SELF-AVOIDING WALKS AND AMENABILITY

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Abstract. The connective constant $\mu(G)$ of an infinite transitive graph G is the exponential growth rate of the number of self-avoiding walks from a given origin. The relationship between connective constants and amenability is explored in the current work.

Various properties of connective constants depend on the existence of so-called 'graph height functions', namely: (i) whether $\mu(G)$ is a local function on certain graphs derived from G, (ii) the equality of $\mu(G)$ and the asymptotic growth rate of bridges, and (iii) whether there exists a terminating algorithm for approximating $\mu(G)$ to a given degree of accuracy.

In the context of amenable groups, it is proved that the Cayley graphs of infinite, finitely generated, elementary amenable groups support graph height functions, which are in addition harmonic. In contrast, the Cayley graph of the Grigorchuk group, which is amenable but not elementary amenable, does not have a graph height function.

In the context of non-amenable, transitive graphs, a lower bound is presented for the connective constant in terms of the spectral bottom of the graph. This is a strengthening of an earlier result of the same authors. Secondly, using a percolation inequality of Benjamini, Nachmias, and Peres, it is explained that the connective constant of a non-amenable, transitive graph with large girth is close to that of a regular tree. Examples are given of non-amenable groups without graph height functions, of which one is the Higman group.

Contents

Part A. Background, and summary of results 2
1. Introduction 2
   1.1. Background 2
   1.2. Amenable groups 3
   1.3. Non-amenable graphs 4
2. Graphs, groups, and self-avoiding walks 4

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2.1. Graphs 4
2.2. Self-avoiding walks 5
2.3. Groups 5
3. Height functions 6

Part B. Results for amenable groups 7
4. Elementary amenable groups 7
5. The Grigorchuk group 8

Part C. Results for non-amenable graphs 9
6. Connective constants of non-amenable graphs 9
7. Graphs with large girth 12
8. The Higman group 12

Part D. Remaining proofs 13
9. Proof of Theorem 4.1 13
10. Criteria for the absence of height functions 17
11. Proof of Theorem 5.1 18
12. Proof of Theorem 8.1 20
13. Proof of Theorem 8.2 23
Acknowledgements 25
References 25

Part A. Background, and summary of results

1. Introduction

1.1. **Background.** A *self-avoiding walk* on a graph $G = (V, E)$ is a path that visits no vertex more than once. The study of the number σ_n of self-avoiding walks of length n from a given initial vertex was initiated by Flory [11] in his work on polymerization, and this topic has acquired an iconic status in the mathematics and physics associated with lattice-graphs. Hammersley and Morton [25] proved in 1954 that, if G is vertex-transitive, there exists a constant $\mu = \mu(G)$, called the *connective constant* of G, such that $\sigma_n = \mu^{n(1+o(1))}$ as $n \to \infty$. This result is important not only for its intrinsic value, but also because its proof contained the introduction of subadditivity to the theory of interacting systems. Subsequent work has concentrated on understanding polynomial corrections in the above asymptotic for σ_n (see, for example, [3, 36]), and on finding exact values and inequalities for connective constants (for example, [10, 18]).

There are several natural questions about connective constants whose answers depend on whether or not the underlying graph admits a so-called graph height function. The first of these is whether $\mu(G)$ is a continuous function of the graph G (see [4, 19]). This so-called *locality* question has received serious attention also in the context of percolation and other disordered systems (see [5, 37, 39]), and has been studied in recent work of the current authors on general transitive graphs, [19], and also on Cayley graphs of finitely generated groups, [22]. Secondly, when G has a graph height function, one may define *bridge* self-avoiding walks on G, and show that their numbers grow asymptotically in the same manner as σ_n (see [19]). The third such question is whether there exists a terminating algorithm to approximate $\mu(G)$ within any given (non-zero) margin of accuracy (see [19, 20]).

Roughly speaking, a graph height function on $G = (V, E)$ is a non-constant function $h : V \to \mathbb{Z}$ whose increments are invariant under the action of a finite-index subgroup of automorphisms (a formal definition may be found at Definition 3.1). It is, therefore, useful to know which transitive graphs support graph height functions.

A method for constructing graph height functions on a certain class of transitive graphs is described in [19], and the question is posed there of deciding whether all transitive graphs support graph height functions. A rich source of interesting examples of transitive graphs is provided by Cayley graphs of finitely generated groups, as studied in [22]. It is proved there that the Cayley graphs of finitely generated, virtually solvable groups support graph height functions, which are in addition harmonic. The question is posed of determining whether or not the Cayley graph of the Grigorchuk group possesses a graph height function.

We are concerned here with the relationship between connective constants and amenability, and we present results for both amenable and for non-amenable graphs. Since these results are fairly distinct, we summarize them here under the two headings of amenable groups and non-amenable graphs.

1.2. **Amenable groups.** This part of the current work has two principal results, one positive and the other negative.

(a) (Theorem 4.1) It is proved that every Cayley graph of an infinite, finitely generated, elementary amenable group supports a graph height function. This extends [22, Thm 5.1] beyond the class of virtually solvable groups.
(b) (Theorem 5.1) It is proved that the Cayley graph of the Grigorchuk group does not support a graph height function. This answers in the negative the above question of [19] (see also [22, Sect. 5]). Since the Grigorchuk group is amenable (but not elementary amenable), possession of a graph height function is not a characteristic of amenable groups. This is in contrast with work of Lee and Peres, [33], who have studied the existence of non-constant, Hilbert space valued, equivariant harmonic maps on amenable graphs.

1.3. **Non-amenable graphs.** In earlier work [21], it was shown that the connective constant μ of a transitive, simple graph with degree Δ satisfies

$$\sqrt{\Delta - 1} \le \mu \le \Delta - 1,$$

and it was asked whether or not the lower bound is sharp. In the first of the following three results, this is answered in the negative for non-amenable graphs.

(a) (Theorem 6.2) It is proved that

$$(\Delta - 1)^{\frac{1}{2}(1+c\lambda)} \le \mu,$$

where $c = c(\Delta)$ is a known constant, and λ is the spectral bottom of the simple random walk on the graph. Kesten [31, 32] and Dodziuk [9] have shown that $\lambda > 0$ if and only if the graph is non-amenable.

(b) (Theorem 7.1) Using a percolation result of Benjamini, Nachmias, and Peres [5], it is explained that the connective constant of a non-amenable, Δ-regular graph with large girth is close to that of the Δ-regular tree.

(c) (Theorem 8.1) It is shown that the Cayley graph of the Higman group of [29] (which is non-amenable) does not support a graph height function. This further example of a transitive graph without a graph height function complements the corresponding statement above for the (amenable) Grigorchuk group.

Relevant notation for graphs, groups, and self-avoiding walks is summarized in Section 2, and three different types of height functions are explained in Section 3. The class EG of elementary amenable groups is described in Sections 4 and 9. The Grigorchuk group is defined in Section 5 and the non-existence of graph height functions thereon is given in Theorem 5.1. The improved lower bound for $\mu(G)$ for non-amenable G is presented at Theorem 6.2, and the remark about non-amenable graphs with large girth at Theorem 7.1. The Higman group is discussed in Section 8. Proofs of theorems appear either immediately after their statements, or are deferred to self-contained sections at the end of the article.

2. Graphs, groups, and self-avoiding walks

2.1. **Graphs.** The graphs $G = (V, E)$ in this paper are *simple*, in that they have neither loops nor multiple edges. The *degree* $\deg(v)$ of vertex $v \in V$ is the number of edges incident to v. We write $u \sim v$ for neighbours u and v, ∂v for the neighbour set of v, and $\partial_{\mathrm{e}} v$ (respectively, $\partial_{\mathrm{e}} W$) for the set of edges incident to v (respectively, between W and $V \setminus W$). The graph is *locally finite* if $|\partial v| < \infty$ for $v \in V$. An edge from u to v is denoted $\langle u, v \rangle$ when undirected, and $[u, v\rangle$ when directed from u to v. The *girth* of G is the infimum of the lengths of its circuits. The infinite Δ-regular tree T_Δ crops up periodically in this paper.

The automorphism group of G is denoted $\mathrm{Aut}(G)$. The subgroup $\Gamma \le \mathrm{Aut}(G)$ is said to act *transitively* on G if, for $u, v \in V$, there exists $\alpha \in \mathrm{Aut}(G)$ with $\alpha(u) = v$. It acts *quasi-transitively* if there exists a finite subset $W \subseteq V$ such that, for $v \in V$, there exists $\alpha \in \Gamma$ and $w \in W$ such that $\alpha(v) = w$. The graph G is said to be *(vertex-)transitive* if $\mathrm{Aut}(G)$ acts transitively on V.

Let $\mathcal{G}$ be the set of infinite, locally finite, connected, transitive, simple graphs, and let $G \in \mathcal{G}$. The *edge-isoperimetric constant* $\phi = \phi(G)$ is defined here as

$$\phi := \inf\left\{\frac{|\partial_{\mathrm{e}} W|}{\Delta |W|} : W \subset V,\ 0 < |W| < \infty\right\}. \tag{2.1}$$

We call G *amenable* if $\phi = 0$ and *non-amenable* otherwise. See [34, Sect. 6] for an account of graph amenability.

2.2. **Self-avoiding walks.** Let $G \in \mathcal{G}$. We choose a vertex of G and call it the *origin*, denoted $\mathbf{1}$. An *n-step self-avoiding walk* (SAW) on G is a walk containing n edges no vertex of which appears more than once. Let Σ_n be the set of n-step SAWs starting at $\mathbf{1}$, with cardinality $\sigma_n := |\Sigma_n|$. We have in the usual way (see [25, 36]) that

$$\sigma_{m+n} \le \sigma_m \sigma_n, \tag{2.2}$$

whence the *connective constant*

$$\mu = \mu(G) := \lim_{n\to\infty} \sigma_n^{1/n} \tag{2.3}$$

exists.

