PERCOLATION OF FINITE CLUSTERS AND INFINITE SURFACES

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Abstract. Two related issues are explored for bond percolation on $\mathbb{Z}^d$ (with $d \geq 3$) and its dual plaquette process. Firstly, for what values of the parameter p does the complement of the infinite open cluster possess an infinite component? The corresponding critical point p_{fin} satisfies $p_{\text{fin}} \geq p_{\text{c}}$, and strict inequality is proved when either d is sufficiently large, or $d \geq 7$ and the model is sufficiently spread out. It is not known whether $d \geq 3$ suffices. Secondly, for what p does there exist an infinite dual surface of plaquettes? The associated critical point p_{surf} satisfies $p_{\text{surf}} \geq p_{\text{fin}}$.

1. Introduction

Bond percolation on the square lattice $\mathbb{Z}^2$ has a natural dual process, which is itself a bond percolation model. This fact has contributed to a detailed understanding of percolation in two dimensions, see for example [11, 18, 31]. The picture is more complicated in d dimensions with $d \geq 3$, in part because the natural dual model is a process on *plaquettes* rather than edges, and these plaquettes form $(d-1)$-dimensional *surfaces*. Perhaps the first systematic study of the plaquette process appeared in [3], where so-called area- and surface-laws were proved. Later papers dealing with plaquettes include [9, 13, 14], and also [21] on first-order phase transition in the random-cluster model (see also [12, Chap. 7]).

We study two related questions concerning bond percolation on $\mathbb{Z}^d$ and its dual process, of which the first is as follows. Suppose we remove all vertices that lie in an infinite open cluster of a bond percolation process with parameter p. The set X that remains is the union of the vertex sets of all finite clusters, and it induces a subgraph of $\mathbb{Z}^d$. We denote this graph by X also. For what values of p does X possess an infinite connected component with positive probability? We may define

Date: 6 March 2013.

1991 *Mathematics Subject Classification.* 60K35, 82B43.

Key words and phrases. Percolation, plaquette model, branching process.

a critical probability p_{fin} such that almost surely X has an infinite component for $p < p_{\mathrm{fin}}$, and not for $p > p_{\mathrm{fin}}$. Let p_{c} be the percolation critical probability. Clearly $p_{\mathrm{fin}} \geq p_{\mathrm{c}}$, since $X = \mathbb{Z}^d$ a.s. for $p < p_{\mathrm{c}}$. In $d = 2$ dimensions, we have $p_{\mathrm{fin}} = p_{\mathrm{c}}$, since self-duality implies that for $p > p_{\mathrm{c}}$ the infinite open cluster contains cycles that enclose every vertex. In $d = 3$ dimensions, it is natural to expect the strict inequality $p_{\mathrm{c}} < p_{\mathrm{fin}}$, since slightly above p_{c} the infinite open cluster should not be sufficiently dense to prevent connections in its complement. We prove the last inequality in high dimensions.

Theorem 1.1. *For $d \geq 19$, we have the strict inequality $p_{\mathrm{c}} < p_{\mathrm{fin}}$.*

Our proof of Theorem 1.1 relies on the recent proof in [20] that the one-arm critical exponent ρ takes its mean-field value $\frac{1}{2}$ in high dimensions. Indeed, we prove that $p_{\mathrm{c}} < p_{\mathrm{fin}}$ provided $\rho < 1$ (see Theorem 4.1 for the precise statement); this is believed to be the case for all $d \geq 5$ but not for $d = 3, 4$. We do not know whether or not $p_{\mathrm{c}} < p_{\mathrm{fin}}$ for $3 \leq d \leq 18$.

Theorem 4.1 is stronger than Theorem 1.1 in two further respects. Firstly, X can be replaced with the complement of the infinite cluster of percolation on the spread-out lattice (while connectedness in X still refers to $\mathbb{Z}^d$). This enables Theorem 1.1 to be extended to sufficiently spread-out lattices for all $d \geq 7$. Secondly, X may be replaced with the set of vertices that are not within distance F of the infinite cluster, for any finite F.

When $p_{\mathrm{c}} < p < p_{\mathrm{fin}}$, there is simultaneous occurrence of two disjoint infinite objects, namely an infinite open cluster and an infinite component of the non-percolating region. This is reminiscent of the result of Campanino and Russo [8] that *site* percolation on $\mathbb{Z}^3$ with $p = \frac{1}{2}$ contains both an infinite open and an infinite closed cluster.

Using fairly standard techniques, we show that $p_{\mathrm{fin}} < 1$ for $d \geq 2$, and that an infinite component of X (when it exists) is (a.s.) unique. When the lattice $\mathbb{Z}^d$ is replaced with a regular tree, p_{fin} may be explicitly computed (as done essentially in [17]) and it satisfies $p_{\mathrm{fin}} < p_{\mathrm{c}}$.

The above results lead to a partial answer to (and were in part motivated by) our second main question, which concerns the plaquette process that is dual to bond percolation on $\mathbb{Z}^d$. A **plaquette** is a $(d-1)$-dimensional face of a unit cube centred at a vertex of $\mathbb{Z}^d$. An edge e of $\mathbb{Z}^d$ crosses a unique plaquette, called the **dual** of e. We declare a plaquette **open** if and only if its dual edge is closed in the bond percolation model. Thus the plaquettes form an i.i.d. percolation process with parameter $1 - p$. Open plaquettes can form surfaces, and one may ask whether these surfaces undergo a phase transition at p_{c}, in the sense that 'infinite surfaces' exist for $p < p_c$ and not for $p > p_c$. Such a statement is of course contingent on a precise definition of 'infinite surface'. We prove that, according to one natural choice of definition, the phase transition does *not* occur at p_c when d is sufficiently large.

We call two plaquettes **adjacent** if their intersection is a $(d-2)$-dimensional cube. We say that a set of plaquettes is **connected** if it induces a connected graph via this adjacency relation, and we say that it has **no boundary** if every $(d-2)$-cube lies in an even number of plaquettes (in other words, if it is a $(d-1)$-cycle in the homology over $\mathbb{Z}/2\mathbb{Z}$. A **surface** is a connected set of plaquettes with no boundary. We define the critical probability p_{surf} such that there exists (a.s.) an infinite surface of open plaquettes for $p < p_{\text{surf}}$, and not for $p > p_{\text{surf}}$.

Theorem 1.2. *For $d \geq 2$, we have $p_{\text{fin}} \leq p_{\text{surf}}$.*

Theorems 1.1 and 1.2 have the following immediate consequence.

Corollary 1.3. *For $d \geq 19$, we have the strict inequality $p_c < p_{\text{surf}}$.*

Thus, infinite dual surfaces of plaquettes (as defined above) exist only *strictly* above p_c in high dimensions. When $d = 2$, an infinite surface is a connected union of doubly infinite dual paths, and therefore $p_{\text{surf}} = p_c$ $(= p_{\text{fin}})$ in this case. We do not know whether $p_c < p_{\text{surf}}$ for $3 \leq d \leq 18$.

One may also impose further topological or other constraints on surfaces. As a preliminary result in this direction, we show in Proposition 6.2 that, for $p > 0$ sufficiently small, there exists a surface of open plaquettes that is uniformly homeomorphic to a hyperplane. In dimension 3, surfaces of zero genus (homeomorphic to planes, spheres or discs) arise as the natural candidates for dual objects that block *entangled* bond-clusters. However, there is additional complication here: existence of (say) a sphere enclosing the origin in the complement of the set of open bonds does not imply existence of a sphere enclosing the origin composed of open plaquettes. For details see [13].

The above questions are presented more formally in Section 2 below. Section 3 contains a key topological lemma concerned with the external boundary of a finite subset of $\mathbb{Z}^d$. This is followed in Sections 4–5 by treatment of the union of the finite clusters X, in the two cases of a lattice and a regular tree respectively. Section 6 is devoted to infinite surfaces, and their connection with X.

