GEOMETRY OF LIPSCHITZ PERCOLATION

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Abstract. We prove several facts concerning Lipschitz percolation, including the following. The critical probability p_{L} for the existence of an open Lipschitz surface in site percolation on $\mathbb{Z}^d$ with $d \geq 2$ satisfies the improved bound $p_{\mathrm{L}} \leq 1-1/[8(d-1)]$. Whenever $p > p_{\mathrm{L}}$, the height of the lowest Lipschitz surface above the origin has an exponentially decaying tail. For p sufficiently close to 1, the connected regions of $\mathbb{Z}^{d-1}$ above which the surface has height 2 or more exhibit stretched-exponential tail behaviour. The last statement is proved via a stochastic inequality stating that the lowest surface is dominated stochastically by the boundary of a union of certain independent, identically distributed random subsets of $\mathbb{Z}^d$.

1. Lipschitz percolation

We consider site percolation with parameter p on the lattice $\mathbb{Z}^d$ with $d \geq 2$, with law denoted $\mathbb{P}_p$. The existence of open Lipschitz surfaces was investigated in [6], the main theorem of which may be summarized as follows. Let $\|\cdot\|$ denote the ℓ^1-norm, and write $\mathbb{Z}^+ = \{1, 2, \dots\}$ and $\mathbb{Z}_0^+ = \{0\} \cup \mathbb{Z}^+$. A function $F : \mathbb{Z}^{d-1} \to \mathbb{Z}^+$ is called *Lipschitz* if:

(1) for any $x, y \in \mathbb{Z}^{d-1}$ with $\|x-y\| = 1$, we have $|F(x)-F(y)| \leq 1$.

Let LIP be the event that there exists a Lipschitz function $F : \mathbb{Z}^{d-1} \to \mathbb{Z}^+$ such that,

$$\text{for each } x \in \mathbb{Z}^{d-1}, \text{ the site } (x, F(x)) \in \mathbb{Z}^d \text{ is open.} \tag{2}$$

The event LIP is clearly increasing. Since it is invariant under translation of $\mathbb{Z}^d$ by the vector $(1, 0, \dots, 0)$, its probability equals either 0 or 1. Therefore, there exists $p_{\mathrm{L}} \in [0, 1]$ such that:

$$\mathbb{P}_p(\text{LIP}) = \begin{cases} 0 & \text{if } p < p_{\mathrm{L}}, \\ 1 & \text{if } p > p_{\mathrm{L}}. \end{cases}$$

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It was proved in [6] that $0 < p_{\mathrm{L}} < 1$, and more concretely that $0 < p_{\mathrm{L}} \le 1 - (2d)^{-2}$. As noted in [6], when $p > p_{\mathrm{L}}$, there exists a Lipschitz function F satisfying (1) with the property that the random field $(F(x) : x \in \mathbb{Z}^{d-1})$ is stationary and ergodic. Applications of these and related statements may be found in [7, 14, 15].

2. Main results

Our first result is an improvement of the upper bound for p_{L} of [6].

Theorem 1. *For $d \ge 2$ we have $p_{\mathrm{L}} \le 1 - [8(d-1)]^{-1}$.*

This is proved in Section 4. The complementary inequality

$$p_{\mathrm{L}} \ge 1 - \frac{1 + \mathrm{o}(1)}{2d} \quad \text{as} \quad d \to \infty$$

is proved in Section 5, yielding that $1/d$ is the correct order of magnitude for $1 - p_{\mathrm{L}}(d)$ in the limit as $d \to \infty$.

A Lipschitz function F satisfying (1) is called *open*. For any family $\mathcal{F}$ of Lipschitz functions, the minimum (or 'lowest') function

$$\overline{F}(x) := \min\{F(x) : F \in \mathcal{F}\}$$

is Lipschitz also. If there exists an open Lipschitz function, there exists necessarily a lowest such function, and we refer to it as the 'lowest open Lipschitz function'. We shall sometimes use the term 'Lipschitz surface' to describe the subset $\{(x, F(x)) : x \in \mathbb{Z}^{d-1}\}$ of $\mathbb{Z}^d$, for some Lipschitz F. See Figure 1. We emphasize that Lipschitz functions are always required to take values in the *positive* integers.

For reasons of exposition, if there exists no open Lipschitz function, we define the lowest open Lipschitz function by $F(x) = \infty$ for all $x \in \mathbb{Z}^{d-1}$. Our second main result is the following.

Theorem 2. *Let $d \ge 2$ and let F be the lowest open Lipschitz function. There exists $\alpha = \alpha(d,p)$ satisfying $\alpha(d,p) > 0$ for $p > p_{\mathrm{L}}$ such that*

$$\mathbb{P}_p(F(0) > n) \le e^{-\alpha n}, \qquad n \ge 0. \tag{3}$$

This is proved in Section 6 by an adaptation of Menshikov's proof of exponential decay for subcritical percolation. Since the law of $F(x)$ is the same for all $x \in \mathbb{Z}^{d-1}$, the choice of the origin 0 in (3) is arbitrary. Theorem 2 extends the exponential-decay theorem of [6] by removing the condition on p that is present in that work.

Our third result is a bound of a different type on the lowest open Lipschitz function F. Recall that, by definition, $F \ge 1$. Let S be the set of all $x \in \mathbb{Z}^{d-1}$ for which $F(x) > 1$. Let S_0 be the vertex-set of

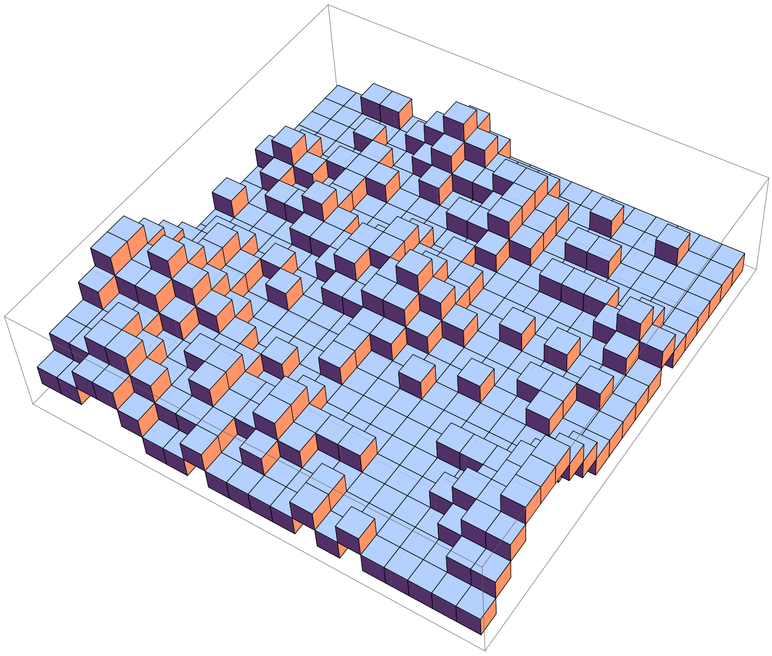

FIGURE 1. Part of the lowest open Lipschitz surface when $d = 3$ and $p = 0.18$. Each cube represents an open site in the surface.

the component containing 0 in the subgraph of the nearest-neighbour lattice of $\mathbb{Z}^{d-1}$ induced by S (and take $S_0 := \varnothing$ if $0 \notin S$).

Theorem 3. *Let $d \geq 2$. There exists $p_{\mathrm{M}} < 1$ such that, for $p > p_{\mathrm{M}}$ and $\epsilon > 0$,*

$$(4)\quad \exp\big(-\lambda n^{1/(d-1)}\big) \leq \mathbb{P}_p(|S_0| > n) \leq \exp\big(-\gamma n^{1/(d-1)-\epsilon}\big), \qquad n \geq 1,$$

where $\lambda = \lambda(d,p)$ and $\gamma = \gamma(d,p,\epsilon)$ are positive and finite. If $d \neq 3$ then (4) holds even with $\epsilon = 0$.

The above statement is similar in spirit to Dobrushin's theorem [8] concerning the existence of 'flat' interfaces in the three-dimensional Ising model with mixed boundary conditions (see also [9] and [12, Chap. 7]). The proof utilizes a bound for the tail of the total progeny in a subcritical branching processes with 'stretched-exponential' family-sizes. (The term 'subexponential' is sometimes used in the literature.) The ϵ of (4) arises from a certain instance in the large-deviation theory of heavy-tailed distributions; see the discussion of Section 8. A mild extension of Theorem 3 is proved in Section 8.

