PERCOLATION CRITICAL PROBABILITIES OF MATCHING LATTICE-PAIRS

GEOFFREY R. GRIMMETT AND ZHONGYANG LI

Abstract. A necessary and sufficient condition is established for the strict inequality $p_c(G_*) < p_c(G)$ between the critical probabilities of site percolation on a quasi-transitive, plane graph G and on its matching graph G_*. It is assumed that G is properly embedded in either the Euclidean or the hyperbolic plane. When G is transitive, strict inequality holds if and only if G is not a triangulation. The basic approach is the standard method of enhancements, but its implemention has complexity arising from the non-Euclidean (hyperbolic) space, the study of site (rather than bond) percolation, and the generality of the assumption of quasi-transitivity. This result is complementary to the work of the authors ("Hyperbolic site percolation", `arXiv:2203.00981`) on the equality $p_u(G) + p_c(G_*) = 1$, where p_u is the critical probability for the existence of a unique infinite open cluster. More specifically, it implies for transitive G that $p_u(G) + p_c(G) \geq 1$, with equality if and only if G is a triangulation.

1. Strict inequalities for percolation probabilities

It is fundamental to the percolation model on a graph G that there exists a 'critical probability' $p_c(G)$ marking the onset of infinite open clusters. Two questions arise immediately.

(a) What can be said about the value of $p_c(G)$?
(b) For what values of the percolation density p is there a *unique* infinite cluster?

These questions have attracted a great deal of attention since percolation was introduced by Broadbent and Hammersley [8] in 1957. They turn out to be more tractable when G is planar.

Amongst exact calculations of $p_c(G)$, those for bond percolation on the square, triangular, and hexagonal lattices have been especially influential (see [14, 17], and also the book [11]). Earlier discussion (falling short of rigorous proof) of these values was provided by Sykes and Essam [16] in 1964. The last paper includes also an

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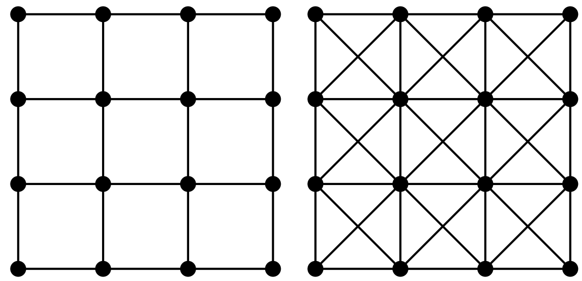

FIGURE 1.1. The square lattice $\mathbb{Z}^2$ and its covering graph.

account of site percolation on the triangular lattice, and a discussion of site percolation on a so-called 'matching pair' of planar lattices. This term is explained in the companion paper [12]; the current work is concerned with the matching pair (G, G_*), where the so-called matching graph G_* is defined as follows.

Let $G = (V, E)$ be a planar graph, embedded in the plane $\mathbb{R}^2$ in such way that two edges may intersect only at their endpoints. A *face* of G is a connected component of $\mathbb{R}^2 \setminus E$. The boundary of a bounded face F is comprised of edges of G. The *matching graph* of G, denoted G_*, is obtained from G by adding all diagonals to all faces. See Figure 1.1. Evidently, $G_* = G$ when G is a triangulation. A graph with connectivity 1 or 2 may have a multiplicity of non-homeomorphic planar embeddings, and therefore there is potential ambiguity over the definition of its matching and dual graphs (see Theorem 2.1(c)).

Sykes and Essam presented motivation for the exact relationship

$$p_c^{\text{site}}(G) + p_c^{\text{site}}(G_*) = 1, \tag{1.1}$$

and this has been verified in a number of cases when G is amenable (see [6, 14]). Note that, since G is a subgraph of G_*, it is trivial that

$$p_c^{\text{site}}(G_*) \le p_c^{\text{site}}(G). \tag{1.2}$$

It is less trivial to prove strict inequality for non-triangulations in (1.2), and indeed this sometimes fails to hold.

Suppose that G is planar, quasi-transitive, one-ended, and possibly non-amenable. If we are to embed G in a plane in a 'proper' fashion, the plane in question may need to be hyperbolic rather than Euclidean. Site percolation in the hyperbolic plane is the subject of the recent companion paper [12], where it is proved, amongst other things, that

$$p_u^{\text{site}}(G) + p_c^{\text{site}}(G_*) = 1, \tag{1.3}$$

where p_u^{site} is the critical probability for the existence of a *unique* infinite open cluster. When G is amenable, we have $p_c^{\text{site}}(G) = p_u^{\text{site}}(G)$, in agreement with (1.1). (When

G is non-amenable, it is proved in [5] that $p_c^{\mathrm{site}}(G) < p_u^{\mathrm{site}}(G)$.) By (1.2), we have $p_u^{\mathrm{site}}(G)+p_c^{\mathrm{site}}(G) \ge 1$, and it becomes desirable to know when strict inequality holds.

Let $\mathcal{Q}$ be the set of all infinite, connected, locally finite, plane, 2-connected, simple graphs that are in addition quasi-transitive. (It is explained in [12, Rem. 3.4] that the assumption of 2-connectedness is innocent in the context of site percolation.) A path $(\dots, x_{-1}, x_0, x_1, \dots)$ of G_* is called *non-self-touching* if, for all i, j, two vertices x_i and x_j are adjacent if and only if $|i-j| = 1$. Here is the main theorem of the current work, followed by a corollary.

Theorem 1.1. *Let $G \in \mathcal{Q}$ be one-ended. Then $p_c^{\mathrm{site}}(G_*) < p_c^{\mathrm{site}}(G)$ if and only if G_* contains some doubly-infinite, non-self-touching path that includes some diagonal of G.*

Corollary 1.2. *Let $G \in \mathcal{Q}$ be one-ended. Then $p_u^{\mathrm{site}}(G) + p_c^{\mathrm{site}}(G) \ge 1$, with strict inequality if and only if the condition of Theorem 1.1 holds.*

Proof of Corollary 1.2. The given (weak) inequality is proved at [12, Thm 1.1(b)], and the strict inequality follows by Theorem 1.1. □

There follow some remarks about the proof of Theorem 1.1. The general approach of the proof is to use the method of enhancements, as introduced and developed in [1] (though there is earlier work of relevance, including [15]). While this approach is fairly standard, and the above result natural, the proof turns out to have substantial complexity arising from the generality of the assumptions on G, and the fact that we are studying site (rather than bond) percolation (see [3]); the proof is, in contrast, fairly immediate for the amenable, planar lattices mentioned above.

We remark that the version of (1.3) for bond percolation, namely

$$p_u^{\mathrm{bond}}(G) + p_c^{\mathrm{bond}}(G^+) = 1, \tag{1.4}$$

was proved by Benjamini and Schramm [5, Thm 3.8] for one-ended, non-amenable, plane, transitive graphs. Here, G^+ denotes the dual graph of G. (The amenable case is standard.) The basic difference between the bond and site problems is the following. In the study of bond percolation, one is interested in open *self-avoiding* paths, whereas for site percolation we study open, *non-self-touching* paths.

Turning to the contents of the current article, after the introductory Section 2, we explain the application of Theorem 1.1 to transitive and quasi-transitive graphs in Section 3. Two methods are given there, the 'metric method' and the 'combinatorial method'. Each can be used to study transitive graphs. When working with quasi-transitive graphs, they lead to different sufficient (but not necessary) conditions for the required strict inequality. The proofs begin with some preliminary observations in Section 4, and the main theorem is proved in Section 5. The claims of Section 3 for quasi-transitive graphs are proved (respectively) by the metric method in Section 6 and by the combinatorial method in Section 7.

2. Notation and basic properties

2.1. Graph embeddings. We shall assume familiarity with basic graph theory and its notation, and refer the reader to [12] for relevant definitions. Let $\mathcal{Q}$ be given as prior to Theorem 1.1, and let $\mathcal{T}$ be the subset of $\mathcal{Q}$ comprising the transitive graphs.

A useful summary of hyperbolic geometry may be found in [9] (see also [13]). Quasi-transitive planar graphs may be embedded as follows in the Euclidean or hyperbolic plane, and *we shall use $\mathcal{H}$ to denote either of these as appropriate for the setting*. An embedding of a graph G in $\mathcal{H}$ is called *proper* if every compact subset of $\mathcal{H}$ contains only finitely many vertices of G and intersects only finitely many edges. Henceforth, all embeddings will be assumed to be proper. Here is a summary of relevant embedding theorems.

An *Archimedean tiling* (or *uniform tiling*) of a two-dimensional Riemannian manifold is a tiling by regular polygons such that its isometry-group acts transitively on its vertex-set. The edges of the tiling are geodesics. The manifolds in question are the Euclidean and hyperbolic planes, always denoted $\mathcal{H}$.

Some known facts concerning embeddings follow. References to proofs of these facts may be found in [12, Sect. 3.1].

Theorem 2.1.

(a) *If $G \in \mathcal{T}$ is one-ended, then G may be embedded in $\mathcal{H}$ as an Archimedean tiling, and all automorphisms of G extend to isometries of $\mathcal{H}$. If $G \in \mathcal{Q}$ is one-ended and 3-connected, then G may be embedded in $\mathcal{H}$ such that all automorphisms of G extend to isometries of $\mathcal{H}$.*

(b) *Let G be a 3-connected graph, cellularly embedded in $\mathcal{H}$ such that all faces are of finite size. Then G is uniquely embeddable in the sense that for any two cellular embeddings $\phi_1 : G \to S_1$, $\phi_2 : G \to S_2$ into planar surfaces S_1, S_2, there is a homeomorphism $\tau : S_1 \to S_2$ such that $\phi_2 = \tau\phi_1$.*

(c) *If $G = (V, E) \in \mathcal{Q}$ is one-ended, there exists some embedding of G in $\mathcal{H}$ such that the edges coincide with geodesics, the dual graph G^+ is quasi-transitive, and all automorphisms of G extend to isometries of $\mathcal{H}$. Such an embedding is called* canonical.

Remark 2.2.

(a) *All one-ended, transitive, planar graphs are 3-connected, and all proper embeddings of a one-ended, quasi-transitive, planar graph have only finite faces.*

(b) *By Theorem 2.1(b), any one-ended $G \in \mathcal{Q}$ that is in addition transitive has a unique proper cellular embedding in $\mathcal{H}$ up to homeomorphism. Hence, the matching and dual graphs of G are independent of the embedding.*
(c) *The conclusion of part (b) holds for any one-ended, 3-connected $G \in \mathcal{Q}$.*
(d) *For a one-ended, 2-connected $G \in \mathcal{Q}$, we fix a canonical embedding (in the sense of Theorem 2.1(c)). With this given, the dual graph G^+ and the matching graph G_* are quasi-transitive, and furthermore the boundary of every face is a cycle of G.*

We give a formal definition of the matching graph of a planar graph $G = (V, E)$. Firstly, one embeds G in the plane in such a way that two edges intersect only at their endpoints; such an embedded graph is called a *plane graph*. A *face* of a plane graph G is a connected component of $\mathcal{H} \setminus E$. In this work we shall treat only one-ended graphs, for which all faces G are bounded with (topological) boundaries ∂F comprised of finitely many edges. A *cycle* C of a simple graph $G = (V, E)$ is a sequence $v_0, v_1, \dots, v_{n+1} = v_0$ of vertices v_i such that $n \geq 3$, $e_i := \langle v_i, v_{i+1} \rangle$ satisfies $e_i \in E$ for $i = 0, 1, \dots, n$, and $v_0, v_1, \dots, v_n$ are distinct. Let G be a plane graph, duly embedded properly in the Euclidean or hyperbolic plane. In this case we write C° for the bounded component of $\mathcal{H} \setminus G$, and $\overline{C}$ for the closure of C°.

For a face F, let $V(\partial F)$ be the set of vertices lying along the boundary of F. We augment G by adding edges between any distinct pair $x, y \in V$ such that (i) there exists a face F with $x, y \in V(\partial F)$ and (ii) $\langle x, y \rangle \notin E$. We write $G_* = (V, E_*)$ for the ensuing *matching graph* of G. An edge $e \in E_* \setminus E$ is called a *diagonal* of G or of G_*, and it is denoted $\delta(a, b)$ where a, b are its endvertices. If $\delta(a, b)$ is a diagonal, a and b are called $*$-*neighbours*.

Note that G_* depends on the particular embedding of G. If G is 3-connected then, by Theorem 2.1(b), it has a unique embedding up to homeomorphism. If G is 2-connected but not 3-connected, we need to be definite about the choice of embedding, and we require it henceforth to be 'canonical' in the sense of Theorem 2.1(c).

2.2. **Further notation.** A plane graph G is called a *triangulation* it every face is bounded by a 3-cycle. The automorphism group of the graph $G = (V, E)$ is denoted $\mathrm{Aut}(G)$. The orbit of $v \in V$ is written $\mathrm{Aut}(G)v$, and we let

$$\Delta = \min\big\{k : \text{for } v, w \in V, \text{ we have } d_G(\mathrm{Aut}(G)v, \mathrm{Aut}(G)w) \leq k\big\}, \tag{2.1}$$

where

$$d_G(A, B) = \min\{d_G(a, b) : a \in A,\ b \in B\}, \qquad A, B \subseteq V,$$

and d_G denotes graph-distance in G. For any G, we fix some vertex denoted v_0.