2. Notation

Let $d \geq 1$, and let $\mathbb{Z}^d$ be the set of d-vectors $x = (x_1, x_2, \ldots, x_d)$ of integers. A **facet** is a subset of $\mathbb{Z}^d$ of the form $F = x + (A_1 \times \cdots \times A_d)$ where $x \in \mathbb{Z}^d$ and each A_i is either $\{0\}$ or $\{0, 1\}$. If $\{0, 1\}$ appears k

times, we call F a k-dimensional facet, or a k-facet. We shall focus on 1-facets and $(d-1)$-facets, called respectively **edges** and **plaquettes**. The set of edges is denoted $\mathbb{E}^d$, and the associated lattice is the graph $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$. An edge $\{x, y\}$ will also be written as the unordered pair $\langle x, y\rangle$.

Two facets of different dimensions are said to be **incident** if one is a subset of the other. Two plaquettes π_1, π_2 are **adjacent**, written $\pi_1 \sim \pi_2$, if some $(d-2)$-facet is incident to both. A set P of plaquettes is called **connected** if the graph with vertex set P and adjacency relation $\sim$ is connected. The **boundary** of a set P of plaquettes is the set of all $(d-2)$-facets that are incident to an odd number of elements of P. A **surface** is a (finite or infinite) connected set of plaquettes with empty boundary.

We make use of another notion of adjacency between plaquettes also, namely $\pi_1 \stackrel{1}{\sim} \pi_2$ if some 1-facet is incident to both.

We introduce the shifted ('dual') set $\widehat{\mathbb{Z}}^d := \mathbb{Z}^d + h$, where $h := (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^d$. A facet of $\widehat{\mathbb{Z}}^d$ is any subset of the form $F+h$ where F is a facet of $\mathbb{Z}^d$, and the concepts of incidence, adjacency, connectedness, boundaries and surfaces are defined as for $\mathbb{Z}^d$. For any k-facet F of $\mathbb{Z}^d$, there is a unique $(d-k)$-facet F' of $\widehat{\mathbb{Z}}^d$ with the same centre of mass as F, and we call F and F' **duals** of one another. (Equivalently: (i) F' is the unique $(d-k)$-facet of $\widehat{\mathbb{Z}}^d$ whose convex hull intersects the convex hull of F; or (ii) if F is expressed as $x + (A_1 \times \cdots \times A_d)$, then F' is obtained by replacing each occurrence of $\{0, 1\}$ with $\{\frac{1}{2}\}$, and each occurrence of $\{0\}$ with $\{-\frac{1}{2}, \frac{1}{2}\}$.) In particular, the dual of an edge $e \in \mathbb{E}^d$ is a plaquette, which we denote $\pi(e)$. Let Π denote the set of all plaquettes of $\widehat{\mathbb{Z}}^d$.

We turn now to probability. Let $\Omega = \{0, 1\}^{\mathbb{E}^d}$ be endowed with the product σ-field. Let $p \in [0, 1]$, and write $\mathbb{P}_p$ for product measure on Ω with parameter p, and let $\mathbb{E}_p$ be the associated expectation operator. For $\omega \in \Omega$, we call the edge e **open** (respectively, **closed**) if $\omega(e) = 1$ (respectively, $\omega(e) = 0$). The plaquette $\pi(e)$ is declared **open** (respectively, **closed**) if e is closed (respectively, open). Thus, each plaquette is open with probability $1-p$. Percolation theory is concerned with the structure of the connected components, or **open clusters**, of the graph $(\mathbb{Z}^d, \eta(\omega))$, where $\eta(\omega) = \{e : \omega(e) = 1\}$ is the set of open edges of the configuration ω. Let $p_c = p_c(d)$ denote the critical probability of bond percolation on $\mathbb{L}^d$, that is, the infimum of p for which there is strictly positive probability of an infinite open cluster. See [11] for a general account of percolation.

We describe next the two events studied in this paper. Let

$$\mathcal{S} := \{\Pi \text{ contains an infinite open surface}\}. \tag{2.1}$$

Since $\mathcal{S}$ is a decreasing subset of Ω, and is invariant under lattice-shifts, there exists $p_{\mathrm{surf}} = p_{\mathrm{surf}}(d) \in [0,1]$ such that

$$\mathbb{P}_p(\mathcal{S}) = \begin{cases} 1 & \text{if } p < p_{\mathrm{surf}}, \\ 0 & \text{if } p > p_{\mathrm{surf}}. \end{cases} \tag{2.2}$$

A **path** of $\mathbb{L}^d$ is an alternating sequence $v_0, e_1, v_1, e_2, \dots$ of distinct vertices v_i and edges $e_i = \langle v_{i-1}, v_i\rangle$. A path is called **open** if all its edges are open. Let $x, y \in \mathbb{Z}^d$. We write $x \leftrightarrow y$, if there exists an open path of $\mathbb{L}^d$ with endpoints x and y. We write $x \leftrightarrow \infty$ if there exists an infinite open path with endpoint x. Let

$$X := \{x \in \mathbb{Z}^3 : x \not\leftrightarrow \infty\},$$

so that X is the union of the vertex-sets of the finite open clusters of the percolation process. We turn X into a graph by adding all edges of $\mathbb{E}^d$ with both endpoints in X. Let

$$\mathcal{T} := \{X \text{ has an infinite connected component}\}. \tag{2.3}$$

As in the case of $\mathcal{S}$, there exists $p_{\mathrm{fin}} = p_{\mathrm{fin}}(d) \in [0,1]$ such that

$$\mathbb{P}_p(\mathcal{T}) = \begin{cases} 1 & \text{if } p < p_{\mathrm{fin}}, \\ 0 & \text{if } p > p_{\mathrm{fin}}. \end{cases} \tag{2.4}$$

3. Topological lemma

This section contains a fundamental lemma concerning the external edge-boundary of a finite subset of $\mathbb{Z}^d$. The case when $d = 2$ appears essentially in [18, App.]. The lemma is closely related to results for d dimensions of [9, 13, 30] and perhaps elsewhere. The proof given here makes use of [12, Thm 7.3].

Let $A \subseteq \mathbb{Z}^d$, and let the **external boundary** ΔA be the set of vertices of $\mathbb{Z}^d \setminus A$ that (i) are adjacent to some element of A, and (ii) lie in some infinite path of $\mathbb{L}^d$ not intersecting A. The (external) **edge-boundary** $\Delta_{\mathrm{e}} A$ is the set of edges $e = \langle x, y\rangle$ such that $x \in A$ and $y \in \Delta A$. For $A \subseteq \mathbb{Z}^d$, we define

$$\Pi(A) := \{\pi(e) : e \in \Delta_{\mathrm{e}} A\},$$

the set of plaquettes dual to edges in its edge-boundary. Let $|A| < \infty$. A set $\Sigma \subseteq \Pi$ of plaquettes is said to **separate** A **from infinity** if every infinite path starting in A contains some edge e with $\pi(e) \in \Sigma$.

Similarly, a set Σ of plaquettes of $\mathbb{Z}^d$ is said to **separate** A **from infinity** if every infinite path starting in A contains some vertex incident to some element of Σ.

Lemma 3.1. *If $A \subset \mathbb{Z}^d$ is finite and connected then $\Pi(A)$ is a surface.*

Proof. We show first that $\Pi(A)$ is connected. Let $W_1, W_2, \dots, W_k$ be the finite, connected components of $\mathbb{Z}^d \setminus A$, and let

$$\overline{A} := A \cup \left(\bigcup_{i=1}^{k} W_k \right).$$

We call the W_k the **holes** of A. It is seen as follows that $\overline{A}$ has no holes. Suppose w lies in a hole of $\overline{A}$. Then any infinite path γ from w has a last vertex $f(\gamma)$ lying in $\overline{A}$. It must be that $f(\gamma) \in A$, since $f(\gamma) \in W_i$ would contradict the definition of the W_i. Therefore, $w \in \overline{A}$, a contradiction.

