SELF-AVOIDING WALKS AND THE FISHER TRANSFORMATION

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Abstract. The Fisher transformation acts on cubic graphs by replacing each vertex by a triangle. We explore the action of the Fisher transformation on the set of self-avoiding walks of a cubic graph. Iteration of the transformation yields a sequence of graphs with common critical exponents, and with connective constants converging geometrically to the golden mean.

We consider the application of the Fisher transformation to one of the two classes of vertices of a bipartite cubic graph. The connective constant of the ensuing graph may be expressed in terms of that of the initial graph. When applied to the hexagonal lattice, this identifies a further lattice whose connective constant may be computed rigorously.

1. Introduction

A *self-avoiding walk* (abbreviated to SAW) on a graph G is a path that visits no point more than once. SAWs were introduced in the chemical theory of polymerization (see Flory [5]), and their critical behaviour has been studied since by mathematicians and physicists (see, for example, the book [13] of Madras and Slade). The exponential rate of growth of the number of SAWs is given by the so-called *connective constant* $\mu = \mu(G)$ of the graph. Only few graphs of interest have connective constants that are known exactly.

We explore the action of the Fisher transformation on the set of SAWs of a cubic graph G. The transformation maps G to a new graph $F(G)$. We have two sets of results. First, the connective constants of G and $F(G)$ satisfy a simple functional relation, and in addition, three of the principal critical exponents are invariant under the transformation. In addition, under repeated applications of the Fisher transformation, the graphs converge to a version of the Sierpinski gasket, and

Date: 24 August 2012, minor revision 5 April 2013.

2010 *Mathematics Subject Classification.* 05C30, 82B20.

Key words and phrases. Self-avoiding walk, connective constant, cubic graph, Fisher transformation, quasi-transitive graph.

the connective constants converge geometrically to the golden mean. See Theorems 3.1 and 3.2 for formal statements of these results.

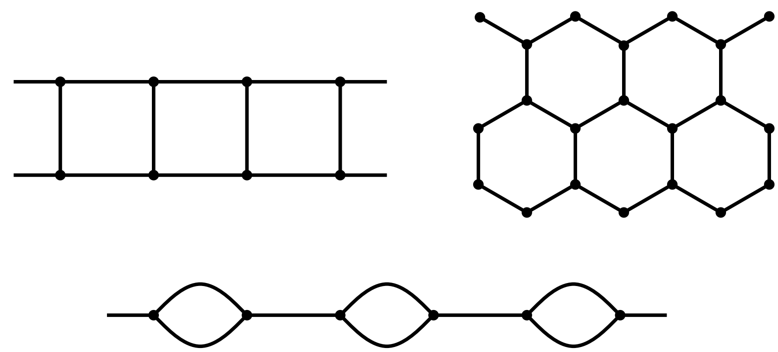

Figure 1.1. Three cubic graphs: the (doubly-infinite) ladder graph $\mathbb{L}$; the hexagonal lattice $\mathbb{H}$; the *bridge graph* $\mathbb{B}_3$ obtained from $\mathbb{Z}$ by joining every alternating pair of consecutive vertices by 2 parallel edges.

Our second set of results concerns the application of the Fisher transformation to a bipartite graph G one of whose vertex-sets is cubic. As before, the ensuing connective constant may be expressed in terms of that of G, and the critical exponents are invariant. When applied to the hexagonal lattice $\mathbb{H}$ (see Figure 1.1), this yields the lattice $\widetilde{\mathbb{H}}$ illustrated in Figure 1.2. Nienhuis's proposed value $\mu(\mathbb{H}) = \sqrt{2+\sqrt{2}}$ has been proved recently by Duminil-Copin and Smirnov [3], and the value of $\mu(\widetilde{\mathbb{H}})$ may be deduced rigorously from this, namely as the root of the equation

$$x^{-3} + x^{-4} = \frac{1}{2+\sqrt{2}}.$$

See Theorem 3.3.

Section 2 is devoted to basic definitions. The Fisher transformation, and its action on counts of SAWs, is described in Section 3, and our Theorems 3.1–3.3 are stated there. The proofs of results are found in Sections 4 and 5.