We shall work with one-ended graphs $G \in \mathcal{Q}$. Since G is assumed one-ended and 2-connected, all its faces are bounded, with boundaries which are cycles of G (see Remark 2.2(d)).

Definition 2.3. *A path* $\pi = (\dots, x_{-1}, x_0, x_1 \dots)$ *of a graph* H *is called* non-self-touching *if* $d_H(x_i, x_j) \geq 2$ *when* $|j - i| \geq 2$. *A cycle* $C = (v_0, v_1, \dots, v_n, v_{n+1} = v_0)$ *of* H *is called non-self-touching if* $d_H(x_i, x_j) \geq 2$ *whenever* $|i - j| \geq 2$ *(with index-arithmetic modulo* $n + 1$*).*

Non-self-touching paths and cycles arise naturally when studying *site* percolation (such paths were called *stiff* in [1], and *self-repelling* in [11, p. 66]).

We shall consider non-self-touching paths in two graphs derived from a given $G \in \mathcal{Q}$, namely its matching graph G_*, and the graph $\widehat{G}$ obtained by adding a site within each face F of size 4 or more, and connecting every vertex of F to this new site. The graph G_* may possess parallel edges. The property of being non-self-touching is indifferent to the existence of parallel edges, since it is given in terms of the vertex-set of π and the adjacency relation of H.

Here is the fundamental property of graphs that implies strict inequality of critical points. This turns out to be equivalent to a more technical 'local' property, as described in Section 4.2; see Theorem 4.7. As a shorthand, henceforth we abbreviate 'doubly-infinite non-self-touching path' to '2∞-nst path'.

Definition 2.4. *The graph* $G \in \mathcal{Q}$ *is said to have property* Π *if* G_* *contains some* 2∞*-nst path that includes some diagonal of* G*.*

For a graph $G = (V, E)$, let

$$\Lambda_n(v) = \Lambda_{G,n}(v) := \{w \in V : d_G(v, w) \leq n\}, \quad \partial\Lambda_n(v) := \Lambda_n(v) \setminus \Lambda_{n-1}(v),$$

and, furthermore, $\Lambda_n = \Lambda_{G,n} := \Lambda_n(v_0)$. The set $\Lambda_n(v)$ will generally have bounded 'holes', which we fill in as follows. Let $\Delta_n(v)$ be the set of all edges $e = \langle u, v \rangle \in E$ such that $u \in \Lambda_n(v)$ and v lies in an infinite path of $G \setminus \Lambda_n(v)$. Let $\overline{\Lambda}_n(v)$ be the bounded subgraph of G after deletion of $\Delta_n(v)$. Let

$$\partial\overline{\Lambda}_n(v) := \overline{\Lambda}_n(v) \setminus \overline{\Lambda}_{n-1}(v),$$

and, furthermore, $\overline{\Lambda}_n = \overline{\Lambda}_{G,n} := \overline{\Lambda}_n(v_0)$. Finally, we write $u \sim v$ if $u, v \in V$ are adjacent.

2.3. **Percolation.** Let $G = (V, E)$ be a connected, locally finite graph with bounded vertex-degrees. A *site percolation* configuration on G is an assignment $\omega \in \Omega := \{0, 1\}^V$ to each vertex of either state 0 or state 1. A vertex is called *open* if it has state 1, and *closed* otherwise. An *open cluster* in ω is a maximal connected set of open vertices.

Let $p \in [0,1]$. We endow Ω with the product measure $\mathbb{P}_p$ with density p. For $v \in V$, let $\theta_v(p)$ be the probability that v lies in an infinite open cluster. It is standard that there exists $p_c(G) \in (0,1]$ such that

$$\text{for } v \in V, \qquad \theta_v(p) \begin{cases} = 0 & \text{if } p < p_c(G), \\ > 0 & \text{if } p > p_c(G), \end{cases}$$

and $p_c(G)$ is called the *critical probability* of G.

For background and notation concerning percolation theory, the reader is referred to the book [11], the article [12], and to Section 5.

3. Applications of Theorem 1.1

3.1. **Transitive graphs have property** Π**.** We investigate two classes of graphs with the property Π of Definition 2.4, and to which Theorem 1.1 may be applied. These are the transitive graphs, and subclasses of quasi-transitive graphs.

Theorem 3.1. *Let $G \in \mathcal{T}$ be one-ended but not a triangulation. Then G has property* Π*, and therefore satisfies $p_c(G_*) < p_c(G)$.*

We shall give two proofs of this result, using what we call the *metric method* and the *combinatorial method*. Each proof may be extended to a certain class of quasi-transitive graphs, the two such classes being different. In each case, the outcome is a sufficient but not necessary condition for a quasi-transitive graph $G \in \mathcal{Q}$ to have property Π, namely Theorems 3.4 and 3.8.

3.2. **The metric method.** The embedding results of Section 2 may be used to show the existence of 2∞-nst paths in *transitive*, one-ended $G \in \mathcal{T}$ that are not triangulations, and for certain *quasi-transitive*, one-ended $G \in \mathcal{Q}$. First, recall the relevant embedding properties. By Theorem 2.1(a), every transitive, one-ended $G \in \mathcal{T}$ may be embedded in $\mathcal{H}$ as an Archimedean tiling. By parts (a, c) of the same theorem, every quasi-transitive, one-ended $G \in \mathcal{Q}$ has a canonical embedding in $\mathcal{H}$.

Throughout this section we shall work with the Poincaré disk model of hyperbolic geometry (also denoted $\mathcal{H}$), and we denote by ρ the corresponding hyperbolic metric. For definiteness, we consider only graphs G embedded in the hyperbolic plane; the Euclidean case is easier.

Let $G \in \mathcal{Q}$ be one-ended and not a triangulation. By 2-connectedness and Remark 2.2(d), the faces of G are bounded by cycles. As before, we restrict ourselves to the case when G is non-amenable, and we embed G canonically in the Poincaré disk $\mathcal{H}$. The edges of G are hyperbolic geodesics, but its diagonals are not generally so. The hyperbolic length of an edge $e \in E_* \setminus E$ does not generally equal the hyperbolic distance between its endvertices, denoted $\rho(e)$.

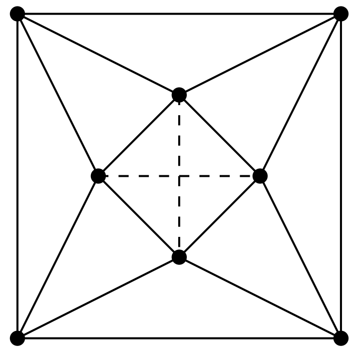

FIGURE 3.1. The graph G is the tiling of the plane with copies of this square. Taking into account the symmetries of the square, this tiling is canonical after a suitable rescaling of the interior square. The diagonals are indicated by dashed lines.

For $e \in E_*$, let Γ_e denote the doubly-infinite hyperbolic geodesic of $\mathcal{H}$ passing though the endvertices of e, and denote by $\pi(x)$ the orthogonal projection of $x \in \mathcal{H}$ onto Γ_e.

Definition 3.2. *An edge $e \in E_*$ is called* maximal *if*

$$\rho(e) \geq \rho(\pi(x), \pi(y)), \qquad f = \langle x, y \rangle \in E. \tag{3.1}$$

It is easily seen that any diagonal whose interior is surrounded by some triangle of G is not maximal; cf. the forthcoming Definition 3.6 of the term $\triangle$-empty. There always exists some maximal edge of E_*, but it is not generally unique. The following lemma is proved in the same manner as the forthcoming Lemma 6.1.

Lemma 3.3. *Let $f \in \operatorname{argmax}\{\rho(e) : e \in E_*\}$. The edge f is maximal.*

Here is the main theorem for quasi-transitive graphs using the metric method.

Theorem 3.4. *Let $G \in \mathcal{Q}$ be one-ended but not a triangulation. Assume that G has a canonical embedding in $\mathcal{H}$ for which some diagonal $d \in E_* \setminus E$ is maximal. Then G has the property Π of Definition 2.4, whence $p_c(G_*) < p_c(G)$.*

See Sections 6.2 and 6.3 for the proofs of Theorems 3.1 and 3.4 by the metric method.

Remark 3.5. *The condition of Theorem 3.4 is sufficient but not necessary, as indicated by the following example. Let G be the canonical tiling of $\mathbb{R}^2$ illustrated in Figure 3.1. By inspection, no diagonal is maximal, whereas G has property Π. The sufficient condition in question can be weakened as explained in Remark 6.4, and the above example satisfes the weaker condition.*

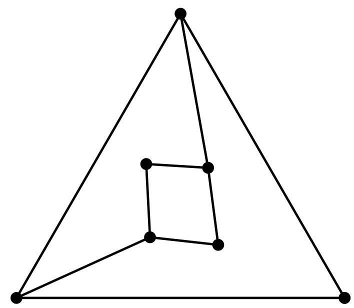

FIGURE 3.2. A doubly periodic family of faces of the triangular lattice are decorated as above, and the resulting graph is not $\triangle$-empty. Since no triangle can be connected to infinity by two paths π_1, π_2 satisfying $d_G(\pi_1, \pi_2) \geq 2$, the configuration on the interor I of this triangle is independent of the existence of an infinite open path starting at a vertex not in I.

3.3. **The combinatorial method.** We begin with some notation.

Definition 3.6. *The plane graph $G = (V, E)$ is said to have property $\square$ if every vertex of G lies in the boundary of some face of size 4 or more. A cycle C is said to* surround *a point $x \in \mathcal{H}$ if $\mathcal{H} \setminus C$ has a bounded component containing x. The graph G is said to be* $\triangle$-empty *if no 3-cycle C surrounds any vertex v.*

Figure 3.2 is an illustration of part of a 2-connected, quasi-transitive graph that is not $\triangle$-empty. It turns out that all transitive graphs are $\triangle$-empty.

Lemma 3.7. *A transitive, properly embedded, plane graph $G = (V, E) \in \mathcal{T}$ is $\triangle$-empty, and furthermore it has property $\square$ if and only if it is not a triangulation.*

Proof. Let $G = (V, E) \in \mathcal{T}$ be properly embedded and plane, but not $\triangle$-empty. Let $v_1 \in V$. By transitivity, v_1 lies in the interior of some 3-cycle C_1. Let v_2 be a vertex of C_1. Then v_2 lies in the interior of some 3-cycle C_2; since G is plane, $C_1 \subseteq \overline{C}_2$. On iterating this construction we obtain an infinite sequence (v_i, C_i) of pairs of vertices and 3-cycles such that: v_i is a vertex of C_i, $C_i \subseteq \overline{C}_{i+1}$, and $v_i \in C_{i+1}^{\circ}$. If the C_i are uniformly bounded, the sequence (v_i) has a limit point, in contradiction of the assumption of proper embedding; if not, it contradicts the fact that the edge-lengths of G are uniformly bounded. From this contradiction we deduce that G is $\triangle$-empty. The second statement of the lemma is immediate. $\square$

We henceforth assume that G is $\triangle$-empty. If this were false, let W be the set of all vertices lying in the interior of some 3-cycle. Let C be a 3-cycle of G that surrounds some vertex. The event that there exists an infinite open path starting in $V \setminus W$ and passing through C is independent of the states of vertices in C°; this holds since every pair of vertices of C are joined by an edge. See Figure 3.2. One may therefore remove all vertices in W without altering the existence or not of an infinite open path.

Here is the main theorem of this section; it is proved in Section 7 by the combinatorial method.

Theorem 3.8. *Let $G \in \mathcal{Q}$ be one-ended and $\triangle$-empty. If G has property $\square$, then G has property Π also.*

Proof of Theorem 3.1 using the combinatorial method. Let $G \in \mathcal{T}$ be one-ended. If G is a triangulation, then $G_* = G$, so that $p_c(G_*) = p_c(G)$. Suppose conversely that G is not a triangulation. By [7, Prop. 2.2] (see Remark 2.2(a)), G is 3-connected. By Lemma 3.7, G is $\triangle$-empty and has property $\square$, and therefore by Theorem 3.8 property Π also. The final claim follows by Theorem 1.1. □

4. Some observations

4.1. **Oxbow-removal.** We begin by describing a technique of loop-removal (henceforth referred to as 'oxbow-removal'). Let H be a simple graph embedded in the Euclidean/hyperbolic plane $\mathcal{H}$ (possibly with crossings).