We prove two facts about the boundaries of A and $\overline{A}$. Firstly,

$$\Pi(\overline{A}) = \Pi(A). \tag{3.1}$$

That $\Pi(A) \subseteq \Pi(\overline{A})$ follows from the definition of $\Pi(A)$. Conversely, if $\pi(e) \in \Pi(\overline{A}) \setminus \Pi(A)$, then e has one endvertex denoted a in $\overline{A} \setminus A$, and another that is joined to infinity off $\overline{A}$. Thus a lies in a hole of A, in contradiction of the fact that a is joined to infinity off A.

Let $\overline{\Pi}(\overline{A})$ be the set of plaquettes whose dual edges have exactly one endvertex in $\overline{A}$. Evidently, $\Pi(\overline{A}) \subseteq \overline{\Pi}(\overline{A})$, and we claim that

$$\Pi(\overline{A}) = \overline{\Pi}(\overline{A}). \tag{3.2}$$

If, on the contrary, $\pi(e) \in \overline{\Pi}(\overline{A}) \setminus \Pi(\overline{A})$, then e has an endvertex lying in a hole of $\overline{A}$. Since $\overline{A}$ has no holes, (3.2) follows.

By [12, Thm 7.3], there exists a subset $Q \subseteq \overline{\Pi}(\overline{A})$ that separates $\overline{A}$ from ∞ and is connected. By (3.1)–(3.2), $Q \subseteq \Pi(A)$. Also, $\Pi(A) \subseteq Q$, since if there exists $\pi \in \Pi(A) \setminus Q$, then Q does not separate A from ∞. Therefore, $\Pi(A) = Q$, implying that $\Pi(A)$ is connected.

Next we show that $\Pi(A)$ has empty boundary. Write u_j for a unit vector in the direction of increasing jth coordinate. Let $e \in \Delta_{\mathrm{e}} A$, and assume without loss of generality that e has the form $\langle a, a+u_1 \rangle$ with $a \in A$; the other cases are treated in the same way after rotation of $\mathbb{Z}^d$. It suffices to take $a = 0$. The plaquette corresponding to the edge $e = \langle 0, u_1 \rangle$ is $\pi(e) = \{\frac{1}{2}\} \times \{-\frac{1}{2}, \frac{1}{2}\}^{d-1}$, and is incident to $2(d-1)$ $(d-2)$-facets of $\widehat{\mathbb{Z}}^d$, of which we shall consider $f := \{\frac{1}{2}\}^2 \times \{-\frac{1}{2}, \frac{1}{2}\}^{d-2}$; the other such $(d-2)$-facets may be treated similarly.

The facet f is incident to exactly four plaquettes in Π, namely the $\pi(e_i)$ with $e_1 = e = \langle 0, u_1\rangle$, $e_2 = \langle u_1, u_1 + u_2\rangle$, $e_3 = \langle u_1 + u_2, u_2\rangle$, $e_4 = \langle u_2, 0\rangle$. As we proceed around the cycle e_1, e_2, e_3, e_4, we encounter vertices that either lie in A or are connected to infinity off A. Each time we pass from a vertex in one category to a vertex in the other, we traverse a plaquette in $\Pi(A)$. Hence we traverse an even number of such plaquettes, and therefore f is incident to an even subset of $\Pi(A)$. ☐

4. Percolation of finite clusters

Let G be an infinite, locally finite graph, and consider bond percolation on G with parameter p. Let X be the set of vertices lying in no infinite cluster of the process. The subgraph of G induced by X is also denoted X. We consider the question of whether X has an infinite connected component. The case of $\mathbb{Z}^d$ is treated in this section, and the corresponding issue for a regular tree is considered briefly in the next section.

Recall the definition (2.4) of the critical point $p_{\text{fin}} = p_{\text{fin}}(d)$ for the hypercubic lattice $\mathbb{L}^d$. When $p < p_c$, all clusters are (a.s.) finite, so that $X = \mathbb{Z}^d$. Hence, $p_c \le p_{\text{fin}}$.

The main goal of this section is to prove Theorem 1.1, which follows from Theorem 4.1 below, by the main result of [20]. Later in this section we also prove that $p_{\text{fin}} < 1$ (Theorem 4.2), and that X has at most one infinite component (Theorem 4.3).

In order to state Theorem 4.1 we introduce next some further notation. Let $\|\cdot\|$ denote L^∞ distance on $\mathbb{Z}^d$. Let $S, F \in \{0, 1, 2, \dots\}$; the parameter S is the range of a spread-out model, and F is a 'fattening' parameter. Consider the **spread-out** percolation model on $\mathbb{Z}^d$ in which edges are placed between any pair $x, y \in \mathbb{Z}^d$ with $0 < \|x - y\| \le S$, and each edge is declared independently open with probability p. The corresponding probability measure is denoted $\mathbb{P}_p^S$, and p_c^S denotes the critical point.

Let I be the set of vertices that lie in infinite open clusters. Write

$$I^F := \{x \in \mathbb{Z}^d : \|x - y\| \le F \text{ for some } y \in I\},$$

and let $X^F = \mathbb{Z}^d \setminus I^F$. With X^F considered as a subgraph of $\mathbb{L}^d$, we let $\mathcal{I}^F$ be the event that X^F has an infinite connected component, and

$$p_{\text{fin}}^S(F) := \sup\{p : \mathbb{P}_p^S(\mathcal{I}^F) > 0\}. \tag{4.1}$$

Let $\text{rad}(C)$ denote the radius of the open cluster C at the origin,

$$\text{rad}(C) := \sup\{\|x\| : 0 \leftrightarrow x\}. \tag{4.2}$$

Theorem 4.1. *Let $d \geq 2$ and $S, F \geq 0$. There exists $c > 0$ such that the following holds. Suppose that, for all large n, say $n \geq N(c)$,*

$$\mathbb{P}^S_{p^S_c}(\mathrm{rad}(C) \geq n) \leq \frac{c}{n}. \tag{4.3}$$

Then $p^F_c < p^S_{\mathrm{fin}}(F)$.

Proof of Theorem 1.1. When $d \geq 19$ and $S = F = 0$, (4.3) is proved in [20]. □

The one-arm critical exponent $\rho = \rho(d)$ is given by

$$\mathbb{P}^S_{p^S_c}(\mathrm{rad}(C) \geq n) \approx n^{-1/\rho}, \tag{4.4}$$

(where various interpretations of the symbol $\approx$ are possible). See for example [11, Sect. 9.1] for a discussion of critical exponents and universality. A relation of the form (4.4) is believed to hold in a wide variety of settings including the spread-out models on $\mathbb{Z}^d$ for all $d \geq 2$. It is known that $\rho(d) = \frac{1}{2}$ for the nearest-neighbour model with $d \geq 19$, and for the sufficiently spread-out model in 7 and more dimensions [20]. On the other hand it is believed that $\rho(2) = \frac{48}{5}$ (> 1), so that (4.3) is expected to fail in two dimensions, as indeed does the conclusion of Theorem 4.1. It is proved in [22] (see also [28]) that ρ exists for site percolation on the triangular lattice and satisfies $\rho(2) = \frac{48}{5}$, and in [19, §5] that $\rho(2) \geq 3$, or more specifically

$$\mathbb{P}_{p_c}(\mathrm{rad}(C) \geq n) \geq cn^{-1/3}, \tag{4.5}$$

for bond percolation on the square lattice. The argument leading to (4.5) may be extended (using [18, Thm 5.1] and [11, eqn (6.56)]) to obtain a lower bound of order $n^{-(d-1)/2}$ on $\mathbb{L}^d$ with $d \geq 2$ (for $d = 2$ the bound is worse than (4.5), but is valid also in the spread-out case).

The critical exponents ρ, η are expected to satisfy the scaling inequality $2 \leq \rho(d - 2 + \eta)$, with equality when $d \leq 6$ (see [11, Sect. 9.1] for a discussion of the scaling relations, and [29] for a proof of the inequality under reasonable assumptions). Assuming the equality for $d \leq 6$, it follows that $\rho < 1$ if and only if

$$\eta > 4 - d. \tag{4.6}$$

Numerical studies of [4, 24] (see also [16, Table 4.1]) suggest that (4.6) fails when $d = 3$ but holds when $d = 5$. The evidence when $d = 4$, while less conclusive, suggests that (4.6) fails in this case also. It thus seems possible that (4.3) fails when $d = 3, 4$ and holds when $d = 5$.

