Random even graphs and the Ising model

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Abstract

We explore the relationship between the Ising model with inverse temperature β, the $q = 2$ random-cluster model with edge-parameter $p = 1 - e^{-2\beta}$, and the random even subgraph with edge-parameter $\frac{1}{2}p$. For a planar graph G, the boundary edges of the $+$ clusters of the Ising model on the planar dual of G forms a random even subgraph of G. A coupling of the random even subgraph of G and the $q = 2$ random-cluster model on G is presented, thus extending the above observation to general graphs. A random even subgraph of a planar lattice undergoes a phase transition at the parameter-value $\frac{1}{2}p_c$, where p_c is the critical point of the $q = 2$ random-cluster model on the dual lattice. These results are motivated in part by an exploration of the so-called random-current method utilised by Aizenman, Barsky, Fernández and others to solve the Ising model on the d-dimensional hypercubic lattice.

1 Introduction

The method of 'random currents' has been immensely valuable in the solution to the ferromagnetic Ising model, see [3, 4] for example. Each edge e of the underlying graph G is replaced by a Poisson-distributed number N_e of parallel edges, and one is interested in the set S of vertices with odd degree in the resulting multigraph. The partition function of the Ising model on G may be expressed as the probability that $S = \varnothing$, and the two-point correlation function between the vertices x and y as the ratio $P(S = \{x, y\})/P(S = \varnothing)$.

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If N_e has the Poisson distribution with parameter λ, then N_e is odd with probability $\pi = \frac{1}{2}(1 - e^{-2\lambda})$, and therefore the distribution of S is unchanged when e is replaced by a single edge with probability π and by no edge otherwise. Thus, $P(S = \varnothing)$ equals the probability that, in a random graph on G in which each edge is retained with probability π, every vertex-degree is even. One is led thus to the study of a random even subgraph of a given graph $G = (V, E)$.

We shall explore here the relationship between the Ising model on G, the $q = 2$ random-cluster model on G, and the random even subgraph of G. The three corresponding probability measures are defined next. We assume here, as elsewhere in the paper unless otherwise stated, that $G = (V, F)$ is a finite graph. (Multiple edges and loops are allowed.)

Let $\beta \in (0, \infty)$, $q \in \{2, 3, \dots\}$, and

$$p = 1 - e^{-q\beta}. \tag{1.1}$$

The Potts model on the graph G has configuration space $\Sigma = \{1, 2, \dots, q\}^V$, and probability measure

$$\pi_{\beta,q}(\sigma) = \frac{1}{Z^{\mathrm{P}}} \exp\left\{ \sum_{e \in E} \beta(q\delta_e(\sigma) - 1) \right\}, \qquad \sigma \in \Sigma, \tag{1.2}$$

where, for $e = \langle x, y \rangle \in E$,

$$\delta_e(\sigma) = \delta_{\sigma_x, \sigma_y} = \begin{cases} 1 & \text{if } \sigma_x = \sigma_y, \\ 0 & \text{if } \sigma_x \neq \sigma_y, \end{cases}$$

and $Z^{\mathrm{P}} = Z^{\mathrm{P}}_G(\beta, q)$ is the partition function

$$Z^{\mathrm{P}} = \sum_{\sigma \in \Sigma} \exp\left\{ \sum_{e \in E} \beta(q\delta_e(\sigma) - 1) \right\}. \tag{1.3}$$

A *spin-cluster* of a configuration $\sigma \in \Sigma$ is a maximal connected subgraph of G each of whose vertices v has the same spin-value σ_v. A spin-cluster is termed a k cluster if $\sigma_v = k$ for all v belonging to the cluster. An important quantity associated with the Potts model is the 'two-point correlation function'

$$\tau_{\beta,q}(x, y) = \pi_{\beta,q}(\sigma_x = \sigma_y) - \frac{1}{q}, \qquad x, y \in V. \tag{1.4}$$

We shall mostly consider the Ising model, for which $q = 2$ and Σ is redefined as $\Sigma = \{-1, +1\}^V$. In this case,

$$2\delta_e(\sigma) - 1 = \sigma_x \sigma_y \qquad \text{if } e = \langle x, y \rangle,$$

so that

$$\pi_{\beta,2}(\sigma) \propto \exp\left\{\beta \sum_{e=\langle x,y\rangle \in E} \sigma_x\sigma_y\right\}, \qquad \sigma \in \Sigma. \tag{1.5}$$

We note that

$$\tau_{\beta,2}(x,y) = \tfrac{1}{2}\pi_{\beta,2}(\sigma_x\sigma_y), \tag{1.6}$$

where $\mu(X)$ denotes the expectation of a random variable X under the probability measure μ.

The random-cluster measure on G is given as follows. Let $p \in (0,1)$ and $q \in (0,\infty)$. The configuration space is $\Omega = \{0,1\}^E$. For $\omega \in \Omega$ and $e \in E$, we say that e is ω-*open* (or, simply, open) if $\omega(e) = 1$, and ω-*closed* otherwise. The random-cluster probability measure on Ω is defined by

$$\begin{aligned}\phi_{p,q}(\omega) &= \frac{1}{Z^{\mathrm{RC}}}\left\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\}q^{k(\omega)}\\ &= \frac{1}{Z^{\mathrm{RC}}}p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}q^{k(\omega)}, \qquad \omega\in\Omega, \end{aligned} \tag{1.7}$$

where $k(\omega)$ denotes the number of ω-open components on the vertex-set V, $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ is the set of open edges, and $Z^{\mathrm{RC}} = Z_G^{\mathrm{RC}}(p,q)$ is the appropriate normalizing factor.