Lemma 4.1. *Let H be a graph embedded in $\mathcal{H}$.*

(a) *Let C be a plane cycle of H that surrounds a point $x \notin H$. There exists a non-empty subset C' of the vertex-set of C that forms a plane, non-self-touching cycle of H and surrounds x.*
(b) *Let π be a finite (respectively, infinite) path with endpoint v. There exists a non-empty subset π' of the vertex-set of π that forms a finite (respectively, infinite) non-self-touching path of H starting at v. If π is finite, then π' can be chosen with the same endpoints as π.*

Proof. (a) Let $C = (v_0, v_1, \dots, v_n, v_{n+1} = v_0)$ be a plane cycle of H that surrounds $x \notin H$; we shall apply an iterative process of 'loop-removal' to C, and may assume $n \geq 4$. We start at v_0 and move around C in increasing order of vertex-index. Let J be the least $j \leq n$ such that there exists $i \in \{1, 2, \dots, j-2\}$ with $v_i \sim v_J$, and let I be the earliest such i. Consider the two cycles $C' = (v_I, v_{I+1}, \dots, v_J, v_I)$ and $C'' = (v_J, v_{J+1}, \dots, v_0, v_1 \dots . v_I, v_J)$. (These cycles are called *oxbows* since they arise through cutting across a bottleneck of the original cycle C.) Since C surrounds x, so does either or both of C' and C'', and we suppose for concreteness that C'' surrounds x. We replace C by C''. This process is iterated until no such oxbows remain.

(b) This part is proved by a similar argument. When the endpoints v_0, v_n of π are not neighbours, we use oxbow-removal as above; otherwise, we set $\pi' = (v_0, v_n)$. □

Path-surgery will be used in the forthcoming proofs: that is, the replacement of certain paths by others. Consider a one-ended $G \in \mathcal{Q}$, embedded properly and canonically in the hyperbolic plane $\mathcal{H}$, which for concreteness we consider here in the Poincaré disk model (see [9]), also denoted $\mathcal{H}$. By Theorem 2.1(c), every automorphism of G extends to an isometry of $\mathcal{H}$. Let $\mathcal{F}$ be the set of faces of G. For $F \in \mathcal{F}$ and $x, y \in V(\partial F)$, let $\mathcal{L}_{x,y}$ be the set of rectifiable curves with endpoints x, y whose interiors are subsets of $F^\circ \setminus E$, and write $l_{x,y}$ for the infimum of the hyperbolic lengths of all $l \in \mathcal{L}_{x,y}$. Let

$$\mathrm{diam}(F) = \sup\{l_{x,y} : x, y \in V(\partial F)\},$$

and

$$\rho = \max\{\mathrm{diam}(F) : F \in \mathcal{F}\}. \tag{4.1}$$

By the properties of G, and in particular Theorem 2.1(c), we have $\rho < \infty$.

Let L be a geodesic of $\mathcal{H}$ with endpoints in the boundary of $\mathcal{H}$. Denote by L_δ the closed, hyperbolic δ-neighbourhood of L (see Figure 4.1); we call L_δ a *hyperbolic tube*, and we say L_δ has *width* 2δ. Write $\partial^+ L_\delta$ and $\partial^- L_\delta$ for the two boundary arcs of L_δ. An arc γ of $\mathcal{H}$ is said to *cross* L_δ *laterally* if it intersects both $\partial^+ L_\delta$ and $\partial^- L_\delta$. A path $\pi = (\dots, x_{-1}, x_0, x_1, \dots)$ of G (or $\widehat{G}$) is said to *cross* L_δ *in the long direction* if, for any arc γ that crosses L_δ laterally and intersects no vertex of G, the number of intersections between γ and π, if finite, is odd.

Lemma 4.2. *Let $G = (V, E) \in \mathcal{Q}$ be one-ended and duly embedded in the Poincaré disk $\mathcal{H}$, and let L_δ be a hyperbolic tube.*

- (a) *If $2\delta > \rho$, then L_δ contains a 2∞-nst path of G, and a 2∞-nst path of G_*, that cross L_δ in the long direction.*
- (b) *There exists $\zeta = \zeta(G)$ (depending on G and its embedding) such that, for $r > \zeta$ and $v \in V$, the annulus $\overline{\Lambda}_r(v) \setminus \overline{\Lambda}_{r-\zeta}(v)$ contains a non-self-touching cycle of G (respectively, G_*) denoted $\sigma_r(v)$ (respectively, $\sigma^*_r(v)$) such that $v \in \sigma_r(v)^\circ$ (respectively, $v \in \sigma^*_r(v)^\circ$).*

A more refined result may be found in Section 6.

Proof. (a) Since all faces of G are bounded, there exist vertices of G in both components of $\mathcal{H} \setminus L_\delta$. Now, L_δ fails to be crossed in the long direction if and only if it contains some arc γ that traverses it laterally and that intersects no edge of G. To see the 'only if' statement, let V^- and V^+ be the subsets of $V \cap L_\delta$ that are joined in $G \cap L_\delta$ to the two boundary points of L, respectively; if $V^- \cap V^+ = \varnothing$, then there exists such γ separating V^+ and V^- in L_δ. For this γ, there exists a face F and points $x, y \in V(\partial F)$, such that $\gamma \subseteq \lambda$ for some $\lambda \in \mathcal{L}_{x,y}$. For $\epsilon \in (0, 2\delta - \rho)$, we may replace γ by some $\gamma' := \lambda' \cap L_\delta$ where $\lambda' \in \mathcal{L}_{x,y}$ has length not exceeding $l_{x,y} + \epsilon$. The

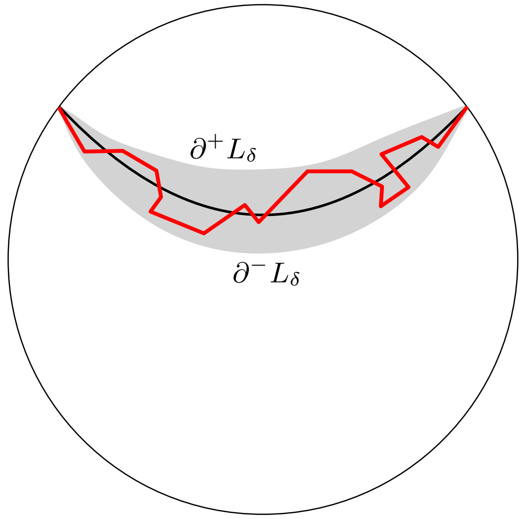

FIGURE 4.1. An illustration of Lemma 4.2. The jagged (red) path crosses L_δ in the long direction.

length of γ' is no greater than $\rho + \epsilon < 2\delta$, a contradiction. Therefore, L_δ contains some path π of G that crosses L_δ in the long direction.

We apply oxbow-removal in G to π as described in the proof of Lemma 4.1. For any arc γ that crosses L_δ laterally and intersects no vertex of G, the number of intersections between γ and π, if finite, decreases by a non-negative, even number whenever an oxbow is removed. It follows that the non-self-touching path π' (obtained after oxbow-removal) crosses L_δ in the long direction. The same conclusion applies to G_* on letting π be a path of G_*.

The proof of (b) is similar. □

4.2. **Graph properties.** The proofs of this article make heavy use of path-surgery which, in turn, relies on planarity of paths.

Lemma 4.3. *Let $G \in \mathcal{Q}$, and let π be a (finite or infinite) non-self-touching path of G_*.*

(a) *For every face F of G, π contains either zero or one or two vertices of F. If π contains two such vertices u, v, then it contains also the corresponding edge $\langle u, v\rangle$, which may be either an edge of G or a diagonal.*
(b) *The path π is plane when viewed as a graph.*

Proof. Let F be a face. The path π cannot contain three or more vertices of F, since that contradicts the non-self-touching property. Similarly, if π contains two such vertices, it must contain also the corresponding edge. If π is non-plane, it contains two or more diagonals of some face, which, by the above, cannot occur. □

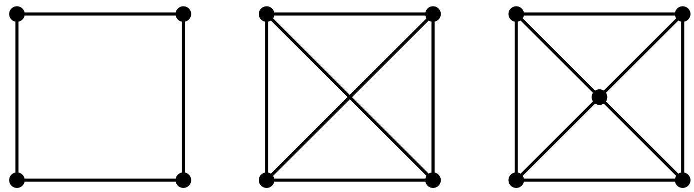

FIGURE 4.2. A square of the square lattice, its matching graph, and with its facial site added.

As a device in the proof of Theorem 1.1, we shall work with the graph $\widehat{G}$ obtained from $G=(V,E)$ by adding a vertex at the centre of each face F, and adding an edge from every vertex in the boundary of F to this central vertex. These new vertices are called *facial sites*, or simply *sites* in order to distinguish them from the *vertices* of G. The facial site in the face F is denoted $\phi(F)$. See [14, Sec. 2.3], and also Figure 4.2. If $\langle v,w\rangle$ is a diagonal of G_*, it lies in some face F, and we write $\phi(v,w)=\phi(F)$ for the corresponding facial site. We note that two vertices $u,v\in V$ are connected in G_* if and only if they are connected in $\widehat{G}$.

The main reason for working with $\widehat{G}$ is that it serves to interpolate between G and G_* in the sense of (5.2): we shall assign a parameter $s\in[0,1]$ to the facial sites in such a way that $s=0$ corresponds to G and $s=1$ to G_*. It will also be useful that $\widehat{G}$ is planar whereas G_* is not.

Next, we specify some desirable properties of the graphs G_* and $\widehat{G}$. The property Π was already the subject of Definition 2.4.

Definition 4.4. *The graph $G\in\mathcal{Q}$ is said to have property*

$$\Pi \quad \text{if } G_* \text{ has a } 2\infty\text{-nst path including some diagonal,}$$

$$\widehat{\Pi} \quad \text{if } \widehat{G} \text{ has a } 2\infty\text{-nst path including some facial site.}$$

Lemma 4.5. *Let $G\in\mathcal{Q}$ be one-ended. Then $\Pi\Rightarrow\widehat{\Pi}$.*

Proof. Let G have property Π and let π be a 2∞-nst path of G_*. For any two consecutive vertices u, v of Π such that $\delta(u,v)$ is a diagonal, we add between u and v the facial site $\phi(u,v)$. The result is a doubly-infinite path π' of $\widehat{G}$. By Lemma 4.3, ν' is non-self-touching in $\widehat{G}$, whence G has property $\widehat{\Pi}$. The converse argument fails. □

The properties of Definition 4.4 are 'global' in that they concern the existence of *infinite* paths. It is sometimes preferable to work in the proofs with *finite* paths, and to that end we introduce corresponding 'local' properties.

Let $\zeta(G)$ be as in Lemma 4.2(b). We shall make reference to the non-self-touching cycles $\sigma_r(v)$, $\sigma_r^*(v)$ given in that lemma. We write $\widehat{\sigma}_r(v)$ for the non-self-touching

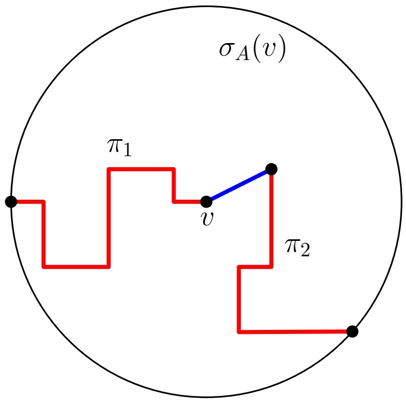

FIGURE 4.3. An illustration of the property Π_A: a non-self-touching path of G_* containing a diagonal near its middle.

cycle of $\widehat{G}$ obtained from $\sigma_r^*(v)$ by replacing any diagonal by a path of length 2 passing via the appropriate facial site of $\widehat{G}$. We abbreviate the closure of the region surrounded by σ_r^* (respectively, $\widehat{\sigma}_r$) to $\overline{\sigma}_r^*$ (respectively, $\overline{\widehat{\sigma}}_r$). Let $A(G)$ be the real number given as

$$A(G) = \zeta(G) + \max\{d_G(u,w) : \langle u,w\rangle \in E_* \setminus E\}. \tag{4.2}$$

Definition 4.6. *Let $A \in \mathbb{Z}$, $A > A(G)$, and let $G \in \mathcal{Q}$ be one-ended.*

(a) *The graph G is said to have property Π_A if there exists a vertex $v \in V$ and a non-self-touching path $\pi = (x_0, x_1, \dots, x_n)$ of G_* such that*
  (i) *every vertex of π lies in $\overline{\sigma}_A^*(v)$, and $x_0, x_n \in \sigma_A^*(v)$,*
  (ii) *there exists i such that $x_i = v$,*
  (iii) *the pair v, x_{i+1} forms a diagonal of G_*, which is to say that $\phi := \phi(v, x_{i+1})$ is a facial site of $\widehat{G}$.*

(b) *The graph G is said to have property $\widehat{\Pi}_A$ if there exist vertices $v, w \in V$ and a non-self-touching path $\pi = (x_0, x_1, \dots, x_n)$ of $\widehat{G}$ such that*
  (i) *every vertex of π lies in $\overline{\widehat{\sigma}}_A(v)$, and $x_0, x_n \in \widehat{\sigma}_A(v)$,*
  (ii) *there exists i such that $x_i = v$, $x_{i+2} = w$,*
  (iii) *x_{i+1} is the facial site $\phi(v,w)$ of $\widehat{G}$.*

That is to say, G has property Π_A (respectively, $\widehat{\Pi}_A$) if G_* (respectively, $\widehat{G}$) contains a finite, non-self-touching path of sufficient length that contains some diagonal (respectively, facial site). This definition is illustrated in Figure 4.3. Note that Π_{A+1} (respectively, $\widehat{\Pi}_{A+1}$) implies Π_A (respectively, $\widehat{\Pi}_A$) for sufficiently large A.

