UNIVERSITY OF CAMBRIDGE

NUMBER THEORY IN DPMMS

Number Theorists in Cambridge

Arithmetic Algebraic Geometry:

- John Coates
- Tom Fisher
- Tony Scholl
- Nick Shepherd-Barron
- Peter Swinnerton-Dyer

Analytic and Additive Number Theory:

- Alan Baker
- Tim Gowers

Research students: 12 in AAG

Postdocs/visitors

Seminars/Study groups for staff and students; student seminars

Recent highlights in Arithmetic Algebraic Geometry:

I Artin's conjecture (Shepherd-Barron + others)

II Special values of L-functions of modular forms (Scholl)

III Non-abelian Iwasawa theory (Coates, Fisher + others)

I: Artin's Conjecture for icosahedral representations

$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{C})$ irreducible, non-trivial representation

Artin's conjecture (1923): $L(\rho, s)$ is entire

$\dim \rho = 1$: classical (Dirichlet L-series)

$\dim \rho = 2$:
– image in $PGL_2(\mathbb{C})$ is

- dihedral (class field theory)
- A_4 or S_4 (Langlands–Tunnell, base-change)
- A_5 (not accessible to base-change: some computational evidence)

Proved in [1,2] for a wide class of *odd* A_5 representations — uses Wiles's method, study of moduli of abelian surfaces and p-adic modular forms.

[1] N. Shepherd-Barron, R. Taylor: *Mod 2 and mod 5 icosahedral representations.* J.A.M.S. 10 (1997) 283-298

[2] K. Buzzard, M. Dickinson, N. Shepherd-Barron, R. Taylor: *On icosahedral Artin representations.* Duke Math. J. 109 (2001), 283-318

II: Special values of L-functions of modular forms

Deep conjectures relating arithmetic phenomena to special values of L-functions: Birch/Swinnerton-Dyer, Beilinson and Bloch/Kato conjectures (modelled on analytic class number formula)

Program: verify as much as possible for L-function of a modular form.

Modular curves (Beilinson, Kato, . . .)

Completed so far:

- modular forms $\Longrightarrow$ "motives"
- their special L-values are (up to rational factors) "higher regulators" of motivic "zeta elements" (generalisations of cyclotomic units)
- zeta elements also related to p-adic L-functions (Euler systems)

[1] *Motives for modular forms.* Invent. math. 100 (1990), 419–430

[2] *Higher regulators and L-functions of modular forms* (book partly written)

III: Non-commutative Iwasawa Theory of Elliptic Curves

Work in progress: Coates with Kato and others.

$E/\mathbb{Q}$ elliptic curve.

Birch/Swinnerton-Dyer Conjecture: links arithmetic of $E/\mathbb{Q}$ and $L(E,s)$.

Classical methods of Iwasawa theory (best results to date by Kato):
— study arithmetic of E over cyclotomic field $F_n = \mathbb{Q}(e^{2\pi i/p^n})$ and $F_\infty = \bigcup F_n$.

Selmer group of E/F_∞ — module over $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F_1)]] \simeq \mathbb{Z}_p[[T]]$ (Iwasawa algebra) $\Longrightarrow$ characteristic power series

Main Conjecture:

- special values $L(E,\chi,1)$ (χ Dirichlet character of p-power conductor)

are interpolated by characteristic power series of Selmer group of E/F_∞.

Nonabelian: Replace F_n by the bigger field $K_n = \mathbb{Q}(E[p^n])$. Typically $\mathrm{Gal}(F_\infty/\mathbb{Q}) = GL_2(\mathbb{Z}_p)$. Iwasawa algebra now non-commutative.

Can now formulate Main Conjecture (p-adic L-function $=$ characteristic element of Selmer group) — numerical evidence! (T & V Dokchitser)

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THE END!