Theorem 4.1 will be proved using a block argument based on a bound for a typical cluster-radius. A similar technique has been used recently in [1] to study a forest-fire problem in seven and more dimensions, also subject to the assumption $\rho < 1$.

Proof of Theorem 4.1. Consider first the unfattened nearest-neighbour model with $S = F = 0$; at the end of the proof, we indicate the necessary changes for the general case. Assume that (4.3) holds for $c > 0$ and $n \geq N(c)$. Rather then deleting the set of points in *infinite* open paths, we shall delete a larger set that is specified in terms of certain *finite* connections. A box argument will then be used to show the existence of an infinite component in the remaining graph.

We specify first the set of points to be deleted. Let $n \in \mathbb{N}$, and write $B_n = (-n, n]^d \cap \mathbb{Z}^d$ and $\partial B_n = B_n \setminus B_{n-1}$. Let R_n be the set of vertices in B_n joined by open paths to ∂B_{2n}, and note that R_n is a superset of the subset of B_n containing points lying in infinite open paths. By (4.3),

$$\mathbb{E}_{p_c}|R_n| \leq c(2n)^d n^{-1} = c2^d n^{d-1}. \tag{4.7}$$

By Markov's inequality,

$$\mathbb{P}_{p_c}(|R_n| \geq \tfrac{1}{2}|B_n|) \leq \frac{2c}{n}. \tag{4.8}$$

Let $C_1, C_2, \dots, C_m$ be the components of B_n after deletion of R_n. Each vertex in the external boundary ΔC_i of some C_i lies either outside B_n or in R_n. Each vertex of R_n is in at most $2d$ such external boundaries. Therefore,

$$2d|R_n| \geq \sum_{i=1}^{m} |\Delta_n C_i|, \tag{4.9}$$

where $\Delta_n C := \Delta C \cap B_n$. By [6, Thm 8], there exists $K > 0$ such that, for any connected subset C of B_n with $|C| \leq \frac{4}{5}|B_n|$, we have $|\Delta_n C| \geq K|C|^{(d-1)/d}$.

Let L_n be the event that there exists i with $|C_i| \geq \frac{4}{5}|B_n|$, and let $M_n = \{|R_n| < \frac{1}{2}|B_n|\}$. On the event $L_n^{\mathrm{c}} \cap M_n$, by (4.9),

$$\begin{aligned} 2d|R_n| &\geq K \sum_{i=1}^{m} |C_i|^{(d-1)/d} \\ &\geq K \left(\sum_{i=1}^{m} |C_i| \right)^{(d-1)/d} \\ &= K\left(|B_n| - |R_n|\right)^{(d-1)/d} \geq \tfrac{1}{2}K|B_n|^{(d-1)/d}. \end{aligned} \tag{4.10}$$

We take expectations to obtain by (4.7) that

$$\begin{aligned} c2^d n^{d-1} &\geq \mathbb{E}_{p_c}\big(|R_n|; L_n^c \cap M_n\big) \\ &\geq \tfrac{1}{4}(K/d)(2n)^{d-1}\mathbb{P}_{p_c}(L_n^c \cap M_n). \end{aligned}$$

By (4.8),

$$\mathbb{P}_{p_c}(L_n) \geq 1 - \frac{8cd}{K} - \frac{2c}{n}. \tag{4.11}$$

On the event L_n, we pick a C_i of largest size, and we colour its vertices 0-**green**.

Inequality (4.11) may be combined with a block argument to show the existence of an infinite component in X, when c and n are chosen suitably. For $z \in \mathbb{Z}^d$, define the block $B_n(z) := B_n + nz$. Two blocks are designated **adjacent** if and only if they have non-empty intersection. Let $R_n(z)$ and $L_n(z)$ be given as above but relative to the block $B_n(z)$. The block $B_n(z)$ is called **good** if $L_n(z)$ occurs. On $L_n(z)$, we pick a largest component in the complement of $R_n(z)$, and colour its vertices z-**green**. Let Γ be the set of all vertices that are z-green for some z.

It may be seen that the set of random block-colours is a $3d$-dependent family of random variables. By [23, Thm 0.0], there exists $\widetilde{p}_c \in (0,1)$ such that any $3d$-dependent site percolation model on $\mathbb{Z}^d$ with all site-marginals exceeding $\widetilde{p}_c$ has (a.s.) an infinite cluster. Choose $c > 0$ and $n \geq N(c)$ such that

$$1 - \frac{8cd}{K} - \frac{2c}{n} > \widetilde{p}_c.$$

By (4.11) and the continuity in p of $\mathbb{P}_p(L_n)$, we may find $p > p_c$ such that $\mathbb{P}_p(L_n) > \widetilde{p}_c$. There exists, $\mathbb{P}_p$-a.s., an infinite connected component of good blocks.

Let $z, z' \in \mathbb{Z}^d$ be adjacent, and note that the intersection of $B_n(z)$ and $B_n(z')$ has cardinality $\frac{1}{2}|B_n|$. Suppose $B_n(z)$ and $B_n(z')$ are both good. Since each contains a component of green vertices of size at least $\frac{4}{5}|B_n|$, at least $\frac{1}{5}$ of the vertices in the intersection $B_n(z) \cap B_n(z')$ are both z-green and z'-green. Therefore, the green sets in $B_n(z)$ and $B_n(z')$ have non-empty intersection. It follows that there exists, $\mathbb{P}_p$-a.s., an infinite component in Γ. Since no vertex of Γ lies in an infinite open cluster, the theorem is proved.

Essentially the same proof is valid for general S, F. The set R_n is defined as the set of $x \in B_n$ such that there exists y with $\|y - x\| \leq F$ and y is joined by an open path to some vertex in $\mathbb{Z}^d \setminus B_{2n+2F-1}$.

Equations (4.7)–(4.8) are replaced by

$$\mathbb{E}^S_{p^S_c}|R_n| \le c(2n+2F)^d n^{-1},$$
$$\mathbb{P}^S_{p^S_c}(|R_n| \ge \tfrac{1}{2}|B_n|) \le \frac{2c}{n}\left(1+\frac{F}{n}\right)^d,$$

and the proof then proceeds as before. □

Theorem 4.2. *Consider bond percolation on the lattice $\mathbb{L}^d$ with $d \ge 3$. We have that $p_{\mathrm{fin}} < 1$.*

The proof below may be made quantitative, in that it provides a calculable bound $p' < 1$ with $p_{\mathrm{fin}} \le p'$. This bound is however too imprecise to be interesting.

Proof of Theorem 4.2. The idea is as follows. When p is sufficiently close to 1, not only is there an infinite open cluster, but this cluster is 'fat' in the sense that it separates the origin from infinity. In order to construct the required cut-surface, we shall first use Lemma 3.1 to show that the origin is a.s. separated from infinity by a surface in $\mathbb{L}^d$ with the property that every constituent edge is open.

Let $0 < \alpha < p_c$, and consider bond percolation on $\mathbb{L}^d$ with parameter α. Let $A \subset \mathbb{Z}^d$ be finite and connected, and let $C = C_A$ be the set of all vertices reached from A by open paths. Since the percolation process is subcritical, C is (a.s.) finite and connected. By Lemma 3.1, $\Pi(C)$ is a surface that separates C (and hence A also) from infinity.

Interchanging the roles of $\mathbb{Z}^d$ and $\widehat{\mathbb{Z}}^d$, the above conclusion may be restated as follows. Suppose that each plaquette of $\mathbb{Z}^d$ is independently declared **occupied** with probability q. If $q > 1 - p_c$, a.s. every finite subset of $\widehat{\mathbb{Z}}^d$ is separated from ∞ by a finite occupied surface of plaquettes of $\mathbb{Z}^d$. If an infinite path ν of $\widehat{\mathbb{Z}}^d$ contains an edge dual to some plaquette π, the shifted path $\nu + h$ contains a vertex incident to π. Therefore, if $q > 1 - p_c$, every finite subset of $\mathbb{Z}^d$ is separated from ∞ by a finite occupied surface of plaquettes of $\mathbb{Z}^d$.

