HYPERBOLIC SITE PERCOLATION

GEOFFREY R. GRIMMETT AND ZHONGYANG LI

Abstract. Several results are presented for site percolation on quasi-transitive, planar graphs G with one end, when properly embedded in either the Euclidean or hyperbolic plane. Firstly, if (G_1, G_2) is a matching pair derived from some quasi-transitive mosaic M, then $p_u(G_1) + p_c(G_2) = 1$, where p_c is the critical probability for the existence of an infinite cluster, and p_u is the critical value for the existence of a *unique* such cluster. This fulfils and extends to the hyperbolic plane an observation of Sykes and Essam in 1964. It follows that $p_u(G)+p_c(G_*) = p_u(G_*) + p_c(G) = 1$, where G_* denotes the matching graph of G. Furthermore, when G is amenable we have $p_c \geq \frac{1}{2}$.

A key technique is a method for expressing a planar site percolation process on a matching pair in terms of a dependent bond process on the corresponding dual pair of graphs. Amongst other things, the results reported here answer positively two conjectures of Benjamini and Schramm (Conjectures 7 and 8, Electron. Comm. Probab. 1 (1996) 71–82) in the case of quasi-transitive graphs.

A necessary and sufficient condition is established for strict inequality between the critical probabilities of site percolation on a quasi-transitive, plane graph G, namely, $p_c(G_*) < p_c(G)$. When G is transitive, strict inequality holds if and only if G is not a triangulation, and thus in this case we have $p_u(G) + p_c(G) > 1$. The basic approach is the method of enhancements, subject to complexities arising from the hyperbolic space, the application to site (rather than bond) percolation, and the generality of the assumption of quasi-transitivity.

1. Introduction

1.1. **Percolation on planar graphs.** Percolation was introduced in 1957 by Broadbent and Hammersley [14] as a model for the spread of fluid through a random medium. Percolation provides a natural mathematical setting for such topics as the study of disordered materials, magnetization, and the spread of disease. See [20, 24, 39] for recent accounts of the theory. We consider here site percolation on a

Date: 28 February 2022.

2010 *Mathematics Subject Classification.* 60K35, 82B20.

Key words and phrases. Percolation, site percolation, critical probability, hyperbolic plane, matching graph, matching pair.

graph $G = (V, E)$, assumed to be infinite, locally finite, connected, and planar. The current work has a number of linked objectives.

Objective I. Our major objective is to study the relationship between the percolation critical point p_c and the critical point p_u marking the existence of a *unique* infinite cluster. More specifically, we establish the formula $p_u^{\text{site}}(G_1) + p_c^{\text{site}}(G_2) = 1$ for a matching pair (G_1, G_2) of graphs arising from a quasi-transitive mosaic, appropriately embedded in either the Euclidean or hyperbolic plane. See Section 1.2.

Objective II. Our second objective, which is achieved in the process of proving the above formula, is to validate Conjectures 7 and 8 of Benjamini and Schramm [8] concerning the existence of infinitely many infinite clusters. Details of these conjectures are found in Section 1.3.

Objective III. Setting $(G_1, G_2) = (G, G_*)$ in I above, with G_* the matching graph of G, we obtain

$$p_u^{\text{site}}(G) + p_c^{\text{site}}(G_*) = p_u^{\text{site}}(G_*) + p_c^{\text{site}}(G) = 1.$$

It follows that $p_u^{\text{site}}(G) + p_c^{\text{site}}(G) > 1$ if and only if the strict inequality $p_c^{\text{site}}(G_*) < p_c^{\text{site}}(G)$ holds. Our third objective is a necessary and sufficient condition for the last inequality. When G is transitive, this implies that $p_u^{\text{site}}(G) + p_c^{\text{site}}(G) > 1$ if and only if G is not a triangulation. See Section 1.4.

The organization of the paper is given in Section 1.5.

1.2. **Critical points of matching pairs.** Since loops and multiple edges have no effect on the existence of infinite clusters in site percolation, the graphs considered in this article are generally assumed to be *simple* (whereas their dual graphs may be non-simple). The main results proved in this paper are as follows (see Sections 2.1–2.2 for explanations of the standard notation used here).

The word 'transitive' shall mean 'vertex-transitive' throughout this work. We denote by

- $\mathcal{G}$: all infinite, locally finite, planar, 2-connected, simple graphs,
- $\mathcal{T}$: the subset of $\mathcal{G}$ containing all such transitive graphs,
- $\mathcal{Q}$: the subset of $\mathcal{G}$ containing all such quasi-transitive graphs.

Since the work reported here concerns matching and dual graphs, the graphs in $\mathcal{G}$ will be considered in their plane embeddings. The most interesting such graphs turn out to be those with one end. We shall recall in Section 3.1 that one-ended graphs in $\mathcal{T}$ have unique proper embeddings in the Euclidean/hyperbolic plane up to homeomorphism, and hence their matching and dual graphs are uniquely defined.

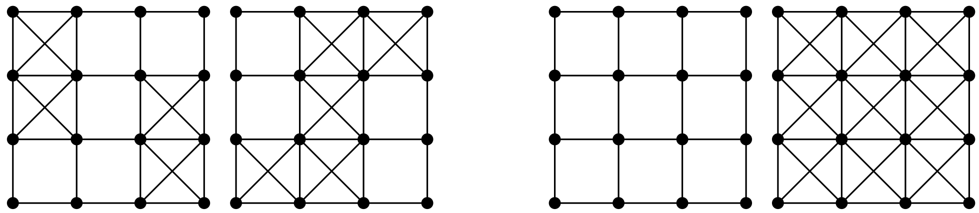

FIGURE 1.1. Two matching pairs derived from the square lattice $\mathbb{Z}^2$. Each 3×3 grid is repeated periodically about $\mathbb{Z}^2$. The pair on the right generates $\mathbb{Z}^2$ and its covering graph.

The situation is more complicated for one-ended graphs in $\mathcal{Q}$, in which case we fix a plane embedding of $G \in \mathcal{Q}$ for which the dual graph G^+ is quasi-transitive. Such an embedding is called *canonical*; if G has connectivity 2, a canonical embedding need not be unique (even up to homeomorphism), but its existence is guaranteed by Theorem 3.1(c).

Matching pairs of graphs were introduced by Sykes and Essam [51] and explored further by Kesten [35]. Let $M \in \mathcal{Q}$ be one-ended and canonically embedded in the plane (we call M a *mosaic* following the earlier literature). Let $\mathcal{F}_4 = \mathcal{F}_4(M)$ be the set of faces of M bounded by n-cycles with $n \geq 4$, and let $\mathcal{F}_4 = F_1 \cup F_2$ be a quasi-transitive partition of $\mathcal{F}_4$. The graph G_i is obtained from M by adding all diagonals to all faces in F_i. The pair (G_1, G_2) is called a *matching pair*. The *matching graph* G_* of a one-ended graph $G \in \mathcal{Q}$ is obtained by adding all diagonals to all faces of G. Thus, (G, G_*) is an instance of a matching pair. Two examples of matching pairs are given in Figure 1.1.

The notation p_{u} denotes the critical value for the existence of a *unique* infinite cluster. Further notation and background for percolation is deferred to Section 2.2.

Theorem 1.1.

(a) *Let (G_1, G_2) be a matching pair derived from the one-ended mosaic $M \in \mathcal{Q}$. We have that*

$$p_{\mathrm{u}}^{\mathrm{site}}(G_1) + p_{\mathrm{c}}^{\mathrm{site}}(G_2) = 1. \tag{1.1}$$

(b) *Let $G \in \mathcal{Q}$ be one-ended. Then*

$$p_{\mathrm{u}}^{\mathrm{site}}(G) + p_{\mathrm{c}}^{\mathrm{site}}(G) \geq 1. \tag{1.2}$$

If G is transitive, equality holds in (1.2) if and only if G is a triangulation.

In the context of (1.1), Sykes and Essam [51, eqn (7.3)] presented motivation for the exact formula

$$p_{\mathrm{c}}^{\mathrm{site}}(G_1) + p_{\mathrm{c}}^{\mathrm{site}}(G_2) = 1, \tag{1.3}$$

and this has been verified in a number of cases when G is amenable (see [10]). This formula does not hold for non-amenable graphs.

Remark 1.2 (Strict inequality). *Equation (1.2) follows from (1.1) with $(G_1, G_2) = (G, G_*)$, by the inequality $p_{\mathrm{c}}^{\mathrm{site}}(G) \geq p_{\mathrm{c}}^{\mathrm{site}}(G_*)$. This weak inequality holds trivially since G is a subgraph of G_*; the corresponding strict inequality $p_{\mathrm{c}}^{\mathrm{site}}(G) > p_{\mathrm{c}}^{\mathrm{site}}(G_*)$ is of a fairly standard type, and is investigated in Section 1.4. A necessary and sufficient condition for strict inequality is presented in Theorem 1.11 for quasi-transitive graphs; see also Theorems 10.1, 10.4. and 10.8. By (1.1),*

$$p_{\mathrm{u}}^{\mathrm{site}}(G) - p_{\mathrm{u}}^{\mathrm{site}}(G_*) = p_{\mathrm{c}}^{\mathrm{site}}(G) - p_{\mathrm{c}}^{\mathrm{site}}(G_*) \geq 0,$$

so that strict inequality for $p_{\mathrm{c}}^{\mathrm{site}}$ is equivalent to strict inequality for $p_{\mathrm{u}}^{\mathrm{site}}$.

Remark 1.3 (Canonical embeddings). *When G has connectivity 2, it may possess more than one canonical embedding; by Theorem 1.1, $p_{\mathrm{c}}^{\mathrm{site}}(G_*)$ and $p_{\mathrm{u}}^{\mathrm{site}}(G_*)$ are independent of the choice of canonical embedding. This may be seen directly.*

Remark 1.4 (Amenability). *If $G \in \mathcal{Q}$ is one-ended and in addition amenable, by the uniqueness of the infinite cluster [2, 15], we have $p_{\mathrm{c}}^{\mathrm{site}}(G) = p_{\mathrm{u}}^{\mathrm{site}}(G)$; in this case, $p_{\mathrm{c}}^{\mathrm{site}}(G) \geq \frac{1}{2}$ by (1.2). If G is transitive, we have $p_{\mathrm{c}}^{\mathrm{site}}(G) = \frac{1}{2}$ if and only if G is the usual amenable, triangular lattice.*

The dual graph of a plane graph G is denoted G^+.

Remark 1.5 (Bond percolation). *Theorem 1.1 may be compared with the corresponding results for bond percolation. It is proved in [9, Thm 3.8] that*

$$p_{\mathrm{c}}^{\mathrm{bond}}(G) + p_{\mathrm{u}}^{\mathrm{bond}}(G^+) = 1$$

for any non-amenable, transitive *$G \in \mathcal{T}$. If, instead, $G \in \mathcal{T}$ is amenable, it is standard that $p_{\mathrm{u}}^{\mathrm{bond}}(G^+) = p_{\mathrm{c}}^{\mathrm{bond}}(G^+) = 1 - p_{\mathrm{c}}^{\mathrm{bond}}(G)$. These facts are extended to* quasi-transitive *graphs in* [39, Thm 8.31].

1.3. **Existence of infinitely many infinite clusters.** A number of problems for percolation on non-amenable graphs were formulated by Benjamini and Schramm in their influential paper [8], including the following two conjectures.

Conjecture 1.6 ([8, Conj. 7]). *Consider site percolation on an infinite, connected, planar graph G with minimal degree at least 7. Then, for any $p \in (p_c^{\text{site}}, 1 - p_c^{\text{site}})$, we have $\mathbb{P}_p(N = \infty) = 1$. Moreover, it is the case that $p_c^{\text{site}} < \frac{1}{2}$, so the above interval is invariably non-empty.*

It was proved in [30, Thm 2] that $p_c^{\text{site}} < \frac{1}{2}$ for planar graphs with vertex-degrees at least 7.

Conjecture 1.7 ([8, Conj. 8]). *Consider site percolation on a planar graph G satisfying $\mathbb{P}_{\frac{1}{2}}(N \geq 1) = 1$. Then $\mathbb{P}_{\frac{1}{2}}(N = \infty) = 1$.*

Percolation in the hyperbolic plane was later studied by Benjamini and Schramm [9]. In the current paper, we extend certain of the results of [9] to amenable planar graphs and to site percolation, and we confirm Conjectures 1.6 and 1.7 for all planar, quasi-transitive graphs.

Conjectures 1.6 and 1.7 were verified in [37] when G is a regular triangular tiling (or 'triangulation') of the hyperbolic plane $\mathcal{H}$ for which each vertex has degree at least 7. A significant property of a triangulation is that its matching graph is the same as the original graph.

The next two theorems establish Conjectures 1.6 and 1.7 for planar, quasi-transitive graphs.

Theorem 1.8. *Consider site percolation on a graph $G \in \mathcal{Q}$, each vertex of which has degree 7 or more.*

(a) *For every $p \in (p_c^{\text{site}}, 1 - p_c^{\text{site}})$, there exist, $\mathbb{P}_p$-a.s., infinitely many infinite 1-clusters and infinitely many infinite 0-clusters.*
(b) *For every $p \in [0, 1]$, there exists, $\mathbb{P}_p$-a.s., at least one infinite cluster that is either a 1-cluster or a 0-cluster.*

Theorem 1.9. *Consider site percolation on a graph $G \in \mathcal{Q}$, and assume that $\mathbb{P}_{\frac{1}{2}}(N \geq 1) = 1$. Then, $\mathbb{P}_{\frac{1}{2}}$-a.s., there exist infinitely many infinite 1-clusters and infinitely many infinite 0-clusters.*

The approach to establishing Conjectures 1.6 and 1.7 is to classify $\mathcal{Q}$ according to amenability and the number of ends, and then prove these conjectures for each such subclass of graphs. We recall the following well-known theorem.

Theorem 1.10 ([32], [5, Prop. 2.1]). *A graph G that is infinite, connected, locally finite, and quasi-transitive has either one or two or infinitely many ends. If it has two ends, then it is amenable. If it has infinitely many ends, then it is non-amenable.*

Let $G \in \mathcal{Q}$. By Theorem 1.10, only the following cases may occur.

(i) G is amenable and one-ended. This case includes the square lattice, for which percolation has been studied extensively; see, for example, [24, 35].
(ii) G is non-amenable and one-ended. It is proved in [9] that $p_c^{\text{site}} < p_u^{\text{site}}$ and $p_c^{\text{bond}} < p_u^{\text{bond}}$ for this case.
(iii) G has two ends, in which case there is no percolation phase transition of interest.
(iv) G has infinitely many ends.

We shall study percolation on each class of graphs listed above. Matching graphs and dual graphs will play important roles in our analysis.

1.4. **Strict inequality for critical points.** Let G be a planar graph with matching graph G_*. Since G is a subgraph of G_*, it is trivial that

$$p_c^{\text{site}}(G_*) \le p_c^{\text{site}}(G). \tag{1.4}$$

The proof of strict inequality in (1.4) for non-triangulations is much more demanding, and indeed this fails to hold for some quasi-transitive graphs.

A path $(\dots, x_{-1}, x_0, x_1, \dots)$ of G_* is called *non-self-touching* if, for all i, j, two vertices x_i and x_j are adjacent if and only if $|i-j| = 1$. Here is the main theorem of the current section.

Theorem 1.11. *Let $G \in \mathcal{Q}$ be one-ended. Then $p_c^{\text{site}}(G_*) < p_c^{\text{site}}(G)$ if and only if G_* contains some doubly-infinite, non-self-touching path that includes some diagonal of G.*

The condition of one-endedness is evidently necessary for the conclusion, since strict inequality fails for a tree. Transitive graphs invariably satisfy the given condition.

Theorem 1.12. *Every one-ended $G \in \mathcal{T}$ has the required property of Theorem 1.11.*

This is restated and proved at Theorem 10.1. The situation is more complicated for quasi-transitive graphs; two sufficient conditions for the required property are given at Theorems 10.4 and 10.8.

The general approach of the proof of Theorem 1.11 is to use the method of enhancements, as introduced and developed in [1] (though there is earlier work of relevance, including [41]). While this approach is fairly standard, and the above result natural, the proof turns out to have substantial complexity arising from the generality of the assumptions on G, and the fact that we are studying site (rather than bond) percolation (see [6]); the proof is, in contrast, fairly immediate for the usual amenable, planar lattices such as the Archimedean tilings.

1.5. **Organization of material.** Section 2 is devoted to basic notation for graphs and percolation. In Section 3, we review certain known results that will be used to prove the main results of Section 1.2. It is explained in Section 4 how a site percolation process on a planar graph may be expressed as a dependent bond process on the dual graph; this allows a connection between site percolation on the matching graph and bond percolation on the dual graph. We prove Theorem 1.1(a) for amenable graphs in Section 5, and for non-amenable graphs in Section 6. Theorem 1.8 is proved in Section 7, and Theorem 1.9 in Section 8.

Turning to strict inequalities between critical points, we explain the application of Theorem 1.11 to transitive and quasi-transitive graphs in Section 10. Two methods are given there, the 'metric method' and the 'combinatorial method'. Each can be used to study transitive graphs. When working with quasi-transitive graphs, they lead to different sufficient (but not necessary) conditions for the required strict inequality. The proofs of these results begin with some preliminary observations in Section 11, and the main theorem is proved in Section 12. The claims of Section 10 for quasi-transitive graphs are proved (respectively) by the metric method in Section 13 and by the combinatorial method in Section 14.

2. Notation

2.1. **Graphical notation.** Let $\mathrm{Aut}(G)$ be the automorphism group of the graph $G = (V, E)$. A graph G is called *vertex-transitive*, or simply *transitive*, if all the vertices lie in the same orbit under the action of $\mathrm{Aut}(G)$. The graph G is called *quasi-transitive* if the action of $\mathrm{Aut}(G)$ on V has only finitely many orbits. It is called *locally finite* if all vertex-degrees are finite. An edge with endpoints u, v is denoted $\langle u, v\rangle$, in which case we call u and v *adjacent* and we write $u \sim v$. The graph-distance $d_G(u, v)$ between vertices u, v is the minimal number of edges in a path from u to v.

A graph G is *planar* if it can be embedded in the plane $\mathbb{R}^2$ in such a way that its edges intersect only at their endpoints; a planar embedding of such G is called a *plane* graph. A *face* of a plane graph G is an (arc-)connected component of the complement $\mathbb{R}^2 \setminus G$. Note that faces are open sets, and may be either bounded or unbounded. With a face F, we associate the set of vertices and edges in its boundary. The *size* of a face is the number of edges in its boundary. While it may be helpful to think of a face as being bounded by a cycle of G, the reality can be more complicated in that faces are not invariably simply connected (if G is disconnected) and their boundaries are not generally self-avoiding cycles or paths (if G is not 2-connected). A plane graph G is called a *triangulation* it every face is bounded by a 3-cycle.

A manifold M is called *plane* if, for every self-avoiding cycle π of M, $M \setminus \pi$ has exactly two connected components. When a graph is drawn in a plane manifold M, the terms embedding and face mean the same as when embedded in the Euclidean plane. We say that an embedded graph $G \subset M$ is *properly embedded* if every compact subset of M contains only finitely many vertices of G and intersects only finitely many edges. Henceforth, all embeddings will be assumed to be proper. The term *plane* shall mean either the Euclidean plane or the hyperbolic plane, and each may be denoted $\mathcal{H}$ when appropriate.

A *cycle* (or *n-cycle*) C of a simple graph $G = (V, E)$ is a sequence $v_0, v_1, \dots, v_{n+1} = v_0$ of vertices v_i such that $n \geq 3$, $e_i := \langle v_i, v_{i+1} \rangle$ satisfies $e_i \in E$ for $i = 0, 1, \dots, n$, and $v_0, v_1, \dots, v_n$ are distinct. Let G be a plane graph, duly embedded properly in $\mathcal{H}$. In this case we write C° for the bounded component of $\mathbb{R}^2 \setminus C$, and $\overline{C}$ for the closure of C°. The 'matching graph' G_* is obtained from G by adding all possible diagonals to every face of G. That is, let F be such a face, and let ∂F be the set of vertices lying in the boundary of F. We augment G by adding edges between any distinct pair $x, y \in V$ such that (i) there exists a face F such that $x, y \in \partial F$ and (ii) $\langle x, y \rangle \notin E$. We write D for the set of diagonals, so that $G_* = (V, E \cup D)$. We recall from [36, Thm 3] (see Remark 3.2(d)) that, for a 2-connected graph G, every face is bounded by either a cycle or a doubly-infinite path, in which case G_* has a simpler form.