In the companion papers [7, 8], we study inequalities for the connective constants of regular graphs. For an infinite, connected, cubic, quasi-transitive graph G (possibly with parallel edges), it is elementary that

$$1 \le \mu(G) \le 2. \tag{1.1}$$

If such G is vertex-transitive and simple (or non-simple and satisfying a certain condition), it is proved in [7, Thms 4.1, 4.3] that $\sqrt{2} \le \mu(G) \le$

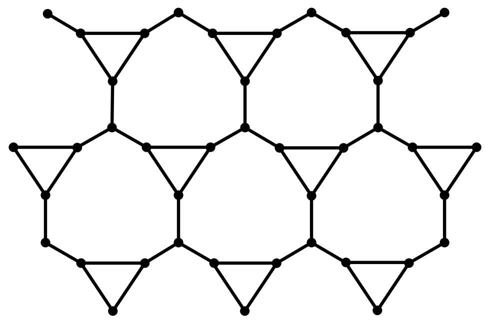

FIGURE 1.2. The lattice $\widetilde{\mathbb{H}}$ derived from the hexagonal lattice $\mathbb{H}$ by applying the Fisher transformation at alternate vertices. Its connective constant $\widetilde{\mu}$ is the root of the equation $x^{-3} + x^{-4} = 1/(2 + \sqrt{2})$.

2, with equalities for the bridge graph $\mathbb{B}_3$ of Figure 1.1 and the 3-regular tree, respectively.

2. Notation

All graphs studied henceforth in this paper will be assumed infinite, connected, and simple (in that they have neither loops nor multiple edges). An edge e with endpoints u, v is written $e = \langle u, v\rangle$. If $\langle u, v\rangle \in E$, we call u and v *adjacent* and write $u \sim v$. The *degree* of vertex v is the number of edges incident to v, denoted $\deg(v)$. A graph is called *cubic* if all vertices have degree 3. The *graph-distance* between two vertices u, v is the number of edges in the shortest path from u to v, denoted $d_G(u, v)$.

The automorphism group of the graph $G = (V, E)$ is denoted $\mathcal{A} = \mathcal{A}(G)$. The graph G is called *quasi-transitive* if there exists a finite subset $W \subseteq V$ such that, for $v \in V$ there exists $\alpha \in \mathcal{A}$ such that $\alpha v \in W$. We call such W a *fundamental domain*, and shall normally (but not invariably) take W to be minimal with this property. The graph is called *vertex-transitive* (or *transitive*) if the singleton set $\{v\}$ is a fundamental domain for some (and hence all) $v \in V$.

A *walk* w on G is an alternating sequence $v_0 e_0 v_1 e_1 \cdots e_{n-1} v_n$ of vertices v_i and edges e_i such that $e_i = \langle v_i, v_{i+1}\rangle$. We write $|w| = n$ for the *length* of w, that is, the number of edges in w.

Let $n \in \mathbb{N}$, the natural numbers. An n-step *self-avoiding walk* (SAW) on G is a walk containing n edges that includes no vertex more than once. Let $\sigma_n(v)$ be the number of n-step SAWs starting at $v \in V$. It was shown by Hammersley [9] that, if G is quasi-transitive, there exists a constant $\mu = \mu(G)$, called the *connective constant* of G, such that

$$\mu = \lim_{n\to\infty} \sigma_n(v)^{1/n}, \qquad v \in V. \tag{2.1}$$

It will be convenient to consider also SAWs starting at 'mid-edges'. We identify the edge e with a point (also denoted e) placed at the middle of e, and then consider walks that start and end at these mid-edges. Such a walk is *self-avoiding* if it visits no vertex or mid-edge more than once, and its *length* is the number of vertices visited.

The minimum of two reals x, y is denoted $x \wedge y$, and the maximum $x \vee y$.

3. Fisher transformation

Let $G = (V, E)$ be a simple graph and let $v \in V$ have degree 3. The so-called *Fisher transformation* acts on v by replacing it by a triangle, as illustrated in Figure 3.1. This transformation has been valuable in the study of the relations between Ising, dimer, and general vertex models (see [2, 4, 11, 12]), and more recently of SAWs on the Archimedean lattice denoted $(3, 12^2)$ (see [6, 10]). In the remainder of this paper, we make use of the Fisher transformation in the context of SAWs and the connective constant. It will be applied to cubic graphs, of which the hexagonal and square/octagon lattices are examples.

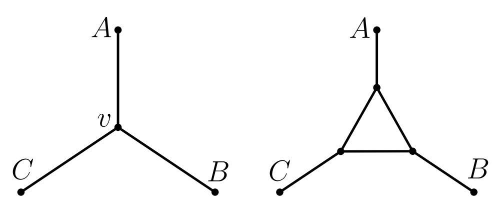

Figure 3.1. The Fisher triangulation of the star. Any triangle thus created is called a *Fisher triangle*.

It is convenient to work with graphs with well-defined connective constants, and to this end we assume that $G = (V, E)$ is quasi-transitive and connected, so that its connective constant is given by (2.1). We write $F(G)$ for the graph obtained from the cubic graph G by applying the Fisher transformation at every vertex. The automorphism group of G induces an automorphism subgroup of $F(G)$. We write $\phi = \frac{1}{2}(\sqrt{5}+1)$ for the golden mean. The next theorem may be known to others.