The relationship between the Potts and random-cluster models on G is well established, and hinges on the fact that, in the notation introduced above,

$$\tau_{\beta,q}(x,y) = (1-q^{-1})\phi_{p,q}(x \leftrightarrow y),$$

where $\{x \leftrightarrow y\}$ is the event that x and y are connected by an open path. See [14] for a recent account of the random-cluster model.

A subset F of the edge-set of $G = (V,E)$ is called *even* if, for all $x \in V$, x is incident to an even number of elements of F. We call the subgraph (V,F) *even* if F is even, and we write $\mathcal{E}$ for the set of all even subsets F of E. It is standard that every even set F may be decomposed as an edge-disjoint union of cycles. Let $p \in [0,1)$. The *random even subgraph* of G with parameter p is that with law

$$\rho_p(F) = \frac{1}{Z^{\mathrm{E}}}p^{|F|}(1-p)^{|E\setminus F|}, \qquad F\in\mathcal{E}, \tag{1.8}$$

where

$$Z^{\mathrm{E}} = \sum_{F\in\mathcal{E}} p^{|F|}(1-p)^{|E\setminus F|}.$$

We may express ρ_p in the following way. Let $\phi_p = \phi_{p,1}$ be product measure with density p on Ω. For $\omega \in \Omega$, let $\partial\omega$ denote the set of vertices $x \in V$ that are incident to an odd number of ω-open edges. Then

$$\rho_p(F) = \frac{\phi_p(\omega_F)}{\phi_p(\partial\omega = \varnothing)}, \qquad F \in \mathcal{E},$$

where ω_F is the edge-configuration whose open set is F. In other words, ϕ_p describes the random subgraph of G obtained by randomly and independently deleting each edge with probability $1-p$, and ρ_p is the law of this random subgraph conditioned on being even.

Theorem 1.9. *Consider the Ising model on G with inverse temperature β, and let $p = 1 - e^{-2\beta}$. Then*

$$2\tau_{\beta,2}(x,y) = \frac{\phi_{p/2}(\partial\omega = \{x,y\})}{\phi_{p/2}(\partial\omega = \varnothing)}, \qquad x, y \in V,\ x \neq y. \tag{1.10}$$

This is proved by the following route. The random-current method of [2, 3] leads to a representation of the left side of (1.10) in terms of ratios of probabilities concerning Poissonian random graphs derived from G by replacing each edge by a Poisson-distributed number (with parameter p) of parallel edges. As remarked at the opening of this paper, such probabilities may be rewritten in terms of a random even subgraph with parameter $\frac{1}{2}p$. See [15] for a summary of the random-current method and the derivation of Theorem 1.9.

In a second paper [17], we study the asymptotic properties of a random even subgraph of the complete graph K_n. Whereas the special relationship with the random-cluster and Ising models is the main feature of the current work, the analysis of [17] is more analytic, and extends to random graphs whose vertex degrees are constrained to lie in any given subsequence of the non-negative integers.

2 The Ising correlation function

The two-point correlation function of the Ising model may be expressed simply in terms of a random even graph. This was noted in Theorem 1.9, but the proof that follows is more direct than that summarised at the end of Section 1. Recall (1.6).

Theorem 2.1. *Let $2p = 1 - e^{-2\beta}$ where $p \in (0, \frac{1}{2})$, and consider the Ising model with inverse temperature β. Then*

$$\pi_{\beta,2}(\sigma_x\sigma_y) = \frac{\phi_p(\partial\omega = \{x,y\})}{\phi_p(\partial\omega = \varnothing)}, \qquad x, y \in V,\ x \neq y.$$

A corresponding conclusion is valid for the product of σ_{x_i} over any even family of distinct $x_i \in V$.

Proof. For $\sigma \in \Sigma$, $\omega \in \Omega$, let

$$\begin{aligned} Z_p(\sigma, \omega) &= \prod_{e=\langle v,w\rangle} \Big\{(1-p)\delta_{\omega(e),0} + p\sigma_v\sigma_w\delta_{\omega(e),1}\Big\} \\ &= p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}\prod_{v\in V}\sigma_v^{\deg(v,\omega)}, \end{aligned} \tag{2.2}$$

where $\deg(v, \omega)$ is the degree of v in the 'open' graph $(V, \eta(\omega))$. Then

$$\begin{aligned} \sum_{\omega\in\Omega} Z_p(\sigma,\omega) &= \prod_{e=\langle v,w\rangle}(1-p+p\sigma_v\sigma_w) = \prod_{e=\langle v,w\rangle} e^{\beta(\sigma_v\sigma_w-1)} \\ &= e^{-\beta|E|}\exp\left(\beta\sum_{e=\langle v,w\rangle}\sigma_v\sigma_w\right), \qquad \sigma\in\Sigma. \end{aligned} \tag{2.3}$$