Next we define a matching pair. Let $M \in \mathcal{Q}$ be one-ended (we follow the earlier literature by calling M a *mosaic* in this context). By the forthcoming Remark 3.2(d), M has an embedding in the plane such that the dual graph M^+ and the matching graph M_* are quasi-transitive, and furthermore every face of M is bounded by a cycle. Let $\mathcal{F}_4 = \mathcal{F}_4(M)$ be the set of faces of M bounded by n-cycles with $n \geq 4$, and let $\mathcal{F}_4 = F_1 \cup F_2$ be a partition of $\mathcal{F}_4$. The graph G_i is obtained from M by adding all diagonals to all faces in F_i, and we assume that $\mathrm{Aut}(M)$ has some subgroup Γ that acts quasi-transitively on each G_i. The pair (G_1, G_2) is said to be a *matching pair* derived from M.

The graph G is called *amenable* if its Cheeger constant satisfies

$$\inf_{K \subseteq V, |K| < \infty} \frac{|\Delta K|}{|K|} = 0, \tag{2.1}$$

where ΔK is the subset of E containing edges with exactly one endpoint in K. If the left side of (2.1) is strictly positive, the graph G is called *non-amenable*.

Each $G \in \mathcal{T}$ is quasi-isometric with one and only one of the following spaces: $\mathbb{Z}$, the 3-regular tree, the Euclidean plane, and the hyperbolic plane; see [5]. See [19, 33] for background on hyperbolic geometry.

Recall that the number of ends of a connected graph is the supremum over its finite subgraphs F of the number of infinite components that remain after removing F, and recall Theorem 1.10. The number of ends of a graph is highly relevant to properties of statistical mechanical models on the graph; see [25, 38], for example, for discussions of the relevance of the number of ends to the number and speed of self-avoiding walks.

2.2. **Percolation notation.** Let $G = (V, E)$ be a connected, simple graph with bounded vertex-degrees. A *site percolation* configuration on G is an assignment $\omega \in \Omega_V := \{0,1\}^V$ to each vertex of either state 0 or state 1. A cluster in ω is a maximal connected set of vertices in which each vertex has the same state. A cluster may be a 0-cluster or a 1-cluster depending on the common state of its vertices, and it may be finite or infinite. We say that 'percolation occurs' in ω if there exists an infinite 1-cluster in ω.

A *bond percolation* configuration $\omega \in \Omega_E := \{0,1\}^E$ is an assignment to each edge in G of either state 0 or state 1. A bond percolation model may be considered as a site percolation model on the so-called *covering graph* (or *line graph*) $\widetilde{G}$ of G. Therefore, we may use the term 1-cluster (respectively, 0-cluster) for a maximal connected set of edges with state 1 (respectively, state 0) in a bond configuration. The *size* of a cluster in site/bond percolation is the number of its vertices.

We call a vertex or an edge *open* if it has state 1, and *closed* otherwise. Let μ be a probability measure on Ω_V endowed with the product σ-field. The corresponding site model is the probability space (Ω_V, μ), with a similar definition for a bond model (Ω_E, μ). The central questions in percolation theory concern the existence and multiplicity of infinite clusters viewed as functions of μ.

A percolation model (Ω, μ) is called *invariant* if μ is invariant under the action of $\mathrm{Aut}(G)$. An invariant measure is called *ergodic* if there exists an automorphism subgroup Γ acting quasi-transitively on G such that $\mu(A) \in \{0,1\}$ for any Γ-invariant event A. See, for example, [39, Prop. 7.3]. It is standard that the product measure $\mathbb{P}_p$ is ergodic if G is infinite and quasi-transitive.

Site and bond configurations induce *open graphs* in the usual way, and we write N for the number of infinite 1-clusters, and $\overline{N}$ for the number of infinite 0-clusters. For site percolation on a graph G, we write N_*, $\overline{N}_*$ for the corresponding quantities on the matching graph G_*. A configuration is in one–one correspondence with the set of elements (vertices or edges, as appropriate) that are open in the configuration.

Let $p \in [0,1]$. We endow Ω_V with the product measure $\mathbb{P}_p$ with density p. For $v \in V$, let $\theta_v(p)$ be the probability that v lies in an infinite open cluster. It is standard that there exists $p_c^{\mathrm{site}}(G) \in (0,1]$ such that

$$\text{for } v \in V, \qquad \theta_v(p) \begin{cases} = 0 & \text{if } p < p_c^{\mathrm{site}}(G), \\ > 0 & \text{if } p > p_c^{\mathrm{site}}(G), \end{cases}$$

and $p_c^{\mathrm{site}}(G)$ is called the *(site) critical probability* of G.

More generally, consider (either bond or site) percolation on a graph G with probability measure $\mathbb{P}_p$. The corresponding critical points may be expressed as follows.

$$p_c^{\mathrm{site}}(G) := \inf\{p \in [0,1] : \mathbb{P}_p(N \geq 1) = 1 \text{ for site percolation}\},$$
$$p_c^{\mathrm{bond}}(G) := \inf\{p \in [0,1] : \mathbb{P}_p(N \geq 1) = 1 \text{ for bond percolation}\},$$

and

$$p_u^{\mathrm{site}}(G) := \inf\{p \in [0,1] : \mathbb{P}_p(N = 1) = 1 \text{ for site percolation}\},$$
$$p_u^{\mathrm{bond}}(G) := \inf\{p \in [0,1] : \mathbb{P}_p(N = 1) = 1 \text{ for bond percolation}\}.$$

By the Kolmogorov zero–one law, $\mathbb{P}_p(N \geq 1)$ equals either 0 or 1.

The notation p_c (respectively, p_u) shall always mean the critical probability p_c^{site} (respectively, p_u^{site}) of the site model. For background and notation concerning percolation theory, the reader is referred to the book [24] and to Section 12.

3. Background

We review certain known results that will be used in the proofs of our main results.

3.1. **Embeddings of one-ended planar graphs.** We say that the 2-sphere, the Euclidean plane, and the hyperbolic plane constitute the *natural geometries* (see, for example, Babai [5, Sect. 3.1]). The natural geometries are two-dimensional Riemannian manifolds. An *Archimedean tiling* of a two-dimensional Riemannian manifold is a tiling by regular polygons such that the group of isometries of the tiling acts transitively on the vertices of the tiling. An infinite, one-ended, transitive planar graph can be characterized as a tiling of either the Euclidean plane or the hyperbolic plane, and we henceforth denote by $\mathcal{H}$ the plane that is appropriate in a given case.

Theorem 3.1.

(a) [5, Thms 3.1, 4.2] *If $G \in \mathcal{T}$ is one-ended, then G may be embedded in $\mathcal{H}$ as an Archimedean tiling, and all automorphisms of G extend to isometries of $\mathcal{H}$. If $G \in \mathcal{Q}$ is one-ended and 3-connected, then G may be embedded in $\mathcal{H}$ such that all automorphisms of G extend to isometries of $\mathcal{H}$.*

(b) [43, p. 42] *Let G be a 3-connected graph, cellularly embedded in $\mathcal{H}$ such that all faces are of finite size. Then G is uniquely embeddable in the sense that for any two cellular embeddings $\phi_1 : G \to S_1$, $\phi_2 : G \to S_2$ into planar surfaces S_1, S_2, there is a homeomorphism $\tau : S_1 \to S_2$ such that $\phi_2 = \tau\phi_1$.*

(c) [39, Thm 8.25 and proof, pp. 288, 298] *If $G = (V, E) \in \mathcal{Q}$ is one-ended, there exists some embedding of G in $\mathcal{H}$ such that the edges coincide with geodesics, the dual graph G^+ is quasi-transitive, and all automorphisms of G extend to isometries of $\mathcal{H}$. Such an embedding is called* canonical.

(d) [48] *The automorphism group $\mathrm{Aut}(G)$ of a quasi-transitive graph G with quadratic growth contains a subgroup isomorphic to $\mathbb{Z}^2$ that acts quasi-transitively on G.*

Remark 3.2. *Some known facts concerning embeddings follow.*

(a) [13, Props 2.2, 2.2] *All one-ended, transitive, planar graphs are 3-connected, and all proper embeddings of a one-ended, quasi-transitive, planar graph have only finite faces.*

(b) *By Theorem 3.1(b), any one-ended $G \in \mathcal{Q}$ that is in addition transitive has a unique proper cellular embedding in $\mathcal{H}$ up to homeomorphism. Hence, the matching and dual graphs of G are independent of the embedding.*

(c) *The conclusion of B holds for any one-ended, 3-connected $G \in \mathcal{Q}$.*

(d) *For a one-ended, 2-connected $G \in \mathcal{Q}$, we fix a canonical embedding (in the sense of Theorem 3.1(c)). With this given, the dual graph G^+ and the matching graph G_* are quasi-transitive, and furthermore (by* [36, Thm 3]*) the boundary of every face is a cycle of G.*

Remark 3.3 (Proper embedding)**.** *Theorem 3.1(a) implies in particular that every such graph may be properly embedded in its natural geometry. Such an embedding is called* topologically locally finite *(TLF) by Renault* [44, Prop. 5.1], [45]*. For a related discussion in the case of non-amenable graphs, see* [9, Prop. 2.1].

Remark 3.4 (Connectivity)**.** *Graphs with connectivity 1 have been excluded from membership of $\mathcal{G}$ (and therefore from $\mathcal{T}$ and $\mathcal{Q}$ also). Percolation on such graphs has little interest since any finite dangling ends may be removed without changing the existence of an infinite cluster. Moreover, let F be a face of a mosaic M, such that F contains some dangling end D. If (G_1, G_2) is a matching pair derived from M, the critical values $p_c(G_i)$ are unchanged if D is deleted.*

The representation of transitive, planar graphs as tilings of natural geometries enables the development of universal techniques to study statistical mechanical models on all such graphs; see, for example, the study [25] of a universal lower bound for connective constants on infinite, connected, transitive, planar, cubic graphs.

3.2. **Percolation.** We assume throughout this subsection that the graph G is infinite, connected, and locally finite.

Lemma 3.5 ([46, Cor. 1.2], [27])**.** *Let G be quasi-transitive, and consider either site or bond percolation on G. Let $0 < p_1 < p_2 \le 1$, and assume that $\mathbb{P}_{p_1}(N = 1) = 1$. Then $\mathbb{P}_{p_2}(N = 1) = 1$.*

Definition 3.6. *Let $G = (V, E)$ be a graph. Given $\omega \in \Omega_V$ and a vertex $v \in V$, write $\Pi_v\omega = \omega \cup \{v\}$ (which is to say that v is declared open). For $A \subseteq \Omega_V$, we write $\Pi_v A = \{\Pi_v\omega : \omega \in A\}$. A site percolation process (Ω_V, μ) on G is called* insertion-tolerant *if $\mu(\Pi_v A) > 0$ for every $v \in V$ and every event $A \subseteq \Omega_V$ satisfying $\mu(A) > 0$.*

A site percolation is called deletion-tolerant *if $\mu(\Pi_{\neg v}A) > 0$ whenever $v \in V$ and $\mu(A) > 0$, where $\Pi_{\neg v}\omega = \omega \setminus \{v\}$ for $\omega \in \Omega_V$, and $\Pi_{\neg v}A = \{\Pi_{\neg v}\omega : \omega \in A\}$.*

Similar definitions apply to bond percolation. We shall encounter weaker definitions in Section 3.3.

Lemma 3.7 ([39, Thm 7.8], [7, Thm 8.1])**.** *Let $G = (V, E)$ be a connected, locally finite, quasi-transitive graph, and let (Ω, μ) be an invariant (site or bond) percolation on G. Assume either or both of the following two conditions hold:*

(a) *(Ω, μ) is insertion-tolerant,*
(b) *G is a non-amenable planar graph with one end.*

Then $\mu(N \in \{0, 1, \infty\}) = 1$. If μ is ergodic, N is μ-a.s. constant.

The sufficiency of (a) is proved in [39, Thm 7.8] for transitive graphs, and the same proof is valid for quasi-transitive graphs. The sufficiency of (b) is proved in [7, Thm 8.1].

3.3. **Planar duality.** Let $G = (V, E)$ be a plane graph, and write $\mathcal{F}$ for the set of its faces. The dual graph $G^+ = (V^+, E^+)$ is defined as follows. The sets V^+ and $\mathcal{F}$ are in one–one correspondence, written $v_f \leftrightarrow f$. Two vertices $v_f, v_g \in V^+$ are joined by $n_{f,g}$ parallel edges where $n_{f,g}$ is the number of edges of E common to the faces $f, g \in \mathcal{F}$. Thus, E^+ and E are in one–one correspondence, written $e^+ \leftrightarrow e$.

For a bond configuration $\omega \in \Omega_E$, we define the dual configuration $\omega^+ \in \Omega_{E^+}$ by: for each dual pair $(e, e^+) \in E \times E^+$ of edges, we have

$$\omega(e) + \omega^+(e^+) = 1. \tag{3.1}$$

In the following, (Ω_E, μ) is a bond percolation model on $G = (V, E)$. Similar definitions apply to site percolation.

Definition 3.8. *A probability measure μ is called* weakly insertion-tolerant *if there exists a function $f : E \times \Omega_E \to \Omega_E$ such that*

(a) *for all e and all $\omega \in \Omega_E$, we have $\omega \cup \{e\} \subseteq f(e, \omega)$,*
(b) *for all e and all ω, the difference $f(e, \omega) \setminus [\omega \cup \{e\}]$ is finite, and*
(c) *for all e and each event A satisfying $\mu(A) > 0$, the image of A under $f(e, \cdot)$ is an event of strictly positive probability.*

Definition 3.9. *A probability measure μ is called* weakly deletion-tolerant *if there exists a function $h : E \times \Omega_E \to \Omega_E$ such that*

(a) *for all e and all $\omega \in \Omega_E$, we have $\omega \setminus \{e\} \supseteq h(e, \omega)$,*
(b) *for all e and all ω, the difference $[\omega \setminus \{e\}] \setminus h(e, \omega)$ is finite, and*
(c) *for all e and each event A satisfying $\mu(A) > 0$, the image of A under $h(e, \cdot)$ is an event of strictly positive probability.*

Lemma 3.10 ([39, Thm 8.30])**.** *Let $G = (V, E) \in \mathcal{Q}$ be non-amenable and one-ended, and consider G embedded canonically in the plane (such an embedding exists by Theorem 3.1(c)). Let (Ω_E, μ) be an invariant, ergodic, bond percolation on G, assumed to be both weakly insertion-tolerant and weakly deletion-tolerant. Let N be the number of infinite open components, and N^+ the number of infinite open components of the dual process. Then*

$$\mu\big((N, N^+) \in \{(0, 1), (1, 0), (\infty, \infty)\}\big) = 1.$$

3.4. **Graphs with two or more ends.** We summarise here the main results for critical percolation probabilities on multiply-ended graphs.

Theorem 3.11 ([28, 47])**.** *Let $G \in \mathcal{Q}$ have two ends. The critical percolation probabilities satisfy*

$$p_{\mathrm{c}}^{\mathrm{bond}}(G) = p_{\mathrm{c}}^{\mathrm{site}}(G) = p_{\mathrm{u}}^{\mathrm{bond}}(G) = p_{\mathrm{u}}^{\mathrm{site}}(G) = 1.$$

Theorem 3.12. *Let $G \in \mathcal{Q}$ have infinitely many ends. Then*

$$p_{\mathrm{c}}^{\mathrm{bond}}(G) \le p_{\mathrm{c}}^{\mathrm{site}}(G) < p_{\mathrm{u}}^{\mathrm{bond}}(G) = p_{\mathrm{u}}^{\mathrm{site}}(G) = 1.$$

The standard inequality $p_c^{\mathrm{bond}} \le p_c^{\mathrm{site}}$ holds for all graphs, and was stated in [29]. The corresponding strict inequality was explored in [26, Thm 2] for bridgeless, quasi-transitive graphs. The equalities $p_u^{\mathrm{bond}} = p_u^{\mathrm{site}} = 1$ were proved for transitive graphs in [47, eqn (3.7)] (see also [28]), and feature in [39, Exer. 7.9] for quasi-transitive graphs. The inequality $p_c^{\mathrm{site}} < 1$ for non-amenable graphs was given in [8, Thm 2].

3.5. **FKG inequality.** For completeness, we state the well-known FKG inequality. See, for example, [24, Sect. 2.2] for further details.

Theorem 3.13 (FKG inequality, [23, 31])**.** *Let μ be a strictly positive probability measure on Ω_V satisfying the* FKG lattice condition*:*

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \ge \mu(\omega_1)\mu(\omega_2), \qquad \omega_1, \omega_2 \in \{0,1\}^V. \tag{3.2}$$

For any increasing events $A, B \subseteq \{0,1\}^V$, we have that $\mu(A \cap B) \ge \mu(A)\mu(B)$.

4. Planar site percolation as a bond model

Let $M = (V, E) \in \mathcal{Q}$ be one-ended, and let (G_1, G_2) be a matching pair derived from M according to the partition $\mathcal{F}_4(M) = F_1 \cup F_2$. If $F_i \ne \varnothing$, then G_i is non-planar. This is an impediment to consideration of the dual graph of G_i, which in turn is overcome by the introduction of so-called facial sites.

Let $\mathcal{F} = \mathcal{F}(M)$ be the set of faces of M (following [35], we include triangular faces). The triangular faces of $\mathcal{F}$ do not appear in $F_1 \cup F_2 = \mathcal{F}_4$, but we allocate each such face arbitrarily to either F_1 of F_2 (for concreteness, we may add them all to F_1). One may replace the mosaic M by the triangulation $\widehat{M}$ obtained by placing a *facial site* $\phi(F)$ inside each face $F \in \mathcal{F}$, and joining $\phi(F)$ to each vertex in the boundary of F. (See [35, Sec. 2.3] and Section 11.2 of the current work.)

When considering site percolation on M (respectively, M_*), one declares the facial sites of $\widehat{M}$ to be invariably closed (respectively, open). Site percolation on G_i is equivalent to site percolation on $\widehat{M}$ subject to:

$$\text{a facial site } \phi(F) \text{ is declared open if } F \in F_i \text{ and closed if } F \in \mathcal{F} \setminus F_i. \tag{4.1}$$

Note that, if F is a triangular face, the state of $\phi(F)$ is independent of the connectivity of other vertices.

Let $\widehat{G}_i$ be obtained by adding to M the facial sites of F_i only, together with their incident edges. We write $\widehat{G}_i = (V \cup \Phi_i, E \cup \eta_i)$ where Φ_i is the set of facial sites of G_i and η_i is the set of edges incident to facial sites. We shall consider two site percolation processes, namely, percolation of open sites on $\widehat{G}_1$ and of closed sites on $\widehat{G}_2$. To this end, for $\omega \in \Omega_V$, let ω_1 (respectively, ω_2) be the site configuration on $\widehat{G}_1$

(respectively, $\widehat{G}_2$) that agrees with ω on V and is *open* on Φ_1 (respectively, *closed* on Φ_2).

Given $\omega \in \Omega_V$, we construct a bond configuration $\beta_{\omega_1} \in \Omega_{E\cup\eta_1}$ by

$$\beta_{\omega_1}(e) = \begin{cases} 1 & \text{if } \omega_1(u) = \omega_1(v) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

where $e = \langle u, v\rangle \in E\cup\eta_1$. Let $\beta^+_{\omega_1} := 1-\beta_{\omega_1}$ be the corresponding dual configuration on the dual graph $\widehat{G}^+_1 = (V^+_1, E^+_1)$ of $\widehat{G}_1$ as in (3.1), and let $\widehat{G}^+_1(\beta^+_{\omega_1})$ be the graph with vertex-set V^+_1 endowed with the open edges of $\beta^+_{\omega_1}$. Note that, if ω has law $\mathbb{P}_p$, then the law of β_{ω_1} is one-dependent. We may identify the vector β_{ω_1} with the set of its open edges.