Returning to bond percolation on $\mathbb{E}^d$, call a plaquette of $\mathbb{Z}^d$ **good** if every incident edge of $\mathbb{E}^d$ is open. Write $\gamma(\pi) = 1$ (respectively, $\gamma(\pi) = 0$) if π is good (respectively, not good). The set of plaquettes of $\mathbb{Z}^d$ may be considered as the vertex-set of a graph G with adjacency relation $\overset{1}{\sim}$ given in Section 2. The vector $\gamma = (\gamma(\pi) : \pi \in \Pi + h)$ is 1-dependent in that, for any subsets Π_1, Π_2 of $\Pi + h$ separated by graph-theoretic distance at least 2, the sub-vectors $(\gamma(\pi) : \pi \in \Pi_1)$ and $(\gamma(\pi) : \pi \in \Pi_2)$ are independent. Let $p' \in (1 - p_c, 1)$. By [23, Thm 0.0] (see also [11, Thm 7.65]), there exists $p'' < 1$ such that: when $p > p''$, the law of γ stochastically dominates product measure with parameter p'.

Let $x \in \mathbb{Z}^d$. Let E_x be the event that x is separated from infinity by some finite surface of good plaquettes of $\mathbb{Z}^d$, and that in addition every vertex incident with this surface lies in the (a.s.) unique infinite open cluster. By the above, for $p > p''$,

$$1 - \mathbb{P}_p(E_0) \leq \mathbb{P}_{p'}(S_n^{\mathrm{c}}) + \mathbb{P}_p(B_n \nleftrightarrow \infty) \to 0 \qquad \text{as } n \to \infty,$$

where S_n is the event that there exists a finite open surface of plaquettes of $\mathbb{Z}^d$ separating $B_n := (-n, n]^d \cap \mathbb{Z}^d$ from infinity. Therefore, $\mathbb{P}_p(E_0) = 1$. By translation-invariance,

$$\mathbb{P}_p\Big(\bigcap_x E_x\Big) = 1.$$

On the last event, every infinite path from X intersects the infinite open cluster. It follows that X has a.s. no infinite component for $p'' < p < 1$. Therefore, $p_{\mathrm{fin}} \leq p''$ as required. □

Theorem 4.3. *Let $p \in [0, 1]$. The set X has either a.s. no infinite component or a.s. a unique infinite component.*

The proof of Theorem 4.3 will be an adaptation of the celebrated argument of Burton and Keane [7], and will proceed via the next two lemmas. The first describes the behaviour of the set X under modifications to the percolation configuration (however, it is easily seen that X does *not* possesses the so-called 'finite-energy property'). The second conveniently encapsulates a sufficiently general consequence of the 'encounter point' argument of [7].

For a percolation configuration $\omega \in \{0, 1\}^{\mathbb{E}^d}$ and an edge $e \in \mathbb{E}^d$, we write ω^e (respectively, ω_e) for the configuration that agrees with ω on $\mathbb{E}^d \setminus \{e\}$ and has $\omega^e(e) = 1$ (respectively, $\omega_e(e) = 0$).

Lemma 4.4. *For any configuration ω and any edge e we have*

$$X(\omega_e) = X(\omega^e) \cup F$$

for some (possibly empty) finite set $F = F(\omega, e) \subset \mathbb{Z}^d$.

Proof. The edge e lies in some open cluster C of ω^e. When the state of e is changed from open to closed, either the set of infinite open clusters is unchanged (in which case we take $F = \varnothing$), or C breaks into an infinite and a finite cluster (in which case we take F to be the vertex-set of the latter). □

The number of **ends** of a connected graph is the supremum over its finite subgraphs of the number of infinite components that remain after removing the subgraph. The following is proved (in the greater generality of amenable transitive graphs) in [5, remark following Cor. 5.5]; see also [25, Ex. 7.28].

Lemma 4.5. *Let H be a random subset of $\mathbb{E}^d$ that is invariant in law under translations of $\mathbb{Z}^d$. Then, almost surely, no component of H has more than 2 ends.*

Proof of Theorem 4.3. Let N be the number of infinite components of X. Since N is a translation-invariant function of a collection of independent random variables, it is a.s. equal to some constant n.

We first show that $n \in \{0, 1, \infty\}$. Suppose on the contrary that $1 < n < \infty$. There exist $x, y \in \mathbb{Z}^d$ that lie in distinct infinite components of X with positive probability. Let U be the vertex set of a finite path in $\mathbb{Z}^d$ connecting x and y. On the event mentioned above, modify the configuration ω by making every edge incident to U closed. In the modified configuration ω', all vertices of U lie in $X(\omega')$. By Lemma 4.4, $X(\omega') \supseteq X(\omega)$, so $X(\omega')$ has an infinite component that contains the original infinite components of x and y. By Lemma 4.4 again, $X(\omega') \setminus X(\omega)$ is finite, so no new infinite components have been created, and thus $X(\omega')$ has strictly fewer that n infinite components. Since the modification involved only a fixed finite set of edges, it follows that $X(\omega)$ has fewer than n components with positive probability, a contradiction.

We employ a similar argument to eliminate the possibility $n = \infty$. If $n = \infty$, there exist $x, y, z \in \mathbb{Z}^d$ that lie in distinct infinite components of X with strictly positive probability. On this event, we can modify ω by making a finite set of edges closed in such a way that x, y and z now lie in a single infinite component of X. By Lemma 4.4, only finitely many vertices are added to X in this process (and none are removed), so the resulting component has at least 3 ends. Therefore X has a component with at least 3 ends with positive probability, in contradiction of Lemma 4.5. □

5. On regular trees

We consider briefly the question of p_{fin} for percolation on a regular tree. Rather than the $(b+1)$-regular tree, for convenience we work with the rooted tree $\mathbb{T}_b$ all of whose vertices other than the root (denoted 0) have degree $b+1$; the root has degree b. Let

$$\rho_b(p) := \mathbb{P}_p(X \text{ possesses an infinite cluster}).$$

Since X is decreasing in the natural coupling of the processes as p varies, ρ_b is a non-increasing function. As before, ρ_b takes only the values 0 and 1, and the two 'phases' are separated by the critical value

$$p_{\text{fin}}(\mathbb{T}_b) = \inf\{p : \rho_b(p) = 0\}.$$

The following theorem is based on an elementary calculation using the theory of branching processes. The result is implicit in [17, Sect. 3.3], and an extension is found at [15, Thm 2.2].

Theorem 5.1. *Let $b \geq 2$. Then*

$$p_{\text{fin}}(\mathbb{T}_b) = \frac{b^{b/(b-1)} - b}{b^{b/(b-1)} - 1}, \quad \textit{and} \quad \rho_b(p_{\text{fin}}) = 0.$$

In particular, $p_{\text{fin}}(\mathbb{T}_2) = \frac{2}{3}$, and

$$p_{\text{fin}}(\mathbb{T}_b) = (1 + \mathrm{o}(1))\frac{1}{b}\log b \qquad \text{as } b \to \infty.$$

We outline below a different proof from those of [15, 17]. The following additional consequence is explained after the proof:

$$\mathbb{P}_{p_{\text{fin}}-\epsilon}\big(\text{the } X\text{-component of 0 is infinite}\big) = c_b\epsilon + \mathrm{O}(\epsilon^2) \tag{5.1}$$

as $\epsilon \downarrow 0$, for some $c_b \in (0, \infty)$ which may be computed explicitly.

Proof of Theorem 5.1. The bond percolation cluster C at the root amounts to a branching process with family-sizes distributed as $\text{bin}(b, p)$ and (probability) generating function $G(s) = (1 - p + ps)^b$. We assume $p > 1/b$, since $X = T$ a.s. otherwise. Conditional on extinction, the family-size generating function is $H(s) = G(s\eta)/\eta$ where the probability $\eta = \eta(p)$ of extinction is the smallest non-negative root of $G(s) = s$.

