Two Griffiths inequalities for the Potts model

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29 July 2007

Abstract

Ganikhodjaev and Razak have proved versions of the two Griffiths inequalities for the ferromagnetic Potts model. We show how their inequalities may be derived via the FKG inequality for the random-cluster representation of the Potts model.

1 Introduction

Ganikhodjaev and Razak have shown in [1] how to formulate and prove two Griffiths-type inequalities for the Potts model with a general number q of local states. Our purpose in this note is to derive their inequalities using the FKG inequality for the random-cluster representation of the Potts model.

2 The inequalities

Let $G = (V, E)$ be a finite graph, and let $J = (J_e : e \in E)$ be a vector of non-negative reals and $q \in \{2, 3, \dots\}$. As in [1], we take as local state space for the q-state Potts model the set $\mathcal{Q} = \{-Q, -Q+1, -Q+2, \dots, Q\}$ where $Q = \frac{1}{2}(q-1)$. The important properties of $\mathcal{Q}$ for what follows are that $|\mathcal{Q}| = q$ and $\mathcal{Q} = -\mathcal{Q}$. The Potts measure on G with parameters J has state space $\Sigma = \mathcal{Q}^V$ and probability measure

$$\pi(\sigma) = \frac{1}{Z} \exp\left(\sum_{e \in E} J_e \delta_e(\sigma) \right), \qquad \sigma = (\sigma_v : v \in V) \in \Sigma,$$

where

$$\delta_e(\sigma) = \delta_{\sigma_x, \sigma_y} \qquad \text{for } e = \langle x, y \rangle \in E,$$

is a Kronecker delta, and Z is the appropriate normalizing constant.

We shall make use of the random-cluster representation in this note, and we refer the reader to [5] for a recent account and bibliography. In particular,

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we shall use the following fact. Let $\Omega = \{0,1\}^E$, and let ϕ denote the random-cluster measure on Ω with edge-parameters $p_e = 1 - e^{-J_e}$ and cluster-weighting factor q. Suppose ω is sampled from Ω according to ϕ, and write $k(\omega)$ for the number of 'open clusters' of ω. To each open cluster of ω we allocate a uniformly chosen spin from $\mathcal{Q}$, such that every vertex in the cluster receives this spin, and the spins of different clusters are independent. The ensuing spin vector σ has law π. See [5], Theorem 1.13 for a proof of this standard fact.

We present next the inequalities of [1]. For $\sigma \in \Sigma$ let

$$\sigma^R = \prod_{v \in R} \sigma_v, \qquad R \subseteq V.$$

Thinking now of σ as a random vector with law π, we write $\langle \sigma^R \rangle$ for the mean value of σ^R.

Theorem 2.1. *For $R, S \subseteq V$, we have that $\langle \sigma^R \rangle \geq 0$ and $\langle \sigma^R \sigma^S \rangle \geq \langle \sigma^R \rangle \langle \sigma^S \rangle$.*

As pointed out in [1], these inequalities generalize the two Griffiths inequalities for the Ising model, see [3, 4]. We do not know if they were known prior to [1]. Check [2].

Several feasible extensions come to mind...

3 Proof of Theorem 2.1

We shall use the coupling of the random-cluster and Potts model described in Section 2. Let $\omega \in \Omega$ and let $A_1, A_2, \ldots, A_k$ be the vertex-sets of the open clusters of ω. Let $R \subseteq V$. We call R *even* (with respect to ω) if $|R \cap A_i|$ is even for every $i \in \{1, 2, \ldots, k\}$. Let $\chi_R(\omega)$ be the indicator function of the event that R is even. Note that $E_R \equiv 0$ if R has odd cardinality.

Let $g_R : \Omega \to \mathbb{R}$ be given by

$$g_R(\omega) = \chi_R(\omega) \prod_{i=1}^{k} E(Q^{|A_i|}). \tag{3.1}$$

where Q is chosen uniformly at random from $\mathcal{Q}$.

Lemma 1. *Let $\omega \in \Omega$ and let σ be chosen at random according to the coupling of Section 2. The conditional expectation of σ^R given ω equals $g_R(\omega)$.*

Proof. Clearly

$$\sigma^R = \prod_{i=1}^{k} Q_i^{|A_i|},$$

where Q_i is the spin allocated to the ith cluster A_i. Assume that R is not even, say that $a = |R \cap A_1|$ is odd. Then Q_1^a has conditional mean satisfying

$$E(Q_1^a) = E((-Q_1)^a) = -E(Q_1^a),$$

whence $E(Q_1^a) = 0$. The claim of the lemma follows. ∎

By Lemma 1, $\langle \sigma^R \rangle = \phi(g_R)$. Now g_R takes values in the non-negative reals, and the first inequality of Theorem 2.1 follows.

Lemma 2. *The function $g_R : \Omega \to \mathbb{R}$ is non-decreasing on the partially ordered set Ω.*

Once this is proved, the second inequality of Theorem 2.1 follows immediately by the FKG property of ϕ, see [5], Theorem 3.8.

Proof. It is clear that χ_R is non-decreasing on Ω, since the addition of new open edges has the effect of joining together open clusters. By (3.1), $g_R(\omega) > 0$ if and only if $\chi_R(\omega) = 1$. It suffices therefore to show (using the above notation) that

$$h_R(\omega) = \prod_{i=1}^{k} E(Q^{|A_i|})$$

is non-decreasing on the event $\{\omega : \chi_R(\omega) = 1\}$ that R is even. By considering the case when ω' is obtained from ω by adding an edge between two clusters of ω, it suffices that

$$E(T^{m+n}) \geq E(T^m)E(T^n), \qquad m, n \geq 1, \tag{3.2}$$

where $T = Q^2$. This trivial inequality may be proved in several ways, of which one is the following. Let T_1, T_2 be independent copies of T. It is trivial that

$$(T_1^m - T_2^m)(T_1^n - T_2^n) \geq 0, \tag{3.3}$$

since either $T_1 \leq T_2$ or $T_1 > T_2$. Inequality (3.2) follows by multiplying out (3.3) and averaging. □

References

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