Lemma 4.1. *Suppose $\omega \in \Omega_V$ has law $\mathbb{P}_p$ where $p \in (0, 1)$. The law μ of β_{ω_1} is weakly deletion-tolerant and weakly insertion-tolerant. Moreover, μ is ergodic.*

Proof. Let $e = \langle u, v\rangle \in E \cup \eta_1$ and $\omega \in \Omega_V$. For $w \in V$, let D_w be the set of edges of $\widehat{G}_1$ of the form $\langle w, x\rangle$ with $\omega(x) = 1$. Select an endvertex, u say, of e that is not a facial site (such a vertex always exists), and define

$$f(e, \beta_{\omega_1}) = \beta_{\omega_1} \cup (D_u \cup D_v \cup \{e\}), \qquad h(e, \beta_{\omega_1}) = \beta_{\omega_1} \setminus (D_u \cup \{e\}).$$

The edge-configuration $f(e, \beta_{\omega_1})$ (respectively, $h(e, \beta_{\omega_1})$) is that obtained by setting u and v to be open (respectively, u to be closed). With these functions f, h, the conditions of Definitions 3.8 and 3.9 hold since G is locally finite. The ergodicity holds by the assumed quasi-transitivity of G_1 and the fact that $\mathbb{P}_p$ is a product measure (see the comment in Section 2.2). $\square$

For $\omega \in \Omega_V$, let $\widehat{G}_1(\omega)$ be the subgraph of $\widehat{G}_1$ induced by the set of ω_1-open vertices (that is, the set of v with $\omega_1(v) = 1$), and define $\widehat{G}_2(\overline{\omega})$ similarly in terms of closed vertices of ω_2 in $\widehat{G}_2$.

We make some notes concerning the relationship between $\widehat{G}_1(\omega)$, $\widehat{G}_2(\overline{\omega})$, and $\widehat{G}^+_1(\beta^+_{\omega_1})$, as illustrated in Figure 4.1. A *cutset* of a graph H is a subset of edges whose removal disconnects some previously connected component of H, and which is minimal with this property. Recall that a *face* of a plane graph H is a connected component of $\mathcal{H} \setminus F$. A face F can be bounded or unbounded, and it need not be simply connected. It has a boundary ΔF comprising edges of H; even when F is bounded and simply connected, the set ΔF of edges need not be cycle of H unless H is 2-connected.

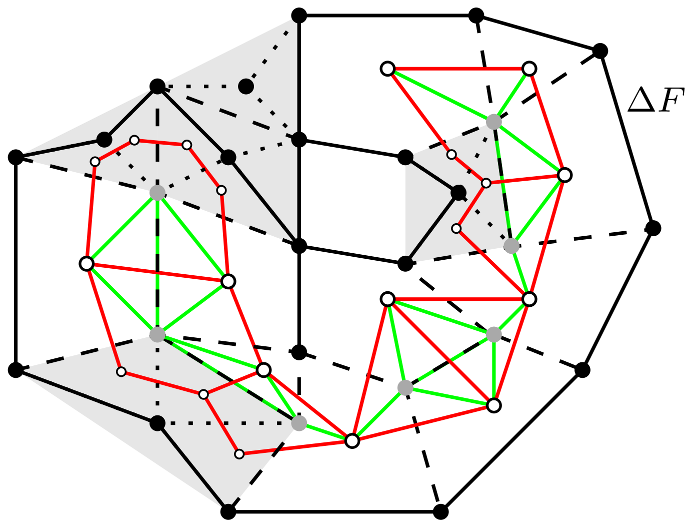

FIGURE 4.1. An illustration of the one–one correspondence between $C_1(F)$ and $C_2(F)$ of Proposition 4.2. The black line is the boundary of the face F; the dashed lines are edges of M inside F; the dotted lines are edges of η_1. The shaded regions are faces of M that belong to F_1; the black points are open vertices; the grey points are closed vertices; the open points are dual vertices of $\widehat{G}_1$. The green graph is the 0-cluster $C_2(F)$ of $\widehat{G}_2(\overline{\omega})$ that corresponds to the red cluster $C_1(F)$ of $\widehat{G}_1^+(\beta^+_{\omega_1})$.

Proposition 4.2. *Let $M = (V, E) \in \mathcal{Q}$ be one-ended and embedded canonically in $\mathcal{H}$. Let $\omega \in \Omega_V$, and let F be a face (either bounded or unbounded) of $\widehat{G}_1(\omega)$.*

(a) *Let C be a cycle (respectively, doubly-infinite path) of $\widehat{G}_1(\omega)$. The set of edges of $\widehat{G}_1^+$ intersecting C forms a finite (respectively, infinite) cutset of $\widehat{G}_1^+$.*
(b) *The set $F \cap V_1^+$ of dual vertices of $\widehat{G}_1$ inside F, together with the set of open edges of $\beta^+_{\omega_1}$ lying inside F, forms a non-empty, connected component $C_1(F)$ of $\widehat{G}_1^+(\beta^+_{\omega_1})$.*
(c) *The set $F \cap (V \cup \Phi_2)$ of vertices of $\widehat{G}_2$ inside F forms a (possibly empty) 0-cluster $C_2(F)$ of $\widehat{G}_2(\overline{\omega})$.*
(d) *Either each of F, $C_1(F)$, $C_2(F)$ is bounded or each is unbounded.*

Proof. (a) This is immediate by the definition (4.2) of β_ω.

(b) Note first that every vertex w of M inside F satisfies $\omega(w) = 0$. Since F is a face of $G(\omega)$, it is a non-empty, disjoint union $F = \bigcup_{i \in I} A_i$ of faces A_i of $\widehat{G}_1$ (more precisely, the two sides of the equality differ on a set of Lebesgue measure 0). Since $\widehat{G}_1$ is one-ended, each A_i is bounded, and therefore contains a unique dual vertex d_i. It is standard that the dual set $D = \{d_i : i \in I\}$ induces a connected graph $C_1(F)$ in F. Since no edge f of $C_1(F)$ intersects ΔF, we have $\beta^+_\omega(f) = 1$ for all such f.

(c) It can be the case that $F \cap (V \cup \Phi_2) = \varnothing$, in which case we take $C_2(F)$ to be the empty graph (this is the situation when F is bounded by a 3-cycle of M). Suppose henceforth that $F \cap (V \cup \Phi_2) \neq \varnothing$ and note as above that $\omega(w) = 0$ for every $w \in F \cap (V \cup \Phi_2)$. It is a standard property of matching pairs of graphs that $F \cap (V \cup \Phi_1)$ induces a connected subgraph $C_2(F)$ of $F \cap \widehat{G}_2$.

Parts (b) and (c) make use of two so-called 'standard' properties, full discussions of which are omitted here. It suffices to prove the 'standard' property of *matching* pairs, since the corresponding property for dual pairs then follows by passing to covering (or line) graphs (see, for example, [35, Sec. 2.6]). For matching pairs, an early reference is [51, App.], and a more detailed account is found in [35, Sec. 3, App.] (see, in particular, Proposition A.1 of [35]). The latter assumes slightly more than here on the mosaic M, but the methods apply notwithstanding.

(d) When F is finite, so must be $C_1(F)$ and $C_2(F)$, since the embedding of G is proper. When F is infinite, the same holds of $C_1(F)$ and $C_2(F)$, since the faces of G are uniformly bounded. □

For a graph H, let $N(H)$ be the number of its infinite components.

Proposition 4.3. *Let $M = (V, E) \in \mathcal{Q}$ be one-ended and embedded canonically in $\mathcal{H}$, and let $\omega \in \Omega_V$. Then,*

$$N(\widehat{G}_1(\omega)) = N(\widehat{G}_1(\beta_{\omega_1})), \qquad N(\widehat{G}_2(\overline{\omega})) = N(\widehat{G}_1^+(\beta^+_{\omega_1})), \tag{4.3}$$

and hence

$$N(G_1(\omega)) = N(G_1(\beta_\omega)), \qquad N(G_2(\overline{\omega})) = N(G_1^+(\beta^+_\omega)). \tag{4.4}$$

Proof. Equation (4.3) holds by the definition of β_ω, and from Proposition 4.2 on noting (for given ω) the one–one correspondence between infinite clusters of $\widehat{G}_2(\overline{\omega})$ and of $\widehat{G}_1^+(\beta^+_\omega)$. Equation (4.4) holds since the facial site in any face F is a surrogate for the diagonals of F. □

Remark 4.4 (Conformality)**.** *It was proved by Smirnov* [50] *that critical site percolation on the triangular lattice $\mathbb{T}$ satisfies Cardy's formula, and moreover has properties of conformal invariance (see also* [17, 18]*). By the above construction, the dependent bond process β_ω on $\mathbb{T}$ has similar properties, and also its dual process on the hexagonal lattice.*

5. Amenable planar graphs with one end

In this section, we prove Theorem 1.1(a) for amenable, one-ended graphs; see Remark 1.2 for an explanation of part (b) of the theorem. It is standard that such graphs are properly embeddable in the Euclidean plane, denoted $\mathcal{H}$ in this section.

Recall first that, for any infinite, quasi-transitive, amenable graph G, and invariant, insertion-tolerant measure μ, the number N of infinite percolation clusters satisfies $\mu(N \le 1) = 1$ (see [39, Thm 7.9] for the transitive case, the quasi-transitive case is similar).

Lemma 5.1. *Let $M = (V, E) \in \mathcal{Q}$ be amenable, one-ended, and embedded canonically in $\mathcal{H}$, and let (G_1, G_2) be a matching pair derived from M. Let (Ω_V, μ) be an ergodic, insertion-tolerant site percolation on M satisfying the FKG lattice condition (3.2). Then*

$$\mu\big((N, \overline{N}) = (1,1)\big) = \mu\big((N(G_1), \overline{N}(G_2)) = (1,1)\big) = 0, \tag{5.1}$$

where $N = N(M)$ and $\overline{N} = \overline{N}(M)$.

A pair γ, γ' of isometries of $\mathbb{R}^2$ is said to act in a *doubly periodic manner* on G (in its canonical embedding) if they generate a subgroup of $\mathrm{Aut}(G)$ that is isomorphic to $\mathbb{Z}^2$, and the embedding is called *doubly periodic* if such a pair exists. In preparation for the proof of Lemma 5.1, we note the following.

Theorem 5.2. *Let $G \in \mathcal{Q}$ be amenable and one-ended. A canonical embedding of G in $\mathbb{R}^2$ is doubly periodic.*

Proof. This may be proved in a number of ways, including using either Bieberbach's theorem on crystalline groups [11, 52] or Selberg's lemma [3]. Instead, we use a more direct route via the main theorem of Seifter and Trofimov [48] (see Theorem 3.1(d)).

Viewed as a graph, G has quadratic growth. This standard fact holds as follows. By [5, Thm 1.1], G has either linear, or quadratic, or exponential growth. As noted at [16, Thm 9.3(b)], being one-ended, it cannot have linear growth. Finally, we rule out exponential growth. Since G is quasi-transitive, there exists $R < \infty$ such that, for all edges $\langle x, y\rangle$ of G, the distance between x and y in $\mathbb{R}^2$ is no greater than R. Therefore, the n-ball centred at vertex v is contained in $B_n(v) := v + [-nR, nR]^2$. By quasi-transitivity again, there exists $A < \infty$ such that, for all v, $B_n(v)$ contains no more than $A(nR)^2$ vertices.

The theorem of [48] may now be applied to find that $\mathrm{Aut}(G)$ has a finite-index subgroup F isomorphic to $\mathbb{Z}^2$. Thus F is generated by a pair of automorphisms which, by Theorem 3.1(c), extend to isometries of the embedding of G. $\square$

Proof of Lemma 5.1. By insertion-tolerance and ergodicity, the random variables N, $\overline{N}$, $N(G_1)$, $\overline{N}(G_2)$ are each μ-a.s. constant and take values in $\{0,1\}$. By Theorem 5.2 and [22, Thm 1.5],

$$\mu\big((N(G_1), \overline{N}(G_2)) = (1,1)\big) = 0. \tag{5.2}$$

Arguments related to but weaker than [22, Thm 1.5] are found in [12, 24, 40, 42, 49]. Note that [22, Thm 1.5] deals with bond percolation, whereas (5.2) is concerned with site percolation. This difference may be handled either by adapting the arguments of [22] to site models, or by applying [22, Thm 1.5] to the one-dependent bond model β_ω constructed in the manner described in Section 4 (see Proposition 4.3). The remaining part of (5.1) follows from the fact that $\overline{N}(G_2) = 1$ μ-a.s. on the event $\{\overline{N} = 1\}$. $\square$

Corollary 5.3. *Let $G \in \mathcal{Q}$ be amenable and one-ended, and consider site percolation on G. Then $\mathbb{P}_{\frac{1}{2}}(N = 0) = 1$.*

Proof. Suppose that $\mathbb{P}_{\frac{1}{2}}(N \geq 1) > 0$, so that $\mathbb{P}_{\frac{1}{2}}(N \geq 1) = 1$ by ergodicity. By amenability and symmetry, we have that $\mathbb{P}_{\frac{1}{2}}(N = \overline{N} = 1) = 1$. This contradicts Lemma 5.1. $\square$

Lemma 5.4. *Let $M = (V, E) \in \mathcal{Q}$ be amenable, one-ended, and embedded canonically in $\mathcal{H}$, and let (G_1, G_2) be a matching pair derived from M. We have for site percolation that $\mathbb{P}_p(\overline{N}(G_2) = 1) = 1$ for $p < p_c^{\text{site}}(G_1)$.*

Proof. Let $p \in (0, p_c^{\text{site}}(G_1))$ be such that $\mathbb{P}_p(\overline{N}(G_2) = 1) < 1$. By amenability and ergodicity, we have that $\mathbb{P}_p(\overline{N}(G_2) = 0) = 1$. Therefore, $\mathbb{P}_p(N(G_1) = \overline{N}(G_2) = 0) = 1$. There is a standard geometrical argument based on exponential decay that leads to a contradiction, as follows.

Fix a vertex v_0 of $M = (V, E)$. Let $n \geq 1$, and let $\Lambda_n = \{u \in V : d_M(u, v_0) \leq n\}$. By [35, Prop. 2.1], if $\partial\Lambda_n \not\leftrightarrow \infty$ in G_2, there exists a closed circuit C of G_1 with Λ_n in its inside. As in [4, Thm 3] for example, there exist $A, a > 0$ such that

$$1 = \mathbb{P}_p(\partial\Lambda_n \not\leftrightarrow \infty \text{ in } G_2) \leq \sum_{k \geq n} A e^{-a(n+k)}.$$

This cannot hold for large n, and the lemma is proved. $\square$

We turn to equation (1.1). In this amenable case, this is equivalent to the following extension of classical results of Sykes and Essam [51] and van den Berg [10].

Theorem 5.5. *Let $M = (V, E) \in \mathcal{Q}$ be amenable, one-ended, and embedded canonically in $\mathcal{H}$, and let (G_1, G_2) be a matching pair derived from M. Then*

$$p_{\mathrm{c}}^{\mathrm{site}}(G_1) + p_{\mathrm{c}}^{\mathrm{site}}(G_2) = 1.$$

Proof. By Lemma 5.1, whenever $p > p_{\mathrm{c}}^{\mathrm{site}}(G_1)$, we have $1 - p \le p_{\mathrm{c}}^{\mathrm{site}}(G_2)$, which implies $p_{\mathrm{c}}^{\mathrm{site}}(G_1) + p_{\mathrm{c}}^{\mathrm{site}}(G_2) \ge 1$. By Lemma 5.4, whenever $p < p_{\mathrm{c}}^{\mathrm{site}}(G_1)$, we have $1 - p \ge p_{\mathrm{c}}^{\mathrm{site}}(G_2)$, which implies $p_{\mathrm{c}}^{\mathrm{site}}(G_1) + p_{\mathrm{c}}^{\mathrm{site}}(G_2) \le 1$. □

6. Non-amenable graphs with one end

In this section, we prove Theorem 1.1(a) for non-amenable, one-ended graphs $G = (V, E) \in \mathcal{Q}$; see Remark 1.2 for an explanation of part (b) of the theorem.

Lemma 6.1. *Let $M \in \mathcal{Q}$ be one-ended and embedded canonically in the hyperbolic plane, and let (G_1, G_2) be a matching pair derived from M. Then*

$$\mathbb{P}_p\big((N(G_1), \overline{N}(G_2)) \in \{(0,1), (1,0), (\infty,\infty)\}\big) = 1.$$

Proof. We fix a canonical embedding of G. By Proposition 4.3,

$$N(G_1(\beta_\omega)) = N(G_1(\omega)), \qquad N(G_2^+(\beta_\omega^+)) = N(G_2(\overline{\omega})).$$

By Lemma 4.1, the law of β_ω is weakly deletion-tolerant, weakly-insertion tolerant, and ergodic, and the claim follows by Lemma 3.10. □

Proof of Theorem 1.1(a). By Lemmas 3.5 and 3.7, we have the following for site percolation on either G_1 or G_2:

$$\begin{aligned} \text{if } p < p_{\mathrm{c}}, &\quad \mathbb{P}_p(N = 0) = 0, \\ \text{if } p_{\mathrm{c}} < p < p_{\mathrm{u}}, &\quad \mathbb{P}_p(N = \infty) = 1, \\ \text{if } p > p_{\mathrm{u}}, &\quad \mathbb{P}_p(N = 1) = 1, \end{aligned}$$

where p_{c}, p_{u} are the critical values appropriate to the graph in question.

By Lemma 6.1, $N(G_1) = 0$ if and only if $\overline{N}(G_2) = 1$, whence $p_{\mathrm{c}}(G) = 1 - p_{\mathrm{u}}(G_*)$. Moreover, $N(G_1) = 1$ if and only if $\overline{N}(G_2) = 0$, whence $p_{\mathrm{u}}(G_1) = 1 - p_{\mathrm{c}}(G_2)$. □

Corollary 6.2. *Let $G \in \mathcal{Q}$ be one-ended and embedded canonically in $\mathcal{H}$, and suppose G is non-amenable. Then*

$$\mathbb{P}_p\big((N, \overline{N}) \in \{(0,0), (0,1), (1,0), (0,\infty), (\infty,0), (\infty,\infty)\}\big) = 1.$$

Proof. We need to eliminate the vectors $(1,1)$, $(1,\infty)$, and $(\infty,1)$. First, by Lemma 3.7, $\mathbb{P}_p$-a.s. the pair $(N, \overline{N})$ takes some given value in the set $\{0, 1, \infty\}^2$. If $\mathbb{P}_p((N, \overline{N}) =$

$(1,1)) > 0$, then $\mathbb{P}_p(N = 1, \overline{N}_* \geq 1) > 0$, in contradiction of Lemma 6.1 applied to the matching pair (G, G_*). Hence, $\mathbb{P}_p((N, \overline{N}) \neq (1,1)) = 1$.

If $\mathbb{P}_p((N, \overline{N}) = (1, \infty)) > 0$, there is strictly positive probability of an infinite component in $G_*(\overline{\omega})$, in contradiction of Lemma 6.1. By symmetry, $\mathbb{P}_p((N, \overline{N}) \neq (\infty, 1)) = 1$, and the corollary follows. □

7. Proof of Theorem 1.8

Let G be a graph satisfying the assumptions of the theorem. We work with the largest finite connected subgraph G_B of G contained in a large box B (with boundary ∂B) of the natural geometry of G, and shall let B expand to fill the space. The numbers of finite faces, vertices, edges of G_B satisfy Euler's formula: $f_B + v_B = e_B + 1$. Since the smallest possible face is a triangle, we have $f_B \leq \frac{2}{3} e_B$; since the degree of interior vertices is 7 or more, there exists $c > 0$ such that $e_B \geq \frac{7}{2}(v_B - c|\partial B|)$. This contradicts Euler's formula unless $e_B/|\partial B|$ is bounded above, which is to say that the natural geometry is the hyperbolic plane. Hence, G is non-amenable. By [30, Thm 2], we have $p_{\mathrm{c}}^{\mathrm{site}} = p_{\mathrm{c}}^{\mathrm{site}}(G) < \frac{1}{2}$.

By the symmetry of the interval $(p_{\mathrm{c}}^{\mathrm{site}}, 1 - p_{\mathrm{c}}^{\mathrm{site}})$ around $\frac{1}{2}$, it suffices to show that $\mathbb{P}_p(N = \infty) = 1$ for $p \in (p_{\mathrm{c}}^{\mathrm{site}}, 1 - p_{\mathrm{c}}^{\mathrm{site}})$. This in turn is implied by Lemma 3.5 and the inequality

$$1 - p_{\mathrm{c}}^{\mathrm{site}} \leq p_{\mathrm{u}}^{\mathrm{site}}. \tag{7.1}$$

Inequality (7.1) holds by (1.2) with G non-amenable and one-ended. In the remaining case when G has infinitely many ends, (7.1) is trivial since $p_{\mathrm{u}}^{\mathrm{site}} = 1$ by Theorem 3.12.