The principal ingredient in the proof of Theorem 3 is the following stochastic inequality, which has other potential applications also. In preparation for this, we introduce the concept of a local cover. For $y \in \mathbb{Z}^{d-1}$, let $\mathcal{F}_y$ be the set of functions $f : \mathbb{Z}^{d-1} \to \mathbb{Z}_0^+$ that are Lipschitz in the sense of (1), and that satisfy:

(a) $f(y) > 0$, and
(b) for all $x \in \mathbb{Z}^{d-1}$, either $f(x) = 0$ or the site $(x, f(x))$ is open.

The *local cover* at y is the Lipschitz function $\overline{f}_y$ given by

$$\overline{f}_y(z) := \min\{f(z) : f \in \mathcal{F}_y\}, \qquad z \in \mathbb{Z}^{d-1}.$$

It is shown in [6] that the lowest open Lipschitz function F is given by

$$F(x) = \sup\{\overline{f}_y(x) : y \in \mathbb{Z}^{d-1}\}. \tag{5}$$

Now let $(g_y : y \in \mathbb{Z}^{d-1})$ be *independent* random functions from $\mathbb{Z}^{d-1}$ to $\mathbb{Z}_0^+$ such that, for each $y \in \mathbb{Z}^{d-1}$, g_y has the same law as $\overline{f}_y$. Let

$$G(x) := \sup\{g_y(x) : y \in \mathbb{Z}^{d-1}\}. \tag{6}$$

Theorem 4. *The lowest open Lipschitz function F is dominated stochastically by G in that, for any increasing subset $A \subseteq [0, \infty]^{\mathbb{Z}^{d-1}}$,*

$$\mathbb{P}_p\Big[(F(x) : x \in \mathbb{Z}^{d-1}) \in A\Big] \le \mathbb{P}\Big[(G(x) : x \in \mathbb{Z}^{d-1}) \in A\Big].$$

Section 3 contains the basic estimate that leads in Section 4 to the proof of Theorem 1. Lower bounds for the critical value p_{L} are found in Section 5. The exponential-decay Theorem 2 is proved in Section 6, followed in Section 7 by the proof of Theorem 4. The final Section 8 contains the proof of Theorem 3.

3. A basic estimate

This section contains a basic estimate (Proposition 5) similar to the principal Lemma 3 of [6], together with a lemma (Lemma 6) that will be useful in the proof of Theorem 1. We begin by introducing some terminology.

Let $d \ge 2$ and $p = 1 - q \in [0, 1]$. The site percolation model on $\mathbb{Z}^d$ is defined as usual by letting each site $x \in \mathbb{Z}^d$ be *open* with probability p, or else *closed*, with the states of different sites independent. The sample space is $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ where 1 represents 'open', and 0 represents 'closed'. The appropriate product probability measure is written $\mathbb{P}_p$, and expectation as $\mathbb{E}_p$. See [11] for a general account of percolation.

As explained in [6], the lowest open Lipschitz function F may be constructed as a blocking surface to certain paths. Let $e_1, e_2, \ldots, e_d \in \mathbb{Z}^d$ be the standard basis vectors of $\mathbb{Z}^d$. We define a Λ-*path* from u to

v to be any finite sequence of distinct sites $u = x_0, x_1, \dots, x_k = v$ of $\mathbb{Z}^d$ such that, for each $i = 1, 2, \dots, k$,

$$x_i - x_{i-1} \in \{\pm e_d\} \cup \{-e_d \pm e_j : j = 1, \dots, d-1\}. \tag{7}$$

The directed step $w := x_i - x_{i-1}$ is called: *upward* (U) if $w = e_d$; *downward vertical* (DV) if $w = -e_d$; *downward diagonal* (DD) otherwise. The notation 'Λ-path' was introduced in [6] to convey an impression of the DD steps available to such a path.

A Λ-path is called *admissible* if every upward step terminates at a closed site, which is to say that, for each $i = 1, 2, \dots, k$,

$$\text{if } x_i - x_{i-1} = e_d \text{ then } x_i \text{ is closed.} \tag{8}$$

For $u = (u_1, u_2, \dots, u_d) \in \mathbb{Z}^d$, we write $h(u) = u_d$ for its *height.* Let

$$L := \mathbb{Z}^{d-1} \times \{0\} \subset \mathbb{Z}^d$$

be the hyperplane of height zero.

A Λ-path is called a *λ-path* if it has no downward vertical steps. Denote by $u \rightarrowtail v$ (respectively, $u \rightarrowtail_\lambda v$) the event that there exists an admissible Λ-path (respectively, λ-path) from u to v. We write $u \overset{+}{\rightarrowtail} v$ and $u \overset{+}{\rightarrowtail}_\lambda v$ for the corresponding events given in terms of paths using no vertex $w \in \mathbb{Z}^d$ with $h(w) < 0$. More generally, for $A, B \subseteq \mathbb{Z}^d$ we write $A \rightarrowtail B$ (and similarly for the other relations) if $a \rightarrowtail b$ for some $a \in A$ and $b \in B$. Similarly, we write '$A \rightarrowtail B$ in C' if such a path exists using only sites in $C \subseteq \mathbb{Z}^d$.

Proposition 5. *Let $d \geq 2$ and let $q = 1 - p$ satisfy $\rho := 8(d-1)q < 1$. Then*

$$\sum_{u \in L:\, \|u\| \geq r} \mathbb{P}_p(0 \overset{+}{\rightarrowtail}_\lambda u) \leq \sum_{n=r}^{\infty} \binom{2n}{n} 2^{-2n} \rho^n, \qquad r \geq 1. \tag{9}$$

Proof. Let $a = 2d - 2$. Any path contributing to the event $0 \overset{+}{\rightarrowtail}_\lambda u$ with $u \in L$ uses some number U of upward steps and some number D of downward diagonal steps. Furthermore, $U = D \geq \|u\|$. Therefore,

$$\begin{aligned} \sum_{u \in L:\, \|u\| \geq r} \mathbb{P}_p(0 \overset{+}{\rightarrowtail}_\lambda u) &\leq \sum_{U = D \geq r} \binom{U+D}{U} q^U a^D \\ &= \sum_{n=r}^{\infty} \binom{2n}{n} (qa)^n, \end{aligned}$$

as required. $\square$

Since $\binom{2n}{n} \le 2^{2n}$, it follows from Proposition 5 when $\rho < 1$ that

$$\sum_{u \in L:\, \|u\| \ge r} \mathbb{P}_p(0 \overset{+}{\rightarrowtail}_\lambda u) \le \frac{\rho^r}{1-\rho}, \qquad r \ge 0. \tag{10}$$

A marginally improved upper bound may be derived by using either Stirling's formula or the local central limit theorem.

Here is a lemma concerning the relationship between Λ-paths and λ-paths. For $S \subseteq \mathbb{Z}^{d-1} \times \mathbb{Z}_0^+$, write

$$\downarrow\!S = \{x \in \mathbb{Z}^{d-1} \times \mathbb{Z}_0^+ : x = s - ke_d \text{ for some } s \in S \text{ and } k \ge 0\}.$$

Lemma 6. *For $\omega \in \Omega$, we have that*

$$\{x \in \mathbb{Z}^{d-1} \times \mathbb{Z}_0^+ : 0 \overset{+}{\rightarrowtail} x\} = \downarrow\!\{x \in \mathbb{Z}^{d-1} \times \mathbb{Z}_0^+ : 0 \overset{+}{\rightarrowtail}_\lambda x\}.$$

Proof. Since every λ-path is a Λ-path, and Λ-paths may end with any number of downward vertical steps without restriction, the right side is a subset of the left side. It remains to show that the left side is a subset of the right side.

Let $x \in \mathbb{Z}^{d-1} \times \mathbb{Z}_0^+$ be such that $0 \overset{+}{\rightarrowtail} x$, and let π be an admissible Λ-path from 0 to x of shortest length. If π contains no DV step, then it is a λ-path, and we are done. Suppose there is a DV step in π, and consider the last one, denoted $\overline{V}$, in the natural order of the path. Then $\overline{V}$ is an ordered pair (x, y) of sites of $\mathbb{Z}^d$ with $y = x - e_d$. Since the sites of the path are distinct, $\overline{V}$ is not followed by a U step. Therefore, $\overline{V}$ is either at the end of the path π, or it is followed by a DD step, which we write as the ordered pair (y, z) with $z = y - e_d + \alpha e_j$ for some $\alpha \in \{-1, 1\}$ and $j \in \{1, 2, \ldots, d-1\}$. In the latter case, we may interchange $\overline{V}$ with this DD step, which is to say that the ordered triple (x, y, z) in π is replaced by (x, y', z) with $y' = y + \alpha e_j$. This change does not alter the admissibility of the path or its endpoints.

A small complication would arise if $y' \in \pi$. If this were to hold, the site sequence thus obtained would contain a loop, and we may erase this loop to obtain an admissible path from 0 to x with fewer steps than π. This would contradict the minimality of the length of π, whence $y' \notin \pi$.