Similarly,

$$\sum_{\sigma\in\Sigma} Z_p(\sigma,\omega) = 2^{|V|}p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}1_{\{\partial\omega=\varnothing\}}, \qquad \omega\in\Omega, \tag{2.4}$$

and

$$\sum_{\sigma\in\Sigma}\sigma_x\sigma_y Z_p(\sigma,\omega) = 2^{|V|}p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}1_{\{\partial\omega=\{x,y\}\}}, \qquad \omega\in\Omega. \tag{2.5}$$

By (2.3),

$$\pi_{\beta,2}(\sigma_x\sigma_y) = \frac{\sum_{\sigma,\omega}\sigma_x\sigma_y Z_p(\sigma,\omega)}{\sum_{\sigma,\omega} Z_p(\sigma,\omega)},$$

and the claim follows by (2.2)–(2.5). □

The proof resembles closely the Edwards–Sokal coupling of the random-cluster and Ising/Potts models, see [9, 14]. Note, however, that the Z_p above are sometimes negative, so that they do not give rise to a probability measure on the product space $\Sigma \times \Omega$. The representation of (2.2) is related to the so-called high temperature expansion of statistical physics.

Remark 2.6. The so-called $\mathrm{O}(N)$ model of statistical physics is constructed as follows. With $G = (V, E)$ a given finite graph, the state space is taken as S^V where S is the space of unit N-vectors and $N \in \{1, 2, \dots\}$. There is more than one way of specifying a Hamiltonian whose associated Gibbs state is invariant under rotations of S, i.e., under the action of orthonormal $N \times N$ matrices. Particular interest has been paid to the measures with densities $Z_i^{-1} B_i(\mathbf{s})$, $\mathbf{s} = (\mathbf{s}_v : v \in V) \in S^V$, with

$$B_1(\mathbf{s}) = \exp\left(\beta \sum_{e=\langle v,w\rangle} \mathbf{s}_v \cdot \mathbf{s}_w\right),$$
$$B_2(\mathbf{s}) = \prod_{e=\langle v,w\rangle} (1 + x\mathbf{s}_v \cdot \mathbf{s}_w),$$

where β and x are real parameters. When $N = 1$, we have $S = \{-1, +1\}$, and the B_i are equivalent to one another and give rise to the Ising model.

Henceforth we consider the case of B_2 only. An expansion similar to the above is given in [8] for the corresponding $\mathrm{O}(N)$ model on (a finite part of) the hexagonal lattice, see also [18, 20]. The argument in [8] may be applied to any finite graph G with maximum vertex degree 3 or fewer, in which case every even subgraph of G consists of disjoint cycles and isolated vertices. In our terminology, the argument in [8] gives, for such graphs, an equivalence between the $\mathrm{O}(N)$ model and a random even subgraph of G with a distribution given by a modified version of (1.8) with an additional factor $N^{k'(F)}$, where $k'(F)$ is the number of components of (V, F) containing at least one edge (and Z^{E} is modified accordingly). Equivalently, the distribution is obtained from a modification of the random-cluster measure (1.7) with $k(\omega)$ replaced by $k'(\omega)$, with $q = N$ (and an appropriate p depending on β), by conditioning on ω being even. (Note that, for the graphs in question, $k'(F)$ equals the number of cycles of (V, F).) For $N = 1$, this yields the random even subgraph of G defined in (1.8). As noted above, the $\mathrm{O}(1)$ model equals the Ising model, and we recover the relation in Theorem 2.1 and its proof.

We point out that, when $N = 1$, the relation of Theorem 2.1 is valid for any finite graph G. Suppose now that $N \geq 2$. When G has maximum degree 4 or more, the $\mathrm{O}(N)$ model no longer corresponds to a modified random even subgraph. There is a similar expansion for general N, see [21], but it gives rise to a more complicated measure on the even subgraphs of G.

3 Uniform random even subgraphs

In the case $p = \frac{1}{2}$ in (1.8), every even subgraph has the same probability, so $\rho_{\frac{1}{2}}$ describes a uniform random even subgraph of G. Such a random subgraph can be obtained as follows.

We identify the family of all spanning subgraphs of $G = (V, E)$ with the family of all subsets of E. This family can further be identified with $\{0,1\}^E = \mathbb{Z}_2^E$, and is thus a vector space over $\mathbb{Z}_2$; the addition is componentwise addition modulo 2 in $\{0,1\}^E$, which translates into taking the symmetric difference of edge-sets: $F_1 + F_2 = F_1 \bigtriangleup F_2$ for $F_1, F_2 \subseteq E$.

The family of even subgraphs of G forms a subspace $\mathcal{E}$ of this vector space $\{0,1\}^E$, since $F_1 + F_2 = F_1 \bigtriangleup F_2$ is even if F_1 and F_2 are even. (In fact, $\mathcal{E}$ is the cycle space Z_1 in the $\mathbb{Z}_2$-homology of G as a simplical complex.) In particular, the number of even subgraphs of G equals $2^{c(G)}$ where $c(G) = \dim(\mathcal{E})$; $c(G)$ is thus the number of independent cycles in G, and, as is well known,

$$c(G) = |E| - |V| + k(G). \tag{3.1}$$

Proposition 3.2. *Let $C_1, \dots, C_c$ be a maximal set of independent cycles in G. Let $\xi_1, \dots, \xi_c$ be independent* $\mathrm{Be}(\frac{1}{2})$ *random variables (i.e., the results of fair coin tosses). Then $\sum_i \xi_i C_i$ is a uniform random even subgraph of G.*

Proof. $C_1, \dots, C_c$ is a basis of the vector space $\mathcal{E}$ over $\mathbb{Z}_2$. □

One standard way of choosing $C_1, \dots, C_c$ is exploited in the next proposition. Another, for planar graphs, is given by the boundaries of the finite faces; this will be used in Section 5. In the following proposition, we use the term *spanning subforest* of G to mean a maximal forest of G, that is, the union of a spanning tree from each component of G.