A SAW is called *extendable* if it is the initial portion of an infinite SAW on G. (An extendable SAW is called 'forward extendable' in [17].)

2.3. **Groups.** Let Γ be a group with generator set S satisfying $|S| < \infty$ and $\mathbf{1} \notin S$, where $\mathbf{1} = \mathbf{1}_\Gamma$ is the identity element. We shall assume that $S^{-1} = S$, while noting that this was not assumed in [22]. We write $\Gamma = \langle S \mid R\rangle$ with R a set of relators (or relations, when convenient). Such a group is called *finitely generated*, and is called *finitely presented* if, in addition, $|R| < \infty$.

The *Cayley graph* of the presentation $\Gamma = \langle S \mid R\rangle$ is the simple graph $G = G(\Gamma, S)$ with vertex-set Γ, and an (undirected) edge $\langle \gamma_1, \gamma_2\rangle$ if and only if $\gamma_2 = \gamma_1 s$ for some $s \in S$. Thus, our Cayley graphs are simple graphs. See [2, 34] for accounts of Cayley graphs, and [27] of geometric group theory.

The amenability of groups was introduced by von Neumann [38]. It is standard that a finitely generated group is amenable if and only if some (and hence every) Cayley graph is amenable (see, for example, [41, Chap. 12A]).

3. Height functions

It was shown in [19] that graphs $G \in \mathcal{G}$ supporting so-called 'graph height functions' have (at least) three properties:

(i) one may define the concept of a 'bridge' SAW on G, as in [26],
(ii) the exponential growth rate for counts of bridges equals the connective constant $\mu(G)$,
(iii) there exists a terminating algorithm for determining $\mu(G)$ to within any prescribed (strictly positive) degree of accuracy.

Several natural sub-classes of $\mathcal{G}$ contain only graphs that support graph height functions, and it was asked in [21] whether or not *every* $G \in \mathcal{G}$ supports a graph height function. This question will be answered in the negative at Theorems 5.1 and 8.1, where it is proved that neither the Grigorchuk nor Higman graphs possess a graph height function. Arguments for proving the non-existence of graph height functions may be found in Section 10.

We review the definitions of the two types of height functions, and introduce a third type. Let $G = (V, E) \in \mathcal{G}$, and let $\mathcal{H} \le \mathrm{Aut}(G)$. A function $F : V \to \mathbb{R}$ is said to be *$\mathcal{H}$-difference-invariant* if

$$F(v) - F(w) = F(\gamma v) - F(\gamma w), \qquad v, w \in V,\ \gamma \in \mathcal{H}. \tag{3.1}$$

Definition 3.1 ([19])**.** *A* graph height function *on G is a pair $(h, \mathcal{H})$, where $\mathcal{H} \le \mathrm{Aut}(G)$ acts quasi-transitively on G and $h : V \to \mathbb{Z}$, such that:*

(a) *$h(\mathbf{1}) = 0$,*
(b) *h is $\mathcal{H}$-difference-invariant,*
(c) *for $v \in V$, there exist $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$.*

Remark 3.2. *By Poincaré's Theorem for subgroups (see* [28, p. 48, Exercise 20]*), it is immaterial whether or not we require the subgroup $\mathcal{H}$ to be* normal *in Definition 3.1.*

We turn to Cayley graphs of finitely generated groups. Let Γ be a finitely generated group with presentation $\langle S \mid R\rangle$. As in Section 2, we assume $S^{-1} = S$ and $\mathbf{1} \notin S$.

Definition 3.3. *A* group height function *on Γ (or on a Cayley graph of Γ) is a function $h : \Gamma \to \mathbb{Z}$ such that:*

(a) *$h(\mathbf{1}) = 0$, and h is not identically zero,*
(b) *if $\gamma = s_1 s_2 \cdots s_m$ with $s_i \in S$, then $h(\gamma) = \sum_{i=1}^m h(s_i)$,*
(c) *the values $(h(s) : s \in S)$ are such that, if $s_1 s_2 \cdots s_n = \mathbf{1}$ is a representation of the identity with $s_i \in S$, then $\sum_{i=1}^n h(s_i) = 0$.*

A necessary and sufficient condition for the existence of a group height function is given in [22, Thm 4.1]. In the language of group theory, this condition amounts to requiring that the first Betti number is strictly positive. It was recalled in [22, Remark 4.2] that (when the non-zero $h(s)$, $s \in S$, are coprime) a group height function is simply a surjective homomorphism from Γ to $\mathbb{Z}$.

We introduce a third type of height function, which may be viewed as an intermediary between a graph height function and group height function.

Definition 3.4. *For a Cayley graph G of a finitely generated group Γ, we say that the pair $(h, \mathcal{H})$ is a* strong graph height function *of the pair (Γ, G) if*

(a) *$\mathcal{H} \trianglelefteq \Gamma$ acts on Γ by left multiplication, and $[\Gamma : \mathcal{H}] < \infty$, and*
(b) *$(h, \mathcal{H})$ is a graph height function.*

It is evident that a group height function h (of Γ) is a strong graph height function of the form (h, Γ), and a strong graph height function is a graph height function. The assumption in (a) above of the normality of $\mathcal{H}$ is benign, as in Remark 3.2.

We recall the definition of a harmonic function. A function $h : V \to \mathbb{R}$ is called *harmonic* on the graph $G = (V, E)$ if

$$h(v) = \frac{1}{\deg(v)} \sum_{u \sim v} h(u), \qquad v \in V.$$

It is an exercise to show that any group height function is harmonic.

Part B. Results for amenable groups

4. Elementary amenable groups

The class EG of elementary amenable groups was introduced by Day in 1957, [8], as the smallest class of groups that contains the set EG_0 of all finite and abelian groups, and is closed under the operations of taking subgroups, and of forming quotients, extensions, and directed unions. Day noted that every group in EG is amenable (see also von Neumann [38]). An important example of an amenable but not elementary amenable group was described by Grigorchuk in 1984, [13]. Grigorchuk's group is important in the study of height functions, and we return to this in Section 5.

Let EFG be the set of infinite, finitely generated members of EG.

Theorem 4.1. *Let $\Gamma \in$ EFG. Any locally finite Cayley graph G of Γ admits a harmonic, strong graph height function.*

We prove a slightly stronger version of this at Theorem 9.1, using transfinite induction. The class EFG includes all virtually solvable groups, and thus Theorem 4.1 extends [22, Thm 5.1]. Since any finitely generated group with polynomial growth is virtually nilpotent, [24], and hence lies in EFG, its locally finite Cayley graphs admit harmonic graph height functions.

5. The Grigorchuk group

The (first) Grigorchuk group is an infinite, finitely generated, amenable group that is not elementary amenable. We show in Theorem 5.1 that there exists a locally finite Cayley graph of the Grigorchuk group with *no* graph height function. This answers in the negative Question 3.3 of [19] (see also [22, Sect. 3]).

Here is the definition of the group in question (see [12, 13, 15]). Let T be the rooted binary tree with root vertex $\varnothing$. The vertex-set of T can be identified with the set of finite strings u having entries 0, 1, where the empty string corresponds to the root $\varnothing$. Let T_u be the subtree of all vertices with root labelled u.

Let $\mathrm{Aut}(T)$ be the automorphism group of T, and let $a \in \mathrm{Aut}(T)$ be the automorphism that, for each string u, interchanges the two vertices $0u$ and $1u$.

Any $\gamma \in \mathrm{Aut}(G)$ may be applied in a natural way to either subtree T_i, $i = 0, 1$. Given two elements $\gamma_0, \gamma_1 \in \mathrm{Aut}(T)$, we define $\gamma = (\gamma_0, \gamma_1)$ to be the automorphism on T obtained by applying γ_0 to T_0 and γ_1 to T_1. Define automorphisms b, c, d of T recursively as follows:

$$b = (a, c), \quad c = (a, d), \quad d = (e, b), \tag{5.1}$$

where e is the identity automorphism. The Grigorchuk group is defined as the subgroup of $\mathrm{Aut}(T)$ generated by the set $\{a, b, c\}$.

Theorem 5.1. *The Cayley graph $G = (V, E)$ of the Grigorchuk group with generator set $\{a, b, c\}$ satisfies:*

(a) *G admits no graph height function,*
(b) *for $\mathcal{H} \trianglelefteq \mathrm{Aut}(G)$ with finite index, any $\mathcal{H}$-difference-invariant function on V is constant on each orbit of $\mathcal{H}$.*

The proof of Theorem 5.1 is given in Section 11. In the preceding Section 10, two approaches are developed for showing the absence of a graph height function within particular classes of Cayley graph. In the case of the Grigorchuk group, two reasons combine to forbid graph height functions, namely, its Cayley group has no automorphisms beyond the action of the group itself, and the group is a torsion group in that every element has finite order.

Since the Grigorchuk group is amenable, Theorems 4.1 and 5.1 yield that: within the class of infinite, finitely generated groups, every elementary amenable group has a graph height function, but there exists an amenable group without a graph height function. The Grigorchuk group is finitely *generated* but not finitely *presented*, [13, Thm 6.2].