Proceeding iteratively, the position in the path of the last DV step may be delayed until: either it is the last step of the path, or it precedes a U step, denoted $\overline{U}$. In the latter case, we may remove $\overline{U}$ together with the previous DV step to obtain a new admissible Λ-path from 0 to x of shorter length than π, a contradiction. Proceeding thus with every DV step of π, we arrive at an admissible Λ-path from 0 to x

comprising an admissible λ-path followed by a number of DV steps. The claim follows. $\square$

4. Hills, mountains, and the proof of Theorem 1

We describe next the use of Proposition 5 in the proof of Theorem 1. For $y \in L$, the *hill* H_y is given by

$$H_y := \{z \in \mathbb{Z}^d : y \overset{+}{\rightarrowtail} z\}. \tag{11}$$

Hills combine as follows to form 'mountains'. For $x \in L$, the *mountain* M_x of x is given by

$$M_x = \bigcup \{H_y : y \in L \text{ such that } x \in H_y\}. \tag{12}$$

Let $x \in L$ and $S \subseteq \mathbb{Z}^d$. The *height* of S at x is defined as

$$l_x(S) = \sup\{k : x + ke_d \in S\},$$

where the supremum of the empty set is interpreted as 0. The *local height* of the mountain M_x is defined as its height $l_x(M_x)$ above x. By the definition of Λ-paths, we have that: either $l_x(M_x) < \infty$ for all $x \in L$, or $l_x(M_x) = \infty$ for all x.

Define the event $I = \bigcap_{x \in L} \{l_x(M_x) < \infty\}$. The event I is invariant under the action of translation of $\mathbb{Z}^d$ by any vector in L, whence it has probability either 0 or 1. By the above, $\mathbb{P}_p(I) = 1$ if and only if $\mathbb{P}_p(l_0(M_0) < \infty) = 1$.

Let the function $F : \mathbb{Z}^{d-1} \to \mathbb{Z}^+ \cup \{\infty\}$ be defined by

$$F(x) = 1 + l_x(M_x), \qquad x \in L, \tag{13}$$

as in (5). By the above discussion, F is finite-valued if and only if I occurs. It is explained in [6] that F is the lowest open Lipschitz function. In conclusion, the lowest open Lipschitz surface is finite if and only if $\mathbb{P}_p(I) = 1$. In particular,

$$p_{\mathrm{L}} = \inf\{p : \mathbb{P}_p(I) = 1\}. \tag{14}$$

The random field $(F(x) : x \in L)$ is stationary and ergodic under the action of translation of L by any e_j with $j \in \{1, 2, \dots, d-1\}$.

We show in the remainder of this section that $\mathbb{P}_p(I) = 1$ under the condition of the following lemma. For $S \subseteq \mathbb{Z}^d$, the *radius* of S with respect to $0 \in \mathbb{Z}^d$ is given by

$$\mathrm{rad}(S) = \sup\{\|s\| : s \in S\}.$$

Lemma 7. *Let $d \geq 2$, and let $\rho := 8(d-1)q < 1$. Then*

$$\mathbb{P}_p(\mathrm{rad}(H_0) \geq r) \leq \frac{\rho^r}{1-\rho}, \qquad r \geq 1, \tag{15}$$

and there exists an absolute constant $c = c(d)$ such that

$$\mathbb{P}_p(\mathrm{rad}(M_0) \geq r) \leq cr^{d-1}\frac{\rho^{r/2}}{1-\rho}, \qquad r \geq 1. \tag{16}$$

Proof of Theorem 1. Since $l_0(M_0) \leq \mathrm{rad}(M_0)$, we have by (16) that $l_0(M_0) < \infty$ a.s. when $\rho < 1$. By (14), $p_{\mathrm{L}} \leq 1 - 1/[8(d-1)]$. □

The following corollary of Lemma 7 will be used in Section 8. The *footprint* $L(S)$ of a subset $S \subseteq \mathbb{Z}^d$ is its projection onto L:

$$L(S) := \{(s_1, s_2, \ldots, s_{d-1}, 0) : (s_1, s_2, \ldots, s_d) \in S\}. \tag{17}$$

Corollary 8. *Let $d \geq 2$, $p = 1 - q \in (0, 1)$, and $\rho := 8(d-1)q$.*

(a) *There exists $\alpha = \alpha(d, p) < \infty$ such that*

$$\mathbb{P}_p(|L(H_0)| \geq n) \geq \exp(-\alpha n^{1/(d-1)}), \qquad n \geq 2.$$

(b) *There exists $\beta = \beta(d, p)$ satisfying $\beta(d, p) > 0$ when $\rho < 1$, such that*

$$\mathbb{P}_p(|L(M_0)| \geq n) \leq \exp(-\beta n^{1/(d-1)}), \qquad n \geq 2.$$

Proof of Lemma 7. Let $\rho < 1$ and $r \geq 1$. Suppose $0 \overset{+}{\rightarrowtail}_\lambda u$ where $u \in \mathbb{Z}^{d-1} \times \mathbb{Z}_0^+$ and $\|u\| \geq r$. By considering all sites v such that there is a λ-path from u to v using downward diagonal steps only, there must exist $v \in L$ with $0 \overset{+}{\rightarrowtail}_\lambda v$ and $\|v\| \geq r$. Therefore, by (10),

$$\mathbb{P}_p(\mathrm{rad}(H_0^\lambda) \geq r) \leq \sum_{u \in L:\, \|u\| \geq r} \mathbb{P}_p(0 \overset{+}{\rightarrowtail}_\lambda u) \leq \frac{\rho^r}{1-\rho},$$

where

$$H_y^\lambda := \{z \in \mathbb{Z}^d : y \overset{+}{\rightarrowtail}_\lambda z\}, \qquad y \in L.$$

By Lemma 6, $H_0 = {\downarrow} H_0^\lambda$, and in particular $\mathrm{rad}(H_0) = \mathrm{rad}(H_0^\lambda)$. Inequality (15) follows.

By the definition of M_0 and the triangle inequality,

$$\mathbb{P}_p(\mathrm{rad}(M_0) \geq r) \leq \sum_{y \in L} \mathbb{P}_p\big(0 \in H_y,\, \mathrm{rad}(H_y) \geq r - \|y\|\big).$$

The last sum equals

$$\sum_{y \in L} \mathbb{P}_p\big(y \in H_0,\, \mathrm{rad}(H_0) \geq r - \|y\|\big),$$

which we split into two sums depending on whether or not $\|y\| \leq r/2$. The first such sum is no larger than $cr^{d-1}\mathbb{P}_p(\mathrm{rad}(H_0) \geq r/2)$ for some constant c. By Lemma 6 and (10), the second satisfies

$$\sum_{y\in L:\,\|y\|>r/2} \mathbb{P}_p(y \in H_0) = \sum_{y\in L:\,\|y\|>r/2} \mathbb{P}_p(0 \overset{+}{\rightarrowtail}_\lambda y) \le \frac{\rho^{r/2}}{1-\rho},$$

as required. $\square$

Proof of Corollary 8. (a) There exists $c > 0$ such that, if every site in $\{ke_d : 1 \le k \le m\}$ is closed, then $|L(H_0)| \ge cm^{d-1}$. Therefore,

$$\mathbb{P}_p(|L(H_0)| \ge cm^{d-1}) \ge q^m, \tag{18}$$

and the first claim follows.
(b) There exists $c > 0$ such that $|L(M_0)| \le c\,\mathrm{rad}(M_0)^{d-1}$, and the second claim follows by (16). $\square$

5. Inequalities for Lipschitz critical points

By Theorem 1, we have $1 - p_{\mathrm{L}}(d) \ge 1/[8(d-1)]$. In this section, we derive further results concerning the values $p_{\mathrm{L}}(d)$. In particular we prove a lower bound for p_{L} that implies that the correct order of magnitude of $1 - p_{\mathrm{L}}(d)$ is indeed $1/d$ in the limit of large d.

Consider the hypercubic lattice $\mathbb{L}^d$ with vertex-set $\mathbb{Z}^d$. Let π be a (finite or infinite) directed self-avoiding path of $\mathbb{L}^d$ with vertices $x_1, x_2, \dots$. We call π *acceptable* if it contains no upward steps, i.e., if $x_{i+1} - x_i \ne e_d$ for all i.

Consider site percolation on $\mathbb{Z}^d$. An acceptable path is called *open* (respectively, *closed*) if all of its sites are open (respectively, closed). Let $p_{\mathrm{c}}^{\downarrow}(d)$ be the critical probability for the existence of an infinite open acceptable path from the origin.

Proposition 9. *The sequence* $(p_{\mathrm{L}}(d) : d \ge 2)$ *is non-decreasing and satisfies* $p_{\mathrm{L}}(d) \ge 1 - p_{\mathrm{c}}^{\downarrow}(d)$.