8. Proof of Theorem 1.9

Let G be a graph satisfying the assumptions of the theorem, and embedded canonically. By Lemma 3.7, symmetry, and the assumption $\mathbb{P}_{\frac{1}{2}}(N \geq 1) = 1$,

$$\mathbb{P}_{\frac{1}{2}}\big((N, \overline{N}) \in \{(1,1), (\infty, \infty)\}\big) = 1. \tag{8.1}$$

By Theorem 1.10, the following four cases may occur:

(a) G is amenable and one-ended. By Lemma 5.1, $\mathbb{P}_{\frac{1}{2}}(N = 0) = 1$. Hence, in this case, the hypothesis of the theorem is invalid.

(b) G is non-amenable and one-ended. By Corollary 6.2 and (8.1), subject to the percolation assumption, we have $\mathbb{P}_{\frac{1}{2}}(N = \overline{N} = \infty) = 1$.

(c) G has two ends. By Theorem 3.11, $p_{\mathrm{c}}^{\mathrm{site}} = 1$. Hence $\mathbb{P}_{\frac{1}{2}}(N = 0) = 1$, and the hypothesis is invalid.

(d) G has infinitely many ends. By Theorem 3.12, $p_{\mathrm{u}}^{\mathrm{site}} = 1$. Under the hypothesis of the theorem, it follows by symmetry that $\mathbb{P}_{\frac{1}{2}}((N, \overline{N}) = (\infty, \infty)) = 1$.

9. Strict inequality: further notation

Recall the matching graph $G_* = (V, E_*)$ of a planar graph $G = (V, E)$; see Section 2.1. An edge $e \in E_* \setminus E$ is called a *diagonal* of G or of G_*, and it is denoted $\delta(a, b)$ where a, b are its endvertices. If $\delta(a, b)$ is a diagonal, a and b are called $*$*-neighbours*. Note that G_* depends on the particular embedding of G. If G is 3-connected then, by Theorem 3.1(b), it has a unique embedding up to homeomorphism. If G is 2-connected but not 3-connected, we need to be definite about the choice of embedding, and we require it henceforth to be 'canonical' in the sense of Theorem 3.1(c).

The orbit of $v \in V$ is written $\mathrm{Aut}(G)v$, and we let

$$\Delta = \min\bigl\{k : \text{for } v, w \in V, \text{ we have } d_G(\mathrm{Aut}(G)v, \mathrm{Aut}(G)w) \le k\bigr\}, \tag{9.1}$$

where

$$d_G(A, B) = \min\{d_G(a, b) : a \in A,\, b \in B\}, \qquad A, B \subseteq V,$$

and d_G denotes graph-distance in G. For any G, we fix some vertex denoted v_0.

We shall work with one-ended graphs $G \in \mathcal{Q}$. Since G is assumed one-ended and 2-connected, all its faces are bounded, with boundaries which are cycles of G (see Remark 3.2(d)).

Definition 9.1. *A path $\pi = (\dots, x_{-1}, x_0, x_1 \dots)$ of a graph H is called* non-self-touching *if $d_H(x_i, x_j) \ge 2$ when $|j - i| \ge 2$. A cycle $C = (v_0, v_1, \dots, v_n, v_{n+1} = v_0)$ of H is called non-self-touching if $d_H(x_i, x_j) \ge 2$ whenever $|i - j| \ge 2$ (with index-arithmetic modulo $n + 1$).*

Non-self-touching paths and cycles arise naturally when studying *site* percolation (such paths were called *stiff* in [1], and *self-repelling* in [24, p. 66]).

We shall consider non-self-touching paths in two graphs derived from a given $G \in \mathcal{Q}$, namely its matching graph G_*, and the graph $\widehat{G}$ obtained by adding a site within each face F of size 4 or more, and connecting every vertex of F to this new site. The graph G_* may possess parallel edges. The property of being non-self-touching is indifferent to the existence of parallel edges, since it is given in terms of the vertex-set of π and the adjacency relation of H.

Here is the fundamental property of graphs that implies strict inequality of critical points. This turns out to be equivalent to a more technical 'local' property, as described in Section 11.2; see Theorem 11.7. As a shorthand, henceforth we abbreviate 'doubly-infinite non-self-touching path' to '2∞-nst path'.

Definition 9.2. *The graph $G \in \mathcal{Q}$ is said to have property* Π *if G_* contains some 2∞-nst path that includes some diagonal of G.*

For a graph $G = (V, E)$, let

$$\Lambda_n(v) = \Lambda_{G,n}(v) := \{w \in V : d_G(v, w) \le n\}, \quad \partial\Lambda_n(v) := \Lambda_n(v) \setminus \Lambda_{n-1}(v),$$

and, furthermore, $\Lambda_n = \Lambda_{G,n} := \Lambda_n(v_0)$. The set $\Lambda_n(v)$ will generally have bounded 'holes', which we fill in as follows. Let $\Delta_n(v)$ be the set of all edges $e = \langle u, v\rangle \in E$ such that $u \in \Lambda_n(v)$ and v lies in an infinite path of $G \setminus \Lambda_n(v)$. Let $\overline{\Lambda}_n(v)$ be the bounded subgraph of G after deletion of $\Delta_n(v)$. Let

$$\partial\overline{\Lambda}_n(v) := \overline{\Lambda}_n(v) \setminus \overline{\Lambda}_{n-1}(v),$$

and, furthermore, $\overline{\Lambda}_n = \overline{\Lambda}_{G,n} := \overline{\Lambda}_n(v_0)$. Finally, we write $u \sim v$ if $u, v \in V$ are adjacent.

10. Applications of Theorem 1.11

10.1. **Transitive graphs have property Π.** We investigate two classes of graphs with the property Π of Definition 9.2, and to which Theorem 1.11 may be applied. These are the transitive graphs, and subclasses of quasi-transitive graphs.

Theorem 10.1. *Let $G \in \mathcal{T}$ be one-ended but not a triangulation. Then G has property* Π, *and therefore satisfies $p_c(G_*) < p_c(G)$.*

We shall give two proofs of this result, using what we call the *metric method* and the *combinatorial method*. Each proof may be extended to a certain class of quasi-transitive graphs, the two such classes being different. In each case, the outcome is a sufficient but not necessary condition for a quasi-transitive graph $G \in \mathcal{Q}$ to have property Π, namely Theorems 10.4 and 10.8.

10.2. **The metric method.** The embedding results of Section 9 may be used to show the existence of 2∞-nst paths in *transitive*, one-ended $G \in \mathcal{T}$ that are not triangulations, and for certain *quasi-transitive*, one-ended $G \in \mathcal{Q}$. First, recall the relevant embedding properties. By Theorem 3.1(a), every transitive, one-ended $G \in \mathcal{T}$ may be embedded in $\mathcal{H}$ as an Archimedean tiling. By parts (a, c) of the same theorem, every quasi-transitive, one-ended $G \in \mathcal{Q}$ has a canonical embedding in $\mathcal{H}$.

Throughout this section we shall work with the Poincaré disk model of hyperbolic geometry (also denoted $\mathcal{H}$), and we denote by ρ the corresponding hyperbolic metric. For definiteness, we consider only graphs G embedded in the hyperbolic plane; the Euclidean case is easier.

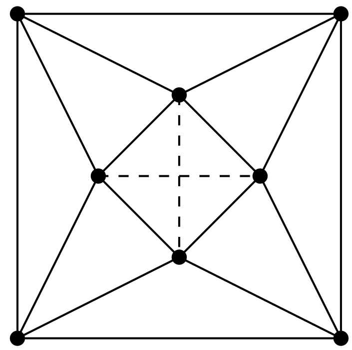

FIGURE 10.1. The graph G is the tiling of the plane with copies of this square. Taking into account the symmetries of the square, this tiling is canonical after a suitable rescaling of the interior square. The diagonals are indicated by dashed lines.

Let $G \in \mathcal{Q}$ be one-ended and not a triangulation. By 2-connectedness and Remark 3.2(d), the faces of G are bounded by cycles. As before, we restrict ourselves to the case when G is non-amenable, and we embed G canonically in the Poincaré disk $\mathcal{H}$. The edges of G are hyperbolic geodesics, but its diagonals are not generally so. The hyperbolic length of an edge $e \in E_* \setminus E$ does not generally equal the hyperbolic distance between its endvertices, denoted $\rho(e)$.

For $e \in E_*$, let Γ_e denote the doubly-infinite hyperbolic geodesic of $\mathcal{H}$ passing though the endvertices of e, and denote by $\pi(x)$ the orthogonal projection of $x \in \mathcal{H}$ onto Γ_e.

Definition 10.2. *An edge $e \in E_*$ is called* maximal *if*

$$\rho(e) \geq \rho(\pi(x), \pi(y)), \qquad f = \langle x, y \rangle \in E. \tag{10.1}$$

It is easily seen that any diagonal whose interior is surrounded by some triangle of G is not maximal; cf. the forthcoming Definition 10.6 of the term $\triangle$-empty. There always exists some maximal edge of E_*, but it is not generally unique. The following lemma is proved in the same manner as the forthcoming Lemma 13.1.

Lemma 10.3. *Let $f \in \operatorname{argmax}\{\rho(e) : e \in E_*\}$. The edge f is maximal.*

Here is the main theorem for quasi-transitive graphs using the metric method.

Theorem 10.4. *Let $G \in \mathcal{Q}$ be one-ended but not a triangulation. Assume that G has a canonical embedding in $\mathcal{H}$ for which some diagonal $d \in E_* \setminus E$ is maximal. Then G has the property Π of Definition 9.2, whence $p_c(G_*) < p_c(G)$.*

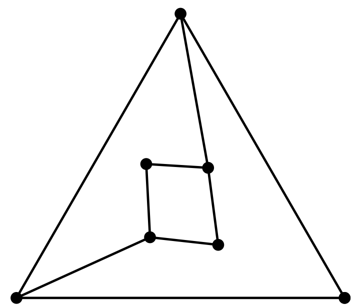

FIGURE 10.2. A doubly periodic family of faces of the triangular lattice are decorated as above, and the resulting graph is not $\triangle$-empty. Since no triangle can be connected to infinity by two paths π_1, π_2 satisfying $d_G(\pi_1, \pi_2) \geq 2$, the configuration on the interor I of this triangle is independent of the existence of an infinite open path starting at a vertex not in I.

See Sections 13.2 and 13.3 for the proofs of Theorems 10.1 and 10.4 by the metric method.

Remark 10.5. *The condition of Theorem 10.4 is sufficient but not necessary, as indicated by the following example. Let G be the canonical tiling of Figure 10.1. By inspection, no diagonal is maximal, whereas G has property Π. The sufficient condition in question can be weakened as explained in Remark 13.4, and the above example satisfes the weaker condition.*

10.3. **The combinatorial method.** We begin with some notation.

Definition 10.6. *The plane graph $G = (V, E)$ is said to have property $\square$ if every vertex of G lies in the boundary of some face of size 4 or more. A cycle C is said to* surround *a point $x \in \mathcal{H}$ if $\mathcal{H} \setminus C$ has a bounded component containing x. The graph G is said to be* $\triangle$-empty *if no 3-cycle C surrounds any vertex v.*

Figure 10.2 is an illustration of part of a 2-connected, quasi-transitive graph that is not $\triangle$-empty. It turns out that all transitive graphs are $\triangle$-empty.

Lemma 10.7. *A transitive, properly embedded, plane graph $G = (V, E) \in \mathcal{T}$ is $\triangle$-empty, and furthermore it has property $\square$ if and only if it is not a triangulation.*

Proof. Let $G = (V, E) \in \mathcal{T}$ be properly embedded and plane, but not $\triangle$-empty. Let $v_1 \in V$. By transitivity, v_1 lies in the interior of some 3-cycle C_1. Let v_2 be a vertex of C_1. Then v_2 lies in the interior of some 3-cycle C_2; since G is plane, $C_1 \subseteq \overline{C}_2$. On iterating this construction we obtain an infinite sequence (v_i, C_i) of pairs of vertices and 3-cycles such that: v_i is a vertex of C_i, $C_i \subseteq \overline{C}_{i+1}$, and $v_i \in C_{i+1}^\circ$. If the C_i are uniformly bounded, the sequence (v_i) has a limit point, in contradiction of the assumption of proper embedding; if not, it contradicts the fact that the edge-lengths of G are uniformly bounded. From this contradiction we deduce that G is $\triangle$-empty. The second statement of the lemma is immediate. $\square$

We henceforth assume that G is $\triangle$-empty. If this were false, let W be the set of all vertices lying in the interior of some 3-cycle. Let C be a 3-cycle of G that surrounds some vertex. The event that there exists an infinite open path starting in $V \setminus W$ and passing through C is independent of the states of vertices in C°; this holds since every pair of vertices of C are joined by an edge. See Figure 10.2. One may therefore remove all vertices in W without altering the existence or not of an infinite open path.

Here is the main theorem of this section; it is proved in Section 14 by the combinatorial method.

Theorem 10.8. *Let $G \in \mathcal{Q}$ be one-ended and $\triangle$-empty. If G has property $\square$, then G has property Π also.*

Proof of Theorem 10.1 using the combinatorial method. Let $G \in \mathcal{T}$ be one-ended. If G is a triangulation, then $G_* = G$, so that $p_c(G_*) = p_c(G)$. Suppose conversely that G is not a triangulation. By [13, Prop. 2.2] (see Remark 3.2(a)), G is 3-connected. By Lemma 10.7, G is $\triangle$-empty and has property $\square$, and therefore by Theorem 10.8 property Π also. The final claim follows by Theorem 1.11. $\square$

11. Some observations

11.1. **Oxbow-removal.** We begin by describing a technique of loop-removal (henceforth referred to as 'oxbow-removal'). Let H be a simple graph embedded in the Euclidean/hyperbolic plane $\mathcal{H}$ (possibly with crossings).

Lemma 11.1. *Let H be a graph embedded in $\mathcal{H}$.*

(a) *Let C be a plane cycle of H that surrounds a point $x \notin H$. There exists a non-empty subset C' of the vertex-set of C that forms a plane, non-self-touching cycle of H and surrounds x.*

(b) *Let π be a finite (respectively, infinite) path with endpoint v. There exists a non-empty subset π' of the vertex-set of π that forms a finite (respectively, infinite) non-self-touching path of H starting at v. If π is finite, then π' can be chosen with the same endpoints as π.*

Proof. (a) Let $C = (v_0, v_1, \dots, v_n, v_{n+1} = v_0)$ be a plane cycle of H that surrounds $x \notin H$; we shall apply an iterative process of 'loop-removal' to C, and may assume $n \geq 4$. We start at v_0 and move around C in increasing order of vertex-index. Let J be the least $j \leq n$ such that there exists $i \in \{1, 2, \dots, j-2\}$ with $v_i \sim v_J$, and let I be the earliest such i. Consider the two cycles $C' = (v_I, v_{I+1}, \dots, v_J, v_I)$ and $C'' = (v_J, v_{J+1}, \dots, v_0, v_1 \dots . v_I, v_J)$. (These cycles are called *oxbows* since they arise through cutting across a bottleneck of the original cycle C.) Since C surrounds x, so does either or both of C' and C'', and we suppose for concreteness that C'' surrounds x. We replace C by C''. This process is iterated until no such oxbows remain.

(b) This part is proved by a similar argument. When the endpoints v_0, v_n of π are not neighbours, we use oxbow-removal as above; otherwise, we set $\pi' = (v_0, v_n)$. □

Path-surgery will be used in the forthcoming proofs: that is, the replacement of certain paths by others. Consider a one-ended $G \in \mathcal{Q}$, embedded properly and canonically in the hyperbolic plane $\mathcal{H}$, which for concreteness we consider here in the Poincaré disk model (see [19]), also denoted $\mathcal{H}$. By Theorem 3.1(c), every automorphism of G extends to an isometry of $\mathcal{H}$. Let $\mathcal{F}$ be the set of faces of G. For $F \in \mathcal{F}$ and $x, y \in V(\partial F)$, let $\mathcal{L}_{x,y}$ be the set of rectifiable curves with endpoints x, y whose interiors are subsets of $F^\circ \setminus E$, and write $l_{x,y}$ for the infimum of the hyperbolic lengths of all $l \in \mathcal{L}_{x,y}$. Let

$$\mathrm{diam}(F) = \sup\{l_{x,y} : x, y \in V(\partial F)\},$$

and

$$\rho = \max\{\mathrm{diam}(F) : F \in \mathcal{F}\}. \tag{11.1}$$

By the properties of G, and in particular Theorem 3.1(c), we have $\rho < \infty$.

Let L be a geodesic of $\mathcal{H}$ with endpoints in the boundary of $\mathcal{H}$. Denote by L_δ the closed, hyperbolic δ-neighbourhood of L (see Figure 11.1); we call L_δ a *hyperbolic tube*, and we say L_δ has *width* 2δ. Write $\partial^+ L_\delta$ and $\partial^- L_\delta$ for the two boundary arcs of L_δ. An arc γ of $\mathcal{H}$ is said to *cross* L_δ *laterally* if it intersects both $\partial^+ L_\delta$ and $\partial^- L_\delta$. A path $\pi = (\dots, x_{-1}, x_0, x_1, \dots)$ of G (or $\widehat{G}$) is said to *cross* L_δ *in the long direction* if, for any arc γ that crosses L_δ laterally and intersects no vertex of G, the number of intersections between γ and π, if finite, is odd.

Lemma 11.2. *Let $G = (V, E) \in \mathcal{Q}$ be one-ended and duly embedded in the Poincaré disk $\mathcal{H}$, and let L_δ be a hyperbolic tube.*

(a) *If $2\delta > \rho$, then L_δ contains a 2∞-nst path of G, and a 2∞-nst path of G_*, that cross L_δ in the long direction.*

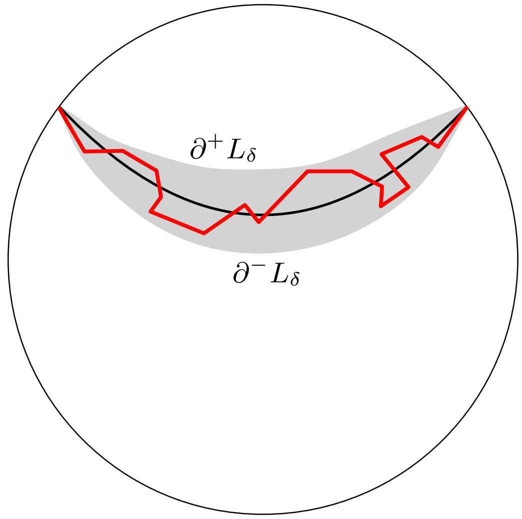

FIGURE 11.1. An illustration of Lemma 11.2. The jagged (red) path crosses L_δ in the long direction.

(b) *There exists $\zeta = \zeta(G)$ (depending on G and its embedding) such that, for $r > \zeta$ and $v \in V$, the annulus $\overline{\Lambda}_r(v) \setminus \overline{\Lambda}_{r-\zeta}(v)$ contains a non-self-touching cycle of G (respectively, G_*) denoted $\sigma_r(v)$ (respectively, $\sigma_r^*(v)$) such that $v \in \sigma_r(v)^\circ$ (respectively, $v \in \sigma_r^*(v)^\circ$).*

A more refined result may be found in Section 13.

Proof. (a) Since all faces of G are bounded, there exist vertices of G in both components of $\mathcal{H} \setminus L_\delta$. Now, L_δ fails to be crossed in the long direction if and only if it contains some arc γ that traverses it laterally and that intersects no edge of G. To see the 'only if' statement, let V^- and V^+ be the subsets of $V \cap L_\delta$ that are joined in $G \cap L_\delta$ to the two boundary points of L, respectively; if $V^- \cap V^+ = \varnothing$, then there exists such γ separating V^+ and V^- in L_δ. For this γ, there exists a face F and points $x, y \in V(\partial F)$, such that $\gamma \subseteq \lambda$ for some $\lambda \in \mathcal{L}_{x,y}$. For $\epsilon \in (0, 2\delta - \rho)$, we may replace γ by some $\gamma' := \lambda' \cap L_\delta$ where $\lambda' \in \mathcal{L}_{x,y}$ has length not exceeding $l_{x,y} + \epsilon$. The length of γ' is no greater than $\rho + \epsilon < 2\delta$, a contradiction. Therefore, L_δ contains some path π of G that crosses L_δ in the long direction.