Consider a branching process with generating function H. The generating function $T(s)$ of the total size satisfies

$$T(s) = sH(T(s)), \tag{5.2}$$

so that $T'(1) = H(1) + H'(1)T'(1)$, and hence

$$T'(1) = \frac{1}{1 - H'(1)} = \frac{1}{1 - G'(\eta)}.$$

Let D be a finite connected subgraph of T, rooted at some vertex v and with the property that every vertex other than v has generation number strictly greater than that of v. A **boundary edge** of D is an edge $\langle x, y\rangle$ of T such that $x \in D$, $y \notin D$, and y is a child of x. If D is *infinite*, it is considered to have *no* boundary edges.

Returning to the original $\text{bin}(n, p)$ branching process, we consider the embedded branching process of boundary edges of rooted finite clusters. The cluster C at the origin has $1+|C|(b-1)$ boundary-edges if finite (and 0 if infinite), with generating function

$$K(s) = 1 - \eta + \eta s T(s^{b-1}). \tag{5.3}$$

A branching process with generating function K survives with strictly positive probability if and only if $K'(1) > 1$, which is to say that

$$\eta[1 + (b-1)T'(1)] > 1,$$

or equivalently

$$G'(\eta) > \frac{1 - b\eta}{1 - \eta}. \tag{5.4}$$

Since $G(s) = (1 - p + ps)^b$,

$$G'(\eta) = bp(1 - p + p\eta)^{b-1} = \frac{bp\eta}{1 - p + p\eta}, \tag{5.5}$$

and (5.4) becomes

$$\eta > \frac{1 - p}{b - p}. \tag{5.6}$$

Suppose that (5.6) were to hold with equality. Since $\eta = G(\eta)$,

$$\left(1 - p + \frac{p(1-p)}{b - p}\right)^b = \frac{1 - p}{b - p},$$

which may be solved to find that

$$p = \frac{b^{b/(b-1)} - b}{b^{b/(b-1)} - 1}.$$

It follows that (5.6) holds if and only if p is strictly smaller than the above value.

In summary, the non-percolating part of the original tree percolates if and only if

$$p < \frac{b^{b/(b-1)} - b}{b^{b/(b-1)} - 1}.$$

That $\rho(p_{\mathrm{fin}}) = 0$ follows from the fact that a critical branching process with non-zero variance dies out almost surely. □

Let $\kappa(p)$ be the probability that the root lies in an infinite component of X. By an elementary analysis based on the above, one may compute the critical exponent of κ as $p \uparrow p_{\mathrm{fin}}$.

Let $b \geq 2$ and $p > 1/b$. By differentiating (5.2),

$$T'(1) = \frac{1}{1 - G'(\eta)}, \quad T''(1) = \frac{2G'(\eta)(1 - G'(\eta)) + \eta G''(\eta)}{(1 - G'(\eta))^3},$$

where η is the extinction probability given earlier. The term $G'(\eta)$ is given in (5.5), and by a similar calculation

$$G''(\eta) = \frac{p^2\eta b(b-1)}{(1-p+p\eta)^2}.$$

The point $p = p_{\text{fin}}$ is characterized by equality in (5.4) and (5.6), and one may use Taylor's theorem to expand $T(s) = T_p(s)$ with p near $p_{\text{fin}}(\mathbb{T}_b)$ and s near 1. With K as in (5.3), the survival probability $\kappa(p)$ is the largest root in $[0,1]$ of the equation $1-\kappa = K(1-\kappa)$. In conclusion, we obtain (5.1).

6. Infinite open surfaces

We return to the lattice $\mathbb{L}^d$ with $d \geq 3$, and consider the existence (or not) of an infinite open surface, that is, an infinite connected set of open plaquettes with empty boundary. Recall the event $\mathcal{S}$ of (2.1), and the critical probability p_{surf}. We indicate first the elementary inequality $p_{\text{surf}} > 0$. Later in this section we prove the inequalities $p_{\text{c}} \leq p_{\text{surf}}$ and $p_{\text{fin}} \leq p_{\text{surf}}$.

Proposition 6.1. *For $d \geq 2$, we have $p_{\text{surf}} > 0$.*

A stronger result holds. We identify a plaquette with the $(d-1)$-dimensional cube in $\mathbb{R}^d$ that is the closed convex hull of its 2^{d-1} points, and we identify a surface with the union of the cubes corresponding to its plaquettes. The surface S is said to be **uniformly homeomorphic to a hyperplane** if there exists a bijection from S to the hyperplane $\{0\} \times \mathbb{R}^{d-1}$ that is uniformly continuous with uniformly continuous inverse. Let $\mu = \mu_d$ be the connective constant of $\mathbb{L}^d$ (see [14] for a definition of μ).

Proposition 6.2. *Let $d \geq 2$. For $p < \mu^{-2}$, there exists (a.s.) an infinite open surface that is uniformly homeomorphic to a hyperplane.*

Although this implies Proposition 6.1, we include a short proof of the first proposition also.

Proof of Proposition 6.1. We call a d-facet B of $\widehat{\mathbb{Z}}^d$ **good** if each of its $2d$ incident plaquettes is open, and we write $\gamma(B) = 1$ if B is good, and $\gamma(B) = 0$ otherwise. As in the proof of Theorem 4.2, the cubes of $\widehat{\mathbb{Z}}^d$ may be considered as the vertices of a graph, and there exists $r < \infty$ such that their states are r-dependent. By [23, Thm 0.0], there exists $p' \in (0,1)$ such that: for $p \leq p'$, the vector γ stochastically dominates product measure with parameter $\vec{p}_{\text{c}}$, the critical probability of oriented site percolation on $\mathbb{L}^d$.

Therefore, for $p < p'$, there exists a.s. an infinite oriented path $v_1, v_2, \dots$ of $\mathbb{Z}^d$ such that the cubes $v_i + \{-\frac{1}{2}, \frac{1}{2}\}^d$ are good. The set $V = \{v_1, v_2 \dots\}$ generates an infinite surface $\Pi(V)$ of open plaquettes. (Note that $\Pi(V)$ is homeomorphic to a hyperplane, but not uniformly.) $\square$

Proof of Proposition 6.2. This relies on the methods developed in [14], and we include only a sketch of the proof.

For $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, let $s(x) = \sum_i x_i$. Define the hyperplane $H = \{x \in \mathbb{Z}^d : s(x) = 0\}$ of $\mathbb{R}^d$. For $r \geq 0$, let $H_r = \{y \in \mathbb{Z}^d : s(y) = r\}$, and $H_+ = \bigcup_{r \geq 0} H_r$.

A path $v_0, v_1, \dots, v_k$ of $\mathbb{L}^d$ is called **good** if (i) $v_i \in H_+$ for all i, and (ii) for every i with $s(v_{i-1}) < s(v_i)$, the edge $\langle v_{i-1}, v_i \rangle$ is open. For $x \in H_0$, let K_x be the set of all $y \in H_+$ such that there exists a good path from x to y, and let $K = \bigcup_{x \in H_0} K_x$.

Let $p < \mu^{-2}$ and $\alpha \in (\mu p, 1)$. By an adaptation of the proof of [14, Lemma 4], there exists $C = C(p, \alpha) < \infty$ such that, for $x \in H_0$,

$$\sum_{y \in H_r} \mathbb{P}_p(y \in K_x) \leq C\alpha^r, \qquad r \geq 1. \tag{6.1}$$

As in [10, eqn (2)], for $y \in H_+$,

$$\begin{aligned} \mathbb{P}_p(y \in K) &\leq \sum_{x \in H_0} \mathbb{P}_p(y \in K_x) \\ &= \sum_{z \in H_{s(y)}} \mathbb{P}_p(z \in K_0) \leq C\alpha^{s(y)}, \end{aligned} \tag{6.2}$$

by (6.1).