Let $p_{\mathrm{c}}(d)$ be the critical probability of site percolation on $\mathbb{L}^d$, and let $\vec{p}_{\mathrm{c}}(d)$ be the critical probability of the oriented site percolation process on $\mathbb{L}^d$ in which every edge is oriented in the direction of increasing coordinate direction. It is elementary by graph inclusion that

$$p_{\mathrm{c}}(d) \le p_{\mathrm{c}}^{\downarrow}(d) \le \min\{p_{\mathrm{c}}(d-1), \vec{p}_{\mathrm{c}}(d)\}.$$

Several proofs are known that $2dp_{\mathrm{c}}(d) \to 1$ as $d \to \infty$ (see [11, p. 30]; indeed the lace expansion permits an expansion of $p_{\mathrm{c}}(d)$ in inverse

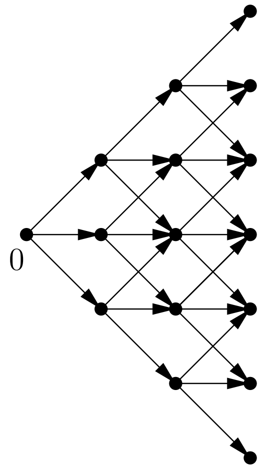

FIGURE 2. Part of the lattice $\vec{\mathbb{L}}^2_{\mathrm{alt}}$ of [13], obtained by adding oriented edges to $\mathbb{Z}^2$.

powers of $2d$). Hence, by Proposition 9,

$$1 - p_{\mathrm{L}}(d) \le \frac{1 + \mathrm{o}(1)}{2d} \qquad \text{as } d \to \infty.$$

The value of $p_{\mathrm{L}}(2)$ may be expressed as the critical value of a certain percolation model. Consider the oriented graph $\vec{\mathbb{L}}^2_{\mathrm{alt}}$ obtained from $\mathbb{Z}^2$ by placing an oriented bond from u to v if and only if $v - u \in \{e_1, e_1 \pm e_2\}$. This graph was used in [13], and is illustrated in Figure 2. Let p_{alt} be the critical probability of oriented site percolation on $\vec{\mathbb{L}}^2_{\mathrm{alt}}$. It is shown in [19, Thm 5.1] that $p_{\mathrm{alt}} \ge \frac{1}{2}$. It is elementary that $p_{\mathrm{alt}} \le \vec{p}_{\mathrm{c}}(2)$, and it was proved in [1] that $\vec{p}_{\mathrm{c}}(2) \le 0.7491$ (see also [17]). In summary,

$$\tfrac{1}{2} \le p_{\mathrm{alt}} \le \vec{p}_{\mathrm{c}}(2) \le 0.7491.$$

Simulations in [19] indicate that $p_{\mathrm{alt}} \approx 0.535$.

Proposition 10. *We have that* $p_{\mathrm{L}}(2) = p_{\mathrm{alt}}$.

The equation $p_{\mathrm{L}}(2) = p_{\mathrm{alt}} \ge \frac{1}{2}$ has been noted in independent work of Berenguer (personal communication).

Proof of Proposition 9. Let $F : \mathbb{Z}^{d-1} \to \mathbb{Z}^+$ be Lipschitz. The restriction $G : \mathbb{Z}^{d-2} \to \mathbb{Z}^+$ given by

$$G(x_1, x_2, \ldots, x_{d-2}) := F(x_1, x_2, \ldots, x_{d-2}, 0)$$

is Lipschitz also, and the monotonicity of $p_{\mathrm{L}}(d)$ follows.

Let $q = 1 - p$ satisfy $q > p_{\mathrm{c}}^{\downarrow}(d)$. Suppose there exists an acceptable closed path from ne_d to some site in L. Fix such a path, and let x

be its earliest site lying in L. Then we see that $x \overset{+}{\rightarrowtail} ne_d$, and hence $0 \in H_x$ and $l_0(M_0) \geq n$. By (13), $F(0) > n$.

On the other hand, by a standard argument of percolation theory, on the event that there exists an infinite acceptable closed path starting from ne_d, there exists a.s. an acceptable closed path from ne_d to some site in L. Hence

$$\mathbb{P}_p(F(0) > n) \geq \theta^{\downarrow}(q) > 0,$$

where $\theta^{\downarrow}(q)$ is the probability that the origin lies in an infinite acceptable closed path. Thus $p \leq p_{\mathrm{L}}$, and hence $1 - p_{\mathrm{c}}^{\downarrow} \leq p_{\mathrm{L}}$. □

Proof of Proposition 10. This is similar to part of the proof of [15, Thm 6]; see also [15, Lemma 7]. Let $d = 2$. If $p > p_{\mathrm{L}}$, there exists a.s. a site z on the 2-coordinate axis of $\mathbb{Z}^2$ such that z is the starting point of some infinite open oriented path of $\vec{\mathbb{L}}^2_{\mathrm{alt}}$. Hence $p \geq p_{\mathrm{alt}}$, so that $p_{\mathrm{L}} \geq p_{\mathrm{alt}}$.

Conversely, let $p > p_{\mathrm{alt}}$. By the block construction of [13] or otherwise (see also [15, Lemma 7]), there is a strictly positive probability $\theta^+(p)$ that the site $(0,1)$ of $\mathbb{Z}^2$ is the starting point of an infinite open oriented path of $\vec{\mathbb{L}}^2_{\mathrm{alt}}$ using no site with 2-coordinate lying in $(-\infty, 0]$. By considering a reflection of $\vec{\mathbb{L}}^2_{\mathrm{alt}}$ in the 2-coordinate axis, there is probability at least $p^{-1}\theta^+(p)^2$ that there exists an open Lipschitz function $F : \mathbb{Z}^2 \to \mathbb{Z}^+$. Therefore, $p \geq p_{\mathrm{L}}$, so that $p_{\mathrm{alt}} \geq p_{\mathrm{L}}$. □

6. Exponential decay

In this section, we prove exponential decay of the tail of $F(0)$ when $p > p_{\mathrm{L}}$, as in Theorem 2. Let $d \geq 2$ and $L_n = \mathbb{Z}^{d-1} \times \{n\}$, so that $L = L_0$. For $x \in L$, let

$$K_x := \sup\{n : x \rightarrowtail L_n\}.$$

Recall the lowest open Lipschitz function F of (13).

Lemma 11. *Let $p \in (0,1)$ and $x \in L$. The random variables K_x and $F(x) - 1 = l_x(M_x)$ have the same distribution.*

Proof. This holds by a process of path-reversal. It is convenient in this proof to work with bond percolation rather than site percolation. Each bond of the hypercubic lattice $\mathbb{Z}^d$ is designated 'open' with probability p and 'closed' otherwise, different bonds receiving independent states. We call a Λ-path *admissible* if any directed step along a bond from some y to $y + e_d$ is closed. It is clear that the set of admissible paths has the same law as in the formulation using site percolation.

A Λ^--path from u to v is defined as any finite sequence of distinct sites $u = x_0, x_1, \ldots, x_k = v$ of $\mathbb{Z}^d$ such that, for each $i = 1, 2, \ldots, k$,

$$x_i - x_{i-1} \in \{\pm e_d\} \cup \{e_d \pm e_j : j = 1, \ldots, d-1\}. \tag{19}$$

Any step in the direction $-e_d$ is called *downward.* A Λ^--path is called $(-)$-*admissible* if every downward step traverses a closed bond.

Let π be a Λ-path of $\mathbb{Z}^d$, and let $\rho\pi$ be the Λ^--path obtained by reversing each step. Note that π is admissible if and only if $\rho\pi$ is $(-)$-admissible.

It suffices to assume $x = 0$. Let Π_n be the union over $y \in L_0$ of the set of Λ-paths from y to ne_d. Then $\rho\Pi_n$ is the set of Λ^--paths from ne_d to L, so that

$$\mathbb{P}_p(L \rightarrowtail ne_d) = \mathbb{P}_p(ne_d \overset{-}{\rightarrowtail} L),$$

where $\overset{-}{\rightarrowtail}$ denotes connection by an admissible Λ^--path. By a reflection of the lattice, the last probability equals $\mathbb{P}_p(0 \rightarrowtail L_n)$, so that

$$\mathbb{P}_p(l_0(M_0) \geq n) = \mathbb{P}_p(0 \rightarrowtail L_n),$$

and the claim follows by (13). $\square$

Theorem 12. *There exists $\alpha = \alpha(d,p)$ satisfying $\alpha(d,p) > 0$ for $p > p_{\mathrm{L}}$ such that*

$$\mathbb{P}_p(0 \rightarrowtail L_n) \leq e^{-\alpha n}, \qquad n \geq 0.$$

Proof of Theorem 2. This is immediate by Theorem 12 and Lemma 11. $\square$

Proof of Theorem 12. The proof is an adaptation of Menshikov's proof of a corresponding fact for percolation, as presented in [11, Sect. 5.2]. We shall use the BK inequality of [3] (see also [11, Sect. 2.3]). The application of the BK inequality (but not the inequality itself) differs slightly in the current context from that of regular percolation, and we illustrate this as follows. Let $a, b, u, v \in \mathbb{Z}^d$. The 'disjoint occurrence' of the events $\{a \rightarrowtail b\}$ and $\{u \rightarrowtail v\}$ is written as usual $\{a \rightarrowtail b\} \circ \{u \rightarrowtail v\}$. In this setting, it comprises the set of configurations such that: there exist admissible Λ-paths $\pi_{a,b}$ from a to b, and $\pi_{u,v}$ from u to v, such that each directed edge from some x to $x + e_d$ lies in no more than one of $\pi_{a,b}$, $\pi_{u,v}$. That is, the paths must have no upward step in common; they are permitted to have downward steps in common.