Theorem 3.1. *Let G be an infinite, quasi-transitive, connected, cubic graph, and consider the sequence $(G_k : k = 0, 1, 2, \dots)$ defined by $G_0 = G$ and $G_{k+1} = F(G_k)$. Then*

(a) *The connective constants* μ_k *of the* G_k *satisfy* $\mu_k^{-1} = g(\mu_{k+1}^{-1})$ *where* $g(x) = x^2 + x^3$.
(b) *The sequence* μ_k *converges monotonely to* ϕ, *and*

$$-\left(\tfrac{4}{7}\right)^k \le \mu_k^{-1} - \phi^{-1} \le \left[\tfrac{1}{2}(7-\sqrt{5})\right]^{-k}, \qquad k \ge 1.$$

Theorem 3.1 provokes the question of the existence of a graph limit of repeated application of the Fisher transformation. It is easily seen that the limiting graph comprises two copies of the Sierpinski gasket, as illustrated in Figure 3.2.

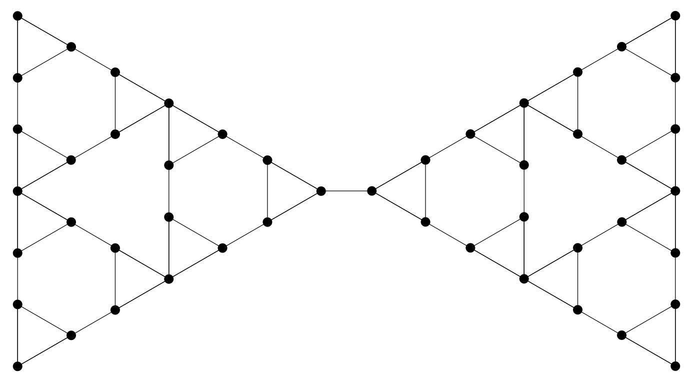

FIGURE 3.2. Through repeated application of the Fisher transformations to a single edge with endvertices of degree three, one arrives at a graph comprising two Sierpinski gaskets.

By Theorem 3.1(b), either $\mu_k \downarrow \phi$ or $\mu_k \uparrow \phi$. The *decreasing* limit holds if and only if $\mu_0 \ge \phi$. We present no satisfactory characterization of graphs G for which $\mu(G) \ge \phi$ beyond noting that this holds whenever G contains as a subgraph a copy of a graph with connective constant ϕ, such as the ladder graph $\mathbb{L}$ (or the semi-infinite ladder graph) of Figure 1.1. Furthermore, if $\mu(G) > \phi$ and $\widetilde{G}$ is obtained from G by a sequence of Fisher transformations, then $\mu(\widetilde{G}) \ge \phi$. We ask whether $\mu(G) \ge \phi$ for any infinite, connected, vertex-transitive, simple, cubic graph G.

We turn to the topic of *critical exponents*, beginning with a general introduction for the case when there exists a periodic, locally finite embedding of G into $\mathbb{R}^d$ with $d \ge 2$. The case of general G has not not been studied extensively, and most attention has been paid to the hypercubic lattice $\mathbb{Z}^d$. It is believed (when $d \ne 4$) that there is a power-order correction, in the sense that there exists $A_v > 0$ and an exponent

$\gamma \in \mathbb{R}$ such that

$$\sigma_n(v) \sim A_v n^{\gamma-1}\mu^n \qquad \text{as } n \to \infty, \qquad v \in V. \tag{3.1}$$

Furthermore, the value of the exponent γ is believed to depend on d and not further on the choice of graph G. When $d = 4$, (3.1) should hold with $\gamma = 1$ and subject to the inclusion on the right side of the logarithmic correction factor $(\log n)^{1/4}$. See [1, 13] for accounts of critical exponents for SAWs.

Let $v \in V$ and

$$Z_{v,w}(x) = \sum_{n=0}^{\infty} \sigma_n(v,w)x^k, \qquad w \in V,\ x > 0, \tag{3.2}$$

where $\sigma_n(v,w)$ is the number of n-step SAWs with endpoints v, w. It is known under certain circumstances that the generating functions $Z_{v,w}$ have radius of convergence μ^{-1} (see [13, Cor. 3.2.6]), and it is believed that there exists an exponent η and constants $A'_v > 0$ such that

$$Z_{v,w}(\mu^{-1}) \sim A'_v d_G(v,w)^{-(d-2+\eta)} \qquad \text{as } d_G(v,w) \to \infty. \tag{3.3}$$

Let $\Sigma_n(v)$ be the set of n-step SAWs from v, and write $\langle \cdot \rangle_n^v$ for expectation with respect to uniform measure on $\Sigma_n(v)$. Let $\|\pi\|$ be the graph-distance between the endpoints of a SAW π. It is believed (when $d \neq 4$) that there exists an exponent ν and constants $A''_v > 0$ such that

$$\langle \|\pi\|^2 \rangle_n^v \sim A''_v n^{2\nu}, \qquad v \in V. \tag{3.4}$$

As above, this should hold for $d = 4$ with $\nu = \frac{1}{2}$ and subject to the inclusion of the correction factor $(\log n)^{1/4}$.