We now follow the proof of [14, Thm 1]. Let $u, v \in H_+$ be such that $e = \langle u, v \rangle$ is an edge of $\mathbb{Z}^d$ with $u \in K$ and $v \notin K$. Then $s(u) < s(v)$ and e is necessarily closed, so that the plaquette $\pi(e)$ is open. The set S of all such plaquettes forms a surface. The required homeomorphism $\phi : S \to H$ is given by the projection onto H,

$$\phi(z) = z - \overline{z}e,$$

where $\overline{z} = (z_1 + z_2 + \cdots + z_d)/d$ and $e = (1, 1, \dots, 1)$. $\square$

We indicate next that $p_{\text{surf}} \geq p_{\text{c}}$, and we defer the proof until later in this section.

Theorem 6.3. *For $d \geq 3$, we have $p_{\text{c}} \leq p_{\text{surf}}$.*

If the complement of the infinite open cluster contains an infinite component, one may deduce the existence of an infinite surface. This is the content of Theorem 1.2, which is a consequence of the following more general proposition.

Proposition 6.4. *Let $A \subset \mathbb{Z}^d$ be infinite and connected, and suppose that its complement A^{c} has an infinite component. There exists an infinite surface of plaquettes that are dual to edges of $\mathbb{L}^d$ having one vertex in A and the other in A^{c}.*

Proof of Theorem 1.2. If $p_{\mathrm{c}} = p_{\mathrm{fin}}$, the claim is a trivial consequence of Theorem 6.3. Assume that $p_{\mathrm{fin}} > p_{\mathrm{c}}$. Let $p_{\mathrm{c}} < p < p_{\mathrm{fin}}$, and apply Proposition 6.4 to the vertex-set A of the infinite open cluster. □

There follows a preliminary lemma that will be useful in the remaining proofs. The convergence of this lemma is in the product topology on $\{0,1\}^{\Pi}$. That is, for a sequence Π_n of subsets of Π, we write $\Pi_n \to \Pi_\infty$ if every $\pi \in \Pi_\infty$ lies in all but finitely many Π_n while every $\pi \notin \Pi_\infty$ lies in only finitely many Π_n.

Lemma 6.5. *Let $W_1 \subseteq W_2 \subseteq \cdots$ be an increasing sequence of subsets of $\mathbb{Z}^d$ that are connected and finite and satisfy $|W_n| \to \infty$ as $n \to \infty$. The limit $\Pi_\infty = \Pi(W_n)$ exists and has empty boundary and no finite components.*

If the set Π_∞ of Lemma 6.5 is non-empty, then it possesses only infinite components, and each such component has empty boundary. Here is an example for which Π_∞ is non-empty. Let $W_n = \{(0,w) \in \mathbb{Z} \times \mathbb{Z}^{d-1} : \|w\|_{d-1} \le n\}$. Then Π_∞ comprises two infinite components.

Proof. Let $\Pi_n = \Pi(W_n)$. Since each W_n is finite and connected in $\mathbb{L}^d$, by Lemma 3.1, the Π_n are connected and have empty boundaries.

We claim first that $\Pi_\infty := \lim_{n\to\infty} \Pi_n$ exists in the sense of the product topology. More specifically, we claim that, for any plaquette $\pi \in \Pi$, exactly one of the following holds:

1. $\pi \notin \Pi_n$ for all n,
2. there exists k such that $\pi \in \Pi_n$ if and only if $n \ge k$.
3. there exist k, m satisfying $k < m$ such that $\pi \in \Pi_n$ if and only if $k \le n < m$.

To prove this, we must show that, as the sequence Π_n is revealed in sequence, if π appears, it may be removed, but if so it never reappears. For given $\pi \in \Pi$, let $k = \min\{n : \pi \in \Pi_n\}$ and assume $k < \infty$. Thus $\pi = \pi(e)$ for some $e = \langle v, w\rangle$ with $v \in W_k$ and w joined to infinity off W_k. Either $\pi \in \Pi_n$ for all $n \ge k$, or $m = \inf\{n > k : \pi \notin \Pi_n\}$ satisfies $m < \infty$. In the latter case, w lies either in W_m or in a hole of W_m (that is, a finite connected component of $\mathbb{Z}^d \setminus W_m$). Therefore, for $n \ge m$, w

lies in either W_n or a hole of W_n. In either case $\pi \notin \Pi_n$, and the claim is shown.

Let f be a $(d-2)$-facet of $\widehat{\mathbb{Z}}^d$. The collection of subsets $F \subset \Pi$ such that f lies in an even number of members of F is a cylinder subset of $\{0,1\}^\Pi$. Since every $(d-2)$-facet lies in an even number of plaquettes in every Π_n, and $\Pi_n \to \Pi_\infty$, Π_∞ has empty boundary. By the same argument, Π_∞ has no finite component (in the plaquette graph with adjacency relation $\sim$). □

Proof of Theorem 6.3. We shall show that $p \le p_{\mathrm{surf}}$ for all $p < p_{\mathrm{c}}$. Let $\omega = (\omega(e) : e \in \mathbb{E}^d) \in \Omega$ be such that

$$\text{every open cluster of } \omega \text{ is finite.} \tag{6.3}$$

From ω, we construct an increasing sequence $V_1, V_2, \dots$ of vertex-sets of $\mathbb{Z}^d$ as follows. Let V_1 be the vertex-set of the open cluster C_0 at $w(0) := 0$. Suppose we have constructed $V_1, V_2, \dots, V_n$, and each is finite. Let $v = (v_1, v_2, \dots, v_d)$ be a rightmost vertex of V_n, in that $v_1 \ge w_1$ for all $w \in V_n$. Let $w(n+1) = v + u_1$ where $u_1 = (1, 0, 0, \dots, 0)$, and let $C_{w(n+1)}$ be the vertex-set of the open cluster of $w(n+1)$. Let $V_{n+1} = V_n \cup C_{w(n+1)}$. Note that V_{n+1} is finite, and $|V_n| \to \infty$ as $n \to \infty$.

We apply Lemma 6.5 to the increasing sequence (V_n) to obtain the limit set $\Pi_\infty = \Pi_\infty(\omega)$ of plaquettes. The proof is completed by showing that, for $p < p_{\mathrm{c}}$,

$$\mathbb{P}_p(\Pi_\infty \ne \varnothing) = 1. \tag{6.4}$$

Let $p < p_{\mathrm{c}}$, so that (6.3) holds almost surely. Let L be the singly-infinite line $[(-\infty, 0] \times \{0\}^{d-1}] \cap \mathbb{Z}^d$. Since Π_n is connected and separates V_n from infinity, there exists an edge $f_n = \langle -r-1, -r \rangle \in L$ such that $\pi(f_n) \in \Pi_n$, and we pick $r = r_n$ maximal with this property. Now, $f_n \ne f_{n+1}$ only if $C_{w(n+1)} \cap L \ne \varnothing$. However,

$$\sum_{n=0}^{\infty} \mathbb{P}_p(C_{w(n+1)} \cap L \ne \varnothing) \le \sum_{n=0}^{\infty} \mathbb{P}_p(\mathrm{rad}(C_0) \ge n) = \mathbb{E}_p(\mathrm{rad}(C_0)),$$

where $\mathrm{rad}(C_0) = \sup\{\|x\| : 0 \leftrightarrow x\}$ is the radius of C_0 as in (4.2).

If $p < p_{\mathrm{c}}$, we have that $\mathbb{P}_p(\mathrm{rad}(C_0)) < \infty$; see [2, 26, 27], and also [11, Chap. 5]. By the Borel–Cantelli lemma, a.s. only finitely many of the $w(n)$ are connected by open paths to L. Therefore, there exists a.s. an edge $f \in L$ such that $\pi(f) \in \Pi_\infty$, whence $\Pi_\infty \ne \varnothing$ a.s. □

We used the fact that $\mathbb{P}_p(\mathrm{rad}(C_0)) < \infty$ when $p < p_{\mathrm{c}}$, at the end of the above proof. Note that the argument cannot be valid when $d = 2$ and $p = p_{\mathrm{c}} = \frac{1}{2}$, since then $\Pi_\infty = \varnothing$ a.s.