We ask if there exists an infinite, finitely *presented*, amenable group with a Cayley graph having no graph height function. A natural candidate might be the group

$\Gamma = \langle S \mid R\rangle$ of [14, Thm 1], with

$$S = \{a, c, d, t\},$$
$$R = \bigl\{a^2 = c^2 = d^2 = (ad)^4 = (adacac)^4 = \mathbf{1}, t^{-1}at = aca, t^{-1}ct = dc, t^{-1}dt = c\bigr\}.$$

This finitely presented, amenable HNN-extension of the Grigorchuk group is not elementary amenable. However, since it contains the free group generated by the stable letter t, it possesses a group height function. More precisely, the function

$$h(\mathbf{1}) = 0, \quad h(t) = 1, \quad h(t^{-1}) = -1, \qquad h(s) = 0 \text{ for } s \in S,\ s \neq t^{\pm 1},$$

defines a group height function.

Part C. Results for non-amenable graphs

6. Connective constants of non-amenable graphs

Let $G \in \mathcal{G}$ have degree Δ. It was proved in [21, Thm 4.1] that

$$\sqrt{\Delta - 1} \le \mu(G) \le \Delta - 1.$$

The upper bound is achieved by the Δ-regular tree T_Δ. It is unknown if the lower bound is sharp for *simple* graphs. This lower bound may however be improved for non-amenable graphs, as follows.

Let P be the transition matrix of simple random walk (SRW) on $G = (V, E)$, and let I be the identity matrix. The *spectral bottom* of $I - P$ is defined to be the largest λ with the property that, for all $f \in l^2(V)$,

$$\langle f, (I - P)f\rangle \ge \lambda \langle f, f\rangle. \tag{6.1}$$

It may be seen that $\lambda(G) = 1 - \rho(G)$ where $\rho(G)$ is the spectral radius of P (see [34, Sect. 6], and [41] for an account of the spectral radius).

Remark 6.1. *It is known that G is a non-amenable if and only if $\rho(G) < 1$, which is equivalent to $\lambda(G) > 0$. This was proved by Kesten* [31, 32] *for Cayley graphs of finitely-presented groups, and extended to general transitive graphs by Dodziuk* [9] *(see also the references in* [34, Sect. 6.10]*).*

Theorem 6.2. *Let $G \in \mathcal{G}$ have degree $\Delta \ge 3$. Let P be the transition matrix of SRW on G, and λ the spectral bottom of P. The connective constant $\mu(G)$ satisfies*

$$\mu(G) \ge (\Delta - 1)^{\frac{1}{2}(1 + c\lambda)}, \tag{6.2}$$

where $c = \Delta(\Delta - 1)/(\Delta - 2)^2$.

The improvement in the lower bound for $\mu(G)$ is strict if and only if $\lambda > 0$, which is to say that G is non-amenable. It is standard (see [34, Thm 6.7]) that

$$\tfrac{1}{2}\phi^2 \le 1 - \sqrt{1 - \phi^2} \le \lambda \le \phi, \tag{6.3}$$

where $\phi = \phi(G)$ is the edge-isoperimetric constant of (2.1). By [1, Thm 3],

$$\lambda(G) \le \lambda(T_\Delta) - \frac{\Delta - 2}{\Delta(\Delta-1)^{g+2}}, \tag{6.4}$$

where g is the girth of G, T_Δ is the Δ-regular tree, and

$$\lambda(T_\Delta) = 1 - \frac{2\sqrt{\Delta - 1}}{\Delta}. \tag{6.5}$$

Remark 6.3. *The spectral bottom (and therefore the spectral radius, also) is not a continuous function of G in the usual graph metric (see* [19, Sect. 5]*). This follows from* [40, Thm 2.4]*, where it is proved that, for all pairs (k, l) with $k \ge 2$ and $l \ge 3$, there exists a group with polynomial growth whose Cayley graph $G_{k,l}$ is $2k$-regular with girth exceeding l. Since $G_{k,l}$ is amenable, we have $\lambda(G_{k,l}) = 0$, whereas $\lambda(T_{2k})$ is given by* (6.5)*.*

Proof of Theorem 6.2. This is achieved by a refinement of the argument used to prove [21, Thm 4.1], and we shall make use of the notation introduced in that proof.

Let $v_0 = \mathbf{1}$, and let $\pi = (v_0, v_1, \dots, v_{2n})$ be an extendable $2n$-step SAW of G. For convenience, we augment π with a mid-edge incident to v_0 and not lying on the edge $\langle v_0, v_1\rangle$. Let E_π be the set of oriented edges $[v, w\rangle$ such that: (i) $v \in \pi$, $v \ne v_{2n}$, and (ii) the (non-oriented) edge $\langle v, w\rangle$ does not lie in π. Note that

$$|E_\pi| = 2n(\Delta - 2). \tag{6.6}$$

Each (oriented) edge in E_π is coloured either red or blue according to the following rule. For $v \in \pi$, let π_v be the sub-path of π joining v_0 to v. The edge $[v, w\rangle \in E_\pi$ is coloured red if $\pi_v \cup [v, w\rangle$ is not an extendable SAW, and is coloured blue otherwise. By (6.6), the number B_π (respectively, R_π) of blue edges (respectively, red edges) satisfies

$$B_\pi + R_\pi = 2n(\Delta - 2). \tag{6.7}$$

We shall make use of the following lemma.

Lemma 6.4. *The number B_π satisfies*

$$B_\pi \ge \frac{n(1 + c\lambda)}{\Delta - 2} - \frac{\Delta - 1}{2}, \tag{6.8}$$

where $c = \Delta(\Delta - 1)/(\Delta - 2)^2$.

We now argue as in [21, Lemma 5.1] to deduce from Lemma 6.4 that the number of extendable $2n$-step SAWs from v_0 is at least $C(\Delta - 1)^{n(1+c\lambda)}$ where $C = C(\Delta)$. Inequality (6.2) follows as required. □

Proof of Lemma 6.4. An edge $[v,w\rangle \in E_\pi$ is said to be *finite* if w lies in a finite component of $G\setminus\pi$, and *infinite* otherwise. If $[v,w\rangle \in E_\pi$ is red, then w is necessarily finite. Blue edges, on the other hand, may be either finite or infinite.

It was explained in the proof of [21, Thm 4.1] that there exists an injection f from the set of red edges to the set of blue edges with the property that, if $e=[v,w\rangle$ is red, and $f(e)=[v',w'\rangle$, then w and w' lie in the same component of $G\setminus\pi_v$. Since e is finite, so is $f(e)$. It follows that

$$B_\pi \geq R_\pi + B_\pi^\infty, \tag{6.9}$$

where R_π is the number of red edges, and B_π^∞ is the number of infinite blue edges.

Let $X=(X_m : m=0,1,2,\dots)$ be a SRW on G, and let $\mathbb{P}_v$ denote the law of X started at $v\in V$. For $[v,w\rangle \in E_\pi$, let

$$\beta_{[v,w\rangle} = \mathbb{P}_v\bigl(X_1=w,\ \text{and } \forall m>0, X_m\notin\pi\bigr).$$

By [5, Lemma 2.1] with $A=\pi$,

$$\lambda \leq \frac{1}{2n+1}\left(\sum_{[v,w\rangle\in E_\pi}\beta_{[v,w\rangle} + \sum_w \beta_{[v_{2n},w\rangle}\right). \tag{6.10}$$

If $[v,w\rangle\in E_\pi$ is finite, then $\beta_{[v,w\rangle}=0$. By (6.10),

$$\lambda \leq \left(\frac{B_\pi^\infty+\Delta-1}{2n}\right)\mathbb{P}_v\bigl(X_1=w,\ \text{and } \forall m>0, X_m\neq v\bigr). \tag{6.11}$$

The last probability depends on the graph G, and it is a maximum when G is the Δ-regular tree T_Δ (since T_Δ is the universal cover of G). Therefore, it is no greater than $1/\Delta$ multiplied by the probability that a random walk on $\mathbb{Z}$, which moves rightwards with probability $p=(\Delta-1)/\Delta$ and leftwards with probability $q=1-p$, never visits 0 having started at 1. By, for example, [23, Example 12.59],

$$\mathbb{P}_v\bigl(X_1=w,\ \text{and } \forall m>0, X_m\neq v\bigr) \leq \frac{1}{\Delta}\left(1-\frac{q}{p}\right) = \frac{\Delta-2}{\Delta(\Delta-1)}.$$

By (6.9) and (6.11),

$$B_\pi \geq R_\pi + 2n\lambda\frac{\Delta(\Delta-1)}{\Delta-2} - \Delta + 1, \tag{6.12}$$

and (6.8) follows by (6.7). □

7. Graphs with large girth

Benjamini, Nachmias, and Peres showed in [5, Thm 1.1] that the critical probability $p_c(G)$ of bond percolation on a Δ-regular, non-amenable graph G with large girth is close to that of the critical probability of the Δ-regular tree T_Δ. Their main result implies the following.

Theorem 7.1. *Let $G \in \mathcal{G}$ be non-amenable with degree $\Delta \geq 3$ and girth $g \leq \infty$. There exists an absolute positive constant C such that*

$$\left[\frac{1}{\Delta - 1} + C\frac{\log(1+\lambda^{-2})}{g\Delta}\right]^{-1} \leq \mu(G) \leq \Delta - 1, \tag{7.1}$$

where $\lambda = \lambda(G)$ is the spectral bottom of SRW on G, as in (6.1). Equality holds in the upper bound of (7.1) if and only if G is a tree, that is, $g = \infty$.