Let $m \geq 1$ and $R_m = [-m, m]^{d-1} \times \mathbb{Z}$. For $x \in R_m \cap L$ and $r \geq 0$, let

$$g_p^{x,m}(r) = \mathbb{P}_p(x \rightarrowtail L_r \text{ in } R_m), \qquad g_p(r) = \mathbb{P}_p(0 \rightarrowtail L_r),$$

and

$$h_p^m(r) = \max\{g_p^{x,m}(r) : x \in R_m \cap L\}. \tag{20}$$

On recalling the definition of an admissible Λ-path, we see that the event $\{x \rightarrowtail L_r \text{ in } R_m\}$ is a finite-dimensional cylinder event, and so each $g_p^{x,m}(r)$ is a polynomial in p. Note that

$$h_p^m(r) \geq g_p^{0,m}(r) \to g_p(r) \quad \text{ as } m \to \infty.$$

Since $g_p^{x,m}(r) \leq g_p^{x,\infty}(r) = g_p(r)$, we have that

$$h_p^m(r) \to g_p(r) \qquad \text{as } m \to \infty. \tag{21}$$

By (20), $h_p^m(r)$ is the maximum of a finite set of polynomials in p. Therefore, $h_p^m(r)$ is a continuous function of p, and there exists a finite set $\mathcal{D}^m(r) \subseteq (0,1)$ such that: for $p \notin \mathcal{D}^m(r)$, there exists $x = x_p^m(r) \in R_m \cap L$ with

$$\frac{d}{dp} h_p^m(r) = \frac{d}{dp} g_p^{x,m}(r). \tag{22}$$

Let $x \in R_m \cap L$ and $A^{x,m}(n) = \{x \rightarrowtail L_n \text{ in } R_m\}$, so that $g_p^{x,m}(n) = \mathbb{P}_p(A^{x,m}(n))$. By Russo's formula, [11, eqn (5.10)],

$$\frac{d}{dp} \log \mathbb{P}_p(A^{x,m}(n)) = -\frac{1}{1-p} \mathbb{E}_p\big(N(A^{x,m}(n)) \,\big|\, A^{x,m}(n)\big), \tag{23}$$

where $N(A)$ is the number of pivotal sites for a decreasing event A. We claim, as in [11, eqn (5.18)], that

$$\mathbb{E}_p(N\big(A^{x,m}(n)) \,\big|\, A^{x,m}(n)\big) \geq \frac{n}{\sum_{r=0}^n h_p^m(r)} - 1. \tag{24}$$

Once this is proved, it follows by (23) that

$$\frac{d}{dp} \log g_p^{x,m}(n) \leq -\frac{1}{1-p} \left(\frac{n}{\sum_{r=0}^n h_p^m(r)} - 1 \right).$$

Therefore, by (22), for $p \notin \mathcal{D}^m(n)$,

$$\frac{d}{dp} \log h_p^m(n) \leq -\frac{1}{1-p} \left(\frac{n}{\sum_{r=0}^n h_p^m(r)} - 1 \right). \tag{25}$$

We integrate (25) to obtain, for $0 < \alpha < \beta < 1$,

$$h_\beta^m(n) \leq h_\alpha^m(n) \exp\left(-(\beta - \alpha) \left[\frac{n}{\sum_{r=0}^n h_\alpha^m(r)} - 1 \right] \right),$$

as in [11, eqn (5.22)]. Let $m \to \infty$, and deduce by (21) that

$$g_\beta(n) \leq g_\alpha(n) \exp\left(-(\beta - \alpha) \left[\frac{n}{\sum_{r=0}^n g_\alpha(r)} - 1 \right] \right). \tag{26}$$

This last inequality may be analyzed just as in [11, Sect. 5.2] to obtain the claim of the theorem.

It remains to prove (24), and the proof is essentially that of [11, Lemma 5.17]. Fix $x \in R_m \cap L$, and let V be a random variable taking values in the non-negative integers with

$$\mathbb{P}(V \geq k) = h_p^m(k), \qquad k \geq 0.$$

Suppose $A^{x,m}(n)$ occurs, and let $z_1, z_2, \dots, z_N$ be the pivotal sites for $A^{x,m}(n)$, listed in the order in which they are encountered beginning at x. Let $\rho_1 := \max\{0, h(z_1) - 1\}$ and

$$\rho_i := \max\big\{0, h(z_i) - h(z_{i-1}) - 1\big\}, \qquad i = 2, 3, \dots, N. \tag{27}$$

Thus, ρ_i measures the positive part of the 'vertical' displacement between the $(i-1)$th and ith pivotal sites.

We claim as in [11, eqn (5.19)] that, for $k \geq 1$,

$$\mathbb{P}_p\big(\rho_1 + \cdots + \rho_k \leq n - k \,\big|\, A^{x,m}(n)\big) \geq \mathbb{P}(V_1 + \cdots + V_k \leq n - k) \tag{28}$$

where the V_i are independent, identically distributed copies of V. Equation (24) follows from this as in [11, Sect. 5.2], and, as in that reference, it suffices for (28) to show the equivalent in the current setting of [11, Lemma 5.12], namely the next lemma. □

Lemma 13. *Let k be a positive integer, and let $r_1, r_2, \dots, r_k$ be non-negative integers with sum not exceeding $n-k$. With the above notation, for $x \in R_m \cap L$,*

$$\begin{aligned}\mathbb{P}_p\big(\rho_k \leq r_k,\ \rho_i = r_i \text{ for } 1 \leq i < k \,\big|\, A^{x,m}(n)\big)&\\ \geq \mathbb{P}(V \leq r_k)\mathbb{P}_p\big(\rho_i = r_i \text{ for } 1 \leq i < k \,\big|\, A^{x,m}(n)\big)&.\end{aligned} \tag{29}$$

Proof. Suppose first that $k = 1$ and $0 \leq r_1 \leq n - 1$. By the BK inequality,

$$\begin{aligned}\mathbb{P}_p(\{\rho_1 > r_1\} \cap A^{x,m}(n)) &\leq \mathbb{P}_p(A^{x,m}(r_1 + 1) \circ A^{x,m}(n))\\ &\leq \mathbb{P}_p(A^{x,m}(r_1 + 1))\mathbb{P}_p(A^{x,m}(n))\\ &\leq \mathbb{P}(V > r_1)\mathbb{P}_p(A^{x,m}(n)),\end{aligned}$$

so that (29) holds with $k = 1$.

Turning to the general case, let $k \geq 1$ and let the r_i satisfy the given conditions. For a site z, let D_z be the set of sites attainable from x along admissible Λ-paths of R_m not containing the upward step from $z - e_d$ to z. Let B_z be the event that the following statements hold:

(a) $z - e_d \in D_z$, and $z \notin D_z$,
(b) z is closed,
(c) D_z contains no vertex of L_n,

(d) the pivotal sites for the event $\{0 \rightarrowtail z\}$ are, taken in order, $z_1, z_2, \dots, z_{k-1} = z$, with the ρ_i of (27) satisfying $\rho_i = r_i$ for $1 \le i < k$.

Let $B = \bigcup_z B_z$, and note that

$$B \cap A^{x,m}(n) = \{\rho_i = r_i \text{ for } 1 \le i < k\} \cap A^{x,m}(n). \tag{30}$$

For $\omega \in A^{x,m}(n) \cap B$, there exists a unique site $\zeta = \zeta(\omega)$ such that B_ζ occurs.