We apply oxbow-removal in G to π as described in the proof of Lemma 11.1. For any arc γ that crosses L_δ laterally and intersects no vertex of G, the number of intersections between γ and π, if finite, decreases by a non-negative, even number

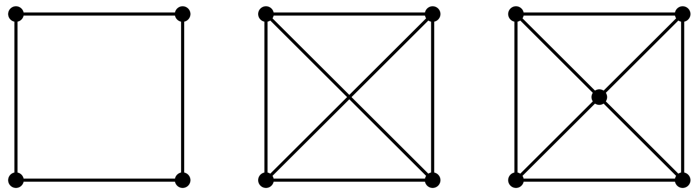

FIGURE 11.2. A square of the square lattice, its matching graph, and with its facial site added.

whenever an oxbow is removed. It follows that the non-self-touching path π' (obtained after oxbow-removal) crosses L_δ in the long direction. The same conclusion applies to G_* on letting π be a path of G_*.

The proof of (b) is similar. $\square$

11.2. **Graph properties.** The proofs of this article make heavy use of path-surgery which, in turn, relies on planarity of paths.

Lemma 11.3. *Let $G \in \mathcal{Q}$, and let π be a (finite or infinite) non-self-touching path of G_*.*

(a) *For every face F of G, π contains either zero or one or two vertices of F. If π contains two such vertices u, v, then it contains also the corresponding edge $\langle u, v\rangle$, which may be either an edge of G or a diagonal.*
(b) *The path π is plane when viewed as a graph.*

Proof. Let F be a face. The path π cannot contain three or more vertices of F, since that contradicts the non-self-touching property. Similarly, if π contains two such vertices, it must contain also the corresponding edge. If π is non-plane, it contains two or more diagonals of some face, which, by the above, cannot occur. $\square$

As a device in the proof of Theorem 1.11, we shall work with the graph $\widehat{G}$ obtained from $G = (V, E)$ by adding a vertex at the centre of each face F, and adding an edge from every vertex in the boundary of F to this central vertex. As in Section 4, these new vertices are called *facial sites*, or simply *sites* in order to distinguish them from the *vertices* of G. The facial site in the face F is denoted $\phi(F)$. See [34, Sec. 2.3], and also Figure 11.2. If $\langle v, w\rangle$ is a diagonal of G_*, it lies in some face F, and we write $\phi(v, w) = \phi(F)$ for the corresponding facial site. We note that two vertices $u, v \in V$ are connected in G_* if and only if they are connected in $\widehat{G}$.

The main reason for working with $\widehat{G}$ is that it serves to interpolate between G and G_* in the sense of (12.2): we shall assign a parameter $s \in [0, 1]$ to the facial sites in such a way that $s = 0$ corresponds to G and $s = 1$ to G_*. It will also be useful that $\widehat{G}$ is planar whereas G_* is not.

Next, we specify some desirable properties of the graphs G_* and $\widehat{G}$. The property Π was already the subject of Definition 9.2.

Definition 11.4. *The graph $G \in \mathcal{Q}$ is said to have property*

Π *if G_* has a 2∞-nst path including some diagonal,*

$\widehat{\Pi}$ *if $\widehat{G}$ has a 2∞-nst path including some facial site.*

Lemma 11.5. *Let $G \in \mathcal{Q}$ be one-ended. Then $\Pi \Rightarrow \widehat{\Pi}$.*

Proof. Let G have property Π and let π be a 2∞-nst path of G_*. For any two consecutive vertices u, v of Π such that $\delta(u, v)$ is a diagonal, we add between u and v the facial site $\phi(u, v)$. The result is a doubly-infinite path π' of $\widehat{G}$. By Lemma 11.3, ν' is non-self-touching in $\widehat{G}$, whence G has property $\widehat{\Pi}$. The converse argument fails. □

The properties of Definition 11.4 are 'global' in that they concern the existence of *infinite* paths. It is sometimes preferable to work in the proofs with *finite* paths, and to that end we introduce corresponding 'local' properties.

Let $\zeta(G)$ be as in Lemma 11.2(b). We shall make reference to the non-self-touching cycles $\sigma_r(v)$, $\sigma_r^*(v)$ given in that lemma. We write $\widehat{\sigma}_r(v)$ for the non-self-touching cycle of $\widehat{G}$ obtained from $\sigma_r^*(v)$ by replacing any diagonal by a path of length 2 passing via the appropriate facial site of $\widehat{G}$. We abbreviate the closure of the region surrounded by σ_r^* (respectively, $\widehat{\sigma}_r$) to $\overline{\sigma}_r^*$ (respectively, $\overline{\widehat{\sigma}}_r$). Let $A(G)$ be the real number given as

$$A(G) = \zeta(G) + \max\{d_G(u, w) : \langle u, w\rangle \in E_* \setminus E\}. \tag{11.2}$$

Definition 11.6. *Let $A \in \mathbb{Z}$, $A > A(G)$, and let $G \in \mathcal{Q}$ be one-ended.*

(a) *The graph G is said to have property Π_A if there exists a vertex $v \in V$ and a non-self-touching path $\pi = (x_0, x_1, \dots, x_n)$ of G_* such that*
   (i) *every vertex of π lies in $\overline{\sigma}_A^*(v)$, and $x_0, x_n \in \sigma_A^*(v)$,*
   (ii) *there exists i such that $x_i = v$,*
   (iii) *the pair v, x_{i+1} forms a diagonal of G_*, which is to say that $\phi := \phi(v, x_{i+1})$ is a facial site of $\widehat{G}$.*

(b) *The graph G is said to have property $\widehat{\Pi}_A$ if there exist vertices $v, w \in V$ and a non-self-touching path $\pi = (x_0, x_1, \dots, x_n)$ of $\widehat{G}$ such that*
   (i) *every vertex of π lies in $\overline{\widehat{\sigma}}_A(v)$, and $x_0, x_n \in \widehat{\sigma}_A(v)$,*

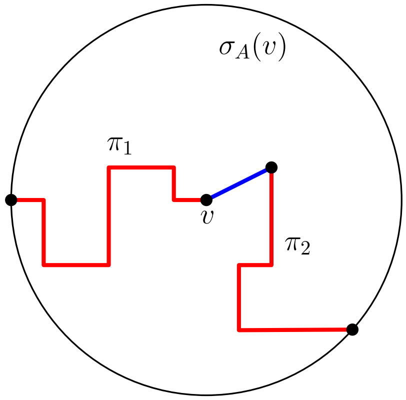

FIGURE 11.3. An illustration of the property Π_A: a non-self-touching path of G_* containing a diagonal near its middle.

(ii) *there exists i such that $x_i = v$, $x_{i+2} = w$,*
(iii) *x_{i+1} is the facial site $\phi(v, w)$ of $\widehat{G}$.*

That is to say, G has property Π_A (respectively, $\widehat{\Pi}_A$) if G_* (respectively, $\widehat{G}$) contains a finite, non-self-touching path of sufficient length that contains some diagonal (respectively, facial site). This definition is illustrated in Figure 11.3. Note that Π_{A+1} (respectively, $\widehat{\Pi}_{A+1}$) implies Π_A (respectively, $\widehat{\Pi}_A$) for sufficiently large A.

Theorem 11.7. *Let $G \in \mathcal{Q}$ be one-ended. There exists $A'(G) \geq A(G)$ such that, for $A > A'(G)$, we have $\Pi \Leftrightarrow \Pi_A$ and $\Pi \Rightarrow \widehat{\Pi}_{A,}$.*

The proof of this useful theorem utilises some methods of path-surgery that will be important later, and it is deferred to Section 11.3.

11.3. **Proof of Theorem 11.7.** (a) First, we prove that $\Pi \Leftrightarrow \Pi_A$. Evidently, $\Pi \Rightarrow \Pi_A$ for all $A > A(G)$, where $A(G)$ is given in (11.2). Assume, conversely, that Π_A holds for some $A > A(G)$. Let the non-self-touching path $\pi = (x_0, x_1, \dots, x_n)$ of G_*, the vertex $v = x_i$, and the diagonal $d = \langle v, x_{i+1} \rangle$ be as in Definition 11.6(a); think of π as a directed path from x_0 to x_n, and note by Lemma 11.3 that π is a plane graph. We abbreviate $\sigma_A^*(v)$ to σ_A^*. Let

$$\partial^- \sigma_A^* = \{y \in (\sigma_A^*)^\circ : d_{G_*}(y, \sigma_A^*) = 1\}.$$

Let π_1 be the subpath of π from v to x_0, and π_2 that from x_{i+1} to x_n. Let a_i be the earliest vertex/site of π_i lying in $\partial^- \sigma_A$. See the central circle of Figure 11.4. We

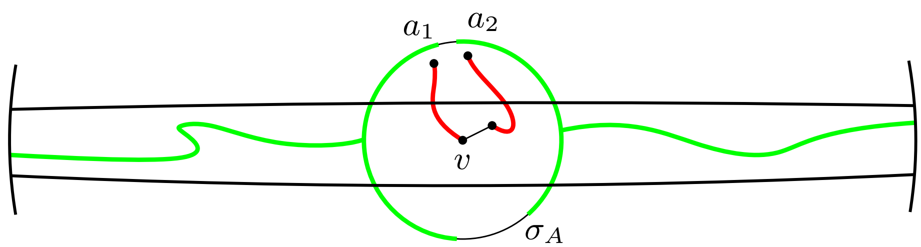

FIGURE 11.4. In the easiest case when $D \geq 2$, one finds (green) non-touching subarcs σ_A^i of σ_A to which v may be connected by non-self-touching paths. These subarcs may be connected to the boundary of $\mathcal{H}$ using subpaths of a doubly-infinite path constructed using Lemma 11.2(a).

claim the following.

(11.3) There exist two non-touching subpaths σ^1, σ^2 of σ_A^*, each of length at least $\frac{1}{2}|\sigma_A^*| - 4$, such that: (i) for $i = 1, 2$, the subpath of π_i leading to a_i may be extended beyond a_i along σ^i to form a non-self-touching path ending at any prescribed $y_i \in \sigma^i$, and (ii) the composite path thus created (after oxbow-removal if necessary) is non-self-touching.

The proof of (11.3) follows. Let

$$A_i = \{b \in \sigma_A^* : d_{G_*}(a_i, b) = 1\}, \quad D = \max\{d_{G_*}(b_1, b_2) : b_1 \in A_1,\, b_2 \in A_2\}. \tag{11.4}$$

Suppose $D \geq 2$. Choose $b_i \in A_i$ such that $d_{G_*}(b_1, b_2) \geq 2$. As illustrated in the centre of Figure 11.4, we may find a non-touching pair of non-self-touching subpaths of σ_A^* such that the conclusion of (11.3) holds. Some oxbow-removal may be needed at the junctions of paths.

Suppose $D = 1$. We may picture σ_A^* as a (topological) circle with centre v, and for concreteness we assume that a_2 lies clockwise of a_1 around σ_A^* (a similar argument holds if not). See Figure 11.5.

A. Suppose the path π_1, when continued beyond a_1, passes at the next step to some $b_1 \in A_1$, and add b_1 to obtain a path denoted π_1'.

   Since $D = 1$, the next step of π_2 beyond a_2 is not into A_2. On following π_2 further, it moves inside $(\sigma_A^*)^\circ$ until it arrives at some point $a_2' \in \partial^- \sigma_A^*$ having some neighbour $b_2' \in \sigma_A^*$ satisfying $d_{G_*}(b_1, b_2') \geq 2$; we then include the subpath of π_2 between a_2 and b_2' to obtain a path denoted π_2'.

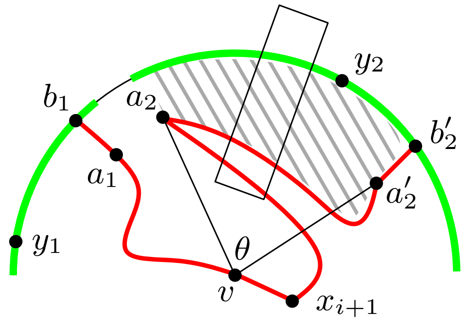

FIGURE 11.5. An illustration of the case $D = 1$. The green lines indicate the subpaths σ_A^i. The rectangle is added in illustration of the case $\theta \geq \frac{3}{4}\pi$.

We declare σ^1 to be the subpath of σ_A^* starting at b_1 and extending a total distance $\frac{1}{2}|\sigma_A^*| - 4$ around σ_A^* anticlockwise. We declare σ^2 similarly to start at distance 2 clockwise of b_1 and to have the same length as σ^1.

Let $\theta \in (0, 2\pi)$ be the angle subtended by the vector $\overrightarrow{a_2 a_2'}$ at the centre v. If $\theta < \frac{3}{4}\pi$, say, each π_i' may be extended along σ^i to end at any prescribed $y_i \in \sigma^i$. Therefore, claim (11.3) holds in this case.

The situation can be more delicate if $\theta \geq \frac{3}{4}\pi$, since then a_2' may be near to σ^1. By the planarity of π, the region R between π_2' and σ_A^* contains no point of π_1' (R is the shaded region in Figure 11.5). We position a hyperbolic tube of width greater than ρ in such a way that it is crossed laterally by both π_2' and the path σ^2 (as illustrated in Figure 11.5). By Lemma 11.2(a), this tube is crossed in the long direction by some path τ of G. The union of π_2' and τ contains a non-self-touching path π_2'' of G_* from x_{i+1} to σ^2 (whose unique vertex in σ^2 is its second endpoint). Claim (11.3) follows in this situation.

B. Suppose the hypothesis of part A does not hold, but instead π_2 passes from a_2 directly into σ_A^*. In this case we follow A above with π_1 and π_2 interchanged.
C. Suppose neither π_i passes from a_i in one step into σ_A^*. We add b_2 to the subpath from x_{i+1} to a_2, and continue as in part A above.

Suppose $D = 0$. Statement (11.3) holds by a similar argument to that above,

Having located the σ^i of (11.3), we position a hyperbolic tube as in Figure 11.4, to deduce (after oxbow-removal) the existence of a 2∞-nst path of G_* that contains the diagonal d. Therefore, G has property Π, as required.

Hyperbolic tubes are superimposed on the graph at two steps of the argument above, and it is for this reason that we need A to be sufficiently large, say $A > A'(G)$.

(b) It remains to show that $\Pi \Rightarrow \widehat{\Pi}_A$. By Lemma 11.5, $\Pi \Rightarrow \widehat{\Pi}$, and it is immediate that $\widehat{\Pi} \Rightarrow \widehat{\Pi}_A$ for large A.

12. Proof of Theorem 1.11

Consider site percolation on G with product measure $\mathbb{P}_p$, and fix some vertex v_0 of G. We write $v \leftrightarrow w$ if there exists a path of G from v to w using only open sites (such a path is called *open*), and $v \leftrightarrow \infty$ if there exists an infinite, open path starting at v. The *percolation probability* is the function θ given by

$$\theta(p) = \theta(p; G) = \mathbb{P}_p(v_0 \leftrightarrow \infty), \tag{12.1}$$

so that the (site) critical probability of G is $p_c(G) := \sup\{p : \theta(p) = 0\}$. The quantities $\theta(p; G_*)$ and $p_c(G_*)$ are defined similarly.

Remark 12.1. *It is an old problem dating back to* [8] *to decide which graphs G satisfy $p_c(G) < 1$, and there has been a series of related results since. It was proved in* [21, Thm 1.3] *that $p_c(G) < 1$ for all quasi-transitive graphs G with super-linear growth. This class includes all $G \in \mathcal{Q}$ with either one or infinitely many ends (see* [5, Sect. 1.4] *and Theorem 3.1).*

Theorem 12.2. *Let $G \in \mathcal{Q}$ be one-ended.*

(a) *Let $A_0 \in \mathbb{Z}$. If G has property Π_A for no $A > A_0$, then $p_c(G_*) = p_c(G)$.*
(b) *There exists $A'(G) \geq A(G)$ such that the following holds. Let $A > A'(G)$. If G has property $\widehat{\Pi}_A$, then $p_c(\widehat{G}) < p_c(G)$.*

The constant $A'(G)$ in part (b) depends on the *embedded graph* G, viewed as a subset of $\mathcal{H}$, rather on the graph G alone.

Proof of Theorem 1.11. If G does not have property Π, by Theorem 11.7 for large A it does not have property Π_A, whence by Theorem 12.2(a), $p_c(G_*) = p_c(G)$. Conversely, if G has property Π, by Theorem 11.7 again it has property $\widehat{\Pi}_A$ for large A, whence by Theorem 12.2(b), $p_c(\widehat{G}) < p_c(G)$. The final claim follows by the elementary inequality $p_c(G_*) \leq p_c(\widehat{G})$; see (12.2). $\square$

Proof of Theorem 12.2(a). Let $A_0 \in \mathbb{Z}$. Assume G has property Π_A for no $A \geq A_0$, and let $p > p_c(G_*)$. Let π be an infinite open path of G_* with some endpoint x. By Lemma 11.1(b), there exists a subset π' of π that forms a non-self-touching path of G_* with endpoint x. Let $A > A_0$. Since Π_A does not hold, every edge of π' at distance $2A$ or more from x is an edge of G, so that there exists an infinite open path in G. Therefore, $p \geq p_c(G)$, whence $p_c(G_*) = p_c(G)$. $\square$

The rest of this section is devoted to the proof of Theorem 12.2(b). Let $\widehat{\Omega} = \Omega_V \times \Omega_\Phi$ where Φ is the set of facial sites and $\Omega_\Phi = \{0,1\}^\Phi$. For $\widehat{\omega} = \omega \times \omega' \in \widehat{\Omega}$ and $\phi \in \Phi$, we call ϕ *open* if $\omega'_\phi = 1$, and *closed* otherwise. Let $\mathbb{P}_{p,s} = \mathbb{P}_p \times \mathbb{P}_s$ be the corresponding product measure on $\Omega_V \times \Omega_\Phi$, and

$$\theta(p,s) = \lim_{n\to\infty} \theta_n(p,s) \qquad \text{where} \qquad \theta_n(p,s) = \mathbb{P}_{p,s}(v_0 \leftrightarrow \partial\overline{\Lambda}_n \text{ in } \widehat{G}),$$

so that

$$\theta(p,0) = \theta(p;G), \quad \theta(p,p) = \theta(p;\widehat{G}), \quad \theta(p,1) = \theta(p;G_*), \tag{12.2}$$

where $\theta(p;H)$ denotes the percolation probability of the graph H. The following proposition implies Theorem 12.2(b).

Proposition 12.3. *There exists $A'(G) < \infty$ such that the following holds. Suppose $G \in \mathcal{Q}$ is one-ended and has property $\widehat{\Pi}_A$ where $A > A'(G)$. Let $s \in (0,1]$. There exists $\epsilon = \epsilon(s) > 0$ such that $\theta(p,s) > 0$ for $p_c(G) - \epsilon < p < p_c(G)$.*

We do not investigate the details of how $A'(G)$ depends on G. An explicit lower bound on $A'(G)$ may be obtained in terms of local properties of the embedding of G, but it is doubtful whether this will be useful in practice.

The rest of this proof is devoted to an outline of that of Proposition 12.3. Full details are not included, since they are very close to established arguments of [1], [24, Sect. 3.3], and elsewhere.

Let n be large, and later we shall let $n \to \infty$. Consider site percolation on $\widehat{G}$ with measure $\mathbb{P}_{p,s}$. We call a vertex (respectively, facial site) z *pivotal* if it is pivotal for the existence of an open path of $\widehat{G}$ from v_0 to $\partial\Lambda_n$ (which is to say that such a path exists if z is open, and not otherwise). Let Pi_n be the set of pivotal vertices, and Di_n the set of pivotal facial sites. Proposition 12.3 follows in the 'usual way' (see [24, Sect. 3.3]) from the following statement.