Proof. The upper bound of (7.1) is from [21, Thm 4.2]. The lower bound is an immediate consequence of [5, Thm 1.1] and the fact that $\mu(G) \geq 1/p_c(G)$ (see, for example, [6, Thm 7] and [16, eqn (1.13)], which hold for general quasi-transitive graphs). □

We recall from Remark 6.1 that $\lambda > 0$ if and only if G is non-amenable. Theorem 7.1 does not, of itself, imply that $\mu(\cdot)$ is continuous at T_Δ, since $\lambda(\cdot)$ is not continuous at T_Δ (in the case when Δ is even, see Remark 6.3). For continuity at T_Δ, it would suffice that $\lambda(\cdot)$ is bounded away from 0 on a neighbourhood of T_Δ. By (6.3), this is valid within any class of graphs whose edge-isoperimetric constants (2.1) are bounded uniformly from 0. See also [19, Thm 5.1].

8. The Higman group

The *Higman group* Γ of [29] is the infinite, finitely presented group with presentation $\Gamma = \langle S \mid R \rangle$ where

$$\begin{aligned} S &= \{a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}\}, \\ R &= \{a^{-1}ba = b^2, b^{-1}cb = c^2, c^{-1}dc = d^2, d^{-1}ad = a^2\}. \end{aligned} \tag{8.1}$$

This group is interesting since it has no proper normal subgroup with finite index, and the quotient of Γ by its maximal proper normal subgroup is an infinite, finitely generated, simple group. By [22, Thm 4.1(b)], Γ has no group height function. The above two reasons conspire to forbid graph height functions.

Theorem 8.1. *The Cayley graph $G = (V, E)$ of the Higman group $\Gamma = \langle S \mid R \rangle$ has no graph height function.*

A further group of Higman type is given as follows. Let S be as above, and let $\Gamma' = \langle S \mid R' \rangle$ be the finitely presented group with

$$R' = \{a^{-1}ba = b^2, b^{-2}cb^2 = c^2, c^{-3}dc^3 = d^2, d^{-4}ad^4 = a^2\}.$$

Note that Γ' is infinite and non-amenable, since the subgroup generated by the set $\{a, c, a^{-1}, c^{-1}\}$ is a free group (as in the corresponding step for the Higman group at [29, pp. 62–63]).

Theorem 8.2. *The Cayley graph $G = (V, E)$ of the above group $\Gamma' = \langle S \mid R' \rangle$ has no graph height function.*

The proofs of the above theorems are given in Sections 12 and 13, respectively.

Part D. Remaining proofs

9. Proof of Theorem 4.1

We shall prove the following stronger form of Theorem 4.1.

Theorem 9.1. *Let $\Gamma \in \mathrm{EFG}$. There exists a normal subgroup $\mathcal{H} \trianglelefteq \Gamma$ with $1 < [\Gamma : \mathcal{H}] < \infty$ such that any locally finite Cayley graph G of Γ possesses a harmonic, strong graph height function of the form $(h, \mathcal{H})$.*

Whereas every member of EFG has a proper, normal subgroup with finite index, it is proved in [30] that there exist amenable *simple* groups.

We review next the structure of EG. Let EG_0 be the class of all groups that are either finite or abelian (or both), and let $\mathcal{O}$ be the class of all ordinals. Let $\alpha \in \mathcal{O}$, $\alpha \neq 0$, and assume we have defined EG_β for each $\beta \in \mathcal{O}$, $\beta < \alpha$. Each $\alpha \in \mathcal{O}$ is either a limit ordinal or a successor ordinal. If α is a limit ordinal, we set

$$\mathrm{EG}_\alpha = \bigcup_{\beta<\alpha} \mathrm{EG}_\beta. \tag{9.1}$$

If α is a successor ordinal, let EG_α be the class of groups which can be obtained from members of $\mathrm{EG}_{\alpha-1}$ by no more than one operation of extension or directed union.

Theorem 9.2 ([7])**.** *We have that* $\mathrm{EG} = \bigcup_{\alpha\in\mathcal{O}} \mathrm{EG}_\alpha$.

Proof of Theorem 9.1. Let $\mathrm{EFG}_\alpha = \mathrm{EFG} \cap \mathrm{EG}_\alpha$. For $\alpha \in \mathcal{O}$, let H_α be the following statement:

H_α : for $\beta \in \mathcal{O}$, $\beta \leq \alpha$, and $\Gamma \in \mathrm{EFG}_\beta$, there exists $\mathcal{H} \trianglelefteq \Gamma$ such that every locally finite Cayley graph of Γ admits a harmonic, strong graph height function of the form $(h, \mathcal{H})$.

Now, EFG_0 is the set of infinite, finitely generated, abelian groups. By [22, Prop. 4.3, Thm 5.2(b)], any locally finite Cayley graph of Γ has a group height function, and hence a harmonic, strong graph height function of the form (h, Γ). Therefore, H_0 holds, and we turn to the induction step.

Let $\alpha \in \mathcal{O}$, $\alpha \neq 0$, and assume H_β holds for all $\beta < \alpha$. Let $\Gamma \in \mathrm{EFG}_\alpha$ with α the smallest such ordinal. There are two cases to consider, depending on whether or not α is a limit ordinal. *If α is a limit ordinal*, by (9.1), there exists $\beta \in \mathcal{O}$, $\beta < \alpha$, such that $\Gamma \in \mathrm{EFG}_\beta$. The claim now follows by H_β.

We assume for the remainder of this proof that α is a successor ordinal. By Theorem 9.2, the group $\Gamma \in \mathrm{EFG}_\alpha$ is obtained from groups in $\mathrm{EFG}_{\alpha-1}$ by exactly one operation of either extension or directed union. That is, there are two sub-cases to consider.

(a) There exist $\mathcal{N}', \mathcal{Q}' \in \mathrm{EFG}_{\alpha-1}$ such that $\mathcal{N}'$ is isomorphic to a normal subgroup $\mathcal{N}$ of Γ, and $\mathcal{Q}' \simeq \mathcal{Q} := \Gamma/\mathcal{N}$.

(b) There exist a directed set Λ and a family $(S_\lambda : \lambda \in \Lambda)$ satisfying
  - (i) $S_\lambda \in \mathrm{EFG}_{\alpha-1}$,
  - (ii) $S_{\lambda_1} \subseteq S_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$,
  - (iii) $\Gamma = \bigcup_{\lambda \in \Lambda} S_\lambda$.

Assume (a) holds. Since Γ is finitely generated, so is $\mathcal{Q}$.

Suppose $\mathcal{Q}$ is infinite. We shall use the fact that $\mathcal{Q} \in \mathrm{EFG}_{\alpha-1}$. Let S be a finite set of generators of Γ with $S = S^{-1}$ and $\mathbf{1} \notin S$, and let $G = G(\Gamma, S)$ be the corresponding Cayley graph of Γ. A locally finite Cayley graph $G_\mathcal{Q}$ of $\mathcal{Q}$ may be constructed as follows. Let

$$\overline{S} = \{\overline{s} = s\mathcal{N} : s \in S\},$$

be the (finite) generator set of $\mathcal{Q}$ derived from S. The vertex-set of $G_\mathcal{Q} = G_\mathcal{Q}(Q, \overline{S})$ is the set of cosets $\{\overline{v} := v\mathcal{N} : v \in \Gamma\}$, and two such vertices $\overline{v}$, $\overline{w}$ are connected by an edge of $G_\mathcal{Q}$ if and only if there exist $v \in \overline{v}$, $w \in \overline{w}$ such that v and w are connected by an edge in G.

By $\mathrm{H}_{\alpha-1}$, there exists $\overline{\mathcal{H}} \trianglelefteq \mathcal{Q}$, not depending on the choice of S, such that $G_\mathcal{Q}$ admits a harmonic, strong graph height function $(h_\mathcal{Q}, \overline{\mathcal{H}})$. Let $h : \Gamma \to \mathbb{Z}$ and $\mathcal{H} \subseteq \Gamma$ be given by

$$h(v) = h_\mathcal{Q}(\overline{v}), \qquad \mathcal{H} = \bigcup_{\gamma\mathcal{N} \in \overline{\mathcal{H}}} \gamma\mathcal{N}. \tag{9.2}$$

The following lemma completes the proof of this case.

Lemma 9.3. *We have that:*

(a) *$\mathcal{H} \trianglelefteq \Gamma$, and $\mathcal{H}$ acts quasi-transitively on G by left-multiplication,*

(b) *the pair $(h, \mathcal{H})$ is a harmonic, strong graph height function of G.*

Proof. (a) Since $\overline{\mathcal{H}} \trianglelefteq \mathcal{Q}$, we have that $(a\mathcal{N})\overline{\mathcal{H}}(a\mathcal{N})^{-1} = \overline{\mathcal{H}}$ for $a \in \Gamma$, whence

$$(a\gamma a^{-1})\mathcal{N} \in \overline{\mathcal{H}} \quad \text{whenever} \quad a, \gamma \in \Gamma,\ \gamma\mathcal{N} \in \overline{\mathcal{H}}. \tag{9.3}$$

It is elementary that, for $a \in \Gamma$ and $\gamma_1\mathcal{N}, \gamma_2\mathcal{N} \in \overline{\mathcal{H}}$,

$$(a\gamma_1 a^{-1})\mathcal{N} = (a\gamma_2 a^{-1})\mathcal{N} \quad \text{if and only if} \quad \gamma_1\mathcal{N} = \gamma_2\mathcal{N}. \tag{9.4}$$

Since $\overline{\mathcal{H}}$ is a group, so is $\mathcal{H}$. For $a \in \Gamma$, by (9.2)–(9.4),

$$\begin{aligned} a\mathcal{H}a^{-1} &= \bigcup_{\gamma\mathcal{N}\in\overline{\mathcal{H}}} a(\gamma\mathcal{N})a^{-1} = \bigcup_{\gamma\mathcal{N}\in\overline{\mathcal{H}}} (a\gamma a^{-1})\mathcal{N} \\ &= \bigcup_{\gamma\mathcal{N}\in\overline{\mathcal{H}}} \gamma\mathcal{N} = \mathcal{H}. \end{aligned}$$

Therefore, $\mathcal{H} \trianglelefteq \Gamma$. We prove next that $[\Gamma : \mathcal{H}] < \infty$.