Let Δ denote the ordered pair (D_ζ, ζ). Now,

$$\mathbb{P}_p(A^{x,m}(n) \cap B) = \sum_\delta \mathbb{P}_p(B \cap \{\Delta = \delta\}) \mathbb{P}_p\big(A^{x,m}(n) \,\big|\, B \cap \{\Delta = \delta\}\big), \tag{31}$$

where the sum is over all possible values $\delta = (\delta_z, z)$ of the random pair Δ. Note that

$$\mathbb{P}_p\big(A^{x,m}(n) \,\big|\, B \cap \{\Delta = \delta\}\big) = \mathbb{P}_p\big(z \rightarrowtail L_n \text{ in } R_m \setminus \delta_z\big).$$

By a similar argument and the BK inequality,

$$\begin{aligned} &\mathbb{P}_p(\{\rho_k > r_k\} \cap A^{x,m}(n) \cap B) \\ &\quad = \sum_\delta \mathbb{P}_p(B \cap \{\Delta = \delta\}) \mathbb{P}_p\big(\{\rho_k > r_k\} \cap A^{x,m}(n) \,\big|\, B \cap \{\Delta = \delta\}\big), \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\mathbb{P}_p\big(\{\rho_k > r_k\} \cap A^{x,m}(n) \,\big|\, B \cap \{\Delta = \delta\}\big) \\ &\qquad\qquad \le \mathbb{P}(V > r_k) \mathbb{P}_p\big(A^{x,m}(n) \,\big|\, B \cap \{\Delta = \delta\}\big). \end{aligned}$$

On dividing (32) by (31), we deduce that

$$\mathbb{P}_p\big(\rho_k > r_k \,\big|\, A^{x,m}(n) \cap B\big) \le \mathbb{P}(V > r_k).$$

The lemma is proved on multiplying through by $\mathbb{P}_p(B \mid A^{x,m}(n))$, and recalling (30). $\square$

7. Decomposition of hill-ranges

This section is devoted to the domination argument used in the proof of Theorem 4. Let $\omega \in \Omega$ be a configuration of the site percolation model on $\mathbb{Z}^d$. Let $v_1, v_2, \dots$ be an arbitrary but fixed ordering of the vertices in L, and write $H_i = H_{v_i}$ for the hill at v_i, as defined in (11). We shall construct the H_i in an iterative manner, and observe the relationship between a hill thus constructed and the previous hills. To this end, we introduce some further notation. For $i \geq 1$, let

$$\overline{H}_i := \bigcup_{j \leq i} H_j.$$

We call a Λ-path *non-negative* if it visits no site w with $h(w) < 0$. For $u, v \in \mathbb{Z}^d \setminus \overline{H}_i$, write $u \overset{+}{\rightarrowtail}_i v$ if there exists an admissible non-negative Λ-path from u to v using no site of $\overline{H}_i$. The 'restricted' hill at v_{i+1} is given by

$$\widetilde{H}_{i+1} := \{z \in \mathbb{Z}^d : v_{i+1} \overset{+}{\rightarrowtail}_i z\}.$$

That is, $\widetilde{H}_{i+1}$ is given as before but in terms of non-negative Λ-paths in the region obtained from $\mathbb{Z}^d$ by removal of all hills already constructed. If $v_{i+1} \in \overline{H}_i$, then $\widetilde{H}_{i+1} := \varnothing$. Finally, let $\widetilde{H}_1 := H_1$.

Lemma 14. *For $\omega \in \Omega$,*

$$\overline{H}_i = \bigcup_{j \leq i} \widetilde{H}_j, \qquad i \geq 1. \tag{33}$$

Note that the above is a *pointwise* statement in that it holds for all configurations ω. Its main application is as follows. In writing that two random variables A, B may be coupled with a certain property Π, we mean that there exists a probability space that supports two random variables A', B' having the same laws as A, B and with property Π. Let $(J_i : i = 1, 2, \dots)$ be independent, identically distributed subsets of $\mathbb{Z}^d$ such that J_i has the same law as H_i.

Theorem 15. *The families $(H_i : i \geq 1)$ and $(J_i : i \geq 1)$ may be coupled in such a way that the following property holds:*

$$\overline{H}_i \subseteq \bigcup_{j \leq i} J_j, \qquad i \geq 1. \tag{34}$$

Proof of Lemma 14. We prove equation (33) by induction on i. It is trivial for $i = 1$.

Suppose (33) holds for $i = I \geq 1$, and consider the case $i = I + 1$. Suppose $\overline{H}_I$ has been found, and consider the vertex v_{I+1}. If $v_{I+1} \in \overline{H}_I$, then $H_{I+1} \subseteq \overline{H}_I$ and $\widetilde{H}_{I+1} = \varnothing$. In this case, (33) holds with $i = I + 1$.

Suppose $v_{I+1} \notin \overline{H}_I$. Since $\widetilde{H}_{I+1}$ is the hill of v_{I+1} within a restricted domain, we have $\widetilde{H}_{I+1} \subseteq H_{I+1}$ and $\overline{H}_{I+1} \supseteq \overline{H}_I \cup \widetilde{H}_{I+1}$. It remains to prove that $\overline{H}_{I+1} \subseteq \overline{H}_I \cup \widetilde{H}_{I+1}$, which holds if $\overline{H}_{I+1} \setminus \overline{H}_I \subseteq \widetilde{H}_{I+1}$. Let $y \in \overline{H}_{I+1} \setminus \overline{H}_I = H_{I+1} \setminus \overline{H}_I$. Since $y \in H_{I+1}$, there exists an admissible non-negative Λ-path π from v_{I+1} to y. If $\pi \cap \overline{H}_I \neq \varnothing$, π has a first vertex z lying in $\overline{H}_I$. Since no admissible non-negative Λ-path can exit $\overline{H}_I$, all points on π beyond z belong to $\overline{H}_I$, and in particular $y \in \overline{H}_I$, a contradiction. Therefore, there exists an admissible non-negative Λ-path from v_{I+1} to y not intersecting $\overline{H}_I$, which is to say that $y \in \widetilde{H}_{I+1}$. In this case, (33) holds with $i = I + 1$.

In all cases, (33) holds with $i = I + 1$, and the induction step is complete. $\square$

Proof of Theorem 15. For $S \subseteq \mathbb{Z}^d$, write

$$\Delta_{\mathrm{u}} S = \{x + e_d : x \in S\} \setminus S.$$

By the definition of the hill H_y of a vertex $y \in L$, every vertex in $\Delta_{\mathrm{u}} H_y$ is open. Since H_y may be constructed by a path-exploration process, an event of the form $\{H_y = S\}$ is an element of the σ-field $\mathcal{F}_S$ generated by the random variables $\omega(s)$, $s \in S \cup \Delta_{\mathrm{u}} S$. Furthermore, for $i \geq 1$, the event $\{\overline{H}_i = S\}$ lies in $\mathcal{F}_S$.

Suppose we are given that $\overline{H}_i = S$ for some i and some S. Any event defined in terms of admissible non-negative Λ-paths of $\mathbb{Z}^d \setminus S$ is independent of the states of $\Delta_{\mathrm{u}} S$, since no such admissible Λ-path contains an upward step with second endvertex in $\Delta_{\mathrm{u}} S$.

Consider a sequence of independent site percolations on $\mathbb{Z}^d$ with parameter p. Let J_i be the hill at v_i in the ith such percolation model. In particular, for every i, J_i has the same law as H_i. We construct as follows a sequence $(\widetilde{H}'_i)$ with the same joint law as $(\widetilde{H}_i)$ and satisfying

$$\bigcup_{j \leq i} \widetilde{H}'_j \subseteq \bigcup_{j \leq i} J_j, \qquad i \geq 1,$$

and the claim of the theorem will follow by Lemma 14. First, we take $\widetilde{H}'_1 = J_1$. Then we let $\widetilde{H}'_2$ be the subset of J_2 containing all endpoints of all admissible non-negative Λ-paths from v_2 in the second percolation model that do not intersect $\widetilde{H}'_1$. More generally, having found $\widetilde{H}'_j$ for $j < i$, we let $\widetilde{H}'_i$ be the subset of J_i containing all endpoints of all admissible non-negative Λ-paths from v_i in the ith percolation model that do not intersect $\widetilde{H}'_1 \cup \cdots \cup \widetilde{H}'_{i-1}$. $\square$

Proof of Theorem 4. By Theorem 15, there exists a probability space on which are defined random variables $(H'_y, J'_y : y \in L)$ such that

(a) the family $(H'_y : y \in L)$ has the same joint law as $(H_y : y \in L)$,
(b) the J'_y are independent, and each J'_y has the same law as H_y,
(c) for all $x \in L$,

$$\bigcup \{H'_y : y \in L \text{ with } x \in H'_y\} \subseteq \bigcup \{J'_y : y \in L \text{ with } x \in J'_y\}.$$

Let

$$F'(x) = 1 + \sup\{l_x(H'_y) : y \in L\},$$
$$G'(x) = 1 + \sup\{l_x(J'_y) : y \in L\}.$$

By (c) above, $F'(x) \le G'(x)$ for all $x \in L$, and the claim follows since F' (respectively, G') has the same law as F (respectively, G). $\square$

8. Finiteness of Mountain-Ranges

We next apply Theorem 4 in order to prove Theorem 3, which states in particular that for p sufficiently close to 1, the lowest open Lipschitz surface is simply the hyperplane $L + e_d$ pierced by mountain-ranges of finite extent. Moreover, we prove an upper bound for the tail of the volume of a mountain-range.