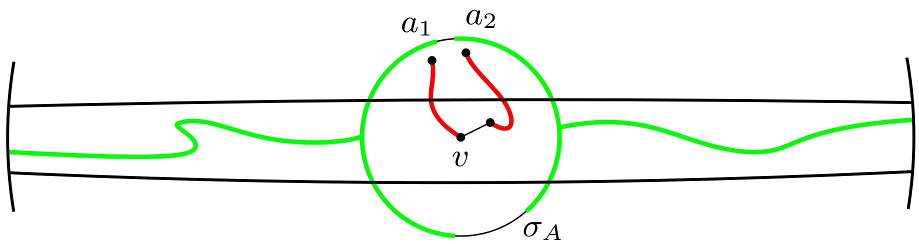

FIGURE 4.4. In the easiest case when $D \geq 2$, one finds (green) non-touching subarcs σ_A^i of σ_A to which v may be connected by non-self-touching paths. These subarcs may be connected to the boundary of $\mathcal{H}$ using subpaths of a doubly-infinite path constructed using Lemma 4.2(a).

Theorem 4.7. *Let $G \in \mathcal{Q}$ be one-ended. There exists $A'(G) \geq A(G)$ such that, for $A > A'(G)$, we have $\Pi \Leftrightarrow \Pi_A$ and $\Pi \Rightarrow \widehat{\Pi}_{A,\cdot}$.*

The proof of this useful theorem utilises some methods of path-surgery that will be important later, and it is deferred to Section 4.3.

4.3. **Proof of Theorem 4.7.** (a) First, we prove that $\Pi \Leftrightarrow \Pi_A$. Evidently, $\Pi \Rightarrow \Pi_A$ for all $A > A(G)$, where $A(G)$ is given in (4.2). Assume, conversely, that Π_A holds for some $A > A(G)$. Let the non-self-touching path $\pi = (x_0, x_1, \dots, x_n)$ of G_*, the vertex $v = x_i$, and the diagonal $d = \langle v, x_{i+1}\rangle$ be as in Definition 4.6(a); think of π as a directed path from x_0 to x_n, and note by Lemma 4.3 that π is a plane graph. We abbreviate $\sigma_A^*(v)$ to σ_A^*. Let

$$\partial^- \sigma_A^* = \{y \in (\sigma_A^*)^\circ : d_{G_*}(y, \sigma_A^*) = 1\}.$$

Let π_1 be the subpath of π from v to x_0, and π_2 that from x_{i+1} to x_n. Let a_i be the earliest vertex/site of π_i lying in $\partial^-\sigma_A$. See the central circle of Figure 4.4. We claim the following.

(4.3) There exist two non-touching subpaths σ^1, σ^2 of σ_A^*, each of length at least $\frac{1}{2}|\sigma_A^*| - 4$, such that: (i) for $i = 1, 2$, the subpath of π_i leading to a_i may be extended beyond a_i along σ^i to form a non-self-touching path ending at any prescribed $y_i \in \sigma^i$, and (ii) the composite path thus created (after oxbow-removal if necessary) is non-self-touching.

The proof of (4.3) follows. Let

$$A_i = \{b \in \sigma_A^* : d_{G_*}(a_i, b) = 1\}, \quad D = \max\{d_{G_*}(b_1, b_2) : b_1 \in A_1,\ b_2 \in A_2\}. \tag{4.4}$$

Suppose $D \geq 2$. Choose $b_i \in A_i$ such that $d_{G_*}(b_1, b_2) \geq 2$. As illustrated in the centre of Figure 4.4, we may find a non-touching pair of non-self-touching subpaths

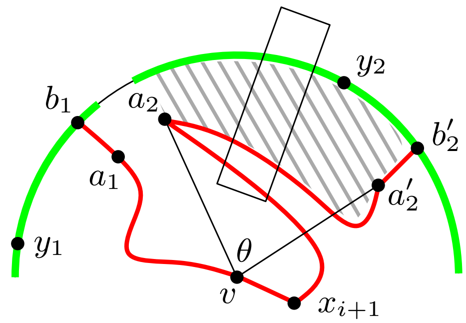

FIGURE 4.5. An illustration of the case $D = 1$. The green lines indicate the subpaths σ_A^i. The rectangle is added in illustration of the case $\theta \geq \frac{3}{4}\pi$.

of σ_A^* such that the conclusion of (4.3) holds. Some oxbow-removal may be needed at the junctions of paths.

Suppose $D = 1$. We may picture σ_A^* as a (topological) circle with centre v, and for concreteness we assume that a_2 lies clockwise of a_1 around σ_A^* (a similar argument holds if not). See Figure 4.5.

A. Suppose the path π_1, when continued beyond a_1, passes at the next step to some $b_1 \in A_1$, and add b_1 to obtain a path denoted π_1'.

Since $D = 1$, the next step of π_2 beyond a_2 is not into A_2. On following π_2 further, it moves inside $(\sigma_A^*)^\circ$ until it arrives at some point $a_2' \in \partial^- \sigma_A^*$ having some neighbour $b_2' \in \sigma_A^*$ satisfying $d_{G_*}(b_1, b_2') \geq 2$; we then include the subpath of π_2 between a_2 and b_2' to obtain a path denoted π_2'.

We declare σ^1 to be the subpath of σ_A^* starting at b_1 and extending a total distance $\frac{1}{2}|\sigma_A^*| - 4$ around σ_A^* anticlockwise. We declare σ^2 similarly to start at distance 2 clockwise of b_1 and to have the same length as σ^1.

Let $\theta \in (0, 2\pi)$ be the angle subtended by the vector $\overrightarrow{a_2 a_2'}$ at the centre v. If $\theta < \frac{3}{4}\pi$, say, each π_i' may be extended along σ^i to end at any prescribed $y_i \in \sigma^i$. Therefore, claim (4.3) holds in this case.

The situation can be more delicate if $\theta \geq \frac{3}{4}\pi$, since then a_2' may be near to σ^1. By the planarity of π, the region R between π_2' and σ_A^* contains no point of π_1' (R is the shaded region in Figure 4.5). We position a hyperbolic tube of width greater than ρ in such a way that it is crossed laterally by both π_2' and the path σ^2 (as illustrated in Figure 4.5). By Lemma 4.2(a), this tube is crossed in the long direction by some path τ of G. The union of π_2' and τ contains a non-self-touching path π_2'' of G_* from x_{i+1} to σ^2 (whose unique vertex in σ^2 is its second endpoint). Claim (4.3) follows in this situation.

B. Suppose the hypothesis of part A does not hold, but instead π_2 passes from a_2 directly into σ_A^*. In this case we follow A above with π_1 and π_2 interchanged.
C. Suppose neither π_i passes from a_i in one step into σ_A^*. We add b_2 to the subpath from x_{i+1} to a_2, and continue as in part A above.

Suppose $D = 0$. Statement (4.3) holds by a similar argument to that above,

Having located the σ^i of (4.3), we position a hyperbolic tube as in Figure 4.4, to deduce (after oxbow-removal) the existence of a 2∞-nst path of G_* that contains the diagonal d. Therefore, G has property Π, as required.

Hyperbolic tubes are superimposed on the graph at two steps of the argument above, and it is for this reason that we need A to be sufficiently large, say $A > A'(G)$.

(b) It remains to show that $\Pi \Rightarrow \widehat{\Pi}_A$. By Lemma 4.5, $\Pi \Rightarrow \widehat{\Pi}$, and it is immediate that $\widehat{\Pi} \Rightarrow \widehat{\Pi}_A$ for large A.

5. Proof of Theorem 1.1

Consider site percolation on G with product measure $\mathbb{P}_p$, and fix some vertex v_0 of G. We write $v \leftrightarrow w$ if there exists a path of G from v to w using only open sites (such a path is called *open*), and $v \leftrightarrow \infty$ if there exists an infinite, open path starting at v. The *percolation probability* is the function θ given by

$$\theta(p) = \theta(p; G) = \mathbb{P}_p(v_0 \leftrightarrow \infty), \tag{5.1}$$

so that the (site) critical probability of G is $p_c(G) := \sup\{p : \theta(p) = 0\}$. The quantities $\theta(p; G_*)$ and $p_c(G_*)$ are defined similarly.

Remark 5.1. *It is an old problem dating back to* [4] *to decide which graphs G satisfy $p_c(G) < 1$, and there has been a series of related results since. It was proved in* [10, Thm 1.3] *that $p_c(G) < 1$ for all quasi-transitive graphs G with super-linear growth. This class includes all $G \in \mathcal{Q}$ with either one or infinitely many ends (see* [2, Sect. 1.4] *and Theorem 2.1).*

Theorem 5.2. *Let $G \in \mathcal{Q}$ be one-ended.*
(a) *Let $A_0 \in \mathbb{Z}$. If G has property Π_A for no $A > A_0$, then $p_c(G_*) = p_c(G)$.*
(b) *There exists $A'(G) \ge A(G)$ such that the following holds. Let $A > A'(G)$. If G has property $\widehat{\Pi}_A$, then $p_c(\widehat{G}) < p_c(G)$.*

The constant $A'(G)$ in part (b) depends on the *embedded graph* G, viewed as a subset of $\mathcal{H}$, rather on the graph G alone.

Proof of Theorem 1.1. If G does not have property Π, by Theorem 4.7 for large A it does not have property Π_A, whence by Theorem 5.2(a), $p_c(G_*) = p_c(G)$. Conversely, if G has property Π, by Theorem 4.7 again it has property $\widehat{\Pi}_A$ for large A, whence by

Theorem 5.2(b), $p_c(\widehat{G}) < p_c(G)$. The final claim follows by the elementary inequality $p_c(G_*) \le p_c(\widehat{G})$; see (5.2). □

Proof of Theorem 5.2(a). Let $A_0 \in \mathbb{Z}$. Assume G has property Π_A for no $A \ge A_0$, and let $p > p_c(G_*)$. Let π be an infinite open path of G_* with some endpoint x. By Lemma 4.1(b), there exists a subset π' of π that forms a non-self-touching path of G_* with endpoint x. Let $A > A_0$. Since Π_A does not hold, every edge of π' at distance $2A$ or more from x is an edge of G, so that there exists an infinite open path in G. Therefore, $p \ge p_c(G)$, whence $p_c(G_*) = p_c(G)$. □

The rest of this section is devoted to the proof of Theorem 5.2(b). Let $\widehat{\Omega} = \Omega_V \times \Omega_\Phi$ where Φ is the set of facial sites and $\Omega_\Phi = \{0,1\}^\Phi$. For $\widehat{\omega} = \omega \times \omega' \in \widehat{\Omega}$ and $\phi \in \Phi$, we call ϕ *open* if $\omega'_\phi = 1$, and *closed* otherwise. Let $\mathbb{P}_{p,s} = \mathbb{P}_p \times \mathbb{P}_s$ be the corresponding product measure on $\Omega_V \times \Omega_\Phi$, and

$$\theta(p,s) = \lim_{n\to\infty} \theta_n(p,s) \qquad \text{where} \qquad \theta_n(p,s) = \mathbb{P}_{p,s}(v_0 \leftrightarrow \partial\overline{\Lambda}_n \text{ in } \widehat{G}),$$

so that

$$\theta(p,0) = \theta(p;G), \quad \theta(p,p) = \theta(p;\widehat{G}), \quad \theta(p,1) = \theta(p;G_*), \tag{5.2}$$

where $\theta(p;H)$ denotes the percolation probability of the graph H. The following proposition implies Theorem 5.2(b).

Proposition 5.3. *There exists $A'(G) < \infty$ such that the following holds. Suppose $G \in \mathcal{Q}$ is one-ended and has property $\widehat{\Pi}_A$ where $A > A'(G)$. Let $s \in (0,1]$. There exists $\epsilon = \epsilon(s) > 0$ such that $\theta(p,s) > 0$ for $p_c(G) - \epsilon < p < p_c(G)$.*

We do not investigate the details of how $A'(G)$ depends on G. An explicit lower bound on $A'(G)$ may be obtained in terms of local properties of the embedding of G, but it is doubtful whether this will be useful in practice.

The rest of this proof is devoted to an outline of that of Proposition 5.3. Full details are not included, since they are very close to established arguments of [1], [11, Sect. 3.3], and elsewhere.

Let n be large, and later we shall let $n \to \infty$. Consider site percolation on $\widehat{G}$ with measure $\mathbb{P}_{p,s}$. We call a vertex (respectively, facial site) z *pivotal* if it is pivotal for the existence of an open path of $\widehat{G}$ from v_0 to $\partial\Lambda_n$ (which is to say that such a path exists if z is open, and not otherwise). Let Pi_n be the set of pivotal vertices, and Di_n the set of pivotal facial sites. Proposition 5.3 follows in the 'usual way' (see [11, Sect. 3.3]) from the following statement.