Since $(h_{\mathcal{Q}}, \overline{\mathcal{H}})$ is a graph height function, we have that $[\mathcal{Q} : \overline{\mathcal{H}}] < \infty$. Let $\overline{W}_1, \overline{W}_2, \dots, \overline{W}_k$ be the cosets of $\overline{\mathcal{H}}$ in $\mathcal{Q}$, and let

$$W_i = \bigcup_{\gamma\mathcal{N}\in\overline{W}_i} \gamma\mathcal{N}.$$

We show next that each W_i is contained in an orbit of $\mathcal{H}$ acting on Γ. (Actually the W_i *are* the orbits.) It follows that $\mathcal{H}$ acts quasi-transitively on G.

Without loss of generality, let $u, v \in W_1$. We shall show that there exists $\nu \in \mathcal{H}$ such that $v = \nu u$. Suppose $u \in a\mathcal{N}$, $v \in b\mathcal{N}$ where $a\mathcal{N}, b\mathcal{N} \in \overline{W}_1$. There exists $\gamma\mathcal{N} \in \overline{\mathcal{H}}$ such that $\gamma\mathcal{N}a\mathcal{N} = b\mathcal{N}$, which is to say that $a\mathcal{N}b^{-1} \in \overline{\mathcal{H}}$.

There exist n_i such that $u = an_1$, $v = bn_2$. Then, $u = (an_1n_2^{-1}b^{-1})v$, and $\nu := a(n_1n_2^{-1})b^{-1} \in \mathcal{H}$ by (9.2).

(b) It is trivial that $h(\mathbf{1}) = h_{\mathcal{Q}}(\overline{\mathbf{1}}) = 0$. For $\gamma \in \mathcal{H}$ and $u, v \in \Gamma$, we have

$$\begin{aligned} h(\gamma u) - h(\gamma v) &= h_{\mathcal{Q}}(\overline{\gamma u}) - h_{\mathcal{Q}}(\overline{\gamma v}) \\ &= h_{\mathcal{Q}}(\overline{\gamma}\,\overline{u}) - h_{\mathcal{Q}}(\overline{\gamma}\,\overline{v}) \quad \text{since } \mathcal{N} \text{ is normal} \\ &= h_{\mathcal{Q}}(\overline{u}) - h_{\mathcal{Q}}(\overline{v}) \qquad \text{since } h_{\mathcal{Q}} \text{ is } \overline{\mathcal{H}}\text{-difference invariant, } \overline{\gamma} \in \overline{\mathcal{H}} \\ &= h(u) - h(v). \end{aligned}$$

Therefore, h is $\mathcal{H}$-difference-invariant.

For $v \in \Gamma$, there exist $\overline{s}_1, \overline{s}_2 \in \overline{S}$ such that

$$h_{\mathcal{Q}}(\overline{v}\,\overline{s}_1) < h_{\mathcal{Q}}(\overline{v}) < h_{\mathcal{Q}}(\overline{v}\,\overline{s}_2),$$

whence, since $\mathcal{N}$ is a normal subgroup of Γ,

$$h(vs_1) < h(v) < h(vs_2).$$

In conclusion, $(h, \mathcal{H})$ is a strong graph height function of G.

We show finally that h is harmonic on the Cayley graph $G = (V, E)$. The edges incident to the vertex labelled $\gamma \in \Gamma$ have the form $\langle \gamma, \gamma s\rangle$ for $s \in S$. Since $h_{\mathcal{Q}}$ is harmonic on the quotient graph, it suffices to show that the cardinality $N_s := |\partial\gamma \cap (\gamma\mathcal{N}s)|$ does not depend on the choice of $s \in S \setminus \mathcal{N}$. For $s \in S \setminus \mathcal{N}$ and $n \in \mathcal{N}$, $\gamma \sim \gamma ns$ if and only if $ns \in S$, which is to say that $n \in Ss^{-1}$, whence $N_s = |S|$. □

Suppose $\mathcal{Q}$ is finite. Since $\mathcal{N} \simeq \mathcal{N}' \in \mathrm{EFG}_{\alpha-1}$, we have that $\mathcal{N} \in \mathrm{EFG}_{\alpha-1}$ and $1 < [\Gamma : \mathcal{N}] < \infty$. By $\mathrm{H}_{\alpha-1}$, there exists $\mathcal{H}' \trianglelefteq \mathcal{N}$ with $[\mathcal{N} : \mathcal{H}'] < \infty$ such that any locally finite Cayley graph $G_{\mathcal{N}}$ of $\mathcal{N}$ admits a strong graph height function of the form $(h_{\mathcal{N}}, \mathcal{H}')$.

Since $|\Gamma/\mathcal{H}'| = |\Gamma/\mathcal{N}| \cdot |\mathcal{N}/\mathcal{H}'| < \infty$, there exists (by Poincaré's Theorem for subgroups) a subgroup $\mathcal{H} \leq \mathcal{H}'$ that is normal in Γ with finite index, that is, $\mathcal{H} \trianglelefteq \Gamma$ and $1 < [\Gamma : \mathcal{H}] < \infty$. Choose a locally finite Cayley graph $G_{\mathcal{N}}$ of $\mathcal{N}$, and find a strong graph height function of the form $(h_{\mathcal{N}}, \mathcal{H}')$. Let $F : \mathcal{H} \to \mathbb{Z}$ be the restriction of $h_{\mathcal{N}}$ to $\mathcal{H}$.

Lemma 9.4. *The function F is a group height function on the group $\mathcal{H}$.*

Proof. As noted in [22, Remark 4.2], a group height function is a homomorphism from $\mathcal{H}$ to $\mathbb{Z}$ that is not identically zero. For $\gamma_1, \gamma_2 \in \mathcal{H}$,

$$\begin{aligned} F(\gamma_1\gamma_2) - F(\gamma_1) &= h_{\mathcal{N}}(\gamma_1\gamma_2) - h_{\mathcal{N}}(\gamma_1) \\ &= h_{\mathcal{N}}(\gamma_2) - h_{\mathcal{N}}(\mathbf{1}) = F(\gamma_2), \end{aligned}$$

since $\gamma_1 \in \mathcal{H}'$ and $h_{\mathcal{N}}$ is $\mathcal{H}'$-difference invariant. Therefore, F is a homomorphism.

It suffices now to show that $F \not\equiv 0$ on $\mathcal{H}$. Assume the converse, that $F \equiv 0$ on $\mathcal{H}$. For $\gamma \in \Gamma$, there exists $a_\gamma \in \mathcal{N}$ such that $\gamma \in a_\gamma\mathcal{H}$, so that $\gamma = a_\gamma\nu$ for $\nu \in \mathcal{H}$. Since $h_{\mathcal{N}}$ is $\mathcal{H}$-difference-invariant,

$$h_{\mathcal{N}}(\gamma) = h_{\mathcal{N}}(a_\gamma) + F(\nu) = h_{\mathcal{N}}(a_\gamma). \tag{9.5}$$

Now $|\mathcal{N}/\mathcal{H}| < \infty$, so we may restrict consideration to only finitely many a_γ. Therefore, $h_{\mathcal{N}}(\gamma)$ is bounded, which is impossible since $h_{\mathcal{N}}$ is a graph height function. We deduce that $F \not\equiv 0$ on $\mathcal{H}$. □

Let $G = G(\Gamma, S)$ be a locally finite Cayley graph of Γ. The triple $(\Gamma, \mathcal{H}, F)$ satisfies the conditions of [22, Thm 3.5] with $\mathcal{H}$ acting by left multiplication, and it follows that G possesses a harmonic graph height function of the form $(h, \mathcal{H})$.

Assume (b) holds. Let Γ be finitely generated with finite generator set $S = \{s_1, s_2, \dots, s_k\}$. Since $\Gamma = \bigcup_{\lambda\in\Lambda} S_\lambda$, there exists $\lambda_i \in \Lambda$ such that $s_i \in S_{\lambda_i}$. Let $L = \max\{\lambda_1, \lambda_2, \dots, \lambda_k\}$, so that $S_L = \Gamma$. Then $\Gamma \in \mathrm{EFG}_{\alpha-1}$, which contradicts the minimality of α. □

10. Criteria for the absence of height functions

This section contains some observations relevant to proofs in Sections 11–13 of the non-existence of graph height functions.

Let $\Gamma = \langle S \mid R\rangle$ where $|S| < \infty$, and let $G = (V, E)$ be the corresponding Cayley graph. Let Π be the set of permutations of S that preserve Γ up to isomorphism, and write $e \in \Pi$ for the identity. Thus, $\pi \in \Pi$ acts on Γ by: for $w = s_1 s_2 \cdots s_m$ with $s_i \in S$, we have $\pi(w) = \pi(s_1)\pi(s_2)\cdots\pi(s_m)$. It follows that $\Pi \subseteq \mathrm{Aut}(G)$. For $\gamma = g_1 g_2 \cdots g_n \in \Gamma$ with $g_i \in S$, and $\pi \in \Pi$, we define $\gamma\pi \in \mathrm{Aut}(G)$ by $\gamma\pi(w) = g_1 g_2 \cdots g_n \pi(w)$, $w \in V$. Write $\Gamma\Pi \subseteq \mathrm{Aut}(G)$ for the subgroup containing all such $\gamma\pi$, and note that γe operates on G in the manner of γ with left-multiplication.