Proposition 3.3. *Let (V, F) be a spanning subforest of G. Each subset X of $E \setminus F$ can be completed by a unique $Y \subseteq F$ to an even edge-set $E_X = X \cup Y \in \mathcal{E}$. Choosing a uniform random subset $X \subseteq E \setminus F$ thus gives a uniform random even subgraph E_X of G.*

Proof. It is easy to see, and well known, that each edge $e_i \in E \setminus F$ can be completed by edges in F to a unique cycle C_i; these cycles form a basis of $\mathcal{E}$ and the result follows by Proposition 3.2. (It is also easy to give a direct proof.) □

There is a natural definition of a uniform random even subgraph of an *infinite*, locally finite graph $G = (V, E)$. In fact, we can define the subspace $\mathcal{E}$ of $\{0,1\}^E$ as before. (Note that we need G to be locally finite in order to do so.) Further, let E_1 be a finite subset of E. The natural projection $\pi_{E_1} : \{0,1\}^E \to \{0,1\}^{E_1}$ given by $\pi_{E_1}(\omega) = (\omega_e)_{e \in E_1}$ maps $\mathcal{E}$ onto a subspace $\mathcal{E}_{E_1} = \pi_{E_1}(\mathcal{E})$ of $\{0,1\}^{E_1}$.

Theorem 3.4. *Let G be a locally finite, countably infinite graph. There exists a unique probability measure ρ on $\Omega = \{0,1\}^E$ such that, for every finite set $E_1 \subset E$, $(\omega_e)_{e\in E_1}$ is uniformly distributed on $\mathcal{E}_{E_1}$, i.e.,*

$$\rho(\pi_{E_1}^{-1}(A)) = |A \cap \mathcal{E}_{E_1}|/|\mathcal{E}_{E_1}|, \qquad A \subseteq \{0,1\}^{E_1}. \tag{3.5}$$

Proof. We have defined ρ on all cylinder events by (3.5). By the Kolmogorov extension theorem, it suffices to show that this definition is consistent as E_1 varies, which amounts to showing that if $E_1 \subseteq E_2 \subset E$ with E_1, E_2 finite, then the projection $\pi_{E_2 E_1} : \{0,1\}^{E_2} \to \{0,1\}^{E_1}$ maps the uniform distribution on $\mathcal{E}_{E_2}$ to the uniform distribution on $\mathcal{E}_{E_1}$. This is an immediate consequence of the fact that $\pi_{E_2 E_1}$ is a linear map of $\mathcal{E}_{E_2}$ onto $\mathcal{E}_{E_1}$. $\square$

4 Random even subgraphs via coupling

We return to the random even subgraph with parameter $p \in [0,1]$ defined by (1.8) for a finite graph $G = (V,E)$. We show next how to couple the $q = 2$ random-cluster model and the random even subgraph of G. Let $p \in [0, \frac{1}{2}]$, and let ω be a realization of the random-cluster model on G with parameters $2p$ and $q = 2$. Let $R = (V, \gamma)$ be a uniform random even subgraph of $(V, \eta(\omega))$.

Theorem 4.1. *The graph $R = (V, \gamma)$ is a random even subgraph of G with parameter p.*

This recipe for random even subgraphs provides a neat method for their simulation, provided $p \le \frac{1}{2}$. One may sample from the random-cluster measure by the method of coupling from the past (see [24]), and then sample a uniform random even subgraph by either Proposition 3.2 or Proposition 3.3. If G itself is even, we can further sample from ρ_p for $p > \frac{1}{2}$ by first sampling a subgraph $(V, \widetilde{F})$ from ρ_{1-p} and then taking the complement $(V, E \setminus \widetilde{F})$, which has the distribution ρ_p. We leave it as an open problem to find an efficient method to sample from ρ_p for $p > \frac{1}{2}$ and general G.

There is a converse to Theorem 4.1. Take a random even subgraph (V, F) of $G = (V,E)$ with parameter $p \le \frac{1}{2}$. To each $e \notin F$, we assign an independent random colour, blue with probability $p/(1-p)$ and red otherwise. Let H be obtained from F by adding in all blue edges.

Theorem 4.2. *The graph (V, H) has law $\phi_{2p,2}$.*

An edge e of a graph is called *cyclic* if it belongs to some cycle of the graph.

Corollary 4.3. *For* $p \in [0, \frac{1}{2}]$ *and* $e \in E$,

$$\rho_p(e \text{ is open}) = \tfrac{1}{2}\phi_{2p,2}(e \text{ is a cyclic edge of the open graph}).$$

By summing over $e \in E$, we deduce that the mean number of open edges under ρ_p is one half of the mean number of cyclic edges under $\phi_{2p,2}$.