Lemma 12.4. *Let $p,s \in (0,1)$. There exists $M \ge 1$ and $f : (0,1)^2 \to (0,\infty)$ such that, for $n > 4M$ and every $z \in \overline{\Lambda}_n$,*

$$\mathbb{P}_{p,s}(z \in \mathrm{Pi}_n) \le f(p,s)\mathbb{P}_{p,s}(\mathrm{Di}_n \cap \overline{\Lambda}_M(z) \neq \varnothing). \tag{12.3}$$

On summing (12.3) over $z \in \overline{\Lambda}_n$, we obtain by Russo's formula (see [24, Sec. 2.4]) that there exists $g(p,s) < \infty$ such that

$$\frac{\partial}{\partial p}\theta_n(p,s) \le g(p,s)\frac{\partial}{\partial s}\theta_n(p,s). \tag{12.4}$$

The derivation of Proposition 12.3 from this differential inequality is explained in [1, 24]. It suffices therefore to prove Lemma 12.4.

Here is an outline of the proof of Lemma 12.4. Let $\widehat{\omega} \in \widehat{\Omega}$, $z \in V \cap \overline{\Lambda}_n$, and suppose

$$z \text{ is open and pivotal in the configuration } \widehat{\omega}. \tag{12.5}$$

By making changes to the configuration $\widehat{\omega}$ within the box $\overline{\Lambda}_{4M}(z)$ for some fixed M,

(12.6) we construct a configuration in which $\overline{\Lambda}_M(z)$ contains a pivotal facial site.

This implies (12.3) with f depending on the choice of z. Since $\overline{\Lambda}_{4M}(z)$ is finite and there are only finitely many types of vertex (by quasi-transitivity), f may be chosen to be independent of z. The above is achieved in five stages.

Assume for now that $\widehat{\omega} \in \widehat{\Omega}$ *and the pivotal vertex* z *satisfies*

$$z \in \overline{\Lambda}_{n-2M} \setminus \overline{\Lambda}_{2M}. \tag{12.7}$$

For clarity of exposition, our illustrations are drawn as if G is duly embedded in the Euclidean rather than hyperbolic plane.

Let G have property $\widehat{\Pi}_A$. Let $\pi = (x_j)$, $v = x_i$, be as in Definition 11.6(b), and write $\phi = x_{i+1} = \phi(v, x_{i+2})$. Find $\alpha \in \mathrm{Aut}(G)$ such that $v' = \alpha v$ satisfies $d_G(z, v') \le \Delta$, where Δ is given in (9.1). Let $M = 2(A + \Delta)$, so that $\overline{\Lambda}_A(v') \subseteq \overline{\Lambda}_{M/2}(z)$. The outline of the proof is as follows.

I. If there exist one or more open facial sites in $\overline{\Lambda}_M(z)$, we declare them one-by-one to be closed. If at some point in this process, some facial site is found to be pivotal, then we have achieved (12.6), by changing $\widehat{\omega}$ within a bounded region. We may therefore assume that this never occurs, or equivalently that

$$\widehat{\omega} \text{ has no open facial site in } \overline{\Lambda}_M(z). \tag{12.8}$$

II. Find a non-self-touching open path ν in $\widehat{\omega}$ from v_0 to $\partial\overline{\Lambda}_n$. This path passes necessarily through the pivotal vertex z.

III. By making changes within $\overline{\Lambda}_{2M}(z)$, we construct non-touching subpaths of ν from v_0 (respectively, $\partial\overline{\Lambda}_n$) to $\partial\overline{\Lambda}_M(z)$, that can be extended inside $\overline{\Lambda}_M(z)$ in a manner to be specified at Stage V. This, and especially the following, stage resembles closely part of the proof in Section 11.3.

IV. We splice a copy (denoted $\pi' = \alpha\pi$) of π inside $\overline{\Lambda}_A(v')$, and we make local changes to obtain paths π_1, π_2 from the two endpoints of $\alpha\phi$, respectively, to $\partial\overline{\Lambda}_A(v')$ that can be extended outside $\overline{\Lambda}_A(v')$ in a manner to be specified at Stage V.

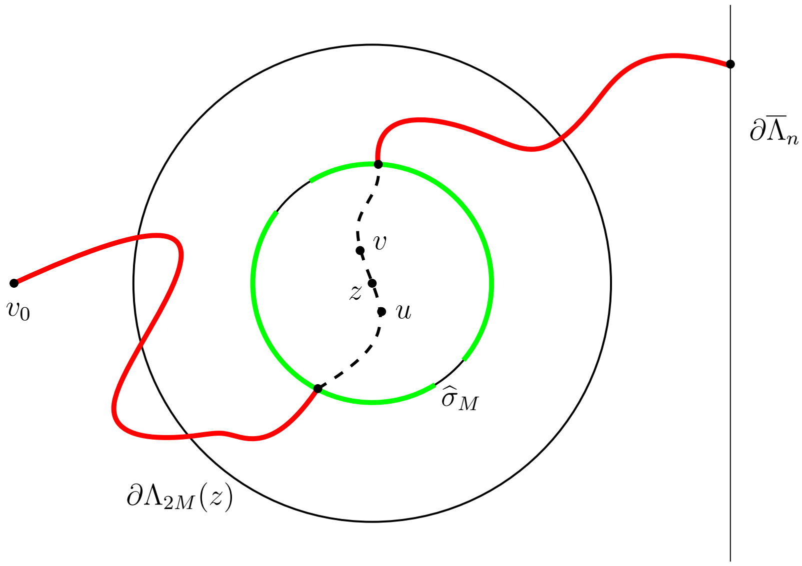

FIGURE 12.1. An illustration of the construction at Stages II/III. The non-self-touching path ν contains subpaths from v_0 to $\widehat{\sigma}_M$, and from the latter set to $\partial\overline{\Lambda}_n$. The subpaths σ_M^i of $\widehat{\sigma}_M$ are indicated in green.

V. Between the contours $\partial\overline{\Lambda}_A(v')$ and $\partial\overline{\Lambda}_M(z)$, we arrange the configuration in such a way that the retained parts of ν hook up with the endpoints of the π_i. In the resulting configuration, the facial site $\phi' := \alpha\phi$ is pivotal.

Some work is needed to ensure that ϕ' can be made pivotal in the final configuration. Lemma 11.2(b) will be used to traverse the annulus between the two contours at Stage V. In making connections at junctions of paths, we shall make use of the planarity of $\widehat{G}$. Rather than working with the boundaries of $\overline{\Lambda}_M(z)$ and $\overline{\Lambda}_A(v')$, we shall work instead with the non-self-touching cycles $\widehat{\sigma}_M := \widehat{\sigma}_M(z)$ and $\widehat{\sigma}_A := \widehat{\sigma}_A(v')$ of $\widehat{G}$ given in Lemma 11.2(b). Let

$$\begin{aligned}\partial^+\widehat{\sigma}_M &= \{y \in \mathcal{H} \setminus \overline{\widehat{\sigma}}_M : d_{\widehat{G}}(y, \widehat{\sigma}_M) = 1\},\\ \partial^-\widehat{\sigma}_A &= \{y \in (\widehat{\sigma}_A)^\circ : d_{\widehat{G}}(y, \widehat{\sigma}_A) = 1\}.\end{aligned}$$

We move to the proof proper. Stage I is first followed as stated above.

Stage II. By (12.5), we may find an open, non-self-touching path ν of $\widehat{G}$ from v_0 to $\partial\overline{\Lambda}_n$, and we consider ν as thus directed. By (12.8), ν includes no facial site of $\overline{\Lambda}_M(z)$. The path ν passes necessarily through z, and we let u (respectively, w) be the preceding (respectively, succeeding) vertex to z.

For $y \in V$, and the given configuration $\widehat{\omega}$ (satisfying (12.8)), let

$$C_y = \{x \in V : y \leftrightarrow x \text{ in } \widehat{G} \setminus \{z\}\},$$

and write C_y also for the corresponding induced subgraph of $\widehat{G}$. By (12.5),

A. C_u and C_w are disjoint (and also non-touching),
B. the subpath of ν, denoted $\nu(u-)$, from v_0 to u contains no facial site of $\overline{\Lambda}_M(z)$,
C. the subpath of ν, denoted $\nu(w+)$, from w to $\partial\overline{\Lambda}_n$ contains no facial site of $\overline{\Lambda}_M(z)$,
D. the pair $\nu(z-)$, $\nu(z+)$ is non-touching.

Stage III. This is closely related to the proof of Theorem 11.7 given in Section 11.3. Note that the intersection of $\nu(u-) \cup \nu(w+)$ and $\overline{\Lambda}_{2M}(z)$ comprises a family of paths rather than two single paths. See Figure 12.1.

We follow $\nu(u-)$ towards u, and $\nu(w+)$ backwards towards w, until we reach the first vertices/sites, denoted a_1, a_2, respectively, lying in $\partial^+\widehat{\sigma}_M$. Let ν_1 be the subpath of $\nu(u-)$ between v_0 and a_1, and ν_2 that of $\nu(w+)$ between $\partial\overline{\Lambda}_n$ and a_2. We now change the states of certain vertices/sites $x \in \overline{\Lambda}_{2M}(z)$ by declaring

(12.9) $\qquad$ every $x \in \overline{\Lambda}_{2M}(z) \setminus \overline{\widehat{\sigma}}_M$ is declared open if and only if $x \in \nu_1 \cup \nu_2$.

We investigate next the subsets of $\widehat{\sigma}_M$ to which the a_i may be connected within σ_M. We shall show that:

(12.10) $\qquad$ there exist two non-touching subpaths σ_M^1, σ_M^2 of $\widehat{\sigma}_M$, each of length at least $\frac{1}{2}|\widehat{\sigma}_M| - 4$, such that, for $i = 1, 2$: (i) a_i has a neighbour $b_i \in \sigma_M^i$, (ii) for $y_i \in \sigma_M^i$, the path ν_i may be extended from b_i to y_i along σ_M^i, thereby creating (after oxbow-removal if necessary) a non-self-touching path from the other endpoint of ν_i, (iii) the composite path ν_i' thus created is non-self-touching, and (iv) the pair ν_1', ν_2' is non-touching.

An explanation follows. Let

(12.11) $$A_i = \{b \in \widehat{\sigma}_M : d_{\widehat{G}}(a_i, b) = 1\}, \quad D = \max\{d_{\widehat{G}}(b_1, b_2) : b_1 \in A_1,\, b_2 \in A_2\}.$$

Suppose $D \geq 2$. Choose $b_i \in A_i$ such that $d_{\widehat{G}}(b_1, b_2) \geq 2$. Statement (12.10) follows as illustrated in Figure 12.1.

Suppose $D = 1$. We may picture σ_M as a circle with centre z, and for concreteness we assume that a_2 lies clockwise of a_1 around $\widehat{\sigma}_M$ (a similar argument holds if not) See Figure 12.2.

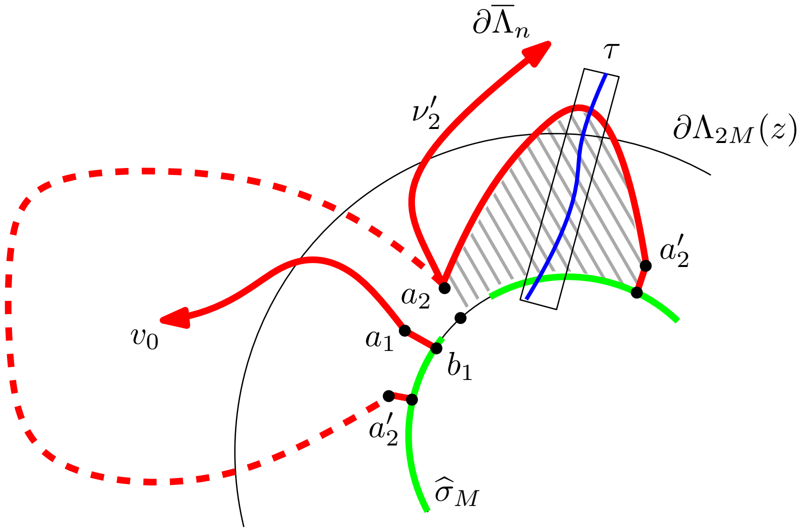

FIGURE 12.2. An illustration of the case $D = 1$ in the Stage III construction. There are two subcases, depending on whether $\theta > 0$ (solid line) or $\theta < 0$ (dashed line). The green lines indicate the subpaths σ_M^i in the subcase $\theta > 0$. The rectangle is added in illustration of the hyperbolic tube used in the case $\theta \geq \frac{3}{4}\pi$.

A. Suppose the path ν_1, when continued along $\nu(z-)$ beyond a_1, passes at the next step to some $b_1 \in A_1$, and add b_1 to ν_1 (to obtain a path denoted ν_1').

Since $D = 1$, the next step of $\nu(w+)$ beyond a_2 is not to A_2. On following $\nu(w+)$ further, it moves inside $\mathcal{H}\backslash\overline{\widehat{\sigma}}_M$ until it arrives at some point $a_2' \in \partial^+\widehat{\sigma}_M$ having some neighbour $b_2' \in \widehat{\sigma}_M$ satisfying $d_{\widehat{G}}(b_1, b_2') \geq 2$; we then add to ν_2 the subpath of $\nu(w+)$ between a_2 and b_2' (to obtain an extended path ν_2'). Let $\theta(a_2')$ be the angle subtended by the vector $\overrightarrow{a_2 a_2'}$ at the centre z, counted positive if $\nu(w+)$ passes clockwise around z of $\widehat{\sigma}_M$, and negative if anticlockwise.

(i) There are two cases, depending on whether $\theta := \theta(a_2')$ is positive or negative. Assume first that $\theta > 0$. If $\theta < \frac{3}{4}\pi$, say, we declare σ_M^1 to be the subpath of $\widehat{\sigma}_M$ starting at b_1 and extending a total distance $\frac{1}{2}|\widehat{\sigma}_M| - 4$ around σ_M anticlockwise. We declare σ_M^2 similarly to start at distance 2 clockwise of b_1 along $\widehat{\sigma}_M$ and to have the same length as σ_M^1. Each ν_i'

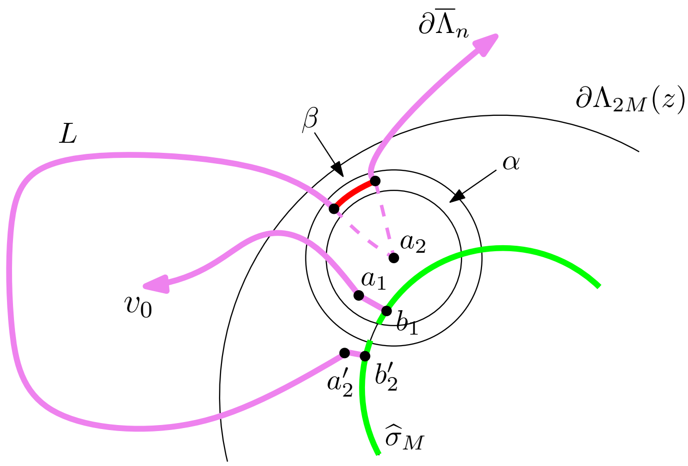

FIGURE 12.3. When $D = 1$ and $\theta < 0$, we adjust the path ν_2 by bypassing a subpath through a_2.

may be extended along σ_M^i to end at any prescribed $y_i \in \sigma_M^i$. Therefore, claim (12.10) holds in this case.

The situation can be more delicate if $\theta \geq \frac{3}{4}\pi$, since then a_2' may be near to σ_M^1. By the planarity of ν, the region R between ν_2' and σ_M contains no point of ν_1' (R is the shaded region in Figure 12.2). We position a hyperbolic tube of width greater than ρ in such a way that it is crossed laterally by both ν_2' and the path σ_M^2 given above. By Lemma 11.2(a), this tube is crossed in the long direction by some path τ of $\widehat{G}$. As illustrated in Figure 12.2, the union of ν_2' and τ contains (after oxbow-removal) a non-self-touching path ν_2'' from $\partial\overline{\Lambda}_n$ to σ_M^2 (whose unique vertex in σ_M^2 is its second endpoint). We now declare each vertex/site of $\overline{\Lambda}_{2M}(z) \setminus (\widehat{\sigma}_M)^\circ$ to be open if and only if it lies in $\nu_1' \cup \nu_2''$. Claim (12.10) follows in this situation, with the σ_M^i given as above.

(ii) Assume $\theta < 0$, in which case there arises a complication in the above construction, as illustrated in Figure 12.3. In this case, there is a subpath L of ν_2' from a_2 to a_2', that passes anticlockwise around v_0, and ν_1' contains no vertex/site outside the closed cycle comprising L followed by the subpath of $\widehat{\sigma}_M$ from b_2' to b_2. In order to overcome this problem, we alter the path ν_2' as follows. Let α denote the annulus $\overline{\Lambda}_M(a_2)\setminus\overline{\Lambda}_{M-\zeta}(a_2)$, with ζ as in Lemma 11.2(b). (We may assume $M \geq 2\zeta$.) By that lemma, α contains a non-self-touching cycle β of $\widehat{G}$ that surrounds a_2. The union of ν_2' and β contains (after oxbow-removal) a non-self-touching path ν_2''

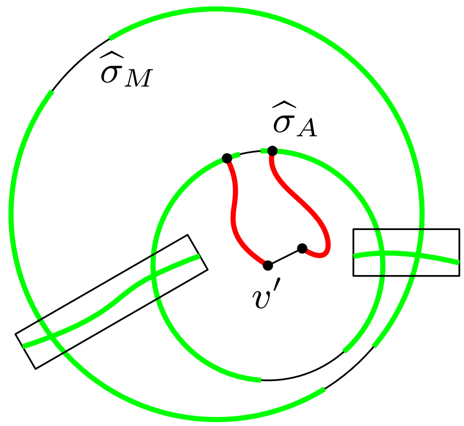

FIGURE 12.4. An illustration of the construction at Stages IV and V.

of $\widehat{G}$ from $\partial\overline{\Lambda}_n$ to a_2' that does not contain a_2 (see Figure 12.3). We declare every $x \in \nu_2''$ open and every $x \in \nu_2' \setminus \nu_2''$ closed. The subpaths σ_M^i of $\widehat{\sigma}_M$ may now be defined as above.

B. Suppose the hypothesis of part A does not hold, but instead ν_2 passes from a_2 into $\widehat{\sigma}_M$. In this case we follow A with $\nu(u-)$ and $\nu(w+)$ interchanged. This case is slightly shorter than A since the above complication cannot occur.

C. Suppose neither ν_i passes from a_i directly into $\widehat{\sigma}_M$. We add b_2 to ν_2 and continue as in A above.

Suppose $D = 0$. Statement (12.10) holds by a similar argument to that of case (ii),

Stage IV. We next pursue a similar strategy within $\overline{\Lambda}_A(v')$. The argument is essentially that in proof of Theorem 11.7 given in Section 11.3, and the details of this are omitted here. See Figures 11.5 and 12.4.

Stage V. Having located the subpaths σ_M^i of $\widehat{\sigma}_M$, and the subpaths σ_A^i of $\widehat{\sigma}_A$, we prove next that there exists $j \in \{1,2\}$, and non-self-touching paths μ_1, μ_2, such that: (i) μ_1, μ_2 is a non-touching pair, (ii) μ_1 has endpoints in σ_M^1 and σ_A^j, and μ_2 has endpoints in σ_M^2 and $\sigma_A^{j'}$, where $j' \in \{1,2\}$, $j' \neq j$, and (iii) apart from their endpoints, μ_1 and μ_2 lie in $(\widehat{\sigma}_M)^\circ \setminus \overline{\overline{\sigma}}_A$. This statement follows as in Figure 12.4 by positioning two hyperbolic tubes of width exceeding ρ, and appealing to Lemma 11.2(a). It may be necessary to remove some oxbows at the junctions of paths.

Hyperbolic tubes are superimposed on $\widehat{\sigma}_A$ above, and it is for this reason that A is assumed to be sufficiently large.