Lemma 5.4. *Let $p, s \in (0,1)$. There exists $M \ge 1$ and $f : (0,1)^2 \to (0,\infty)$ such that, for $n > 4M$ and every $z \in \overline{\Lambda}_n$,*

$$\mathbb{P}_{p,s}(z \in \mathrm{Pi}_n) \le f(p,s)\mathbb{P}_{p,s}(\mathrm{Di}_n \cap \overline{\Lambda}_M(z) \ne \varnothing). \tag{5.3}$$

On summing (5.3) over $z \in \overline{\Lambda}_n$, we obtain by Russo's formula (see [11, Sec. 2.4]) that there exists $g(p,s) < \infty$ such that

$$\frac{\partial}{\partial p}\theta_n(p,s) \le g(p,s)\frac{\partial}{\partial s}\theta_n(p,s). \tag{5.4}$$

The derivation of Proposition 5.3 from this differential inequality is explained in [1, 11]. It suffices therefore to prove Lemma 5.4.

Here is an outline of the proof of Lemma 5.4. Let $\widehat{\omega} \in \widehat{\Omega}$, $z \in V \cap \overline{\Lambda}_n$, and suppose

(5.5) z is open and pivotal in the configuration $\widehat{\omega}$.

By making changes to the configuration $\widehat{\omega}$ within the box $\overline{\Lambda}_{4M}(z)$ for some fixed M,

(5.6) we construct a configuration in which $\overline{\Lambda}_M(z)$ contains a pivotal facial site.

This implies (5.3) with f depending on the choice of z. Since $\overline{\Lambda}_{4M}(z)$ is finite and there are only finitely many types of vertex (by quasi-transitivity), f may be chosen to be independent of z. The above is achieved in five stages.

Assume for now that $\widehat{\omega} \in \widehat{\Omega}$ *and the pivotal vertex* z *satisfies*

$$z \in \overline{\Lambda}_{n-2M} \setminus \overline{\Lambda}_{2M}. \tag{5.7}$$

For clarity of exposition, our illustrations are drawn as if G is duly embedded in the Euclidean rather than hyperbolic plane.

Let G have property $\widehat{\Pi}_A$. Let $\pi = (x_j)$, $v = x_i$, be as in Definition 4.6(b), and write $\phi = x_{i+1} = \phi(v, x_{i+2})$. Find $\alpha \in \mathrm{Aut}(G)$ such that $v' = \alpha v$ satisfies $d_G(z, v') \le \Delta$, where Δ is given in (2.1). Let $M = 2(A + \Delta)$, so that $\overline{\Lambda}_A(v') \subseteq \overline{\Lambda}_{M/2}(z)$. The outline of the proof is as follows.

I. If there exist one or more open facial sites in $\overline{\Lambda}_M(z)$, we declare them one-by-one to be closed. If at some point in this process, some facial site is found to be pivotal, then we have achieved (5.6), by changing $\widehat{\omega}$ within a bounded region. We may therefore assume that this never occurs, or equivalently that

$$\widehat{\omega} \text{ has no open facial site in } \overline{\Lambda}_M(z). \tag{5.8}$$

II. Find a non-self-touching open path ν in $\widehat{\omega}$ from v_0 to $\partial\overline{\Lambda}_n$. This path passes necessarily through the pivotal vertex z.

III. By making changes within $\overline{\Lambda}_{2M}(z)$, we construct non-touching subpaths of ν from v_0 (respectively, $\partial\overline{\Lambda}_n$) to $\partial\overline{\Lambda}_M(z)$, that can be extended inside $\overline{\Lambda}_M(z)$ in a manner to be specified at Stage V. This, and especially the following, stage resembles closely part of the proof in Section 4.3.

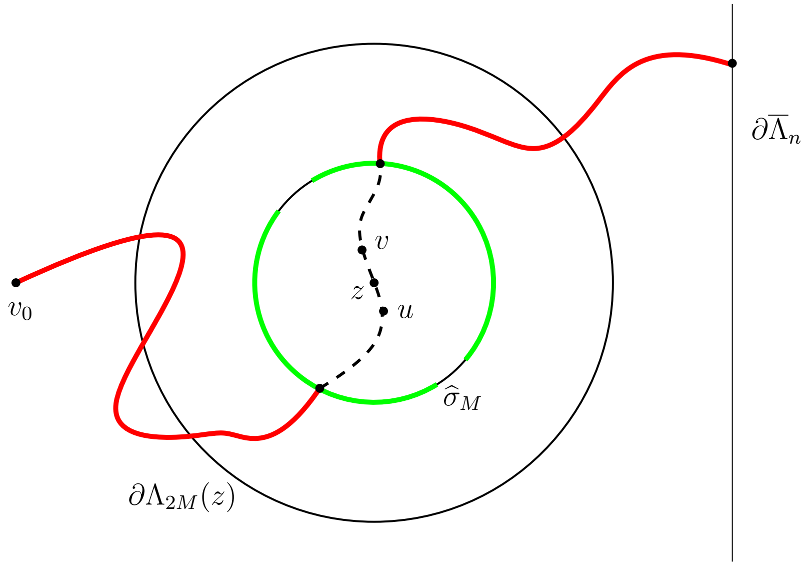

FIGURE 5.1. An illustration of the construction at Stages II/III. The non-self-touching path ν contains subpaths from v_0 to $\widehat{\sigma}_M$, and from the latter set to $\partial\overline{\Lambda}_n$. The subpaths σ_M^i of $\widehat{\sigma}_M$ are indicated in green.

IV. We splice a copy (denoted $\pi' = \alpha\pi$) of π inside $\overline{\Lambda}_A(v')$, and we make local changes to obtain paths π_1, π_2 from the two endpoints of $\alpha\phi$, respectively, to $\partial\overline{\Lambda}_A(v')$ that can be extended outside $\overline{\Lambda}_A(v')$ in a manner to be specified at Stage V.

V. Between the contours $\partial\overline{\Lambda}_A(v')$ and $\partial\overline{\Lambda}_M(z)$, we arrange the configuration in such a way that the retained parts of ν hook up with the endpoints of the π_i. In the resulting configuration, the facial site $\phi' := \alpha\phi$ is pivotal.

Some work is needed to ensure that ϕ' can be made pivotal in the final configuration. Lemma 4.2(b) will be used to traverse the annulus between the two contours at Stage V. In making connections at junctions of paths, we shall make use of the planarity of $\widehat{G}$. Rather than working with the boundaries of $\overline{\Lambda}_M(z)$ and $\overline{\Lambda}_A(v')$, we shall work instead with the non-self-touching cycles $\widehat{\sigma}_M := \widehat{\sigma}_M(z)$ and $\widehat{\sigma}_A := \widehat{\sigma}_A(v')$ of $\widehat{G}$ given in Lemma 4.2(b). Let

$$\partial^+\widehat{\sigma}_M = \{y \in \mathcal{H} \setminus \overline{\widehat{\sigma}}_M : d_{\widehat{G}}(y, \widehat{\sigma}_M) = 1\},$$
$$\partial^-\widehat{\sigma}_A = \{y \in (\widehat{\sigma}_A)^\circ : d_{\widehat{G}}(y, \widehat{\sigma}_A) = 1\}.$$

We move to the proof proper. Stage I is first followed as stated above.

Stage II. By (5.5), we may find an open, non-self-touching path ν of $\widehat{G}$ from v_0 to $\partial\overline{\Lambda}_n$, and we consider ν as thus directed. By (5.8), ν includes no facial site of

$\overline{\Lambda}_M(z)$. The path ν passes necessarily through z, and we let u (respectively, w) be the preceding (respectively, succeeding) vertex to z.

For $y \in V$, and the given configuration $\widehat{\omega}$ (satisfying (5.8)), let

$$C_y = \{x \in V : y \leftrightarrow x \text{ in } \widehat{G} \setminus \{z\}\},$$

and write C_y also for the corresponding induced subgraph of $\widehat{G}$. By (5.5),

A. C_u and C_w are disjoint (and also non-touching),
B. the subpath of ν, denoted $\nu(u-)$, from v_0 to u contains no facial site of $\overline{\Lambda}_M(z)$,
C. the subpath of ν, denoted $\nu(w+)$, from w to $\partial\overline{\Lambda}_n$ contains no facial site of $\overline{\Lambda}_M(z)$,
D. the pair $\nu(z-)$, $\nu(z+)$ is non-touching.

Stage III. This is closely related to the proof of Theorem 4.7 given in Section 4.3. Note that the intersection of $\nu(u-)\cup\nu(w+)$ and $\overline{\Lambda}_{2M}(z)$ comprises a family of paths rather than two single paths. See Figure 5.1.

We follow $\nu(u-)$ towards u, and $\nu(w+)$ backwards towards w, until we reach the first vertices/sites, denoted a_1, a_2, respectively, lying in $\partial^+\widehat{\sigma}_M$. Let ν_1 be the subpath of $\nu(u-)$ between v_0 and a_1, and ν_2 that of $\nu(w+)$ between $\partial\overline{\Lambda}_n$ and a_2. We now change the states of certain vertices/sites $x \in \overline{\Lambda}_{2M}(z)$ by declaring

$$\text{every } x \in \overline{\Lambda}_{2M}(z) \setminus \overline{\widehat{\sigma}}_M \text{ is declared open if and only if } x \in \nu_1 \cup \nu_2. \tag{5.9}$$

We investigate next the subsets of $\widehat{\sigma}_M$ to which the a_i may be connected within σ_M. We shall show that:

(5.10) there exist two non-touching subpaths σ_M^1, σ_M^2 of $\widehat{\sigma}_M$, each of length at least $\frac{1}{2}|\widehat{\sigma}_M| - 4$, such that, for $i = 1, 2$: (i) a_i has a neighbour $b_i \in \sigma_M^i$, (ii) for $y_i \in \sigma_M^i$, the path ν_i may be extended from b_i to y_i along σ_M^i, thereby creating (after oxbow-removal if necessary) a non-self-touching path from the other endpoint of ν_i, (iii) the composite path ν_i' thus created is non-self-touching, and (iv) the pair ν_1', ν_2' is non-touching.

An explanation follows. Let

$$A_i = \{b \in \widehat{\sigma}_M : d_{\widehat{G}}(a_i, b) = 1\}, \quad D = \max\{d_{\widehat{G}}(b_1, b_2) : b_1 \in A_1,\ b_2 \in A_2\}. \tag{5.11}$$

Suppose $D \geq 2$. Choose $b_i \in A_i$ such that $d_{\widehat{G}}(b_1, b_2) \geq 2$. Statement (5.10) follows as illustrated in Figure 5.1.

Suppose $D = 1$. We may picture σ_M as a circle with centre z, and for concreteness we assume that a_2 lies clockwise of a_1 around $\widehat{\sigma}_M$ (a similar argument holds if not) See Figure 5.2.

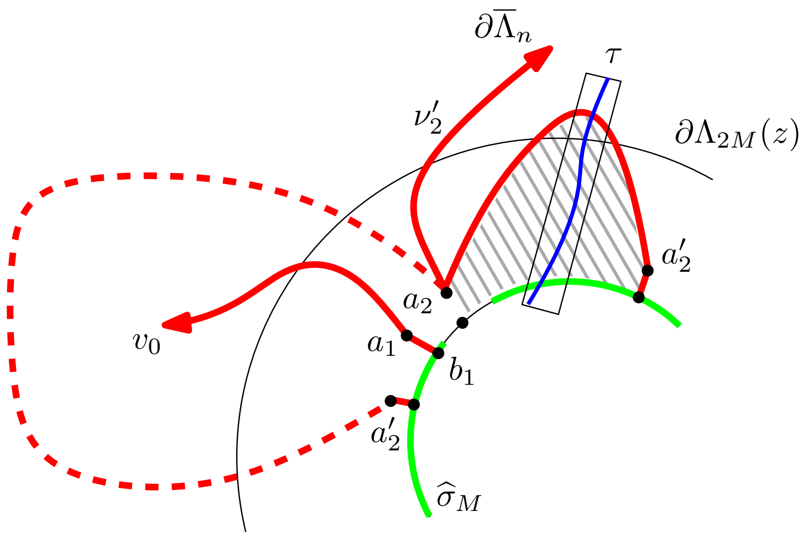

FIGURE 5.2. An illustration of the case $D = 1$ in the Stage III construction. There are two subcases, depending on whether $\theta > 0$ (solid line) or $\theta < 0$ (dashed line). The green lines indicate the subpaths σ_M^i in the subcase $\theta > 0$. The rectangle is added in illustration of the hyperbolic tube used in the case $\theta \geq \frac{3}{4}\pi$.

A. Suppose the path ν_1, when continued along $\nu(z-)$ beyond a_1, passes at the next step to some $b_1 \in A_1$, and add b_1 to ν_1 (to obtain a path denoted ν_1').

Since $D = 1$, the next step of $\nu(w+)$ beyond a_2 is not to A_2. On following $\nu(w+)$ further, it moves inside $\mathcal{H}\setminus\overline{\widehat{\sigma}}_M$ until it arrives at some point $a_2' \in \partial^+\widehat{\sigma}_M$ having some neighbour $b_2' \in \widehat{\sigma}_M$ satisfying $d_{\widehat{G}}(b_1, b_2') \geq 2$; we then add to ν_2 the subpath of $\nu(w+)$ between a_2 and b_2' (to obtain an extended path ν_2'). Let $\theta(a_2')$ be the angle subtended by the vector $\overrightarrow{a_2 a_2'}$ at the centre z, counted positive if $\nu(w+)$ passes clockwise around z of $\widehat{\sigma}_M$, and negative if anticlockwise.