The above three exponents are believed to be related to one another through the so-called *Fisher relation*

$$\gamma = \nu(2 - \eta). \tag{3.5}$$

It is convenient to work with definitions of critical exponents that do not depend on an assumption of dimensionality, and thus we proceed as follows. Let G be an infinite, connected, quasi-transitive graph with connective constant μ and fundamental domain W. Let X be the set of edges incident to vertices in W, and let Σ be the set of SAWs on G starting at mid-edges in X. We define the function

$$Y(x,y) = \sum_{\pi \in \Sigma} \frac{x^{|\pi|}}{|\pi|^y}, \qquad x > 0,\ y \in \mathbb{R}.$$

(The denominator is interpreted as 1 when $|\pi| = 0$.) For fixed x, $Y(x,y)$ is non-increasing in y. Let $\gamma = \gamma(G) \in [-\infty, \infty]$ be such that

$$Y(\mu^{-1}, y) \begin{cases} = \infty & \text{if } y < \gamma, \\ < \infty & \text{if } y > \gamma. \end{cases}$$

We shall assume that $-\infty < \gamma < \infty$. It will be convenient at times to assume more about the number σ_n of n-step SAWs from mid-edges in X, namely that there exist constants $C_i = C_i(W) \in (0, \infty)$ and a slowly varying function L such that

$$C_1 L(n) n^{\gamma-1} \mu^n \le \sigma_n \le C_2 L(n) n^{\gamma-1} \mu^n, \qquad n \ge 1. \tag{3.6}$$

Let

$$V(z) = \sum_{n=1}^{\infty} \frac{1}{n^{2z+1}} \langle \|\pi\|^2 \rangle_n, \qquad z \in [-\infty, \infty], \tag{3.7}$$

where $\langle \cdot \rangle_n$ denotes the uniform average over the set Σ_n of n-step SAWs in Σ. Thus, $V(z)$ is non-increasing in z, and we let $\nu = \nu(G) \in [-\infty, \infty]$ be such that

$$V(z) \begin{cases} = \infty & \text{if } z < \nu, \\ < \infty & \text{if } z > \nu. \end{cases}$$

Let αW denote the image of W under an automorphism $\alpha \in \mathcal{A}$, with incident edges αX, and let

$$Z_\alpha(x) = \sum_{\pi \in \Sigma(\alpha)} x^{|\pi|},$$

where $\Sigma(\alpha)$ is the subset of Σ containing SAWs ending at mid-edges in αX. We assume there exists $\eta = \eta(G) \in [-\infty, \infty]$ such that, for any sequence of automorphisms α satisfying $d_G(W, \alpha W) \to \infty$,

$$Z_\alpha(\mu^{-1}) d_G(W, \alpha W)^w \begin{cases} \to 0 & \text{if } w < \eta, \\ \to \infty & \text{if } w > \eta. \end{cases} \tag{3.8}$$

The η of (3.3) should agree with that defined here when $d = 2$.

It is easily seen that the values of γ, η, ν do not depend on the choice of fundamental domain W.

We consider now the effect on critical exponents of the Fisher transformation. Let W_0 be a minimal fundamental domain of $G_0 := G$, with incident edge-set $X_0 := X$ as above. Write $W_1 = F(W_0)$, the set of vertices of the triangles formed by the Fisher transformation at vertices in W_0, and X_1 for the set of edges of G_1 incident to vertices in W_1. It may be seen that W_1 is a fundamental domain of G_1.

Theorem 3.2. *Let G_0 be an infinite, quasi-transitive, connected, cubic graph. Assume that $|\gamma(G_0)| < \infty$ and that $\eta(G_1)$ exists.*

(a) *The exponents γ, η of G_0 and G_1 are equal.*
(b) *Let $\sigma_{n,k}$ be the number of n-step SAWs on G_k from mid-edges in X_k. Assume the $\sigma_{n,k}$ satisfy (3.6) for constants $C_{i,k}$ and a common slowly varying function L. Then the exponents ν of G_0 and G_1 are equal.*

Our final result concerns the effect of the Fisher transformation when applied to just one of the vertex-sets of a bipartite graph. Let $G = (V, E)$ be bipartite with vertex-sets V_1, V_2 coloured white and black, respectively. We think of G as a graph together with a colouring χ, and the *coloured-automorphism group* $\mathcal{A}_c = \mathcal{A}_c(G)$ of the pair (G, χ) is the set of maps $\phi : V \to V$ which preserve both graph structure and colouring. The coloured graph is *quasi-transitive* if there exists a finite subset $W \subseteq V$ such that: for all $v \in V$, there exists $\alpha \in \mathcal{A}_c$ such that $\alpha v \in W$ and $\chi(v) = \chi(\alpha v)$. As before, such a set W is called a *fundamental domain*.