Having satisfied (12.6) subject to (12.7), we next explain how to remove the assumption (12.7). Let the pivotal vertex v satisfy $v \in \overline{\Lambda}_{2M}$; a similar argument

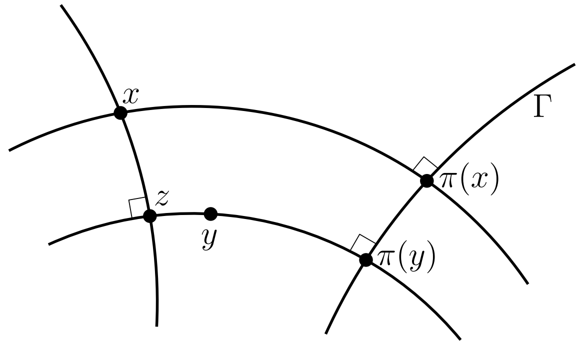

Figure 13.1. An illustration of the proof of Lemma 13.1. The four curved lines are geodesics.

applies if $v \in \overline{\Lambda}_n \setminus \overline{\Lambda}_{n-2M}$. Let π be an infinite, non-self-touching open path of $\widehat{G}$ starting at v_0, and declare closed every vertex of $\overline{\Lambda}_{4M}$ not lying in π. (Such a π exists by connectivity and oxbow-removal.) In the resulting configuration, every vertex/site in the subpath of π from $\partial\overline{\Lambda}_{2M}$ to $\partial\overline{\Lambda}_{4M}$ is pivotal. We pick one such vertex and apply the above arguments to obtain a pivotal facial site lying in $\overline{\Lambda}_{4M}$.

13. Strict inequality using the metric method

13.1. **Embeddings in the Poincaré disk.** Throughout this section we shall work with the Poincaré disk model of hyperbolic geometry (also denoted $\mathcal{H}$), and we denote by ρ the corresponding hyperbolic metric.

13.2. **Proof of Theorem 10.1 by the metric method.** Let Γ be a doubly-infinite geodesic in the Poincaré disk. Pick a fixed but arbitrary total ordering $<$ of Γ. Then Γ may be parametrized by any function $p : \Gamma \to \mathbb{R}$ satisfying $p(v) = p(u) + \rho(u, v)$ for $u, v \in \Gamma$, $u < v$, and we fix such p.

Here is a lemma. Any $x \notin \Gamma$ has an orthogonal projection $\pi(x)$ onto Γ (for $x \in \Gamma$, we set $\pi(x) = x$).

Lemma 13.1. *For $x, y \in \mathcal{H}$, we have $\rho(\pi(x), \pi(y)) \le \rho(x, y)$.*

Proof. We assume for simplicity that x and y are distinct and lie in the same connected component of $\mathcal{H} \setminus \Gamma$; a similar proof holds if not. The points $x, \pi(x), \pi(y), y$ form a quadrilateral with two consecutive right angles (see Figure 13.1). Let z be the orthogonal projection of x onto the geodesic containing y and $\pi(y)$. The triple x, z, y forms a right-angled triangle, and the quadruple $x, z, \pi(y), \pi(x)$ forms a Lambert quadrilateral. By the geometry of such shapes (see, for example, [33, Sect. III.5]), we have that $\rho(x,y) \ge \rho(x,z) \ge \rho(\pi(x),\pi(y))$. $\square$

Let $G = (V,E) \in \mathcal{T}$ be one-ended but not a triangulation. We shall consider only the case when G is non-amenable, so that it is embedded as an Archimedean tiling in the Poincaré disk; the Euclidean case is similar and easier. For an edge e of $G_* = (V, E_*)$, let $\rho(e)$ denote the hyperbolic distance between its endvertices; since every e of G_* (in its embedding) is a geodesic, $\rho(e)$ equals the hyperbolic length of e. Since the embedding is Archimedean, every edge of G has the same hyperbolic length, and we may therefore assume for simplicity that

$$\rho(e) = 1, \qquad e \in E. \tag{13.1}$$

Each $e \in E_*$ is a sub-arc of a unique doubly-infinite geodesic, denoted Γ_e, of $\mathcal{H}$.

Let r be the maximal number of edges in a face of G, and let F be a face of size r. Since F is a regular r-gon, by (13.1), F has some diagonal d satisfying

$$\rho(d) \ge \rho(e) \ge 1, \qquad e \in E_*, \tag{13.2}$$

and we choose d accordingly. By Lemma 13.1 applied to the geodesic Γ_d,

$$\rho(\pi(e)) \le \rho(e) \le \rho(d), \qquad e \in E_*, \tag{13.3}$$

where π denote orthogonal projection onto Γ_d, and $\rho(\gamma)$ is the hyperbolic distance between the endpoints of an arc γ.

Let $<$ and p be the ordering and parametrization of Γ_d given at the start of this subsection. We extend the domain of p by setting

$$p(x) = p(\pi(x)), \qquad x \in \mathcal{H}.$$

We construct next a doubly-infinite path of G_* containing d and lying 'close' to Γ_d. Write $d = \langle a, b\rangle$ where $a < b$. Let Γ_d^+ (respectively, Γ_d^-) be the sub-geodesic obtained by proceeding along Γ_d from b in the positive direction (respectively, from a in the negative direction). As we proceed along Γ_d^+, we encounter edges and faces of G. If $e \in E$ is such that $e \cap \Gamma_d^+ \ne \varnothing$, then the intersection is either a point or the entire edge e (this holds since both e and Γ_d are geodesics).

Lemma 13.2. *Let $e = \langle x, y\rangle \in E$ be an edge whose interior e° intersects Γ_d^+ at a singleton g only, so that $e^\circ \cap \Gamma_d^+ = \{g\}$. Then,*

(a) *either $p(x) = p(g) = p(y)$, or*
(b) *some endvertex $z \in \{x, y\}$ of e satisfies $p(z) > p(g)$.*

Proof. The first case arises when e, viewed as a geodesic, is perpendicular to Γ_d^+, and the second when it is not. See Figure 13.2. $\square$

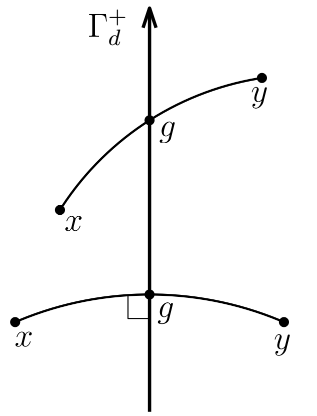

FIGURE 13.2. The two cases that arise when Γ_d^+ meets an edge which is either perpendicular or not.

In proceeding along Γ_d^+, we make an ordered list (w_i) of vertices as follows.

(a) Set $w_0 = b$.
(b) Every time Γ_d passes into the interior of a face F', it exits either at a vertex v' or across the interior of some edge e'. In the first case we add v' to the list, and in the second, we add to the list an endvertex of e' with maximal p-value.
(c) If Γ_d^+ passes along an edge $e \in E$, we add both its endvertices to the list in the order in which they are encountered.

The following lemma is proved after the end of the current proof.

Lemma 13.3. *The infinite ordered list $w = (w_0, w_1, \dots)$ is a path of G_* with the property that $p(w_i)$ is strictly increasing in i.*

We apply oxbow-removal, Lemma 11.1(b), to w to obtain an infinite, non-self-touching path $\nu^+ = (\nu_0, \nu_1, \dots)$ of G_* satisfying

$$\nu_0 = b, \qquad p(\nu_0) < p(\nu_1) < \cdots . \tag{13.4}$$

By the same argument applied to Γ_d^-, there exists an infinite, non-self-touching path $\nu^- = (\nu_{-1}, \nu_{-2}, \dots)$ of G_* satisfying

$$\nu_{-1} = a, \qquad p(\nu_{-1}) > p(\nu_{-2}) > \cdots . \tag{13.5}$$

The composite path ν obtained by following ν^- towards a, then d, then ν_+, fails to be non-self-touching in G_* if and only if there exists $s < 0$ and $t \geq 0$ with $(s,t) \neq (-1,0)$

such that $e'' := \langle \nu_s, \nu_t \rangle \in E_*$. If the last were to occur, by (13.4)–(13.5),

$$\rho(\pi(e'')) = p(\nu_t) - p(\nu_s) > p(b) - p(a) = \rho(d),$$

in contradiction of (13.3). Thus ν is the required non-self-touching path. The above may be regarded as a more refined version of part of Proposition 11.2.

Proof of Lemma 13.3. That w is a path of G_* follows from its construction, and we turn to the second claim. Let $m \geq 0$, and consider $w_0, w_1, \dots, w_m$ as having been identified. We claim that

$$p(w_m) < p(w_{m+1}). \tag{13.6}$$

(a) Suppose $w_m \in \Gamma_d^+$.
   (i) If Γ_d^+ includes next an entire edge of the form $\langle w_m, g \rangle \in E$, then $w_{m+1} = g$ and (13.6) holds.
   (ii) Suppose Γ_d^+ enters next the interior of some face F'. If it exits F' at a vertex, then this vertex is w_{m+1} and (13.6) holds. Suppose it exits by crossing the interior of an edge e'. If w_m is an endvertex of e', then w_{m+1} is its other endvertex and (13.6) holds; if not, then w_{m+1} is an endvertex of e' with maximal p-value (recall Lemma 13.2).

(b) Suppose w_m is the endvertex of an edge e that is crossed (but not traversed in its entirety) by Γ_d^+, and let F' be the face thus entered. The next vertex w_{m+1} is given as in (a)(ii) above, and (13.6) holds.

The proof is complete. □

13.3. **The case of quasi-transitive graphs.** Certain complexities arise in applying the techniques of Section 13.2 to quasi-transitive graphs. In contrast to transitive graphs, the faces are not generally regular polygons, and the longest edge need not be a diagonal.

Let $G \in \mathcal{Q}$ be one-ended and not a triangulation. As before, we restrict ourselves to the case when G is non-amenable, and we embed G canonically in the Poincaré disk $\mathcal{H}$. The edges of G are hyperbolic geodesics, but its diagonals need not be so. The hyperbolic length of an edge $e \in E_* \setminus E$ does not generally equal the hyperbolic distance $\rho(e)$ between its endvertices.

The proof is an adaptation of that of Section 13.2, and full details are omitted. In identifying a path corresponding to the path w of Lemma 13.3, we use the fact that edges of E are geodesics, and concentrate on the *final* departures of Γ_d^+ from the faces whose interiors it enters.

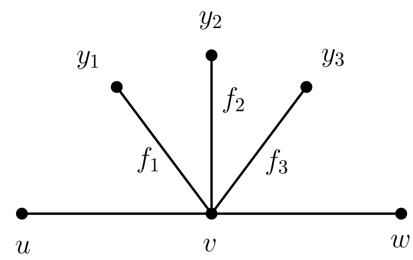

FIGURE 14.1. An illustration with $r = 3$.

Remark 13.4. *The condition of Theorem 10.4 may be weakened as follows. In the above proof of Theorem 10.1 is constructed a* 2∞*-nst path* ν *of* G_* *(see the discussion following Lemma 13.3). It suffices that, in the notation of that discussion, there exist a diagonal* d *and* $s < 0$, $t \geq 1$ *such that (i) the path* $(\nu_s, \nu_{s+1}, \dots, \nu_t)$ *is non-self-touching in* G_**, and (ii) for all* $e \in E$ *we have* $p(\nu_t) - p(\nu_s) > p(\pi(e))$*. Cf. Theorem 11.7.*

14. Strict inequality using the combinatorial method

We prove Theorem 10.8 in this section. Let G have the given properties, and let $\nu = (\dots, \nu_{-1}, \nu_0, \nu_1, \dots)$ be a 2∞-nst path of G_*. Such a path exists by Lemma 11.2(a) since G is connected. If ν contains some diagonal, then we are done. Assume therefore that

$$\nu \text{ contains no diagonal.}$$

We shall make local changes to ν to obtain a 2∞-nst path $\overline{\nu}$ containing some diagonal. The following analysis is 'case-by-case'.

In the various steps and figures that illustrate this construction, we write

$$u = \nu_{-1}, \quad v = \nu_0, \quad w = \nu_1.$$

Draw the triple u, v, w in the planar embedding of G as in Figure 14.1. Let $f_i = \langle v, y_i \rangle$, $i = 1, 2, \dots, r$, be the edges of G incident to v in the sector obtained by rotating $\langle u, v \rangle$ clockwise about v until it coincides with $\langle w, v \rangle$; the f_i are listed in clockwise order. Let $\nu(u-)$ (respectively, $\nu(w+)$) be the subpath of ν prior to and including u (respectively, after and including w).

Assume first that G has no triangular faces. For clarity, we begin with this simpler situation. If $r = 0$, the edges $\langle u, v \rangle$, $\langle v, w \rangle$ lie in some face F of G which, by assumption, is not a triangle. In this case, we remove v from ν and add the diagonal $\delta(u, w)$. The ensuing path $\overline{\nu}$ has the required properties.

Suppose henceforth that $r \ge 1$. Since ν is assumed non-self-touching, no y_i lies in $\nu(u-) \cup \nu(w+)$. For $i = 1, 2, \dots, r$, denote the neighbours of y_i other than v as $z_{i,1}, z_{i,2}, \dots, z_{i,\delta_i}$, listed in clockwise order of the planar embedding. Note that, while the $z_{i,1}, z_{i,2}, \dots, z_{i,\delta_i}$ are distinct for given i, there may exist values of i, j and $1 \le a \le \delta_i$, $1 \le b \le \delta_j$ with $z_{i,a} = z_{j,b}$. By the assumed absence of triangles, we have $z_{i,j} \ne y_k$ for all i, j, k.

We list the labels $z_{i,j}$ in lexicographic order (that is, $z_{a,b} < z_{c,d}$ if either $a < c$, or $a = c$ and $b < d$) as $z_1 < z_2 < \cdots < z_s$; this is a total order of the *label-set* Z but not of the underlying *vertices* since a given vertex may occur multiply. If $a < b$ we speak of z_a as preceding, or being to the *left* of z_b (and z_b succeeding, or being to the *right* of z_a). For $1 \le i \le r$, let

$$S_i = (z_{i,j} : j = 1, 2, \dots, \delta_i), \text{ viewed as an ordered subsequence of } Z. \tag{14.1}$$

In making changes to the path ν, it is useful to first record which vertices lie in either $\nu(u-)$ or $\nu(w+)$, or in neither. We label each vertex z as

$$\begin{cases} U & \text{if } z \in \nu(u-), \\ W & \text{if } z \in \nu(w+), \\ Q & \text{if } z \notin \nu(u-) \cup \nu(w+). \end{cases}$$

Write N_L be the number of z_i with label L. Here is a technical lemma.

Lemma 14.1. *Suppose $N_U \ge 1$, and let z_T be the leftmost vertex labelled U. Let $\nu''(u-)$ be the subpath of $\nu(u-)$ from z_T to u, and $\nu'(u-)$ that obtained from $\nu(u-)$ by deleting the edges of $\nu''(u-)$. Let $\alpha = \min\{j : y_j \sim z_T\}$ and $S = (z_t, z_{t+1}, \dots, z_T)$ be the z_i adjacent to y_α that precede or equal z_T.*

(a) *For $t \le i < j \le T$, we have that $z_i \nsim z_j$.*
(b) *For $1 \le i \le T-1$, z_i is labelled Q.*
(c) *For $1 \le i \le T-2$, z_i has no $*$-neighbour lying in $\nu'(u-)$. Furthermore, z_T is the unique $*$-neighbour of z_{T-1} lying in $\nu'(u-)$.*
(d) *For $1 \le i \le T$, z_i has no $*$-neighbour lying in $\nu(w+)$.*

Proof. (a) If $z_i \sim z_j$ for some $t \le i < j \le T$, then (y_α, z_i, z_j) forms a triangle, which is forbidden by assumption.

(b) By the planarity of ν (see Lemma 11.3), $\nu''(u-)$ moves around v in an anticlockwise direction, in the sense that the directed cycle obtained by traversing $\nu''(u-)$ from z_T to u, followed by the edges $\langle u, v\rangle$, $\langle v, y_\alpha\rangle$, $\langle y_\alpha, z_T\rangle$, has winding number -1. If, on the contrary, it has winding number 1, then $\nu''(u-)$ intersects $\nu(w+)$ in contradiction of the planarity of ν. See Figure 14.2.

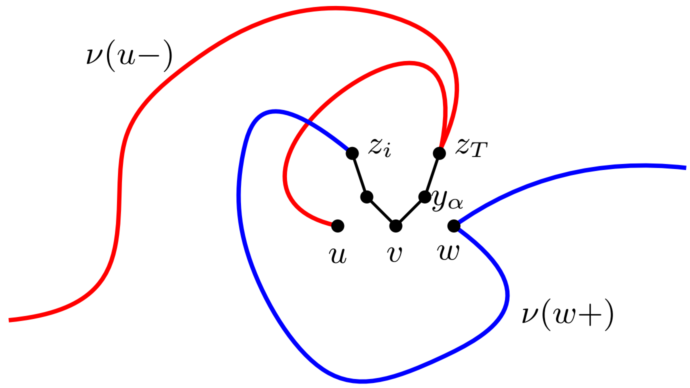

FIGURE 14.2. If $z_i \in \nu(w+)$ and $z_T \in \nu(u-)$, then the pair $\nu(u-)$, $\nu(w+)$ fails to be non-touching.

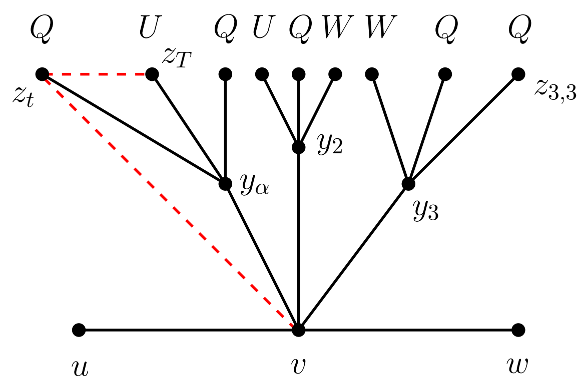

FIGURE 14.3. The dashed red line contains the diagonal $\delta(v, z_t)$.

Let $1 \le i \le T-1$. By assumption, z_i is not labelled U. If $z_i \in \nu(w+)$, then (as illustrated in the figure), $\nu(u-)$ and $\nu(w+)$ must intersect (when viewed as arcs in $\mathcal{H}$). This is a contradiction by Lemma 11.3(b).

(c) If $1 \le i \le T-2$ and z_i has a $*$-neighbour x in $\nu'(u-)$, then $d_{G_*}(x, \nu''(u-)) \le 1$, which (as above) contradicts the assumption that $\nu(u-)$ is non-self-touching in G_*. The second statement holds similarly.

(d) This is similar to the above. □

We consider the various cases, and use the notation of Lemma 14.1.

(a) Suppose $N_U \ge 1$. Start with the path $\nu'(u-)$, and consider the pairs

$$P = \big\{(z_T, z_{T-1}), (z_{T-1}, z_{T-2}), \dots, (z_t, v)\big\}.$$

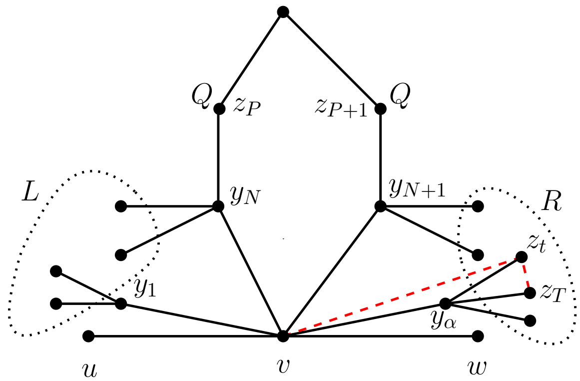

FIGURE 14.4. The path ν passes through a vertex v that lies in a 6-face F. With $z_T \in \nu(u-)$ as given, when $y_\alpha \nsim y_{\alpha-1}$ we may adjust ν to obtain a non-self-touching path ν' passing along the diagonal $\delta(v, z_t)$.

Since G has no triangles (see also Lemma 14.1(a)), every such pair forms a diagonal. We add to $\nu'(u-)$ the vertices $v, z_t, \dots, z_{T-1}$. Let $\overline{\nu}$ be the path of G_* obtained by following $\nu'(u-)$, then the pairs in P, and then $\nu(w+)$. By Lemma 14.1(b, c, d), $\overline{\nu}$ is non-self-touching, and furthermore it contains a diagonal. See Figure 14.3.

(b) If $N_W \geq 1$, we perform a similar construction to the above, utilizing the rightmost appearance of W.

(c) If $N_U = N_W = 0$, we remove v from ν, and replace it by the sequence of sites $y_1, y_2, \dots, y_r$ (joined by their intermediate diagonals). The ensuing path $\overline{\nu}$ is non-self-touching and contains a diagonal.