We remark that an alternative proof of Theorem 6.3 proceeds by applying Lemma 6.5 to the sequence (W_n), where W_n is the set of sites of $\mathbb{Z}^d$ connected by open paths to $\{v \in \mathbb{Z}^d : v_1 = 0, \|v\| \le n\}$.

Proof of Proposition 6.4. Let $B_n = (-n,n]^d \cap \mathbb{Z}^d$. Fix $x \in A$, and let A_n be the component of $A \cap B_n$ containing x; we set $A_n = \varnothing$ if $x \notin B_n$. Let $\Pi_n = \Pi(A_n)$. By Lemma 6.5, the limit $\Pi_\infty := \lim_{n\to\infty} \Pi_n$ exists, and (if non-empty) has empty boundary and only infinite components. Moreover, it is independent of the choice of x since, for $x, y \in A$, there exists a path of A joining x to y and, for all sufficiently large n, this path lies in B_n.

We argue as follows to show that $\Pi_\infty \ne \varnothing$. There exists an edge $f = \langle a, b\rangle$ with $a \in A$ and b joined to infinity off A, so that $\pi(f) \in \Pi_n$ for all large n. $\square$

Open Questions

(1) Does $p_c < p_{\mathrm{surf}}$ hold for $\mathbb{Z}^d$ with $3 \le d \le 18$?
(2) Does $p_c < p_{\mathrm{fin}}$ hold for $\mathbb{Z}^d$ with $3 \le d \le 18$?
(3) Does $p_{\mathrm{fin}} < p_{\mathrm{surf}}$ hold for $\mathbb{Z}^d$ with $d \ge 3$?

Acknowledgements

GRG acknowledges the hospitality of the Department of Mathematics at the University of British Columbia, the Section de Mathématiques at the University of Geneva, and the Theory Group at Microsoft Research. He was supported in part by the Swiss National Science Foundation, and the EPSRC under grant EP/103372X/1. GK was supported in part by the Israel Science Foundation.

References

[1] D. Ahlberg, H. Duminil-Copin, G. Kozma, and V. Sidoravicius, *Seven-dimensional forest fires*, (2012), available at: arXiv:1302.6872.
[2] M. Aizenman and D. J. Barsky, *Sharpness of the phase transition in percolation models*, Commun. Math. Phys. **108** (1987), no. 3, 489–526, available at: projecteuclid.org.
[3] M. Aizenman, J. T. Chayes, L. Chayes, J. Fröhlich, and L. Russo, *On a sharp transition from area law to perimeter law in a system of random surfaces*, Commun. Math. Phys. **92** (1983), no. 1, 19–69, available at: projecteuclid.org.
[4] H. G. Ballasteros, A. Fernández, V. Martín-Major, M. Muñoz Sudupe, G. Parisi, and J. J. Ruiz-Lorenzo, *Measures of critical exponents in the four dimensional site percolation*, Phys. Lett. B **400** (1997), no. 3-4, 346–351, available at: sciencedirect.com.

[5] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Group-invariant percolation on graphs*, Geom. Funct. Anal. **9** (1999), no. 1, 29–66, available at: springer.com.

[6] B. Bollobás and I. Leader, *Compressions and isoperimetric inequalities*, J. Combin. Theory Ser. A **56** (1991), no. 1, 47–62, available at: sciencedirect.com.

[7] R. M. Burton and M. Keane, *Density and uniqueness in percolation*, Commun. Math. Phys. **121** (1989), no. 3, 501–505, available at projecteuclid.org.

[8] M. Campanino and L. Russo, *An upper bound on the critical percolation probability for the three-dimensional cubic lattice*, Ann. Probab. **13** (1985), no. 2, 478–491, available at: jstor.org.

[9] J.-D. Deuschel and A. Pisztora, *Surface order large deviations for high-density percolation*, Probab. Th. Rel. Fields **104** (1996), no. 4, 467–482, available at: springer.com.

[10] N. Dirr, P. W. Dondl, G. R. Grimmett, A. E. Holroyd, and M. Scheutzow, *Lipschitz percolation*, Electron. Commun. Probab. **15** (2010), 14–21, available at: ejpecp.org.

[11] G. R. Grimmett, *Percolation*, 2nd ed., Springer, Berlin, 1999.

[12] ———, *The Random-Cluster Model*, Springer, Berlin, 2006, available at: cam.ac.uk/~grg.

[13] G. R. Grimmett and A. E. Holroyd, *Entanglement in percolation*, Proc. London Math. Soc. **81** (2000), 485–512.

[14] ———, *Plaquettes, spheres, and entanglement*, Electron. J. Probab. **15** (2010), 1415–1428, available at: ejpecp.org.

[15] G. R. Grimmett and S. Janson, *Branching processes, and random-cluster measures on trees*, J. European Math. Soc. **7** (2005), no. 2, 253–281, available at: ems-ph.org.

[16] B. D. Hughes, *Random Walks and Random Environments. Volume 2: Random Environments*, Clarendon Press, Oxford, 1996.

[17] J. Jonasson, *The random cluster model on a general graph and a phase transition characterization of nonamenability*, Stoch. Proc. Appl. **79** (1999), no. 2, 335–354, available at: sciencedirect.com.

[18] H. Kesten, *Percolation Theory for Mathematicians*, Birkhäuser Boston, Mass., 1982, available at: cornell.edu/~kesten.

[19] ———, *Scaling relations for 2D-percolation*, Commun. Math. Phys. **109** (1987), 109–156, available at: projecteuclid.org.

[20] G. Kozma and A. Nachmias, *Arm exponents in high dimensional percolation*, J. Amer. Math. Soc. **24** (2011), 375–409, available at: ams.org, arXiv:0911.0871.

[21] L. Laanait, A. Messager, S. Miracle-Solé, J. Ruiz, and S. Shlosman, *Interfaces in the Potts model I: Pirogov–Sinai theory of the Fortuin–Kasteleyn representation*, Commun. Math. Phys. **140** (1991), no. 1, 81–91, available at: projecteuclid.org.

[22] G. F. Lawler, O. Schramm, and W. Werner, *One-arm exponent for 2D critical percolation*, Electron. J. Probab. **7** (2002), 1–13, available at: ejpecp.org.

[23] T. M. Liggett, R. H. Schonmann, and A. M. Stacey, *Domination by product measures*, Ann. Probab. **25** (1997), no. 1, 71–95, available at: jstor.org.

[24] C. D. Lorenz and R. M. Ziff, *Precise determination of the bond percolation thresholds and finite-size scaling corrections for the sc, fcc, and bcc lattices*, Phys. Rev. E **57** (1998), no. 1, 230–236, available at: aps.org.

[25] R. Lyons with Y. Peres, *Probability on Trees and Networks*, 2012, in preparation, available at: iu.edu/~rdlyons.

[26] M. V. Menshikov, *Coincidence of critical points in percolation problems*, Dokl. Akad. Nauk SSSR **288** (1986), 1308–1311.

[27] M. V. Menshikov, S. A. Molchanov, and A. F. Sidorenko, *Percolation theory and some applications*, Probability Theory. Mathematical Statistics. Theoretical Cybernetics, Vol. 24 (Russian), Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Moscow, 1986, pp. 53–110.

[28] S. Smirnov and W. Werner, *Critical exponents for two-dimensional percolation*, Math. Res. Lett. **8** (2001), no. 6, 729–744, available at: intlpress.com, arXiv:math/0109120.

[29] H. Tasaki, *Hyperscaling inequalities for percolation*, Commun. Math. Phys. **113** (1987), no. 1, 49–65, available at: projecteuclid.org.

[30] Á. Timár, *Boundary-connectivity via graph theory*, Proc. Amer. Math. Soc. **141** (2013), no. 2, 475–480, available at: ams.org, arXiv:0711.1713.

[31] W. Werner, *Lectures on two-dimensional critical percolation*, Statistical Mechanics, IAS/Park City Math. Ser., vol. 16, Amer. Math. Soc., Providence, RI, 2009, pp. 297–360.

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