Theorem 3.3. *Let G be an infinite, connected, bipartite graph with vertex-sets coloured black and white, and suppose that the coloured graph G is quasi-transitive, and every black vertex has degree 3. Let $\widetilde{G}$ be obtained by applying the Fisher transformation at each black vertex.*

(a) *The connective constants μ and $\widetilde{\mu}$ of G and $\widetilde{G}$, respectively, satisfy $\mu^{-2} = h(\widetilde{\mu}^{-1})$ where $h(x) = x^3 + x^4$.*
(b) *Under the corresponding assumptions of Theorem 3.2, the exponents γ, η, ν are the same for G as for $\widetilde{G}$.*

Theorem 3.3(a) implies an exact value of a connective constant that does not appear to have been noted previously. Take $G = \mathbb{H}$, the hexagonal lattice with connective constant $\mu = \sqrt{2+\sqrt{2}} \approx 1.84776$, see [3]. The decorated lattice $\widetilde{\mathbb{H}}$ is illustrated in Figure 1.2, and has connective constant $\widetilde{\mu}$ satisfying $\mu^{-2} = h(\widetilde{\mu}^{-1})$, which may be solved to obtain $\widetilde{\mu} \approx 1.75056$.

The proofs of Theorems 3.1–3.2 and 3.3 are found in Sections 4 and 5, respectively.

4. Proof of Theorems 3.1–3.2

Proof of Theorem 3.1. Let $G_0 = (V_0, E_0)$ be an infinite, connected, quasi-transitive, cubic graph. The graph $G_1 = F(G_0)$ is also quasi-transitive and cubic. It suffices for part (a) to show that the connective constants μ_k of the G_k satisfy

$$g(\mu_1^{-1}) = \mu_0^{-1}. \tag{4.1}$$

By (1.1), $\mu_k \in [1, 2]$ for $k = 1, 2$.

Let W_0 be a minimal fundamental domain of G_0, and let X_0 be the subset of E_0 comprising all edges incident to vertices in W_0. Write $W_1 = F(W_0)$, the set of vertices of the triangles formed by the Fisher transformation at vertices in W_0, and X_1 for the set of edges of G_1 incident to vertices in W_1. It may be seen that W_1 is a fundamental domain of G_1.

It is convenient to work with SAWs that start and end at mid-edges. Note that the mid-edges of E_0 (respectively, X_0) may be viewed as mid-edges of E_1 (respectively, X_1).

For $k = 0, 1$, the partition functions of SAWs on G_k are the polynomials

$$Z_k(x) = \sum_{\pi \in \Sigma_k} x^{|\pi|}, \qquad x > 0,$$

where the sum is over the set Σ_k of SAWs starting at mid-edges of X_k. Similarly, we set

$$Z_1^*(x) = \sum_{\pi \in \Sigma_1^*} x^{|\pi|},$$

where the sum is over the set Σ_1^* of SAWs on G_1 starting at mid-edges of X_0 and ending at mid-edges of E_0. For $k = 0, 1$,

$$Z_k(x) \begin{cases} < \infty & \text{if } x < \mu_k^{-1}, \\ = \infty & \text{if } x > \mu_k^{-1}. \end{cases} \tag{4.2}$$

The following basic argument formalizes a method known already in the special case of the hexagonal lattice, see for example [6, 10]. Since $\Sigma_1^* \subseteq \Sigma_1$, we have

$$Z_1^*(x) \le Z_1(x). \tag{4.3}$$

Let $N(\pi)$ be the number of endpoints of a SAW $\pi \in \Sigma_1$ that are mid-edges of E_0. The set Σ_1 may be partitioned into three sets.

(a) If $N(\pi) = 2$, then π contributes to Z_1^*.
(b) π may be a walk within a single Fisher triangle.
(c) If (b) does not hold and $N(\pi) \le 1$, any endpoint not in E_0 may be moved by one, two, or three steps along π to obtain a shorter SAW in Σ_1^*.