The *footprint* $L(S)$ of a subset $S \subseteq \mathbb{Z}^d$ was defined at (17). Let H_x (respectively, M_x) be the hill (respectively, mountain) at the site $x \in L$, as in (11) (respectively, (12)), and note that $M_x \cap M_y \neq \varnothing$ if and only if $L(M_x) \cap L(M_y) \neq \varnothing$. For $x, y \in L$ with $x \neq y$, write $x \overset{M}{\longleftrightarrow} y$ if $M_x \cap M_y \neq \varnothing$. Let $G(\overset{M}{\longleftrightarrow})$ denote the graph having vertex set L, and an edge between vertices x and y if and only if $x \overset{M}{\longleftrightarrow} y$. Connected components of $G(\overset{M}{\longleftrightarrow})$ are called *mountain-ranges*. The mountain-range R_x at the vertex $x \in L$ is the set of all vertices $z \in L$ such that: there exists $k \ge 1$ and $x_1, x_2, \ldots, x_k \in L$ such that $x \in M_{x_1}$, $z \in M_{x_k}$, and $M_{x_i} \cap M_{x_{i+1}} \neq \varnothing$ for $1 \le i < k$. For $z \in R_x$ with $z \neq x$, the minimal value of k above is denoted $d_M(x, z)$. Note that $x \in R_x$, and set $d_M(x, x) = 0$.

As a measure of the extent of the mountain-range at x, we shall study its volume $|R_x|$ and its 'mountain radius' given by

$$\rho_M(x) := \sup\{d_M(x, z) : z \in R_x\}.$$

The results of this section are valid for p sufficiently large. Let

$$\sigma(d, p) := \sum_{r \ge 1} c(2r+1)^{d-2} r^{d-1} \frac{\rho^{r/2}}{1-\rho},$$

where $\rho = 8(d-1)q < 1$ and c is given as in Lemma 7. The function σ is decreasing in p. Let

$$p_{\mathrm{M}}(d) := \inf\{p : \sigma(d, p) < 1\}, \tag{35}$$

and note that $p_{\mathrm{M}} < 1$. We shall work with $p > p_{\mathrm{M}}$, and have not attempted to weaken this assumption.

Theorem 16. *Let $d \geq 2$ and $p_{\mathrm{M}} < p < 1$, so that $\sigma = \sigma(d,p) < 1$.*

(a) *We have $\mathbb{E}_p|R_0| \leq (1-\sigma)^{-1}$, and*
$$\mathbb{P}_p(\rho_M(0) \geq n) \leq \sigma^n, \qquad n \geq 0.$$

(b) *Let $d \neq 3$. There exists $\gamma = \gamma(d,p) > 0$ such that*
$$\mathbb{P}_p(|R_0| \geq n) \leq \exp(-\gamma n^{1/(d-1)}), \qquad n \geq 2.$$

(c) *Let $d = 3$. For every $\epsilon > 0$, there exists $\gamma = \gamma(p,\epsilon) > 0$ such that*
$$\mathbb{P}_p(|R_0| \geq n) \leq \exp(-\gamma n^{\frac{1}{2}-\epsilon}), \qquad n \geq 2.$$

A slightly more precise estimate may be obtained when $d = 3$, but we omit this since our methods, in their simplest form, will not deliver the anticipated tail $\exp(-\gamma n^{1/2})$. The reason for this is that the exponent $n^{1/2}$ is a boundary case of the large-deviation theory of random variables with stretched-exponential tails, as explained for example in [5]. With the exception of this case, the bounds of Theorem 16(b, c) have optimal order; see Corollary 8, and note that $L(H_0) \subseteq R_0$.

Proof of Theorem 3. From (13) we have $S_0 \times \{0\} \subseteq R_0$, so the upper bound is immediate from Theorem 16(b,c). The lower bound may be proved by a minor modification of (18), or may be deduced directly from Corollary 8(a) as follows. Let Y be the set of $y \in L(H_0)$ for which $l_y(H_0) \geq 1$. Then $S_0 \times \{0\} \supseteq Y$, and, since every site in $L(H_0) \setminus Y$ must have a neighbour in Y, we have $|S_0| \geq |Y| \geq (2d-1)^{-1}|L(H_0)|$. $\square$

Theorem 16 is proved by a comparison with an independent family, and an appeal to results for branching processes. Let $d \geq 2$, and let $J = (J_y : y \in L)$ be a vector of random subsets of L such that:

(a) for each $y \in L$ we have $y \in J_y$,
(b) the sets J_y are independent,
(c) the distribution of the translated set $J_y - y$, does not depend on the choice of y.

We shall impose a further condition on J, namely the following. We say that J is 0-*monotone* if, for $S \subseteq L$ with $0 \notin S$, the conditional distribution of J_0, given $S \cap J_0 = \varnothing$, is stochastically smaller than J_0. Let $\mathbb{P}$ denote the appropriate probability measure.

The random set J_0 satisfies the above condition whenever its law, considered as a probability measure on $\{0,1\}^L$, is positively associated (a discussion of positive association may be found in [12, Sect. 2.2]). An example of this arises in a commonly studied instance of the continuum percolation model, namely when J_0 has support in a given *increasing* sequence of subsets of L.

For $x \in L$, let

$$K_x := \bigcup \{J_y : y \in L \text{ is such that } x \in J_y\}. \tag{36}$$

For $x, y \in L$, we write $x \overset{K}{\longleftrightarrow} y$ if there exist $k \geq 1$ and $z_1, z_2, \dots, z_k \in L$ such that $x \in K_{z_1}$, $y \in K_{z_k}$, and $K_{z_i} \cap K_{z_{i+1}} \neq \varnothing$ for $1 \leq i < k$. When $z \neq x$, we write $d_K(x, z)$ for the minimal value of k above, and we set $d_K(x, x) = 0$. Let the *cluster* at $x \in L$ be the set

$$C_x := \{y : x \overset{K}{\longleftrightarrow} y\},$$

and let its 'radius' be

$$\rho_K(x) = \sup\{d_K(x, z) : z \in C_x\}.$$

We seek conditions under which $\mathbb{P}(|C_0| < \infty) = 1$ or, stronger, $\rho_K(0)$ and $|C_0|$ have stretched-exponential tails.

Theorem 17. *Let $d \geq 2$.*

(a) *If J is 0-monotone and $\mu := \mathbb{E}|K_0| - 1$ satisfies $\mu < 1$, then $\mathbb{E}|C_0| \leq (1 - \mu)^{-1}$ and*

$$\mathbb{P}(\rho_K(0) \geq n) \leq \mu^n, \qquad n \geq 0.$$

(b) *If, in addition, there exist $\zeta > 0$ and $a \in (0, \frac{1}{2}) \cup \{1\}$ such that*

$$\mathbb{P}(|K_0| \geq n) \leq \exp(-\zeta n^a), \qquad n \geq 2,$$

then there exists $\zeta' \in (0, \infty)$ such that

$$\mathbb{P}(|C_0| \geq n) \leq \exp(-\zeta' n^a), \qquad n \geq 2.$$

This theorem is related to certain results for continuum percolation to be found in [10, 18]. The proof of part (b) will make use of the following theorem for the tail of the total progeny in a branching process with stretched-exponential family-size distribution.

Theorem 18. *Let T be the total progeny in a branching process with typical family-size F satisfying $\mathbb{E}F < 1$ and*

$$\mathbb{P}(F > n) \leq \exp(-\gamma n^a), \qquad n \geq 2, \tag{37}$$

for constants $\gamma \in (0, \infty)$ and $a \in (0, \frac{1}{2}) \cup \{1\}$. There exists $\gamma' \in (0, \infty)$ such that

$$\mathbb{P}(T > n) \leq \exp(-\gamma' n^a), \qquad n \geq 2.$$

Proof of Theorem 17. Suppose that J is 0-monotone, and let L be ordered in some arbitrary but fixed manner starting with the origin. We shall construct the cluster C_0 in an iterative manner, and shall compare certain features of C_0 with those of a branching process. This branching process will be subcritical if $\mu := \mathbb{E}|K_0 \setminus \{0\}| < 1$, and the claims will follow by Theorem 18.