The *stabilizer* Stab_v of $v \in V$ is the set of automorphisms of G that preserve v, that is,

$$\mathrm{Stab}_v = \{\eta \in \mathrm{Aut}(G) : \eta(v) = v\}.$$

Proposition 10.1. *Suppose* $\mathrm{Stab}_{\mathbf{1}} = \Pi$.

(a) $\mathrm{Aut}(G) = \Gamma\Pi$.

(b) *If* $\mathcal{M} \trianglelefteq \mathrm{Aut}(G)$ *has finite index, the subgroup* $\mathcal{N} = \mathcal{M} \cap \Gamma$ *satisfies* $\mathcal{N} \trianglelefteq \Gamma$ *and* $[\Gamma : \mathcal{N}] < \infty$.

(c) *If G has a graph height function, then it has a strong graph height function.*

Proof. Assume $\mathrm{Stab}_{\mathbf{1}} = \Pi$.

(a) Let $\eta \in \mathrm{Aut}(G)$, and write $\gamma = \eta(\mathbf{1})$. Then $\gamma^{-1}\eta \in \mathrm{Stab}_{\mathbf{1}}$, which is to say that $\gamma^{-1}\eta = \pi \in \Pi$, and thus $\eta = \gamma\pi \in \Gamma\Pi$ so that $\mathrm{Aut}(G) = \Gamma\Pi$. Note for future use that

$$[\mathrm{Aut}(G) : \Gamma] = |\Pi| < \infty.$$

(b) Let $\mathcal{M} \trianglelefteq \mathrm{Aut}(G)$ be a finite-index normal subgroup, and let $\mathcal{N} = \{\gamma e : \gamma e \in \mathcal{M}\}$. Viewed as automorphisms, we have that $\gamma e = \gamma$, and hence $\mathcal{N} \le \Gamma \le \mathrm{Aut}(G)$. For $\alpha \in \Gamma$, $\nu \in \mathcal{N}$, we have that $(\alpha^{-1}\nu\alpha)e = \alpha^{-1}(\nu e)\alpha \in \mathcal{M}$, since $\mathcal{M} \trianglelefteq \mathrm{Aut}(G)$. Therefore, $\mathcal{N} \trianglelefteq \Gamma$.

Since $\Gamma, \mathcal{M} \le \mathrm{Aut}(G)$ and $\mathcal{N} = \Gamma \cap \mathcal{M}$, we have that

$$[\mathrm{Aut}(G) : \mathcal{N}] \le [\mathrm{Aut}(G) : \Gamma] \cdot [\mathrm{Aut}(G) : \mathcal{M}] < \infty,$$

which implies $[\Gamma : \mathcal{N}] < \infty$, as required.

(c) Let $(h, \mathcal{H})$ be a graph height function of G. Since $\mathcal{H}$ is a finite-index normal subgroup of $\mathrm{Aut}(G)$, by part (b), there exists $\mathcal{N} \le \mathcal{H}$ that is a finite-index normal subgroup of Γ. Since $\mathcal{N} \le \mathcal{H}$, h acts on Γ and is $\mathcal{N}$-difference invariant, whence $(h, \mathcal{N})$ is a strong graph height function. □

Corollary 10.2. *Let* $\Gamma = \langle S \mid R\rangle$ *have Cayley graph G satisfying* $\mathrm{Stab}_{\mathbf{1}} = \Pi$.

(a) *If Γ has no proper, normal subgroup with finite index, any graph height function of G is also a group height function of Γ.*

(b) *If every element in* Γ *has finite order, then* G *has no graph height function.*

Proof. (a) Let $(h, \mathcal{M})$ be a graph height function of G. If Γ satisfies the given condition then, by Proposition 10.1(b), $\mathcal{M} \supseteq \Gamma$. Therefore, (h, Γ) is a graph height function and hence a group height function.

(b) If G has a graph height function, by Proposition 10.1(c), G has a strong graph height function $(h, \mathcal{N})$. Assume every element of Γ has finite order. For $\gamma \in \mathcal{N}$ with $\gamma^n = \mathbf{1}$, we have that $h(\gamma^n) = nh(\gamma) = 0$, whence $h \equiv 0$ on $\mathcal{N}$.

We now use the argument around (9.5). For $\gamma \in \Gamma$, find α_γ such that $\gamma \in \alpha_\gamma \mathcal{N}$. Since h is $\mathcal{N}$-difference-invariant, there exists $\nu \in \mathcal{N}$ such that

$$h(\gamma) = h(\alpha_\gamma) + h(\nu) = h(\alpha_\gamma). \tag{10.1}$$

Now $[\Gamma : \mathcal{N}] < \infty$, so we may consider only finitely many choices for α_γ. Therefore, h is bounded on Γ, in contradiction of the assumption that it is a graph height function. □

11. Proof of Theorem 5.1

The main step is to show that

$$\mathrm{Stab}_{\mathbf{1}} = \{e\}, \tag{11.1}$$

where e is the identity of $\mathrm{Aut}(G)$. Once this is shown, claim (a) follows from Corollary 10.2(b) and the fact that every element of the Grigorchuk group has finite order, [27]. It therefore suffices for (a) to show (11.1), and to this end we study the structure of the Cayley graph $G = (V, E)$.

It was shown in [35] (see also [15, eqn (4.7)]) that $\Gamma = \langle S \mid R \rangle$ where $S = \{a, b, c, d\}$, R is the following set of relations

$$\begin{aligned} \mathbf{1} &= a^2 = b^2 = c^2 = d^2 = bcd \\ &= \sigma^k((ad)^4) = \sigma^k((adacac)^4), \qquad k = 0, 1, 2, \dots, \end{aligned} \tag{11.2}$$

and σ is the substitution

$$\sigma : \begin{cases} a \mapsto aca, \\ b \mapsto d, \\ c \mapsto b, \\ d \mapsto c. \end{cases}$$

It follows that the following, written in terms of the reduced generator set $\{a, b, c\}$ after elimination of d, are valid relations:

$$\mathbf{1} = a^2 = b^2 = c^2 = (bc)^2 = (abc)^4 = (ac)^8 = (abcacac)^4 = (acab)^8 = (ab)^{16}, \tag{11.3}$$

(see also [15, Sect. 1]). Note the asymmetry between b and c in that ab (respectively, ac) has order 16 (respectively, 8).

Let

$$V_n = \{v \in \Gamma : \mathrm{dist}(v, \mathbf{1}) = n\},$$

where dist denotes graph-distance on G. Since G is locally finite, $|V_n| < \infty$. For $\eta \in \mathrm{Stab}_{\mathbf{1}}$, η restricted to V_n is a permutation of V_n. As illustrated in Figure 11.1,

$$V_0 = \{\mathbf{1}\}, \qquad V_1 = \{a, b, c\}, \qquad V_2 = \{ab, ac, ba, bc = cb, ca\}.$$

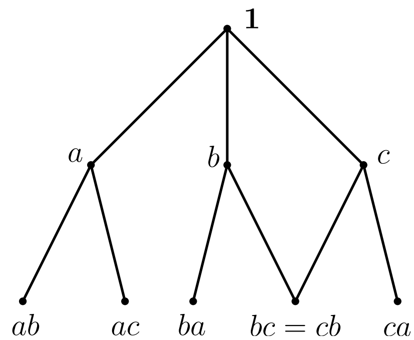

FIGURE 11.1. The subgraph of G on $V_0 \cup V_1 \cup V_2$.

Let $\eta \in \mathrm{Stab}_{\mathbf{1}}$, so that $\eta(a) \in V_1$. Since the shortest cycles using the edges $\langle \mathbf{1}, b\rangle$ and $\langle \mathbf{1}, c\rangle$ have length 4, and using $\langle \mathbf{1}, a\rangle$ greater than 4 (see Figure 11.1), we have that $\eta(a) = a$. By a similar argument, we obtain that, for $n \ge 1$,

$$\eta(va) = \eta(v)a, \qquad v \in V_n,\ va \in V_{n+1}, \tag{11.4}$$

which we express by saying that η maps a-type edges to a-type edges.

We show next that

$$\eta(vc) = \eta(v)c, \qquad v \in V,\ \eta \in \mathrm{Stab}_{\mathbf{1}}, \tag{11.5}$$

which is to say that η maps c-type edges to c-type edges. By (11.4)–(11.5), $\eta \in \mathrm{Stab}_{\mathbf{1}}$ maps b-type edges to b-type edges also, whence $\eta = e$ as required. It remains to prove (11.5).

Assume, in contradiction of (11.5), that there exists $v \in V$, $\eta \in \mathrm{Stab}_{\mathbf{1}}$ such that $\eta(vc) = \eta(v)b$. Since ac has order 8, we have that $(ca)^8 = \mathbf{1}$. Let C be the directed cycle corresponding to the word $v(ca)^8$; thus, C includes the edge $[v, vc\rangle$. Then $\eta(C)$ is a cycle of length 16 including the edge $[\eta(v), \eta(v)b\rangle$. Since C contains exactly 8 a-type edges at alternating positions, by (11.4), so does $\eta(C)$. Therefore, $\eta(C)$ has the form $\eta(v)ba\prod_{i=2}^{8}(x_i a)$, where $x_i \in \{b, c\}$ for $i = 2, 3, \dots, 8$. In particular,

$$ba\prod_{i=2}^{8}(x_i a) = \mathbf{1}, \qquad x_i \in \{b, c\},\ i = 2, 3, \dots, 8. \tag{11.6}$$

The word problem of the Grigorchuk group is solvable (see [13] and [15, Sect. 4]), in that there exists an algorithm to determine whether or not $w = \mathbf{1}$ for any given word $w \in \{a, b, c\}^*$ (where S^* denotes the set of finite ordered sequences of elements of S). By applying this algorithm (see below), we deduce that (11.6) has no solution. Equation (11.5) follows, and the proof of part (a) is complete.