(i) There are two cases, depending on whether $\theta := \theta(a_2')$ is positive or negative. Assume first that $\theta > 0$. If $\theta < \frac{3}{4}\pi$, say, we declare σ_M^1 to be the subpath of $\widehat{\sigma}_M$ starting at b_1 and extending a total distance $\frac{1}{2}|\widehat{\sigma}_M| - 4$ around σ_M anticlockwise. We declare σ_M^2 similarly to start at distance 2 clockwise of b_1 along $\widehat{\sigma}_M$ and to have the same length as σ_M^1. Each ν_i' may be extended along σ_M^i to end at any prescribed $y_i \in \sigma_M^i$. Therefore, claim (5.10) holds in this case.

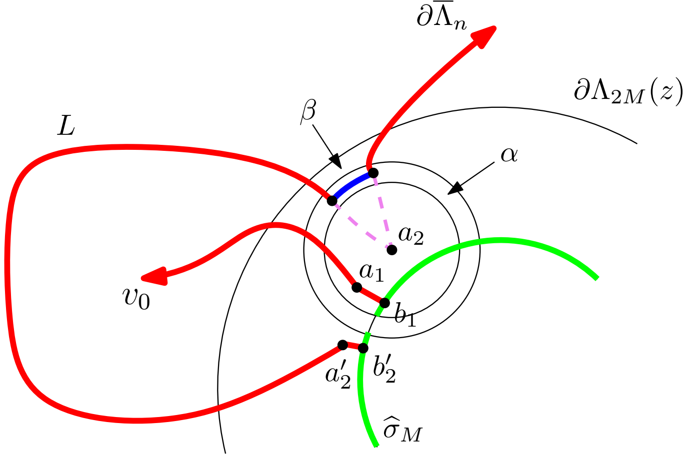

FIGURE 5.3. When $D = 1$ and $\theta < 0$, we adjust the path ν_2 by bypassing a subpath through a_2.

The situation can be more delicate if $\theta \geq \frac{3}{4}\pi$, since then a_2' may be near to σ_M^1. By the planarity of ν, the region R between ν_2' and σ_M contains no point of ν_1' (R is the shaded region in Figure 5.2). We position a hyperbolic tube of width greater than ρ in such a way that it is crossed laterally by both ν_2' and the path σ_M^2 given above. By Lemma 4.2(a), this tube is crossed in the long direction by some path τ of $\widehat{G}$. As illustrated in Figure 5.2, the union of ν_2' and τ contains (after oxbow-removal) a non-self-touching path ν_2'' from $\partial\overline{\Lambda}_n$ to σ_M^2 (whose unique vertex in σ_M^2 is its second endpoint). We now declare each vertex/site of $\overline{\Lambda}_{2M}(z) \setminus (\widehat{\sigma}_M)^\circ$ to be open if and only if it lies in $\nu_1' \cup \nu_2''$. Claim (5.10) follows in this situation, with the σ_M^i given as above.

(ii) Assume $\theta < 0$, in which case there arises a complication in the above construction, as illustrated in Figure 5.3. In this case, there is a subpath L of ν_2' from a_2 to a_2', that passes anticlockwise around v_0, and ν_1' contains no vertex/site outside the closed cycle comprising L followed by the subpath of $\widehat{\sigma}_M$ from b_2' to b_2. In order to overcome this problem, we alter the path ν_2' as follows. Let α denote the annulus $\overline{\Lambda}_M(a_2)\setminus\overline{\Lambda}_{M-\zeta}(a_2)$, with ζ as in Lemma 4.2(b). (We may assume $M \geq 2\zeta$.) By that lemma, α contains a non-self-touching cycle β of $\widehat{G}$ that surrounds a_2. The union of ν_2' and β contains (after oxbow-removal) a non-self-touching path ν_2'' of $\widehat{G}$ from $\partial\overline{\Lambda}_n$ to a_2' that does not contain a_2 (see Figure 5.3). We declare every $x \in \nu_2''$ open and every $x \in \nu_2' \setminus \nu_2''$ closed. The subpaths σ_M^i of $\widehat{\sigma}_M$ may now be defined as above.

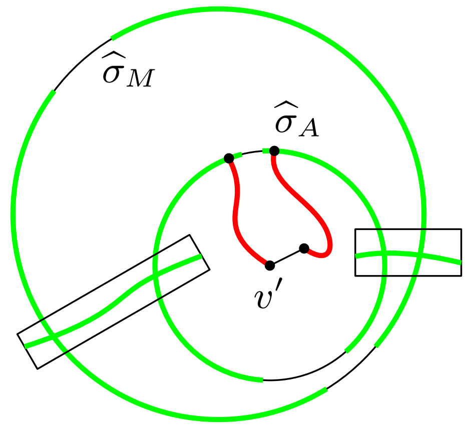

FIGURE 5.4. An illustration of the construction at Stages IV and V.

B. Suppose the hypothesis of part A does not hold, but instead ν_2 passes from a_2 into $\widehat{\sigma}_M$. In this case we follow A with $\nu(u-)$ and $\nu(w+)$ interchanged. This case is slightly shorter than A since the above complication cannot occur.
C. Suppose neither ν_i passes from a_i directly into $\widehat{\sigma}_M$. We add b_2 to ν_2 and continue as in A above.

Suppose $D = 0$. Statement (5.10) holds by a similar argument to that of case (ii),

Stage IV. We next pursue a similar strategy within $\overline{\Lambda}_A(v')$. The argument is essentially that in proof of Theorem 4.7 given in Section 4.3, and the details of this are omitted here. See Figures 4.5 and 5.4.

Stage V. Having located the subpaths σ_M^i of $\widehat{\sigma}_M$, and the subpaths σ_A^i of $\widehat{\sigma}_A$, we prove next that there exists $j \in \{1, 2\}$, and non-self-touching paths μ_1, μ_2, such that: (i) μ_1, μ_2 is a non-touching pair, (ii) μ_1 has endpoints in σ_M^1 and σ_A^j, and μ_2 has endpoints in σ_M^2 and $\sigma_A^{j'}$, where $j' \in \{1, 2\}$, $j' \neq j$, and (iii) apart from their endpoints, μ_1 and μ_2 lie in $(\widehat{\sigma}_M)^\circ \setminus \overline{\widehat{\sigma}}_A$. This statement follows as in Figure 5.4 by positioning two hyperbolic tubes of width exceeding ρ, and appealing to Lemma 4.2(a). It may be necessary to remove some oxbows at the junctions of paths.

Hyperbolic tubes are superimposed on $\widehat{\sigma}_A$ above, and it is for this reason that A is assumed to be sufficiently large.

Having satisfied (5.6) subject to (5.7), we next explain how to remove the assumption (5.7). Let the pivotal vertex v satisfy $v \in \overline{\Lambda}_{2M}$; a similar argument applies if $v \in \overline{\Lambda}_n \setminus \overline{\Lambda}_{n-2M}$. Let π be an infinite, non-self-touching open path of $\widehat{G}$ starting at v_0, and declare closed every vertex of $\overline{\Lambda}_{4M}$ not lying in π. (Such a π exists by connectivity and oxbow-removal.) In the resulting configuration, every vertex/site

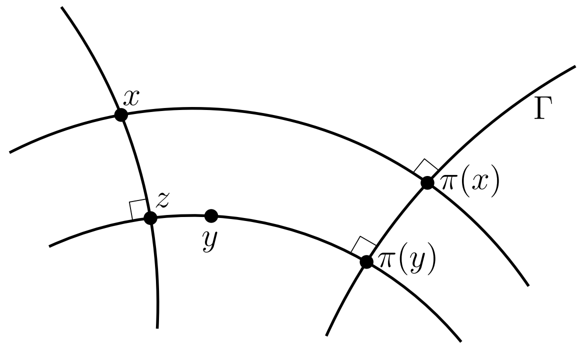

FIGURE 6.1. An illustration of the proof of Lemma 6.1. The four curved lines are geodesics.

in the subpath of π from $\partial\overline{\Lambda}_{2M}$ to $\partial\overline{\Lambda}_{4M}$ is pivotal. We pick one such vertex and apply the above arguments to obtain a pivotal facial site lying in $\overline{\Lambda}_{4M}$.

6. Strict inequality using the metric method

6.1. **Embeddings in the Poincaré disk.** Throughout this section we shall work with the Poincaré disk model of hyperbolic geometry (also denoted $\mathcal{H}$), and we denote by ρ the corresponding hyperbolic metric.

6.2. **Proof of Theorem 3.1 by the metric method.** Let Γ be a doubly-infinite geodesic in the Poincaré disk. Pick a fixed but arbitrary total ordering $<$ of Γ. Then Γ may be parametrized by any function $p : \Gamma \to \mathbb{R}$ satisfying $p(v) = p(u) + \rho(u, v)$ for $u, v \in \Gamma$, $u < v$, and we fix such p.

Here is a lemma. Any $x \notin \Gamma$ has an orthogonal projection $\pi(x)$ onto Γ (for $x \in \Gamma$, we set $\pi(x) = x$).

Lemma 6.1. *For $x, y \in \mathcal{H}$, we have $\rho(\pi(x), \pi(y)) \le \rho(x, y)$.*

Proof. We assume for simplicity that x and y are distinct and lie in the same connected component of $\mathcal{H} \setminus \Gamma$; a similar proof holds if not. The points $x, \pi(x), \pi(y), y$ form a quadrilateral with two consecutive right angles (see Figure 6.1). Let z be the orthogonal projection of x onto the geodesic containing y and $\pi(y)$. The triple x, z, y forms a right-angled triangle, and the quadruple $x, z, \pi(y), \pi(x)$ forms a Lambert quadrilateral. By the geometry of such shapes (see, for example, [13, Sect. III.5]), we have that $\rho(x, y) \ge \rho(x, z) \ge \rho(\pi(x), \pi(y))$. □

Let $G = (V, E) \in \mathcal{T}$ be one-ended but not a triangulation. We shall consider only the case when G is non-amenable, so that it is embedded as an Archimedean tiling in the Poincaré disk; the Euclidean case is similar and easier. For an edge e of

$G_* = (V, E_*)$, let $\rho(e)$ denote the hyperbolic distance between its endvertices; since every e of G_* (in its embedding) is a geodesic, $\rho(e)$ equals the hyperbolic length of e. Since the embedding is Archimedean, every edge of G has the same hyperbolic length, and we may therefore assume for simplicity that

$$\rho(e) = 1, \qquad e \in E. \tag{6.1}$$

Each $e \in E_*$ is a sub-arc of a unique doubly-infinite geodesic, denoted Γ_e, of $\mathcal{H}$.

Let r be the maximal number of edges in a face of G, and let F be a face of size r. Since F is a regular r-gon, by (6.1), F has some diagonal d satisfying

$$\rho(d) \ge \rho(e) \ge 1, \qquad e \in E_*, \tag{6.2}$$

and we choose d accordingly. By Lemma 6.1 applied to the geodesic Γ_d,

$$\rho(\pi(e)) \le \rho(e) \le \rho(d), \qquad e \in E_*, \tag{6.3}$$

where π denote orthogonal projection onto Γ_d, and $\rho(\gamma)$ is the hyperbolic distance between the endpoints of an arc γ.

Let $<$ and p be the ordering and parametrization of Γ_d given at the start of this subsection. We extend the domain of p by setting

$$p(x) = p(\pi(x)), \qquad x \in \mathcal{H}.$$

We construct next a doubly-infinite path of G_* containing d and lying 'close' to Γ_d. Write $d = \langle a, b\rangle$ where $a < b$. Let Γ_d^+ (respectively, Γ_d^-) be the sub-geodesic obtained by proceeding along Γ_d from b in the positive direction (respectively, from a in the negative direction). As we proceed along Γ_d^+, we encounter edges and faces of G. If $e \in E$ is such that $e \cap \Gamma_d^+ \ne \varnothing$, then the intersection is either a point or the entire edge e (this holds since both e and Γ_d are geodesics).

Lemma 6.2. *Let $e = \langle x, y\rangle \in E$ be an edge whose interior e° intersects Γ_d^+ at a singleton g only, so that $e^\circ \cap \Gamma_d^+ = \{g\}$. Then,*

(a) *either $p(x) = p(g) = p(y)$, or*
(b) *some endvertex $z \in \{x, y\}$ of e satisfies $p(z) > p(g)$.*

Proof. The first case arises when e, viewed as a geodesic, is perpendicular to Γ_d^+, and the second when it is not. See Figure 6.2. □

In proceeding along Γ_d^+, we make an ordered list (w_i) of vertices as follows.

(a) Set $w_0 = b$.
(b) Every time Γ_d passes into the interior of a face F', it exits either at a vertex v' or across the interior of some edge e'. In the first case we add v' to the list, and in the second, we add to the list an endvertex of e' with maximal p-value.

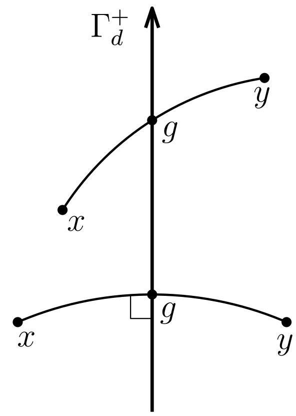

FIGURE 6.2. The two cases that arise when Γ_d^+ meets an edge which is either perpendicular or not.