Proof of Theorem 4.1. Let $g \subseteq E$ be even. By the observations in Section 3, with $c(\omega) = c(V, \eta(\omega))$ denoting the number of independent cycles in the open subgraph,

$$\mathbb{P}(\gamma = g \mid \omega) = \begin{cases} 2^{-c(\omega)} & \text{if } g \subseteq \eta(\omega), \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\mathbb{P}(\gamma = g) = \sum_{\omega: g \subseteq \eta(\omega)} 2^{-c(\omega)} \phi_{2p,2}(\omega).$$

Now $c(\omega) = |\eta(\omega)| - |V| + k(\omega)$, so that, by (1.7),

$$\begin{aligned} \mathbb{P}(\gamma = g) &\propto \sum_{\omega: g \subseteq \eta(\omega)} (2p)^{|\eta(\omega)|}(1-2p)^{|E \setminus \eta(\omega)|} 2^{k(\omega)} \frac{1}{2^{|\eta(\omega)|-|V|+k(\omega)}} \\ &\propto \sum_{\omega: g \subseteq \eta(\omega)} p^{|\eta(\omega)|}(1-2p)^{|E \setminus \eta(\omega)|} \\ &= [p + (1-2p)]^{|E \setminus g|} p^{|g|} \\ &= p^{|g|}(1-p)^{|E \setminus g|}, \qquad g \subseteq E. \end{aligned}$$

The claim follows. □

Proof of Theorem 4.2. For $h \subseteq E$,

$$\begin{aligned} \mathbb{P}(H = h) &\propto \sum_{J \subseteq h,\ J \text{ even}} \left(\frac{p}{1-p}\right)^{|J|} \left(\frac{p}{1-p}\right)^{|h \setminus J|} \left(\frac{1-2p}{1-p}\right)^{|E \setminus h|} \\ &\propto p^{|h|}(1-2p)^{|E \setminus h|} N(h), \end{aligned}$$

where $N(h)$ is the number of even subgraphs of (V, h). As in the above proof, $N(h) = 2^{|h|-|V|+k(h)}$ where $k(h)$ is the number of components of (V, h), and the proof is complete. □

Proof of Corollary 4.3. Let $\omega \in \Omega$ and let $\mathcal{C}$ be a maximal family of independent cycles of ω. Let $R = (V, \gamma)$ be a uniform random even subgraph of $(V, \eta(\omega))$, constructed using Proposition 3.2 and $\mathcal{C}$. For $e \in E$, let M_e be the number of elements of $\mathcal{C}$ that include e. If $M_e \geq 1$, the number of these M_e

cycles of γ that are selected in the construction of γ is equally likely to be even as odd. Therefore,

$$\mathbb{P}(e \in \gamma \mid \omega) = \begin{cases} \frac{1}{2} & \text{if } M_e \geq 1, \\ 0 & \text{if } M_e = 0. \end{cases}$$

The claim follows by Theorem 4.1. □

5 Random even subgraphs of planar lattices

When G is planar, the recipe of the last section may be recast in terms of the Ising model on the dual graph of G, via the so-called Edwards–Sokal coupling of the Ising and random-cluster models, [9].

Let $G = (V, E)$ be a planar graph embedded in $\mathbb{R}^2$, with dual graph $G_\mathrm{d} = (V_\mathrm{d}, E_\mathrm{d})$, and write e_d for the dual edge corresponding to the primal edge $e \in E$. Let $p \in (0, \frac{1}{2}]$ and let $\omega \in \Omega = \{0,1\}^E$ have law $\phi_{2p,2}$. There is a one–one correspondence between Ω and $\Omega_\mathrm{d} = \{0,1\}^{E_\mathrm{d}}$ given by $\omega(e) + \omega_\mathrm{d}(e_\mathrm{d}) = 1$. It is well known that ω_d has the law of the random-cluster model on G_d with parameters $(1-2p)/(1-p)$ and 2, see [14] for example.

For any $\omega \in \Omega$, let $f_0, f_1, \dots, f_c$ be the faces of $(V, \eta(\omega))$, with f_0 the infinite face. These faces are in one-to-one correspondence with the clusters of $(V_\mathrm{d}, \eta(\omega_\mathrm{d}))$, which we thus denote by $K_0, K_1, \dots, K_c$, and the boundaries of the finite faces form a basis of $\mathcal{E} = \mathcal{E}(V, \eta(\omega))$. More precisely, the boundary of each finite face f_i consists of an 'outer boundary' and zero, one or several 'inner boundaries'; each of these parts is a cycle (and two parts may have up to one vertex in common). If we orient the outer boundary cycle counter-clockwise (positive) and the inner boundary cycles clockwise (negative), the face will always be on the left side along the boundaries, and the winding numbers of the boundary cycles sum up to 1 at every point inside the face and to 0 outside the face. It is easy to see that the outer boundary cycles form a maximal family of independent cycles of $(V, \eta(\omega))$, and thus a basis of $\mathcal{E}$; another basis is obtained by the complete boundaries C_i of the finite faces. We use the latter basis, and select a random subset of the basis by randomly assigning (by fair coin tosses) $+$ and $-$ to each cluster in the dual graph $(V_\mathrm{d}, \eta(\omega_\mathrm{d}))$, or equivalently to each face f_i of $(V, \eta(\omega))$. We then select the boundaries C_i of the finite faces f_i that have been given a sign different from the sign of the infinite face f_0. The union (modulo 2) of the selected boundaries is by Proposition 3.2 and Theorem 4.1 a random even subgraph of G with parameter p. On the other hand, this union is exactly the dual boundary of the $+$ clusters of G_d, that is, the set of open edges $e \in E$ with the property that one endpoint of the corresponding dual edge e_{d} is labelled $+$ and the other is labelled $-$. [Such an edge e_{d} is called a $+/-$ edge.]