Finally, here is a short amplification of the analysis of (11.6). The word in (11.6) has the form $b(ay_1a)z_1(ay_2a)z_2(ay_3a)z_3(ay_4a)$, where $y_i, z_j \in \{b, c\}$. By (5.1), the effect of such a word on the right sub-tree T_1 is $\gamma_1 := ca(c/d)a(c/d)a(c/d)a$, where each term of the form (y/z) is to be interpreted as 'either y or z'. The effect of γ_1 on the left sub-tree T_{10} of T_1 is $\gamma_{10} := a(d/b)(a/e)(d/b)$. If there is an odd number of appearances of a in γ_{10}, then γ_{10} is not the identity, and thus we may assume $\gamma_{10} := a(d/b)a(d/b)$. It is immediate that none of the four possibilities is the identity, and the claim follows.

Part (b) holds as follows. Suppose there exists $\mathcal{H} \trianglelefteq \Gamma$, $\gamma \in \Gamma$, and a non-constant $\mathcal{H}$-difference-invariant function $F : \gamma\mathcal{H} \to \mathbb{Z}$. It is elementary that $\mathcal{H}$ is unimodular and symmetric (see, for example, [20, Sect. 4]). By [22, Thm 3.5] and the comment near the beginning of [22, Sect. 8], G has a graph height function, in contradiction of part (a).

12. Proof of Theorem 8.1

We shall prove three statements:

(i) Γ has no group height function,

(ii) Π is the cyclic group generated by the permutation $(abcd)$, with the convention that $\eta(x^{-1}) = \eta(x)^{-1}$, for $\eta \in \Pi$, $x \in \{a, b, c, d\}$,

(iii) $\text{Stab}_{\mathbf{1}} = \Pi$.

It is proved in [29] that the Higman group has no proper, finite-index, normal subgroup, and the result follows from the above statements by Corollary 10.2(a).

Proof of (i). The absence of a group height function is immediate by [22, Example 6.3].

Proof of (ii). Evidently, Π contains the given cyclic group, and we turn to the converse. Since elements of Π preserve Γ up to isomorphism,

$$\eta(x^{-1}) = \eta(x)^{-1}, \qquad x \in S. \tag{12.1}$$

We next rule out the possibility that $\eta(x) = y^{-1}$ for some $x, y \in \{a, b, c, d\}$. Suppose, for illustration, that $\eta(a) = b^{-1}$. By (12.1), the relation $a^{-1}ba = b^2$ becomes $b\beta b^{-1} = \beta^2$ where $\beta = \eta(b)$. The Higman group has no such relation with $\beta \in S$. In summary,

$$\eta(x) \in \{a, b, c, d\}, \quad \eta(x^{-1}) = \eta(x)^{-1}, \qquad x \in \{a, b, c, d\}. \tag{12.2}$$

The shortest cycles containing the edge $\langle \mathbf{1}, a\rangle$, modulo rotation and reversal, arise from the relations $ab^2a^{-1}b^{-1} = \mathbf{1}$ and $ada^{-2}d^{-1} = \mathbf{1}$ (see Figure 12.2). The first uses

$a^{\pm 1}$ twice and $b^{\pm 1}$ thrice, and the second uses $a^{\pm 1}$ thrice and $d^{\pm 1}$ twice. Let $\eta \in \Pi$, and suppose for illustration that $\eta(a) = b$ (the same argument is valid for any $\eta(x)$, $x \in \{a, b, c, d\}$). By considering the cycles starting $\langle \mathbf{1}, b\rangle$, $\langle \mathbf{1}, c\rangle$, $\langle \mathbf{1}, d\rangle$, and using (12.2), we deduce that

$$\eta(b) = c, \quad \eta(c) = d, \quad \eta(d) = a,$$

and the claim is proved.

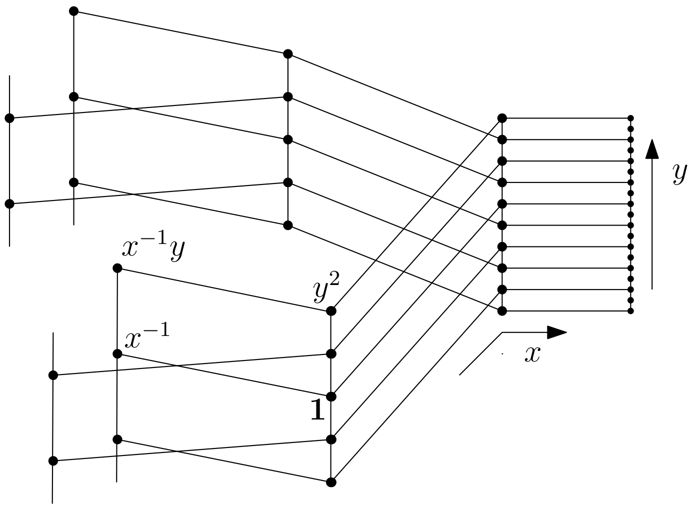

FIGURE 12.1. Part of the Cayley graph of the Baumslag–Solitar group $\mathrm{BS}(x, y)$.

Proof of (iii). We begin with some observations concerning the Baumslag–Solitar (BS) group $\mathrm{BS}(x, y)$ with presentation $\langle x, y, x^{-1}, y^{-1} \mid x^{-1}yx = y^2\rangle$, of which the Cayley graph is sketched in Figure 12.1. Edges of the form $\langle\gamma, \gamma x^{\pm 1}\rangle$ have *type* x, and of the form $\langle\gamma, \gamma y^{\pm 1}\rangle$ *type* y. By inspection, the shortest cycles have length 5 (see Figure 12.2), and, for $\gamma \in \mathrm{BS}(x, y)$,

(12.3) for $p, q = \pm 1$, the edges $\langle\gamma, \gamma x^p\rangle$ and $\langle\gamma, \gamma y^q\rangle$ lie in a common 5-cycle,

(12.4) the third edge of any directed 5-cycle beginning $[\gamma, \gamma x\rangle$ has type y,

(12.5) the third edge of any directed 5-cycle beginning $[\gamma, \gamma x^{-1}\rangle$ has type x,

(12.6) every 5-cycle contains two consecutive edges of type y, and not of type x,

(12.7) a type x (respectively, type y) edge lies in 2 (respectively, 3) 5-cycles.

Returning to the Higman group, for convenience, we relabel the vector (a, b, c, d) as (s_0, s_1, s_2, s_3), with addition and subtraction of indices modulo 4. Let G be the Cayley graph of the Higman group $\Gamma = \langle S \mid R\rangle$, rooted at $\mathbf{1}$. An edge of G is said to be of *type* s_i if it has the form $\langle\gamma, \gamma s_i^{\pm 1}\rangle$ with $\gamma \in \Gamma$. We explain next how to

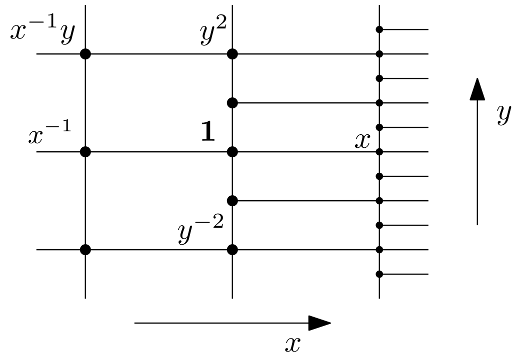

FIGURE 12.2. Part of one 'sheet' of the Cayley graph of $\mathrm{BS}(x, y)$.

obtain information about the types of the edges of G, by examination of G only, and without further information about the vertex-labellings as elements of Γ.

We consider first the set $\partial_e \mathbf{1}$ of edges of G incident to $\mathbf{1}$. Let $e_1 = \langle \mathbf{1}, v\rangle$, $t \in \{0, 1, 2, 3\}$, and $p \in \{-1, 1\}$. Assume that

$$v = s_t^p, \tag{12.8}$$

so that, in particular, e_1 has type s_t. By (12.3), for $j = \pm 1$, e_1 lies in a 5-cycle of $\mathrm{BS}(s_{t-1}, s_t)$ (respectively, $\mathrm{BS}(s_t, s_{t+1})$) containing $\langle \mathbf{1}, s_{t-1}^j\rangle$ (respectively, $\langle \mathbf{1}, s_{t+1}^j\rangle$). On the other hand, by consideration of the relator set R, e_1 lies in no 5-cycle including an edge of type s_{t+2}. Therefore, the edges of the form $\langle \mathbf{1}, s_{t+2}^{\pm 1}\rangle$ may be identified by examination of G, and we denote these as g_1, g_2. There is exactly one further edge of $\partial_e \mathbf{1}$ that lies in no 5-cycle containing either g_1 or g_2, and we denote this edge as e_2. In summary,

$$\{e_1, e_2\} = \big\{\langle \mathbf{1}, s_t^{-1}\rangle, \langle \mathbf{1}, s_t\rangle\big\}, \qquad \{g_1, g_2\} = \big\{\langle \mathbf{1}, s_{t+2}^{-1}\rangle, \langle \mathbf{1}, s_{t+2}\rangle\big\}.$$

Having identified the edges of $\partial_e \mathbf{1}$ with types s_t and s_{t+2}, we move to the other endpoint $v = s_t^p$ of e_1, and apply the same argument. Let e_1, e_1' be the two type-s_t edges incident to v.