By considering the numbers of SAWs in each subcase of (c), we find that

$$Z_1(x) \le [1 + 2x + 2x^2 + 2x^3]^2 Z_1^*(x) + 6|W_0|(1 + x + x^2). \tag{4.4}$$

where the last term corresponds to case (b). By (4.3)–(4.4),

$$Z_1(x) < \infty \quad \Leftrightarrow \quad Z_1^*(x) < \infty,$$

so that, by (4.2),

$$Z_1^*(x) \begin{cases} < \infty & \text{if } x < \mu_1^{-1}, \\ = \infty & \text{if } x > \mu_1^{-1}. \end{cases} \tag{4.5}$$

With a SAW in Σ_1^* we associate a SAW in Σ_0 by shrinking each Fisher triangle to a vertex. Each n-step SAW in Σ_0 arises thus from 2^n SAWs in Σ_1^*, because each triangle may be circumnavigated in either of 2 directions. Therefore,

$$Z_0(x^2(1+x)) = Z_1^*(x), \tag{4.6}$$

and (4.1) follows by (4.2) and (4.5).

We turn to Theorem 3.1(b). By (1.1), $\mu_0^{-1} \in [\frac{1}{2}, 1]$. The function g is a bijection from $[\frac{1}{2}, 1]$ to $[\frac{3}{8}, 2]$. Furthermore, g is strictly convex on $[\frac{1}{2}, 1]$ with fixed point ϕ^{-1}. By (4.1) applied iteratively, $\mu_k^{-1} \to \phi^{-1}$ as $k \to \infty$, and the limit is monotone. The bounds on $\mu_k^{-1} - \phi^{-1}$ follow from the facts that $g'(\frac{1}{2}) = \frac{7}{4}$ and $g'(\phi^{-1}) = \frac{1}{2}(7 - \sqrt{5})$. □

Proof of Theorem 3.2. Let

$$Y_k(x, y) = \sum_{\pi \in \Sigma_k} \frac{x^{|\pi|}}{|\pi|^y}, \quad Y_k^*(x, y) = \sum_{\pi \in \Sigma_k^*} \frac{x^{|\pi|}}{|\pi|^y}, \qquad x > 0,\ y \in \mathbb{R},$$

where the denominator is interpreted as 1 when $|\pi| = 0$. Since $\Sigma_1^* \subseteq \Sigma_1$,

$$Y_1^*(x, y) \le Y_1(x, y).$$

Since every SAW in $\Sigma_1 \setminus \Sigma_1^*$ either is an extension of a SAW in Σ_1^* at the starting point, or endpoint (or both), by at most 3 steps, or is a short walk in a single Fisher triangle,

$$Y_1(x, y) \le 7^{|y|}[1 + 2x + 2x^2 + 2x^3]^2 Y_1^*(x, y) + 6|W_0| \left(1 + x + \frac{x^2}{2^y}\right).$$

Therefore,

$$Y_1^*(x, y) < \infty \quad \Leftrightarrow \quad Y_1(x, y) < \infty. \tag{4.7}$$

As in the previous proof, any n-step SAW in Σ_0 gives rise to 2^n SAWs in Σ_1^*, and conversely any SAW in Σ_1^* gives rise to a SAW in Σ_0 by shrinking each triangle to a vertex. For $n \ge 1$, the contribution of an n-step SAW $\pi \in \Sigma_0$ to $Y_0(x, y)$ is x^n / n^y, and to $Y_1^*(x, y)$ is

$$T_n := \sum_{l=0}^{n} \binom{n}{l} \frac{x^{2n+l}}{(2n+l)^y}.$$

Since

$$C\frac{[x^2(1+x)]^n}{n^y} \le T_n \le D\frac{[x^2(1+x)]^n}{n^y},$$

where $C = 2^{-y} \wedge 3^{-y}$ and $D = 2^{-y} \vee 3^{-y}$, we have that

$$C\widetilde{Y}_0(x^2(1+x), y) \le \widetilde{Y}_1^*(x,y) \le D\widetilde{Y}_0(x^2(1+x), y),$$

where $\widetilde{S}$ denotes the summation S without the $n = 0$ term. Therefore,

$$Y_1^*(x,y) < \infty \quad \Leftrightarrow \quad Y_0(x^2(1+x), y) < \infty.$$

By (4.7) and Theorem 3.1(a), $\gamma(G_1) = \gamma(G_0)$.

Let $\|\pi\|_k$ be the graph-distance between the endpoints of the walk π on G_k. Assume $|\gamma| = |\gamma(G_0)| < \infty$, and write

$$V_k(z) = \sum_{n=1}^{\infty} \frac{1}{n^{2z+1}} \langle \|\pi\|_k^2 \rangle_{n,k}, \tag{4.8}$$

where $\langle \cdot \rangle_{n,k}$ denotes the uniform average over the set $\Sigma_{n,k}$ of n-step SAWs of G_k starting at mid-edges of X_k. Similarly,

$$V_1^*(z) = \sum_{n=1}^{\infty} \frac{1}{n^{2z+1}} \frac{\sigma_{n,1}^*}{\sigma_{n,1}} \langle \|\pi\|_1^2 \rangle_{n,1}^*, \tag{4.9}$$

where $\sigma_{\cdot}^{\cdot} = |\Sigma_{\cdot}^{\cdot}|$, and $\langle \cdot \rangle_{n,1}^*$ averages over the subset $\Sigma_{n,1}^*$ of $\Sigma_{n,1}$ containing n-step SAWs of G_1 that start in X_0 and end in E_0. We assume there exist constants $C_{i,k} \in (0,\infty)$ and a slowly varying function L such that

$$C_{1,k}L(n)n^{\gamma-1}\mu_k^n \le \sigma_{n,k} \le C_{2,k}L(n)n^{\gamma-1}\mu_k^n, \qquad k = 1,2. \tag{4.10}$$

We shall in fact use slightly less than this.