(c) If Γ_d^+ passes along an edge $e \in E$, we add both its endvertices to the list in the order in which they are encountered.

The following lemma is proved after the end of the current proof.

Lemma 6.3. *The infinite ordered list $w = (w_0, w_1, \dots)$ is a path of G_* with the property that $p(w_i)$ is strictly increasing in i.*

We apply oxbow-removal, Lemma 4.1(b), to w to obtain an infinite, non-self-touching path $\nu^+ = (\nu_0, \nu_1, \dots)$ of G_* satisfying

$$\nu_0 = b, \qquad p(\nu_0) < p(\nu_1) < \cdots . \tag{6.4}$$

By the same argument applied to Γ_d^-, there exists an infinite, non-self-touching path $\nu^- = (\nu_{-1}, \nu_{-2}, \dots)$ of G_* satisfying

$$\nu_{-1} = a, \qquad p(\nu_{-1}) > p(\nu_{-2}) > \cdots . \tag{6.5}$$

The composite path ν obtained by following ν^- towards a, then d, then ν_+, fails to be non-self-touching in G_* if and only if there exists $s < 0$ and $t \ge 0$ with $(s,t) \ne (-1,0)$ such that $e'' := \langle \nu_s, \nu_t \rangle \in E_*$. If the last were to occur, by (6.4)–(6.5),

$$\rho(\pi(e'')) = p(\nu_t) - p(\nu_s) > p(b) - p(a) = \rho(d),$$

in contradiction of (6.3). Thus ν is the required non-self-touching path. The above may be regarded as a more refined version of part of Proposition 4.2.

Proof of Lemma 6.3. That w is a path of G_* follows from its construction, and we turn to the second claim. Let $m \ge 0$, and consider $w_0, w_1, \dots, w_m$ as having been identified. We claim that

$$p(w_m) < p(w_{m+1}). \tag{6.6}$$

(a) Suppose $w_m \in \Gamma_d^+$.
   (i) If Γ_d^+ includes next an entire edge of the form $\langle w_m, g\rangle \in E$, then $w_{m+1} = g$ and (6.6) holds.
   (ii) Suppose Γ_d^+ enters next the interior of some face F'. If it exits F' at a vertex, then this vertex is w_{m+1} and (6.6) holds. Suppose it exits by crossing the interior of an edge e'. If w_m is an endvertex of e', then w_{m+1} is its other endvertex and (6.6) holds; if not, then w_{m+1} is an endvertex of e' with maximal p-value (recall Lemma 6.2).

(b) Suppose w_m is the endvertex of an edge e that is crossed (but not traversed in its entirety) by Γ_d^+, and let F' be the face thus entered. The next vertex w_{m+1} is given as in (a)(ii) above, and (6.6) holds.

The proof is complete. □

6.3. **The case of quasi-transitive graphs.** Certain complexities arise in applying the techniques of Section 6.2 to quasi-transitive graphs. In contrast to transitive graphs, the faces are not generally regular polygons, and the longest edge need not be a diagonal.

Let $G \in \mathcal{Q}$ be one-ended and not a triangulation. As before, we restrict ourselves to the case when G is non-amenable, and we embed G canonically in the Poincaré disk $\mathcal{H}$. The edges of G are hyperbolic geodesics, but its diagonals need not be so. The hyperbolic length of an edge $e \in E_* \setminus E$ does not generally equal the hyperbolic distance $\rho(e)$ between its endvertices.

The proof is an adaptation of that of Section 6.2, and full details are omitted. In identifying a path corresponding to the path w of Lemma 6.3, we use the fact that edges of E are geodesics, and concentrate on the *final* departures of Γ_d^+ from the faces whose interiors it enters.

Remark 6.4. *The condition of Theorem 3.4 may be weakened as follows. In the above proof of Theorem 3.1 is constructed a 2∞-nst path ν of G_* (see the discussion following Lemma 6.3). It suffices that, in the notation of that discussion, there exist a diagonal d and $s < 0$, $t \geq 1$ such that (i) the path $(\nu_s, \nu_{s+1}, \dots, \nu_t)$ is non-self-touching in G_*, and (ii) for all $e \in E$ we have $p(\nu_t) - p(\nu_s) > p(\pi(e))$. Cf. Theorem 4.7.*

7. Strict inequality using the combinatorial method

We prove Theorem 3.8 in this section. Let G have the given properties, and let $\nu = (\dots, \nu_{-1}, \nu_0, \nu_1, \dots)$ be a 2∞-nst path of G_*. Such a path exists by Lemma 4.2(a) since G is connected. If ν contains some diagonal, then we are done. Assume therefore that

$$\nu \text{ contains no diagonal.}$$

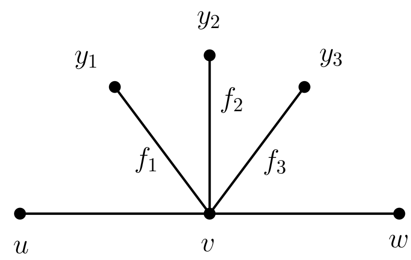

FIGURE 7.1. An illustration with $r = 3$.

We shall make local changes to ν to obtain a 2∞-nst path $\overline{\nu}$ containing some diagonal. The following analysis is 'case-by-case'.

In the various steps and figures that illustrate this construction, we write

$$u = \nu_{-1}, \quad v = \nu_0, \quad w = \nu_1.$$

Draw the triple u, v, w in the planar embedding of G as in Figure 7.1. Let $f_i = \langle v, y_i \rangle$, $i = 1, 2, \dots, r$, be the edges of G incident to v in the sector obtained by rotating $\langle u, v \rangle$ clockwise about v until it coincides with $\langle w, v \rangle$; the f_i are listed in clockwise order. Let $\nu(u-)$ (respectively, $\nu(w+)$) be the subpath of ν prior to and including u (respectively, after and including w).

Assume first that G has no triangular faces. For clarity, we begin with this simpler situation. If $r = 0$, the edges $\langle u, v \rangle$, $\langle v, w \rangle$ lie in some face F of G which, by assumption, is not a triangle. In this case, we remove v from ν and add the diagonal $\delta(u, w)$. The ensuing path $\overline{\nu}$ has the required properties.

Suppose henceforth that $r \geq 1$. Since ν is assumed non-self-touching, no y_i lies in $\nu(u-) \cup \nu(w+)$. For $i = 1, 2, \dots, r$, denote the neighbours of y_i other than v as $z_{i,1}, z_{i,2}, \dots, z_{i,\delta_i}$, listed in clockwise order of the planar embedding. Note that, while the $z_{i,1}, z_{i,2}, \dots, z_{i,\delta_i}$ are distinct for given i, there may exist values of i, j and $1 \leq a \leq \delta_i$, $1 \leq b \leq \delta_j$ with $z_{i,a} = z_{j,b}$. By the assumed absence of triangles, we have $z_{i,j} \neq y_k$ for all i, j, k.

We list the labels $z_{i,j}$ in lexicographic order (that is, $z_{a,b} < z_{c,d}$ if either $a < c$, or $a = c$ and $b < d$) as $z_1 < z_2 < \cdots < z_s$; this is a total order of the *label-set* Z but not of the underlying *vertices* since a given vertex may occur multiply. If $a < b$ we speak of z_a as preceding, or being to the *left* of z_b (and z_b succeeding, or being to the *right* of z_a). For $1 \leq i \leq r$, let

(7.1) $$S_i = (z_{i,j} : j = 1, 2, \dots, \delta_i), \text{ viewed as an ordered subsequence of } Z.$$

In making changes to the path ν, it is useful to first record which vertices lie in either $\nu(u-)$ or $\nu(w+)$, or in neither. We label each vertex z as

$$\begin{cases} U & \text{if } z \in \nu(u-), \\ W & \text{if } z \in \nu(w+), \\ Q & \text{if } z \notin \nu(u-) \cup \nu(w+). \end{cases}$$

Write N_L be the number of z_i with label L. Here is a technical lemma.

Lemma 7.1. *Suppose $N_U \geq 1$, and let z_T be the leftmost vertex labelled U. Let $\nu''(u-)$ be the subpath of $\nu(u-)$ from z_T to u, and $\nu'(u-)$ that obtained from $\nu(u-)$ by deleting the edges of $\nu''(u-)$. Let $\alpha = \min\{j : y_j \sim z_T\}$ and $S = (z_t, z_{t+1}, \dots, z_T)$ be the z_i adjacent to y_α that precede or equal z_T.*

(a) *For $t \leq i < j \leq T$, we have that $z_i \nsim z_j$.*
(b) *For $1 \leq i \leq T-1$, z_i is labelled Q.*
(c) *For $1 \leq i \leq T-2$, z_i has no $*$-neighbour lying in $\nu'(u-)$. Furthermore, z_T is the unique $*$-neighbour of z_{T-1} lying in $\nu'(u-)$.*
(d) *For $1 \leq i \leq T$, z_i has no $*$-neighbour lying in $\nu(w+)$.*

Proof. (a) If $z_i \sim z_j$ for some $t \leq i < j \leq T$, then (y_α, z_i, z_j) forms a triangle, which is forbidden by assumption.

(b) By the planarity of ν (see Lemma 4.3), $\nu''(u-)$ moves around v in an anticlockwise direction, in the sense that the directed cycle obtained by traversing $\nu''(u-)$ from z_T to u, followed by the edges $\langle u, v\rangle$, $\langle v, y_\alpha\rangle$, $\langle y_\alpha, z_T\rangle$, has winding number -1. If, on the contrary, it has winding number 1, then $\nu''(u-)$ intersects $\nu(w+)$ in contradiction of the planarity of ν. See Figure 7.2.

Let $1 \leq i \leq T-1$. By assumption, z_i is not labelled U. If $z_i \in \nu(w+)$, then (as illustrated in the figure), $\nu(u-)$ and $\nu(w+)$ must intersect (when viewed as arcs in $\mathcal{H}$). This is a contradiction by Lemma 4.3(b).

(c) If $1 \leq i \leq T-2$ and z_i has a $*$-neighbour x in $\nu'(u-)$, then $d_{G_*}(x, \nu''(u-)) \leq 1$, which (as above) contradicts the assumption that $\nu(u-)$ is non-self-touching in G_*. The second statement holds similarly.

(d) This is similar to the above. □

We consider the various cases, and use the notation of Lemma 7.1.

(a) Suppose $N_U \geq 1$. Start with the path $\nu'(u-)$, and consider the pairs

$$P = \big\{(z_T, z_{T-1}), (z_{T-1}, z_{T-2}), \dots, (z_t, v)\big\}.$$

Since G has no triangles (see also Lemma 7.1(a)), every such pair forms a diagonal. We add to $\nu'(u-)$ the vertices $v, z_t, \dots, z_{T-1}$. Let $\overline{\nu}$ be the path of G_* obtained by following $\nu'(u-)$, then the pairs in P, and then $\nu(w+)$.

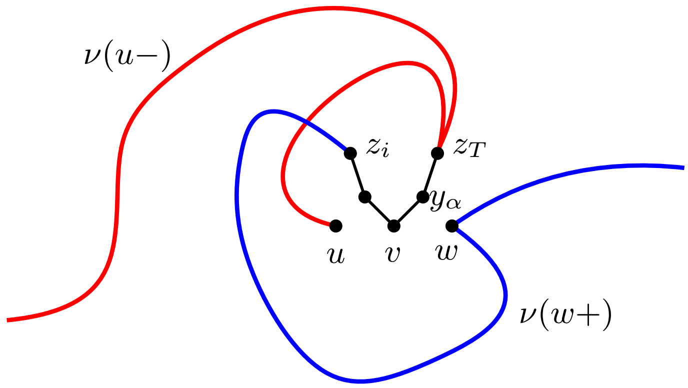

FIGURE 7.2. If $z_i \in \nu(w+)$ and $z_T \in \nu(u-)$, then the pair $\nu(u-)$, $\nu(w+)$ fails to be non-touching.

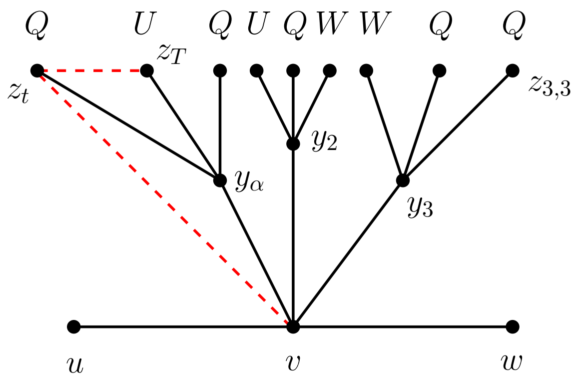

FIGURE 7.3. The dashed red line contains the diagonal $\delta(v, z_t)$.

By Lemma 7.1(b, c, d), $\overline{\nu}$ is non-self-touching, and furthermore it contains a diagonal. See Figure 7.3.

(b) If $N_W \geq 1$, we perform a similar construction to the above, utilizing the rightmost appearance of W.

