reflecting the fact that the distance function of (2.13) is sublinear in time. The circuit illustrated in Figure 2 is generated by translation, discretisation, and reflection of the Cartesian line $y = 2x$. In the disordered setting, we work instead with the curve $y = e^x$, and we assume $\beta > 5e^{m+\frac{1}{2}L}$. We define two further events that depend on the environment. Assume for simplicity that m is even, write $k = \frac{1}{2}m$, and let

$$C_{L,m} = \bigcap_{x=0}^{L} \big\{D_x < \tfrac{1}{2}\min\{k+x, L+k-x\}\big\},$$

$$D_{L,m} = \bigcap_{x=-k}^{L+k} \big\{D_x < \min\{m+x, L+m-x\}\big\}.$$

In the current setting, (3.8) becomes

$$t_1 \leq C_1 e^{-\frac{1}{4}\gamma m} \quad \text{on the event} \quad C_{L,m},$$

for some constant C_1 depending on γ. Similarly, (3.9) is replaced by

$$t_2^2 \leq C_2 e^{-\frac{1}{2}\gamma m} \quad \text{on the event} \quad D_{L,m}.$$

An amended version of Lemma 2.10 thus holds, so long as the event $C_{L,m} \cap D_{L,m}$ occurs.

We estimate $P(C_{L,m} \cap D_{L,m})$ as follows. First, since $P(D) < \infty$,

$$P(C_{L,m}) \geq 1 - 2\sum_{x=0}^{\lfloor\frac{1}{2}L\rfloor} P(D_x \geq \tfrac{1}{2}(k+x)) \to 1 \qquad \text{as } k = \tfrac{1}{2}m \to \infty. \tag{4.6}$$

Similarly,

$$P(D_{L,m}) \geq 1 - 2\sum_{x=-k}^{\lfloor\frac{1}{2}L\rfloor} P(D_x \geq m+x) \to 1 \qquad \text{as } m \to \infty. \tag{4.7}$$

[Note: The following is subject to editing.] Let $a_K \to \infty$ as $K \to \infty$, and let $E_{K,L} = A_{K,L} \cap B_{K,L}(a_K) \cap C_{L,m} \cap D_{L,m}$, noting from (4.3), (4.5), and (4.6)–(4.7) that $\mathbb{P}_{\boldsymbol{\lambda},\boldsymbol{\delta}}(E_{K,L}) \to 1$ as $K \to \infty$. On the event $E_{K,L}$, the estimate (1.7) holds with C replaced by $Ce^{ca_K\sqrt{K}}$ for some absolute constant $c > 0$. The proof of Theorem 1.9 may be followed to obtain that there exists a random variable $c_1 < \infty$ such that $S(\rho_m^L) \leq c_1$. ★

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