Similarly to the proof of (4.7), by (4.9)–(4.10), there exists $C_1 < \infty$ such that

$$V_1^*(z) \le V_1(z) \le C_1 V_1^*(z),$$

whence

$$V_1(z) < \infty \quad \Leftrightarrow \quad V_1^*(z) < \infty. \tag{4.11}$$

The contribution of $\pi \in \Sigma_{n,0}$ to $V_0(z)$ is

$$\frac{1}{\sigma_{n,0} n^{2z+1}} \|\pi\|_0^2.$$

As explained previously, π gives rise to 2^n SAWs on G_1, making an aggregate contribution of

$$\sum_{l=0}^{n} \binom{n}{l} \frac{1}{\sigma_{2n+l,1}(2n+l)^{2z+1}} (2\|\pi\|_0)^2$$

to $V_1^*(z)$. By (4.10), there exist constants $C_i > 0$ such that

$$\begin{aligned}\frac{C_2}{n^{\gamma-1}L(n)}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{1}{\mu_1}\right)^{2n+l} &\le \sum_{l=0}^{n}\binom{n}{l}\frac{1}{\sigma_{2n+l,1}}\\ &\le \frac{C_3}{n^{\gamma-1}L(n)}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{1}{\mu_1}\right)^{2n+l}.\end{aligned}$$

By Theorem 3.1(a),

$$\sum_{l=0}^{n}\binom{n}{l}\left(\frac{1}{\mu_1}\right)^{2n+l} = \left(\frac{1}{\mu_0}\right)^{n},$$

so that

$$C_4(2^{-2z}\wedge 3^{-2z})V_0(z) \le V_1^*(z) \le C_5(2^{-2z}\vee 3^{-2z})V_0(z).$$

Therefore, for $|z| < \infty$,

$$V_1^*(z) < \infty \quad\Leftrightarrow\quad V_0(z) < \infty.$$

By (4.11), $\nu(G_0) = \nu(G_1)$.

Any $\alpha \in \mathcal{A}(G_0)$ acts in a natural way on $G_1 = F(G_0)$. For $k = 0, 1$ and $\alpha \in \mathcal{A}$, let

$$Z_{\alpha,k}(x) = \sum_{\pi\in\Sigma_k(\alpha)} x^{|\pi|}, \quad Z^*_{\alpha,k}(x) = \sum_{\pi\in\Sigma^*_k(\alpha)} x^{|\pi|}, \qquad x > 0,$$

where $\Sigma_k(\alpha)$ (respectively, $\Sigma^*_k(\alpha)$) is the set of SAWs of G_k from mid-edges of X_k (respectively, X_{k-1}) to mid-edges of αX_k (respectively, αX_{k-1}). Assume X_0 and αX_0 are disjoint. As before,

$$Z^*_{\alpha,1}(x) = Z_{\alpha,0}(x^2(1+x)) \tag{4.12}$$

and, as in (4.4),

$$Z^*_{\alpha,1}(x) \le Z_{\alpha,1}(x) \le [1+2x+2x^2+2x^3]^2 Z^*_{\alpha,1}(x). \tag{4.13}$$

By (4.12) and Theorem 3.1(a), for $w \in \mathbb{R}$,

$$\lim_{d_0(W_0,\alpha W_0)\to\infty}\Big[Z_{\alpha,0}(\mu_0^{-1})d_0(W_0,\alpha W_0)^w\Big] = \infty$$

if and only if

$$\lim_{d_1(W_1,\alpha W_1)\to\infty}\Big[Z_{\alpha,1}(\mu_1^{-1})d_1(W_1,\alpha W_1)^w\Big] = \infty,$$

where $d_k = d_{G_k}$. It follows that $\eta(G_0) = \eta(G_1)$. □

5. Proof of Theorem 3.3

Let $G = (V, E)$ be a coloured bipartite graph satisfying the given assumptions. The vertices of any SAW on G are alternately black and white. The decorated graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ is obtained from G by replacing each black vertex by a triangle, as illustrated in Figure 3.1. The set $\widetilde{V}$ is coloured in the natural way: white vertices remain white, and vertices of Fisher triangles are coloured black.