(c) If $N_U = N_W = 0$, we remove v from ν, and replace it by the sequence of sites $y_1, y_2, \ldots, y_r$ (joined by their intermediate diagonals). The ensuing path $\overline{\nu}$ is non-self-touching and contains a diagonal.

Next we lift the no-triangle assumption. We now permit G to have triangular faces, but assume it has property $\square$. By $\square$, the vertex v is incident to some face denoted F whose boundary has four or more edges. Let u, w, $\nu(u-)$, $\nu(w+)$ be as before. We draw the triple u, v, w as in Figure 7.4, and assume without loss of generality that F lies above the line drawn horizontally in the illustration. We shall use much of the notation introduced above.

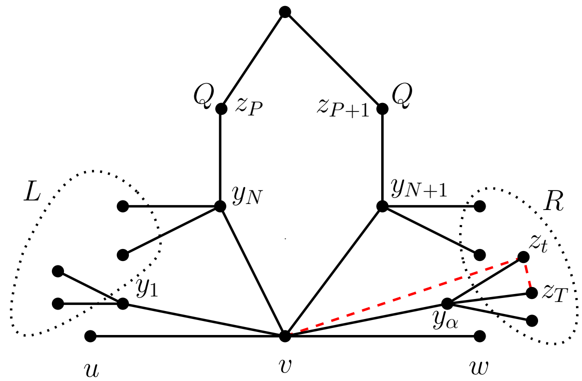

FIGURE 7.4. The path ν passes through a vertex v that lies in a 6-face F. With $z_T \in \nu(u-)$ as given, when $y_\alpha \nsim y_{\alpha-1}$ we may adjust ν to obtain a non-self-touching path ν' passing along the diagonal $\delta(v, z_t)$.

Let $y_1, y_2, \dots, y_r$ be the neighbours of v other than u and w, considered clockwise from u to w, as in Figure 7.4, and let $z_1, z_2 \dots, z_s$ be as before (we exclude the y_j from the sequence (z_i)). Let $r \geq 1$. The following technical lemma is related to the earlier Lemma 7.1. With ν as above, let $\nu'(u-)$ and $\nu''(u-)$ be as in Lemma 7.1, and S_i as in (7.1).

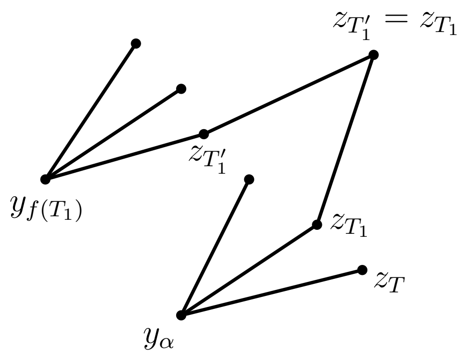

FIGURE 7.5. An illustration of the function K in the proof of Theorem 3.8. For $z_T \in S_\alpha$, we track backwards through S_α from z_T until we find some z_{T_1} representing a vertex that appears in some S_γ with $N_1 \leq \gamma < \alpha$. In this example, we have $K(T) = T_1'$.

Lemma 7.2.

(a) *Let $s_0 = u$, $s_{r+1} = v$, and $s_i = y_i$ for $i = 1, 2, \ldots, r$. If $s_i \sim s_j$ then $|i - j| = 1$.*
(b) *Suppose $1 \leq T \leq s$ and z_T is labelled U. Let α be such that $z_T \in S_\alpha$, and let $S = (z_t, z_{t+1}, \ldots, z_T)$ be the z_i adjacent to y_α that precede or equal z_T. Assume $z_t, z_{t+1}, \ldots, z_{T-1}$ are not labelled U.*
  (i) *For $t \leq i \leq T - 1$, z_i is labelled Q. For $1 \leq i < t$, z_i is labelled either Q or U.*
  (ii) *For $1 \leq i \leq T - 2$, z_i has no $*$-neighbour lying in $\nu'(u-)$. Furthermore, z_T is the unique $*$-neighbour of z_{T-1} lying in $\nu'(u-)$.*
  (iii) *For $1 \leq i \leq T$, z_i has no $*$-neighbour lying in $\nu(w+)$.*

Proof. (a) Suppose $s_i \sim s_j$ where $j \geq i + 2$. Then (v, s_i, s_j) forms a triangle C of G that intersects the interior of the edge $\langle v, s_{i+1} \rangle$ (viewed as a 1-dimensional simplex). Since G is planar, it follows that $s_{i+1} \in C^\circ$. This is a contradiction since G is assumed $\triangle$-empty.

Part (b) is proved as in the proof of Lemma 7.1. □

Let y_N, y_{N+1} be the neighbours of v in F, and z_P, z_{P+1} their further neighbours in F (if F is a quadrilateral, we have $z_P = z_{P+1}$). We assume that $y_N \neq u$ and $y_{N+1} \neq w$; similar arguments are valid otherwise.

Suppose $z_i \in \nu(u-)$ for some $i \in \{P, P+1\}$. Either $z_i \sim v$ or $\delta(z_i, v)$ is a diagonal. In either case there is a contradiction with the fact that ν is non-self-touching in G_*. A similar argument holds if one of z_P, z_{P+1} lies in $\nu(w+)$. Therefore, neither z_P nor z_{P+1} lies in $\nu(u-) \cup \nu(w+)$, and we label them Q accordingly as in Figure 7.4.

Let $L = \{z_1, z_2, \dots, z_{P-1}\}$ (respectively, $R = \{z_{P+2}, z_{P+2}, \dots, z_s\}$) denote the set of neighbours of y_N and the y_j to its left (respectively, y_{N+1} and the y_j to its right) other than u, v, w and z_P, z_{P+1}. We do not assume that L and R are disjoint when viewed as sets of vertices.

Next, we define an iterative construction. For $P + 2 \le a \le s$, let

$$f(a) = \min\{\beta \ge N + 1 : y_\beta \sim z_a\}.$$

Let $T \ge P+2$ and let $\alpha \ge N+1$ be such that $z_T \in S_\alpha$, where S_α is given in (7.1). We define $K(T)$ as follows. Let $T_1 = \max\{a \in [\phi(\alpha), T] : f(a) < \alpha\}$ with the convention that the maximum of the empty set is 0.

1. If $T_1 = 0$, let $K(T) = 0$.
2. Assume $T_1 > 0$, so that $S_{f(T_1)}$ contains the vertex represented by the label z_{T_1}, say with label $z_{T_1'} \in S_{f(T_1)}$. We set $K(T) = T_1'$.

The motivation for the function K is as follows. A difficulty arises from the fact that each z_j is a label rather than a vertex, and different labels can correspond to the same vertex. For an initial label $z_T \in S_\alpha$, we examine its predecessors in S_α. If no such predecessor (including z_T itself) represents a vertex that appears also in some earlier $S_{N+1}, \dots, S_{\alpha-1}$, we declare $K(T) = 0$. If such a predecessor exists, find the first such $z_{T_1} \in S_\alpha$, and find the earliest z_j (with $j \ge P + 2$) that represents the same vertex as z_{T_1}. Then $K(T)$ is the index of this z_j.

We move now to the argument proper. The idea is to replace a subpath of ν by another set of vertices, thus creating a non-self-touching path $\overline{\nu}$ that includes a diagonal.

(a) Assume some $z_\gamma \in R$ is labelled U, and let z_T be the earliest such z_γ. We remove $\nu''(u-)$ from ν (while retaining its endvertex z_T but not its other endvertex u), noting by Lemma 7.2 that

$$\text{no } *\text{-neighbour of } z_{P+1} \text{ lies in either } \nu''(u-) \text{ or } \nu(w+). \tag{7.2}$$

Next, we add some further vertices in a set A determined according to which of the following cases applies. Let S and α be given as in (7.1) and Lemma 7.2(b).

Case I. Suppose $\alpha = N+1$. Then $A = \{z_{P+1}\} \cup S$, By (7.2) and Lemma 7.2, the ensuing path $\overline{\nu}$ is non-self-touching and traverses the diagonal $\delta(z_{P+1}, v)$.

Case II. Suppose $\alpha > N + 1$.

1. If $K(T) = 0$, we take $A = S$. If $y_\alpha \nsim y_{\alpha-1}$ we stop. The ensuing path $\overline{\nu}$ is non-self-touching and traverses the diagonal $\delta(z_t, v)$. See Figure 7.4.
2. Let $K(T) = 0$, and assume that $y_\alpha \sim y_{\alpha-1}$. If $z_t \nsim y_{\alpha-1}$ we take $A = \{y_{\alpha-1}\} \cup S$. The construction of $\overline{\nu}$ is complete on noting that $\delta(z_t, y_{\alpha-1})$ is a diagonal.

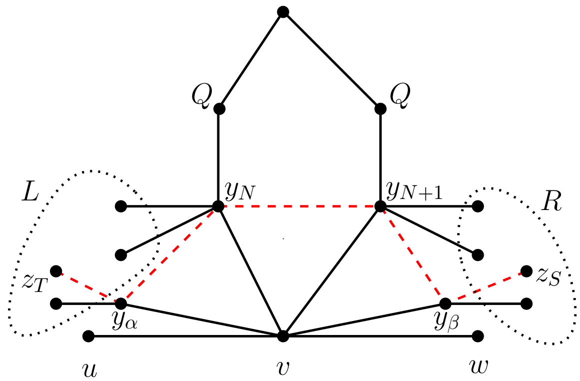

FIGURE 7.6. When the rightmost U is on the left, and the leftmost W is on the right, we replace the subpath of ν from z_T to z_S by the dashed edges.

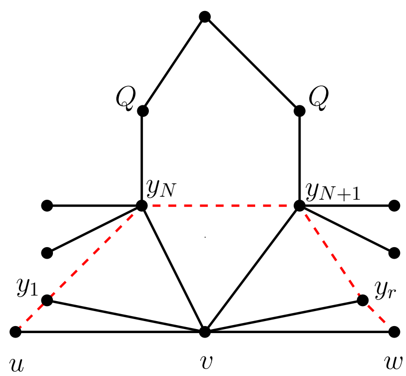

FIGURE 7.7. This is the picture when neither U nor W is represented in the set $R \cup L$.

3. Let $K(T) = 0$, and assume that $y_\alpha \sim y_{\alpha-1}$ and $z_t \sim y_{\alpha-1}$. Take $A = \{z_{t-1}\} \cup S$, and repeat the above with (α, T) replaced by $(\alpha - 1, t - 1)$.
4. If $K(T) = T_1' > 0$, repeat the above with (α, T) replaced by $(f(T_1'), T_1')$. See Figure 7.5.

This iterative process terminates with a path $\overline{\nu}$ containing a diagonal of the form either $\delta(z_k, v)$ or $\delta(z_k, y_\beta)$ for some $P+1 \le k < T$ and $N+1 \le \beta < \alpha$. If $\overline{\nu}$ is not non-self-touching, one may apply oxbow-removal (by Lemma 4.1(b)) to obtain a path $\overline{\overline{\nu}}$ containing the above diagonal.

A similar construction is valid if some vertex in L is labelled W.

(b) Assume U appears in $L \setminus R$ but not in R, and W appears in $R \setminus L$ but not in L. By Lemma 7.2(b),

$$\begin{aligned} &\text{no } y_i \text{ with } i \le N \text{ has a neighbour labelled } W; \\ &\text{no } y_i \text{ with } i > N \text{ has a neighbour labelled } U. \end{aligned} \tag{7.3}$$

Let $z_T \in L$ be the rightmost U and $z_S \in R$ the leftmost W, and let $\alpha = \min\{i : y_i \sim z_T\}$ and $\beta = \max\{i : y_i \sim z_S\}$. The z_i between z_T and z_S are labelled Q. We remove from ν the part of $\nu(u-)$ between z_T and v, and similarly that of $\nu(w+)$ between z_S and w (we retain the endpoints z_T and z_S). See Figure 7.6.

Next we add $y_\alpha, y_{\alpha+1}, \dots, y_N$ and also $y_\beta, y_{\beta+1}, \dots, y_{N+1}$. By Lemma 7.2(a), the ensuing $\overline{\nu}$ is non-self-touching, and includes the diagonal $\delta(y_N, y_{N+1})$.

(c) Assume that U appears in $L \setminus R$ but not in R, and W appears nowhere in $L \cup R$. The argument of part (b) applies with the sequence $y_\beta, y_{\beta+1}, \dots, y_{N+1}$ replaced by $y_{N+1}, y_{N+2}, \dots, y_r$.

(d) Finally, if neither U nor W is represented in $L \cup R$, then all members of $L \cup R$ are labelled Q. In this case, we remove v, and we add the points $\{y_i : i = 1, 2, \dots, r\}$. See Figure 7.7. By Lemma 7.2(a), the ensuing $\overline{\nu}$ is non-self-touching and traverses the diagonal $\delta(y_N, y_{N+1})$.

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(GRG) Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WB, UK
Email address: g.r.grimmett@statslab.cam.ac.uk
URL: http://www.statslab.cam.ac.uk/~grg/

(ZL) Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269-3009, USA
Email address: zhongyang.li@uconn.edu
URL: http://www.math.uconn.edu/~zhongyang/