It is standard that the $+/-$ configuration on G_{d} is distributed as the Ising model on G_{d} with parameter β satisfying

$$\frac{1-2p}{1-p} = 1 - e^{-2\beta}. \tag{5.1}$$

In summary, we have found the following.

Theorem 5.2. *Let G be a finite planar graph with dual G_{d}. A random even subgraph of G with parameter $p \in (0, \frac{1}{2}]$ is dual to the $+/-$ edges of the Ising model on G_{d} with β satisfying* (5.1)*.*

Much is known about the Ising model on subsets of two-dimensional lattices, and the above fact permits an analysis of random even subgraphs of their dual lattices.

Consider the case when G is a box in the square lattice $\mathbb{Z}^2$. That is, $G = G_{m,n}$ is the subgraph of $\mathbb{Z}^2$ induced by the vertex-set $[-m, m] \times [-n, n]$, where $m, n \in \mathbb{Z}_+$ and $[a, b]$ is to be interpreted as $[a, b] \cap \mathbb{Z}$. For convenience, we assign 'periodic boundary conditions' to $G_{m,n}$, which is to say that we add further edges joining $(-m, y)$ to (m, y) for $-n \le y \le n$, and joining $(x, -n)$ to (x, n) for $-m \le x \le m$. Thus, $G_{m,n}$ may be considered as a vertex-transitive graph embedded in a torus.

The dual of $G_{m,n}$ is isomorphic to $G_{m,n}$. The Ising model on $\mathbb{Z}^2$ with parameter β is *critical* when $e^{2\beta} = 1 + \sqrt{2}$, or equivalently when the above random-cluster model on the dual lattice has parameter satisfying

$$\frac{1-2p}{1-p} = \frac{\sqrt{2}}{1+\sqrt{2}} = 2 - \sqrt{2},$$

i.e., $p = p_{\mathrm{c}}$ where

$$p_{\mathrm{c}} = \frac{1}{2+\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}. \tag{5.3}$$

The Ising model has been studied extensively in the physics literature, and physicists have a detailed knowledge of the two-dimensional case particularly. There is a host of 'exact calculations', rigorous proofs of which can present challenges to mathematicians, see [5, 23]. The random-current representation of [2, 3, 4], referred to above, has permitted a rigorous qualitative analysis of the Ising model in all dimensions. Further results for the critical Ising model are imminent, see [25, 28].

We shall use the established facts that: the critical value satisfies $\beta_{\mathrm{c}} = \beta_{\mathrm{sd}}$ where $\beta_{\mathrm{sd}} = \frac{1}{2}\log(1+\sqrt{2})$ is the 'self-dual point', and the magnetization (and therefore the corresponding random-cluster percolation-probability also) is a continuous function of β, even at the critical point. These facts are 'classical' and have received much attention; they may be proved as follows using 'modern' arguments. Recall first that the magnetization equals the percolation probability of the corresponding wired random-cluster model, and the two-point correlation function of the Ising model equals the two-point connectivity function of the random-cluster model (see [14]). We have that $\beta_{\mathrm{sd}} \le \beta_{\mathrm{c}}$, by Theorem 6.17(a) of [14] or otherwise, and similarly the random-cluster model with free boundary conditions has percolation-probability 0 whenever either $\beta \le \beta_{\mathrm{sd}}$ or $\beta < \beta_{\mathrm{c}}$. One may deduce by a coupling argument that the Ising model with $\beta < \beta_{\mathrm{c}}$ has a unique Gibbs state which we denote by π_β.

By the results of [3, 22, 26], the two-point correlation function $\pi_\beta(\sigma_x\sigma_y)$ of the spins at x and y decays exponentially as $|x-y| \to \infty$ when $\beta < \beta_{\mathrm{c}}$, and it follows by the final statement of [16] or otherwise that $\beta_{\mathrm{c}} = \beta_{\mathrm{sd}}$. The continuity of the magnetization at $\beta \ne \beta_{\mathrm{c}}$ is standard, see for example [14], Theorems 5.16 and 6.17(b). When $\beta = \beta_{\mathrm{c}}$, it suffices to show that the $\pm$ boundary-condition Gibbs states $\pi^{\pm}_{\beta_{\mathrm{c}}}$ and the free boundary-condition Gibbs state $\pi^0_{\beta_{\mathrm{c}}}$ satisfy $\pi^+_{\beta_{\mathrm{c}}} = \pi^-_{\beta_{\mathrm{c}}} = \pi^0_{\beta_{\mathrm{c}}}$. Suppose this does not hold, so that $\pi^+_{\beta_{\mathrm{c}}} \ne \pi^-_{\beta_{\mathrm{c}}} \ne \pi^0_{\beta_{\mathrm{c}}}$. By the random-cluster representation or otherwise, the two-point correlation functions $\pi^{\pm}_{\beta_{\mathrm{c}}}(\sigma_x\sigma_y)$ are bounded away from 0 for all pairs x, y of vertices. By the main result of [1, 19] (see also [12]) and the symmetry of $\pi^0_{\beta_{\mathrm{c}}}$, we have that $\pi^0_{\beta_{\mathrm{c}}} = \frac{1}{2}\pi^+_{\beta_{\mathrm{c}}} + \frac{1}{2}\pi^-_{\beta_{\mathrm{c}}}$, whence $\pi^0_{\beta_{\mathrm{c}}}(\sigma_x\sigma_y)$ is bounded away from 0. By [14], Theorem 5.17, this contradicts the above remark that the percolation-probability of the free-boundary condition random-cluster measure is 0 at $\beta = \beta_{\mathrm{sd}} = \beta_{\mathrm{c}}$.