Let W be a minimal fundamental domain of G, and let X be the subset of E comprising all edges incident to vertices in W. Write $\widetilde{W} = F(W)$, the set of vertices of the triangles formed by the Fisher transformations at black vertices in W, and $\widetilde{X}$ for the set of edges of $\widetilde{G}$ incident to vertices in $\widetilde{W}$. It may be seen that $\widetilde{W}$ is a fundamental domain for the coloured graph $\widetilde{G}$. Recall that the mid-edges of E may be viewed as a subset of mid-edges of $\widetilde{E}$, and thus E may be viewed as a subset of $\widetilde{E}$.

Let s_n be the number of n-step SAWs of $\widetilde{G}$ starting at mid-edges in $\widetilde{X}$, and let c_n be the number of n-step SAWs of $\widetilde{G}$ starting at a mid-edge of X and ending at a mid-edge of E. It is immediate that

$$c_n \le s_n. \tag{5.1}$$

Any SAW counted in s_n either lies within a single Fisher triangle, or may be obtained by a k-step extension (with some $k \le 3$) at one or both endpoints of some SAW counted in one of c_n, c_{n-1}, c_{n-2}, c_{n-3}. Therefore,

$$s_n \le c_n + 4c_{n-1} + 8c_{n-2} + 12c_{n-3} + 18|W|. \tag{5.2}$$

By (2.1), the limits $\lim_{n\to\infty} s_n^{1/n}$ and $\lim_{n\to\infty} c_n^{1/n}$ exist and, by (5.1)–(5.2), these limits are equal.

A SAW is called *even* if it has even length. Let $\mathcal{E}$ be the set of SAWs on G starting at mid-edges of X, and let $\mathcal{E}_\mathrm{e}$ be the subset of $\mathcal{E}$ comprising the even SAWs. Let $x, y > 0$. Each step of a SAW on G is assigned weight x at a black vertex, and weight y at a white vertex. Let

$$Z(x, y) = \sum_{\pi\in\mathcal{E}} x^{|\pi_\mathrm{b}|} y^{|\pi_\mathrm{w}|},$$

where $|\pi_\mathrm{b}|$ and $|\pi_\mathrm{w}|$ are the numbers of black and white vertices visited by π. Similarly, let

$$Z_\mathrm{e}(x, y) = \sum_{\pi\in\mathcal{E}_\mathrm{e}} (xy)^{|\pi|/2}. \tag{5.3}$$

It is clear by a decomposition of paths that

$$Z_{\mathrm{e}}(x,y) \le Z(x,y),$$
$$Z(x,y) - Z_{\mathrm{e}}(x,y) \le (2x+2y)(1+Z_{\mathrm{e}}(x,y)).$$

Hence,

$$Z_{\mathrm{e}}(x,y) < \infty \quad \Leftrightarrow \quad Z(x,y) < \infty. \tag{5.4}$$

We now introduce a third partition function $\widetilde{Z}$, namely of the set $\widetilde{\mathcal{E}}$ of SAWs on $\widetilde{G}$ starting at the mid-edges of X and ending at mid-edges of E. Each step of such a SAW traverses two half-edges, and is allocated a weight which depends on these half-edges. Let $p,q,r>0$. Whenever both half-edges belong to $\widetilde{E}\setminus E$, the weight is p; if one half-edge is in E and the other in $\widetilde{E}\setminus E$, the weight is q; if both half-edges are in E, the weight is r. Then

$$\widetilde{Z}(p,q,r) := \sum_{\pi\in\widetilde{\mathcal{E}}} p^{|\pi_p|} q^{|\pi_q|} r^{|\pi_r|},$$

where $|\pi_p|$ is the number of p-steps, etc. By counting edges of the different types,

$$\widetilde{Z}(p,q,r) = Z(q^2(1+p), r).$$

By (5.4),

$$\widetilde{Z}(p,q,r) < \infty \quad \Leftrightarrow \quad Z_{\mathrm{e}}(q^2(1+p), r) < \infty. \tag{5.5}$$

By (5.3),

$$Z_{\mathrm{e}}(q^2(1+p), r) \begin{cases} < \infty & \text{if } q^2(1+p)r < \mu^{-2}, \\ = \infty & \text{if } q^2(1+p)r > \mu^{-2}, \end{cases}$$

whence the radius of convergence of $\widetilde{Z}(x,x,x) = \sum_{n\ge 0} c_n x^n$ is the root of the equation

$$x^3(1+x) = \frac{1}{\mu^2}.$$

Theorem 3.3(a) follows. Part (b) is proved is a similar manner to the proof of Theorem 3.2.

Acknowledgement

This work was supported in part by the Engineering and Physical Sciences Research Council under grant EP/103372X/1.

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