We turn next to the remaining four edges of $\partial_e \mathbf{1}$. Let k be such an edge, and consider the property: *k lies in a 5-cycle of G containing both e_1 and e_1'.* By (12.6) and examination of the Cayley graphs of the four groups $\mathrm{BS}(s_i, s_{i+1})$, $0 \le i < 4$, we see that k has this property if it has type $t - 1$, and not if it has type $t + 1$. Thus we may identify the types of the four remaining edges of $\partial_e \mathbf{1}$, which we write as

$$\{f_1, f_2\} = \big\{\langle \mathbf{1}, s_{t+1}^{-1}\rangle, \langle \mathbf{1}, s_{t+1}\rangle\big\}, \qquad \{h_1, h_2\} = \big\{\langle \mathbf{1}, s_{t+3}^{-1}\rangle, \langle \mathbf{1}, s_{t+3}\rangle\big\}.$$

Having determined the types of edges in $\partial_e \mathbf{1}$ (relative to the type t of the initial edge e_1), we move to an endpoint of such an edge other than $\mathbf{1}$, and apply the same argument. By iteration, we deduce the types of all edges of G. Let $T(k)$ denote the type of edge k. It follows from the above that

$$T(k) - T(e_1) \text{ is independent of } t = T(e_1), \tag{12.9}$$

with arithmetic on indices, modulo 4.

We explain next how to identify the value of $p = p(v)$ in (12.8) from the graphical structure of G. Let S_i be the subgraph of G containing all edges with type either s_i or s_{i+1}, so that each component of S_i is isomorphic to the Cayley graph of $\mathrm{BS}(s_i, s_{i+1})$. By (12.4)–(12.5), every directed 5-cycle of $\mathrm{BS}(s_t, s_{t+1})$ starting with the edge $[\mathbf{1}, s_t\rangle$ has third edge with type s_{t+1}, whereas every directed 5-cycle starting with $[\mathbf{1}, s_t^{-1}\rangle$ has third edge with type s_t. We examine S_t to determine which of these two cases holds, and the outcome determines the value of $p = p(v)$.

The above argument is applied to each directed edge $[\gamma, \gamma s_i^{\pm 1}\rangle$ of G, and the power of s_i is thus determined from the graphical structure of G.

Let $\eta \in \mathrm{Stab}_{\mathbf{1}}$. By (12.9), the effect of η is to change the edge-types by

$$T(k) \mapsto T(k) + T(\eta(e_1)) - t.$$

Now, $\eta(v)$ is adjacent to $\mathbf{1}$ and, by the above, once $\eta(v)$ is known, the action of η on the rest of G is determined. Since $\eta \in \mathrm{Aut}(G)$, $\eta(v)$ may be any neighbour w of $\mathbf{1}$ with the property that $p(w) = p(v)$. There are exactly four such neighbours (including v) and we deduce from (12.9) that η lies in the cyclic group generated by the permutation $(s_0 s_1 s_2 s_3)$.

13. Proof of Theorem 8.2

We shall prove three statements:

(i) Γ has no group height function,
(ii) $\mathrm{Stab}_{\mathbf{1}} = \Pi$ where $\Pi = \{e\}$,
(iii) Γ has no proper normal subgroup with finite index.

The result follows from these statements by Corollary 10.2(a), and we turn to their proofs.

Proof of (i). The absence of a group height function is immediate by [22, Thm 4.1(b)].

Proof of (ii). Let $\eta \in \mathrm{Stab}_{\mathbf{1}}$ and $\gamma \in \Gamma$. We consider the action of η on directed edges of G. By inspection of the set R' of relations, an edge of the type $\langle\gamma, \gamma x\rangle$ lies in shortest cycles of length

$$\begin{cases} 5,\ 8 & \text{if } x = a^{\pm 1}, \\ 5,\ 7 & \text{if } x = b^{\pm 1}, \\ 7,\ 8 & \text{if } x = c^{\pm 1}, \\ 9,\ 11 & \text{if } x = d^{\pm 1}. \end{cases}$$

Since the four combinations are distinct, it must be that

$$\eta([\gamma,\gamma x\rangle) = [\gamma',\gamma' x^{\pm 1}\rangle, \qquad \gamma\in\Gamma,\ x\in S, \tag{13.1}$$

where $\gamma'=\eta(\gamma)$. We show next that

$$\eta([\gamma,\gamma x\rangle) \neq [\gamma',\gamma' x^{-1}\rangle, \qquad \gamma\in\Gamma,\ x\in S, \tag{13.2}$$

which combines with (13.1) to imply $\eta=e$ as required.

It suffices to consider the case $x=a$ in (13.2), since a similar proof holds in the other cases. Suppose $\eta([\gamma,\gamma a\rangle) = [\gamma',\gamma' a^{-1}\rangle$, and consider the cycle corresponding to $\gamma ab^{-2}a^{-1}b^{-1}$, that is $(\gamma,\gamma a,\gamma ab^{-1},\gamma ab^{-2},\gamma ab^{-2}a^{-1},\gamma ab^{-2}a^{-1}b^{-1}=\gamma)$. By (13.1), this is mapped under η to the cycle corresponding to $\gamma' a^{-1}b^{\pm 2}a^{\pm 1}b^{\pm 1}$. By examining the relation set R', the only cycles beginning $\gamma' a^{-1}b^{\pm 1}$ with length not exceeding 5 are $\gamma' a^{-1}bab^{-2}$ and $\gamma' a^{-1}b^{-1}ab^2$, in contradiction of the above (since the third step of these two cycles is a rather than the required $b^{\pm 1}$).

Proof of (iii). Suppose $\mathcal{N}$ is a proper normal subgroup of Γ with finite index. The quotient group $\Gamma/\mathcal{N}$ is non-trivial and finite with generators $\overline{s}=s\mathcal{N}$, $s\in S$, satisfying

$$\begin{aligned} \overline{a}^{-1}\overline{b}\overline{a} &= \overline{b}^2, & \overline{b}^{-2}\overline{c}\overline{b}^2 &= \overline{c}^2,\\ \overline{c}^{-3}\overline{d}\overline{c}^3 &= \overline{d}^2, & \overline{d}^{-4}\overline{a}\overline{d}^4 &= \overline{a}^2. \end{aligned} \tag{13.3}$$

Since $\Gamma/\mathcal{N}$ is finite, each $\overline{s}$ has finite order, denoted $\operatorname{ord}(\overline{s})$. It follows from (13.3) that

$$\operatorname{ord}(\overline{s})>1, \qquad s=a,b,c,d. \tag{13.4}$$

To see this, suppose for illustration that $\operatorname{ord}(\overline{c})=1$, so that $\overline{c}=\overline{\mathbf{1}}$. By the third equation of (13.3), $\operatorname{ord}(\overline{d})=1$, so that $\overline{d}=\overline{\mathbf{1}}$, and similarly for $\overline{a}$ and $\overline{b}$, implying that $\Gamma/\mathcal{N}$ is trivial, a contradiction.

By induction, for $n\ge 1$,

$$\begin{aligned} \overline{a}^{-n}\overline{b}\overline{a}^n &= \overline{b}^{2^n}, & \overline{b}^{-2n}\overline{c}\overline{b}^{2n} &= \overline{c}^{2^n},\\ \overline{c}^{-3n}\overline{d}\overline{c}^{3n} &= \overline{d}^{2^n}, & \overline{d}^{-4n}\overline{a}\overline{d}^{4n} &= \overline{a}^{2^n}, \end{aligned}$$

whence, by setting $n=\operatorname{ord}(\overline{a})$, etc,

$$\begin{aligned} \operatorname{ord}(\overline{b}) &\,\big|\, (2^{\operatorname{ord}(\overline{a})}-1), & \operatorname{ord}(\overline{c}) &\,\big|\, (2^{\operatorname{ord}(\overline{b})}-1),\\ \operatorname{ord}(\overline{d}) &\,\big|\, (2^{\operatorname{ord}(\overline{c})}-1), & \operatorname{ord}(\overline{a}) &\,\big|\, (2^{\operatorname{ord}(\overline{d})}-1), \end{aligned} \tag{13.5}$$

where $u\mid v$ means that v is a multiple of u. We shall deduce a contradiction from (13.4) and (13.5). This is done as in [29], of which we reproduce the proof for completeness.

Let p be the least prime factor of the four integers $\mathrm{ord}(\overline{s})$, $s \in \{a,b,c,d\}$. By (13.4), $p > 1$. Suppose that $p \mid \mathrm{ord}(\overline{a})$ (with a similar argument if $p \mid \mathrm{ord}(\overline{s})$ for some other parameter s). Then $p \mid 2^{\mathrm{ord}(\overline{d})} - 1$ by (13.5), and in particular p is odd and therefore coprime with 2. Let r be the multiplicative order of 2 mod p, that is, the least positive integer r such that $p \mid 2^r - 1$. In particular, $r > 1$, so that r has a prime factor q. By Fermat's little theorem, $r \mid p-1$ so that $q < p$. Furthermore, $r \mid \mathrm{ord}(\overline{d})$ so that $q \mid \mathrm{ord}(\overline{d})$, in contradiction of the minimality of p. We deduce that Γ has no proper, normal subgroup with finite index.

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