Next we lift the no-triangle assumption. We now permit G to have triangular faces, but assume it has property $\Box$. By $\Box$, the vertex v is incident to some face denoted F whose boundary has four or more edges. Let u, w, $\nu(u-)$, $\nu(w+)$ be as before. We draw the triple u, v, w as in Figure 14.4, and assume without loss of generality that F lies above the line drawn horizontally in the illustration. We shall use much of the notation introduced above.

Let $y_1, y_2, \dots, y_r$ be the neighbours of v other than u and w, considered clockwise from u to w, as in Figure 14.4, and let $z_1, z_2 \dots, z_s$ be as before (we exclude the y_j from the sequence (z_i)). Let $r \geq 1$. The following technical lemma is related to the earlier Lemma 14.1. With ν as above, let $\nu'(u-)$ and $\nu''(u-)$ be as in Lemma 14.1, and S_i as in (14.1).

Lemma 14.2.

(a) *Let* $s_0 = u$, $s_{r+1} = v$, *and* $s_i = y_i$ *for* $i = 1, 2, \dots, r$. *If* $s_i \sim s_j$ *then* $|i-j| = 1$.

(b) *Suppose* $1 \le T \le s$ *and* z_T *is labelled* U. *Let* α *be such that* $z_T \in S_\alpha$, *and let* $S = (z_t, z_{t+1}, \dots, z_T)$ *be the* z_i *adjacent to* y_α *that precede or equal* z_T. *Assume* $z_t, z_{t+1}, \dots, z_{T-1}$ *are not labelled* U.

 (i) *For* $t \le i \le T-1$, z_i *is labelled* Q. *For* $1 \le i < t$, z_i *is labelled either* Q *or* U.

 (ii) *For* $1 \le i \le T-2$, z_i *has no* $*$*-neighbour lying in* $\nu'(u-)$. *Furthermore,* z_T *is the unique* $*$*-neighbour of* z_{T-1} *lying in* $\nu'(u-)$.

 (iii) *For* $1 \le i \le T$, z_i *has no* $*$*-neighbour lying in* $\nu(w+)$.

Proof. (a) Suppose $s_i \sim s_j$ where $j \ge i+2$. Then (v, s_i, s_j) forms a triangle C of G that intersects the interior of the edge $\langle v, s_{i+1}\rangle$ (viewed as a 1-dimensional simplex). Since G is planar, it follows that $s_{i+1} \in C^\circ$. This is a contradiction since G is assumed $\triangle$-empty.

Part (b) is proved as in the proof of Lemma 14.1. □

Let y_N, y_{N+1} be the neighbours of v in F, and z_P, z_{P+1} their further neighbours in F (if F is a quadrilateral, we have $z_P = z_{P+1}$). We assume that $y_N \ne u$ and $y_{N+1} \ne w$; similar arguments are valid otherwise.

Suppose $z_i \in \nu(u-)$ for some $i \in \{P, P+1\}$. Either $z_i \sim v$ or $\delta(z_i, v)$ is a diagonal. In either case there is a contradiction with the fact that ν is non-self-touching in G_*. A similar argument holds if one of z_P, z_{P+1} lies in $\nu(w+)$. Therefore, neither z_P nor z_{P+1} lies in $\nu(u-) \cup \nu(w+)$, and we label them Q accordingly as in Figure 14.4.

Let $L = \{z_1, z_2, \dots, z_{P-1}\}$ (respectively, $R = \{z_{P+2}, z_{P+2}, \dots, z_s\}$) denote the set of neighbours of y_N and the y_j to its left (respectively, y_{N+1} and the y_j to its right) other than u, v, w and z_P, z_{P+1}. We do not assume that L and R are disjoint when viewed as sets of vertices.

Next, we define an iterative construction. For $P+2 \le a \le s$, let

$$f(a) = \min\{\beta \ge N+1 : y_\beta \sim z_a\}.$$

Let $T \ge P+2$ and let $\alpha \ge N+1$ be such that $z_T \in S_\alpha$, where S_α is given in (14.1). We define $K(T)$ as follows. Let $T_1 = \max\{a \in [\phi(\alpha), T] : f(a) < \alpha\}$ with the convention that the maximum of the empty set is 0.

1. If $T_1 = 0$, let $K(T) = 0$.
2. Assume $T_1 > 0$, so that $S_{f(T_1)}$ contains the vertex represented by the label z_{T_1}, say with label $z_{T_1'} \in S_{f(T_1)}$. We set $K(T) = T_1'$.

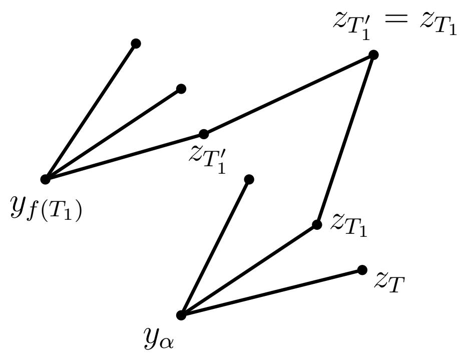

FIGURE 14.5. An illustration of the function K in the proof of Theorem 10.8. For $z_T \in S_\alpha$, we track backwards through S_α from z_T until we find some z_{T_1} representing a vertex that appears in some S_γ with $N_1 \le \gamma < \alpha$. In this example, we have $K(T) = T_1'$.

The motivation for the function K is as follows. A difficulty arises from the fact that each z_j is a label rather than a vertex, and different labels can correspond to the same vertex. For an initial label $z_T \in S_\alpha$, we examine its predecessors in S_α. If no such predecessor (including z_T itself) represents a vertex that appears also in some earlier $S_{N+1}, \dots, S_{\alpha-1}$, we declare $K(T) = 0$. If such a predecessor exists, find the first such $z_{T_1} \in S_\alpha$, and find the earliest z_j (with $j \ge P + 2$) that represents the same vertex as z_{T_1}. Then $K(T)$ is the index of this z_j.

We move now to the argument proper. The idea is to replace a subpath of ν by another set of vertices, thus creating a non-self-touching path $\overline{\nu}$ that includes a diagonal.

(a) Assume some $z_\gamma \in R$ is labelled U, and let z_T be the earliest such z_γ. We remove $\nu''(u-)$ from ν (while retaining its endvertex z_T but not its other endvertex u), noting by Lemma 14.2 that

$$\text{no } *\text{-neighbour of } z_{P+1} \text{ lies in either } \nu''(u-) \text{ or } \nu(w+). \tag{14.2}$$

Next, we add some further vertices in a set A determined according to which of the following cases applies. Let S and α be given as in (14.1) and Lemma 14.2(b).

Case I. Suppose $\alpha = N + 1$. Then $A = \{z_{P+1}\} \cup S$, By (14.2) and Lemma 14.2, the ensuing path $\overline{\nu}$ is non-self-touching and traverses the diagonal $\delta(z_{P+1}, v)$.

Case II. Suppose $\alpha > N + 1$.

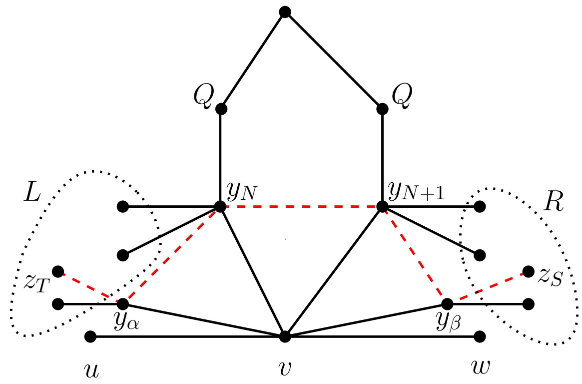

FIGURE 14.6. When the rightmost U is on the left, and the leftmost W is on the right, we replace the subpath of ν from z_T to z_S by the dashed edges.

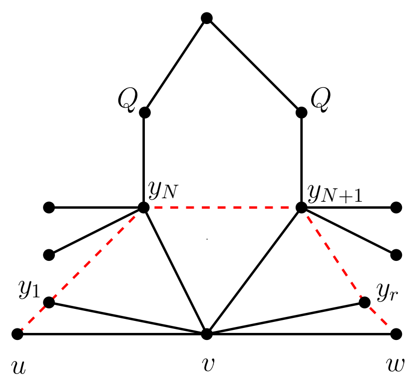

FIGURE 14.7. This is the picture when neither U nor W is represented in the set $R \cup L$.

1. If $K(T) = 0$, we take $A = S$. If $y_\alpha \nsim y_{\alpha-1}$ we stop. The ensuing path $\overline{\nu}$ is non-self-touching and traverses the diagonal $\delta(z_t, v)$. See Figure 14.4.
2. Let $K(T) = 0$, and assume that $y_\alpha \sim y_{\alpha-1}$. If $z_t \nsim y_{\alpha-1}$ we take $A = \{y_{\alpha-1}\} \cup S$. The construction of $\overline{\nu}$ is complete on noting that $\delta(z_t, y_{\alpha-1})$ is a diagonal.
3. Let $K(T) = 0$, and assume that $y_\alpha \sim y_{\alpha-1}$ and $z_t \sim y_{\alpha-1}$. Take $A = \{z_{t-1}\} \cup S$, and repeat the above with (α, T) replaced by $(\alpha - 1, t - 1)$.
4. If $K(T) = T_1' > 0$, repeat the above with (α, T) replaced by $(f(T_1'), T_1')$. See Figure 14.5.

This iterative process terminates with a path $\overline{\nu}$ containing a diagonal of the form either $\delta(z_k, v)$ or $\delta(z_k, y_\beta)$ for some $P+1 \le k < T$ and $N+1 \le \beta < \alpha$. If $\overline{\nu}$ is not non-self-touching, one may apply oxbow-removal (by Lemma 11.1(b)) to obtain a path $\overline{\overline{\nu}}$ containing the above diagonal.

A similar construction is valid if some vertex in L is labelled W.

(e) Assume U appears in $L \setminus R$ but not in R, and W appears in $R \setminus L$ but not in L. By Lemma 14.2(b),

$$\begin{aligned} &\text{no } y_i \text{ with } i \le N \text{ has a neighbour labelled } W; \\ &\text{no } y_i \text{ with } i > N \text{ has a neighbour labelled } U. \end{aligned} \tag{14.3}$$

Let $z_T \in L$ be the rightmost U and $z_S \in R$ the leftmost W, and let $\alpha = \min\{i : y_i \sim z_T\}$ and $\beta = \max\{i : y_i \sim z_S\}$. The z_i between z_T and z_S are labelled Q. We remove from ν the part of $\nu(u-)$ between z_T and v, and similarly that of $\nu(w+)$ between z_S and w (we retain the endpoints z_T and z_S). See Figure 14.6.

Next we add $y_\alpha, y_{\alpha+1}, \dots, y_N$ and also $y_\beta, y_{\beta+1}, \dots, y_{N+1}$. By Lemma 14.2(a), the ensuing $\overline{\nu}$ is non-self-touching, and includes the diagonal $\delta(y_N, y_{N+1})$.

(f) Assume that U appears in $L \setminus R$ but not in R, and W appears nowhere in $L \cup R$. The argument of part (b) applies with the sequence $y_\beta, y_{\beta+1}, \dots, y_{N+1}$ replaced by $y_{N+1}, y_{N+2}, \dots, y_r$.

(g) Finally, if neither U nor W is represented in $L \cup R$, then all members of $L \cup R$ are labelled Q. In this case, we remove v, and we add the points $\{y_i : i = 1, 2, \dots, r\}$. See Figure 14.7. By Lemma 14.2(a), the ensuing $\overline{\nu}$ is non-self-touching and traverses the diagonal $\delta(y_N, y_{N+1})$.

Acknowledgements

ZL's research was supported by National Science Foundation grant 1608896 and Simons Collaboration Grant 638143.

References

[1] M. Aizenman and G. R. Grimmett, *Strict monotonicity for critical points in percolation and ferromagnetic models*, J. Statist. Phys. **63** (1991), 817–835.

[2] M. Aizenman, H. Kesten, and C. M. Newman, *Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation*, Commun. Math. Phys. **111** (1987), 505–531.

[3] R. C. Alperin, *An elementary account of Selberg's lemma*, Enseign. Math. (2) **33** (1987), 269–273.

[4] T. Antunović and I. Veselić, *Sharpness of the phase transition and exponential decay of the subcritical cluster size for percolation on quasi-transitive graphs*, J. Statist. Phys. **130** (2008), 983–1009.

[5] L. Babai, *The growth rate of vertex-transitive planar graphs.*, Proceedings of the Eighth Annual ACM–SIAM Symposium on Discrete Algorithms (New Orleans, LA, 1997), New York, 1997, pp. 564–573.

[6] P. Balister, B. Bollobás, and O. Riordan, *Essential enhancements revisited*, (2014), http://arxiv.org/abs/1402.0834.

[7] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Group-invariant percolation on graphs*, Geom. Funct. Anal. **9** (1999), 29–66.

[8] I. Benjamini and O. Schramm, *Percolation beyond $\mathbb{Z}^d$, many questions and a few answers*, Electron. Commun. Probab. **1** (1996), 71–82.

[9] ———, *Percolation in the hyperbolic plane*, J. Amer. Math. Soc. **14** (2001), 487–507.

[10] J. van den Berg, *Percolation theory on pairs of matching lattices*, J. Math. Phys. **22** (1981), 152–157.

[11] L. Bieberbach, *Über die Bewegungsgruppen der Euklidischen Raumen*, Math. Ann. **71** (1911), 400–412.

[12] B. Bollobás and O. Riordan, *Percolation on dual lattices with k-fold symmetry*, Random Struct. Alg. **32** (2008), 463–472.

[13] C. P. Bonnington, W. Imrich, and M. E. Watkins, *Separating double rays in locally finite planar graphs*, Discrete Math. **145** (1995), 61–72.

[14] S. R. Broadbent and J. M. Hammersley, *Percolation processes: I. Crystals and mazes*, Math. Proc. Cam. Phil. Soc. **53** (1957), 479–497.

[15] R. M. Burton and M. Keane, *Density and uniqueness in percolation*, Commun. Math. Phys. **121** (1989), 501–505.

[16] P. J. Cameron, *Automorphisms of graphs*, Topics in Algebraic Graph Theory (L. W. Beineke and R. J. Wilson, eds.), Cambridge Univ. Press, Cambridge, 2004, pp. 137–155.

[17] F. Camia and C. M. Newman, *Continuum nonsimple loops and 2D critical percolation*, J. Statist. Phys. **116** (2004), 157–173.

[18] ———, *Two-dimensional critical percolation: the full scaling limit*, Commun. Math. Phys. **268** (2006), 1–38.

[19] J. W. Cannon, W. J. Floyd, R. Kenyon, and W. R. Parry, *Hyperbolic geometry*, Flavors of Geometry, Cambridge Univ. Press, Cambridge, 1997, pp. 59–115.

[20] H. Duminil-Copin, *Sixty years of percolation*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018, vol. IV, World Sci. Publ., Hackensack, NJ, 2018, pp. 2829–2856.

[21] H. Duminil-Copin, S. Goswami, A. Raoufi, F. Severo, and A. Yadin, *Existence of phase transition for percolation using the Gaussian Free Field*, Duke Math. J. (2018), to appear.

[22] H. Duminil-Copin, A. Raoufi, and V. Tassion, *Sharp phase transition for the random-cluster and Potts models via decision trees*, Ann. of Math. (2) **189** (2019), 75–99.

[23] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre, *Correlation inequalities on some partially ordered sets*, Commun. Math. Phys. **22** (1971), 89–103.

[24] G. R. Grimmett, *Percolation*, 2nd ed., Springer, Berlin, 1999.

[25] G. R. Grimmett and Z. Li, *Cubic graphs and the golden mean*, Discrete Math. **343** (2020), paper 11638.

[26] G. R. Grimmett and A. M. Stacey, *Critical probabilities for site and bond percolation models*, Ann. Probab. **26** (1998), 1788–1812.

[27] O. Häggström and Y. Peres, *Monotonicity of uniqueness for percolation on Cayley graphs: All infinite clusters are born simultaneously*, Probab. Th. Rel. Fields **113** (1999), 273–285.

[28] O. Häggström, Y. Peres, and R. H. Schonmann, *Percolation on transitive graphs as a coalescent process: Relentless merging followed by simultaneous uniqueness*, Perplexing Problems in Probability, Birkhäuser, Basel–Boston, 1999, pp. 69–90.

[29] J. M. Hammersley, *Comparison of atom and bond percolation processes*, J. Math. Phys. **2** (1961), 728–733.

[30] J. Haslegrave and C. Panagiotis, *Site percolation and isoperimetric inequalities for plane graphs*, Random Struct. Alg. **58** (2021), 150–163.

[31] R. Holley, *Remarks on the FKG inequalities*, Commun. Math. Phys. **36** (1974), 227–231.

[32] H. Hopf, *Enden offener Räume und unendliche diskontinuierliche Gruppen*, Comment. Math. Helv. **16** (1944), 81–100.

[33] B. Iversen, *Hyperbolic Geometry*, Cambridge Univ. Press, Cambridge, 1993.

[34] H. Kesten, *The critical probability of bond percolation on the square lattice equals* $\frac{1}{2}$, Commun. Math. Phys. **74** (1980), 41–59.

[35] ———, *Percolation Theory for Mathematicians*, Birkhäuser, Boston, 1982, https://pi.math.cornell.edu/~kesten/kesten-book.html.

[36] B. Krön, *Infinite faces and ends of almost transitive plane graphs*, Hamburger Beiträge zur Mathematik **257** (2006), https://preprint.math.uni-hamburg.de/public/hbm.html.

[37] Z. Li, *Constrained percolation, Ising model, and XOR Ising model on planar lattices*, Rand. Struct. Alg. **57** (2020), 474–525.

[38] ———, *Positive speed self-avoiding walks on graphs with more than one end*, J. Combin. Th., Ser. A **175** (2020), paper 105257.

[39] R. Lyons and Y. Peres, *Probability on Trees and Networks*, Cambridge Univ. Press, 2016, https://rdlyons.pages.iu.edu/prbtree/.

[40] M. V. Menshikov, *Coincidence of critical points in percolation problems*, Dokl. Akad. Nauk SSSR **288** (1986), 1308–1311, Transl: Soviet Math. Dokl. 33 (1986), 856–859.

[41] ———, *Quantitative estimates and strong inequalities for the critical points of a graph and its subgraph*, Teor. Veroyatnost. i Primenen. **32** (1987), 599–601, Transl: Th. Probab. Appl. 32, 544–547.

[42] M. V. Menshikov, S. A. Molchanov, and A. F. Sidorenko, *Percolation theory and some applications*, Probability theory. Mathematical statistics. Theoretical cybernetics, Itogi Nauki i Tekhniki, vol. 24, 1986, pp. 53–110, Transl: J. Soviet Math. 42 (1988), 1766–1810.

[43] B. Mohar, *Embeddings of infinite graphs*, J. Combin. Th. Ser. B **44** (1988), 29–43.

[44] D. Renault, *Étude des graphes planaires cofinis selon leurs groupes de symétries*, Ph.D. thesis, 2004, Université de Bordeaux.

[45] ———, *The vertex-transitive TLF-planar graphs*, Discrete Math. **309** (2009), 2815–2833.

[46] R. H. Schonmann, *Stability of infinite clusters in supercritical percolation*, Probab. Th. Rel. Fields **113** (1999), 287–300.

[47] ———, *Multiplicity of phase transitions and mean-field criticality on highly non-amenable graphs*, Commun. Math. Phys. **219** (2001), 271–322.

[48] N. Seifter and V. I. Trofimov, *Automorphism groups of graphs with quadratic growth*, J. Combin. Theory Ser. B **71** (1997), 205–210.

[49] S. Sheffield, *Random Surfaces*, Astérisque, no. 304, Société Mathématique de France, 2005.

[50] S. Smirnov, *Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 239–244.

[51] M. F. Sykes and J. W. Essam, *Exact critical percolation probabilities for site and bond problems in two dimensions*, J. Math. Phys. **5** (1964), 1117–1127.

[52] A. Vince, *Periodicity, quasiperiodicity, and Bieberbach's theorem on crystallographic groups*, Amer. Math. Monthly **104** (1997), 27–35.

(GRG) Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WB, UK

Email address: g.r.grimmett@statslab.cam.ac.uk

URL: http://www.statslab.cam.ac.uk/~grg/

(ZL) Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269-3009, USA

Email address: zhongyang.li@uconn.edu

URL: http://www.math.uconn.edu/~zhongyang/