We consider now the so-called thermodynamic limit of the random even graph on the torus $G_{m,n}$, as $m, n \to \infty$. It is long established that the Ising measure on $G_{m,n}$ converges weakly (in the product topology) to an infinite-volume limit measure denoted π_β. This may be seen as follows using the theory of the corresponding random-cluster model on $\mathbb{Z}^2$ (see [14]). When $\beta \le \beta_{\mathrm{c}}$, the existence of the limit follows more or less as discussed above, using the coupling with the random-cluster measure, and the fact that the percolation probability of the latter measure is 0 whenever $\beta \le \beta_{\mathrm{c}}$. We write π_β for the limit Ising measure as $m, n \to \infty$.

The thermodynamic limit is slightly more subtle when $\beta > \beta_{\mathrm{c}}$, since the infinite-volume Ising model has a multiplicity of Gibbs states in this case. The random-cluster model on the torus $G_{m,n}$ lies (stochastically) between the free and the wired measures on the graph obtained from $G_{m,n}$ by overlooking the extra boundary edges. By the uniqueness of infinite-volume random-cluster measures, the limit Ising measure is obtained by allocating random spins to the clusters of the infinite-volume random-cluster model (see Section 4.6 of

[14]). Once again, we write π_β for the ensuing measure on $\{-1,+1\}^{\mathbb{Z}^2}$, and we note that $\pi_\beta = \frac{1}{2}\pi_\beta^+ + \frac{1}{2}\pi_\beta^-$ where $\pi_\beta^\pm$ denotes the infinite-volume Ising measure with $\pm$ boundary conditions.

It has been shown in [7] (see also [11], Corollary 8.4) that there exists (with strictly positive π_β-probability) an infinite spin-cluster in the Ising model if and only if $\beta > \beta_c$. More precisely:

(a) if $\beta \le \beta_c$, there is π_β-probability 1 that all spin-clusters are finite,

(b) if $\beta > \beta_c$, there is π_β-probability 1 that there exists a unique infinite spin-cluster, which is equally likely to be a $+$ cluster as a $-$ cluster. Furthermore, by the main theorem of [10] or otherwise, for any given finite set S of vertices, the infinite spin-cluster contains, π_β-a.s., a cycle containing S in its interior.

On passing to the dual graph, one finds that the random even subgraph of $G_{m,n}$ with parameter $p \in (0, \frac{1}{2}]$ converges weakly to a probability measure ρ_p that is concentrated on even subgraphs of $\mathbb{Z}^2$ and satisfies:

(a′) if $p \ge p_c$, there is ρ_p-probability 1 that all faces of the graph are bounded,

(b′) if $p < p_c$, there is ρ_p-probability 1 that the graph is the vertex-disjoint union of finite clusters.

(Note that (5.1) defines β as a decreasing function of p, so the order relations are reversed.)

We have thus obtained a description of the weak-limit measure ρ_p when $p \le \frac{1}{2}$, and we note the phase transition at the parameter-value $p = p_c$. When $p > \frac{1}{2}$, a random even subgraph of $G_{m,n}$ is the complement of a random even subgraph with parameter $1-p$. [It is a convenience at this point that $G_{m,n}$ is itself an even graph.] Hence the weak-limit measure ρ_p exists for all $p \in [0,1]$ and gives meaning to the expression "a random even subgraph on $\mathbb{Z}^2$ with parameter p". [It is easily verified that $\rho_{\frac{1}{2}}$ equals the measure defined in Theorem 3.4 for $\mathbb{Z}^2$, and thus describes a uniform random even subgraph of $\mathbb{Z}^2$.] There is a sense in which the random even subgraph on $\mathbb{Z}^2$ has two points of phase transition, corresponding to the values p_c and $1-p_c$.

We consider finally the question of the size of a typical face of the random even graph on $\mathbb{Z}^2$ when $p_c \le p \le \frac{1}{2}$. This amounts to asking about the size of a (sub)critical Ising spin-cluster. The extremal cases are informative. When $p = p_c$, the (dual) Ising model is critical. The Ising spin-cluster C_x containing a given vertex x has volume and radius whose laws have polynomial decay, and it is widely accepted that the boundary of a large spin-cluster converges in a suitable manner to a Schramm–Löwner curve SLE_3 as the size of the cluster approaches infinity. The proof of the latter has been announced and summarised in [28], and rigorous proofs of polynomial decay and the associated scaling theory for the critical Ising model on $\mathbb{Z}^2$, long accepted in the physics literature, will follow.