At stage 0, we write $l_0 := 0$; the *family* of l_0 is defined to be the set $F_0 := K_{l_0} \setminus \{l_0\}$, and we declare l_0 to be *dead* and sites in F_0 to be *live*. At stage 1, we let l_1 be the earliest live site, and define its family as the set $F_1 := K_{l_1} \setminus F_0$; we declare sites in F_1 to be live, and l_1 to be *dead*. Suppose that, after stage $n-1$, we have defined the families $F_0, F_1, \dots, F_{n-1}$, and have a current live set G_{n-1} and dead set D_{n-1}. At stage n, we let l_n be a live site (chosen in a way that we describe next), and declare

$$\begin{aligned}
F_n &:= K_{l_n} \setminus (\{l_0\} \cup F_0 \cup \dots \cup F_{n-1}), \\
G_n &:= (G_{n-1} \cup F_n) \setminus \{l_n\}, \\
D_n &:= D_{n-1} \cup \{l_n\}.
\end{aligned}$$

The site l_n is chosen as follows. Let $m := \min\{k : F_k \cap G_{n-1} \neq \varnothing\}$, the index of the earliest family containing a live site, and let l_n be the earliest live site in F_m.

This process either terminates or continues forever. In the former case, let N be the greatest value of n for which F_n is defined, and set $N = \infty$ in the latter case. It is easily seen that $C_0 = \bigcup_{n=0}^{N} F_n$.

We claim that the above process is dominated stochastically by a branching process with family-sizes distributed as $X := |K_0 \setminus \{0\}|$, and we explain this by a recursive argument. The size of the family F_0 of l_0 is evidently distributed as X, and is given in terms of the sequence J as follows. Let

$$Y_0 := \{y \in L : l_0 \in J_y\}.$$

Then $F_0 = \left(\bigcup_{y \in Y_0} J_y\right) \setminus \{l_0\}$. In determining the family F_1 of l_1, we set

$$Y_1 := \{y \in L \setminus Y_0 : l_1 \in J_y\},$$

and we have that

$$F_1 = \left(\bigcup_{y \in Y_1} J_y\right) \setminus (\{l_0\} \cup F_0).$$

Since J is 0-monotone, given Y_0 the family $(J_y : y \notin Y_0)$ is stochastically dominated by a family $(J'_y : y \notin Y_0)$ of independent random sets such that J'_y has the distribution of J_y. It follows that, given F_0, the conditional distribution of $|F_1|$ is no greater than that of X.

Let $n \geq 1$. Suppose stage $n-1$ is complete, and write

$$Y_i := \big\{ y \in L \setminus (Y_0 \cup \cdots \cup Y_{i-1}) : l_i \in J_y \big\},$$

$$F_i := \left(\bigcup_{y \in Y_i} J_y \right) \setminus (\{l_0\} \cup F_0 \cup \cdots \cup F_{i-1}),$$

for $0 \leq i \leq n$. Given (l_i, Y_i, F_i) for $0 \leq i < n$, the set $(J_y : y \notin Y_0 \cup \cdots \cup Y_{n-1})$ is stochastically dominated by an independent family $(J'_y : y \notin Y_0 \cup \cdots \cup Y_{n-1})$ where each J'_y has the unconditional law of J_y. Therefore, the conditional law of $|F_n|$ is stochastically smaller than that of X. This completes the proof of domination by a branching process.

The dominating branching process has mean family-size $\mu := \mathbb{E}|K_0 \setminus \{0\}|$ which, by assumption, satisfies $\mu < 1$. The process is therefore subcritical. It follows in particular that $\mathbb{E}|C_0| \leq (\mathbb{E}|K_0| - 1)^{-1}$, and $\mathbb{P}(|C_0| < \infty) = 1$. The radius $\rho_K(0)$ of C_0 is stochastically smaller than the number of generations of the branching process, so that

$$\mathbb{P}(\rho_K(0) \geq n) \leq \mu^n,$$

and part (a) of the theorem is proved.

Part (b) is a consequence of Theorem 18. □

Proof of Theorem 18. The total progeny of a branching process is connected to the length of a busy period of a certain queue, and to a first-passage time of a certain random walk. One may construct a proof that exploits the connection to queueing theory and makes use of [2, Thm 1.2], but instead we shall use the representation in terms of random walks.

Let X be a random variable taking values in the non-negative integers, such that $\mathbb{E}X < 1$. Consider a random walk $(S_n : n \geq 0)$ on the integers with $S_0 = 1$ and steps distributed as $X - 1$. Let

$$T := \inf\{n : S_n = 0\}.$$

It is standard that T has the same distribution as the size of the total progeny of a branching process with family-sizes distributed as X. The relationship between the branching process and the random walk is as follows. Suppose the elements of the branching process are ordered within families in some arbitrary way. When an element having a family of size F arrives, the walker is displaced by a step $F - 1$.

Let F satisfy $\mathbb{E}F < 1$ and (37) with some $\gamma > 0$ and $a \in (0, \frac{1}{2})$. Let F' be a random variable taking values in the non-negative integers such that $\mathbb{E}F' < 1$ and

$$\mathbb{P}(F' \geq n) = \exp(-\gamma n^a), \qquad n \geq N, \tag{38}$$

for some $N \geq 1$. Let T' be the first-passage time to 0 of a random walk with steps distributed as $F' - 1$, starting at 1.

If $0 < a < \frac{1}{2}$, by [4, Thms 8.2.3–8.2.4], there exists $c \in (0, \infty)$ such that

$$\mathbb{P}(T' \geq n) \sim c \exp\bigl(-[1 - \mathbb{E}F']n\bigr) \quad \text{as} \quad n \to \infty. \tag{39}$$

Let $0 < \epsilon < 1 - \mathbb{E}F$. It is an elementary exercise to find a positive integer N and a random variable F', taking values in the non-negative integers, such that:

(a) F is stochastically smaller than F',
(b) $\mathbb{E}F' \leq \mathbb{E}F + \epsilon$,
(c) (38) holds.

With F' chosen thus, we have $\mathbb{E}F' < 1$ by (b). Therefore, the corresponding first-passage time T' satisfies (39). The claim of the theorem follows by the fact that T is stochastically smaller than T'.

When $a = 1$, basically the same argument is valid. When the moment generating function $M(\theta) = \mathbb{E}(e^{\theta F'})$ satisfies

$$\theta_0 := \sup\{\theta : M(\theta) < \infty\} > 0,$$

quite precise estimates are known for the tail of T' in terms of the minimum of M. Complications arise when the minimum is achieved at $\theta = \theta_0$, as discussed in [4, Sect. 8.2.3]. We avoid these details here by citing [16, Thm 1] in place of [4, Thms 8.2.3–8.2.4] above. □

Proof of Theorem 16. Let $p_{\mathrm{M}} < p < 1$, implying in particular that $\rho < 1$. Let $J = (J_y : y \in L)$ be independent subsets of L such that, for each y, J_y has the same law as $L(H_y)$. The corresponding sets K_x are given as in (36). We shall apply Theorem 17 to the sequence J.

By Lemma 7, $\mathbb{E}|L(M_0)| \leq 1 + \sigma(d, p)$. Since J_0 has the same law as $L(H_0)$, and $\mathrm{rad}(H_0) = \mathrm{rad}(L(H_0))$, the proof of (16) may be followed with M_0 and H_y replaced by K_0 and J_y respectively. This yields

$$\mathbb{P}_p(\mathrm{rad}(K_0) \geq r) \leq c r^{d-1} \frac{\rho^{r/2}}{1 - \rho}, \qquad r \geq 1, \tag{40}$$

and hence $\mu := \mathbb{E}|K_0| - 1$ satisfies

$$\mu \leq \sigma(d, p) < 1.$$

We claim that J is 0-monotone. Let $S \subseteq L$ with $0 \notin S$, and let $A \subseteq L$. The events $\{L(H_0) \subseteq A\}$ and $\{L(H_0) \cap S = \varnothing\}$ are increasing, and therefore

$$\mathbb{P}_p(L(H_0) \subseteq A \mid L(H_0) \cap S = \varnothing) \geq \mathbb{P}_p(L(H_0) \subseteq A), \tag{41}$$

by the FKG inequality. The claim follows.

By either Theorem 4 or Theorem 15, $|R_0|$ and $\rho_M(0)$ are bounded above (stochastically) by $|C_0|$ and $\rho_K(0)$. The claim of part (a) follows by Theorem 17(a).

Let $d \geq 2$. The proof of Corollary 8(b) holds with M_0 and $L(M_0)$ replaced by K_0, and with (40) in place of (16), yielding that

$$\mathbb{P}(|K_0| \geq n) \leq \exp(-\beta n^{1/(d-1)}), \qquad n \geq 2, \tag{42}$$

where $\beta > 0$.

When $d \neq 3$, claim (b) holds by (42) and Theorem 17(b). Let $d = 3$. By (42), for all $a \in (0, \frac{1}{2})$ there exists $\zeta > 0$ such that

$$\mathbb{P}_p(|K_0| \geq n) \leq \exp(-\zeta n^a), \qquad n \geq 2,$$

and claim (c) follows by Theorem 17(b). $\square$

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