When $p = \frac{1}{2}$, and thus $\beta = 0$, the Ising model amounts to site percolation with density $\frac{1}{2}$. The critical point of site percolation on $\mathbb{Z}^2$ is strictly greater than $\frac{1}{2}$, whence the volume and radius of each C_x have tails that decay exponentially. See, for example, Theorems 3.28, 5.4, and 6.75 of [13]. This is believed to be typical of the situation when $p_c < p \leq \frac{1}{2}$, namely that the volume and diameter of a given face of a random even subgraph of $\mathbb{Z}^2$ have laws with exponentially decaying tails. We know no proper proof of this for all $\beta < \beta_c$, but, for sufficiently small values of β, one may derive as follows exponentially-decaying estimates for the tails of the volume and diameter distributions by way of a comparison inequality.

One first adds an external field to the Ising measure of (1.5) on a finite graph $G = (V, E)$ to obtain the measure

$$\pi_{\beta,h}(\sigma) \propto \exp\left\{\beta \sum_{e=\langle x,y\rangle \in E} \sigma_x \sigma_y + h \sum_{x \in V} \sigma_x\right\}, \qquad \sigma \in \Sigma. \tag{5.4}$$

We write $\mu_1 \leq_{\mathrm{st}} \mu_2$ to mean that μ_1 is dominated stochastically by μ_2.

Theorem 5.5. *We have that $\pi_{\beta,0} \leq_{\mathrm{st}} \pi_{0,\beta N}$, where $\beta \geq 0$ and N is the maximum vertex degree of G.*

Proof. This follows the standard route of [14], Theorem 2.6, or [11], Theorem 4.8 and Proposition 4.16. □

Now, $\pi_{0,h}$ is site percolation on G with density $e^{2h}/(1 + e^{2h})$. Letting $G \uparrow \mathbb{Z}^2$, and noting that the critical probability of site percolation on $\mathbb{Z}^2$ is strictly greater than $\frac{1}{2}$, we deduce that, for sufficiently small positive β, the + cluster at the origin (under $\pi_{\beta,0}$) is dominated by a subcritical percolation cluster. The above claims follow.

The picture is quite different when the square lattice is replaced by the hexagonal lattice $\mathbb{H}$. Any even subgraph of $\mathbb{H}$ has vertex degrees 0 and/or 2, and thus comprises a vertex-disjoint union of cycles, doubly infinite paths, and isolated vertices. The (dual) Ising model inhabits the (Whitney) dual lattice of $\mathbb{H}$, namely the triangular lattice $\mathbb{T}$. Once again there exists a critical point $p_c = p_c(\mathbb{T}) < \frac{1}{2}$ such that the random even subgraph of $\mathbb{H}$ satisfies (a′) and (b′) above. In particular, the random even subgraph has a.s. only cycles and isolated vertices but no infinite paths. Recall that site percolation on $\mathbb{T}$ has critical value $\frac{1}{2}$. Therefore, for $p = \frac{1}{2}$, the face F_x of the random even subgraph containing the dual vertex x corresponds to a *critical* percolation cluster. It follows that its volume and radius have polynomially decaying tails, and that the boundary of F_x, when conditioned to be increasingly large, approaches SLE_6. See [27, 28] and [6]. The spin-clusters of the Ising model on $\mathbb{T}$ are 'critical' (in a certain sense described below) for all $p \in (p_c(\mathbb{T}), \frac{1}{2}]$, and this suggests the possibility that the boundary of F_x, when conditioned to be increasingly large, approaches SLE_6 for any such p. This is supported by the belief in the physics community that the so-called universality class of the spin-clusters of the subcritical Ising model on $\mathbb{T}$ is the same as that of critical percolation.

The 'criticality' of such Ising spin-clusters (mentioned above) may be obtained as follows. Note first that, since $\beta < \beta_c$, there is a unique Gibbs state π_β for the Ising model. Therefore, π_β is invariant under the interchange of spin-values $-1 \leftrightarrow +1$. Let R_n be a rhombus of the lattice with side-lengths n and axes parallel to the horizontal and one of the diagonal lattice directions, and consider the event A_n that R_n is traversed from left to right by a $+$ path (i.e., a path ν satisfying $\sigma_y = +1$ for all $y \in \nu$). It is easily seen that the complement of A_n is the event that R_n is crossed from top to bottom by a $-$ path (see Lemma 11.21 of [13] for the analogous case of bond percolation on the square lattice). Therefore,

$$\pi_\beta(A_n) = \tfrac{1}{2}, \qquad 0 \le \beta < \beta_c. \tag{5.6}$$

For $x \in \mathbb{Z}^2$, let S_x denote the spin-cluster containing x, and define

$$\mathrm{rad}(S_x) = \max\{|z - x| : z \in S\},$$

where $|y|$ is the graph-theoretic distance from 0 to y. By (5.6), there exists a vertex x such that $\pi_\beta(\mathrm{rad}(S_x) \ge n) \ge (2n)^{-1}$. By the translation-invariance of π_β,

$$\pi_\beta\big(\mathrm{rad}(S_0) \ge n\big) \ge \frac{1}{2n}, \qquad 0 \le \beta < \beta_c, \tag{5.7}$$

where 0 denotes the origin of the lattice. The left side of (5.7) tends to 0 as $n \to \infty$, and the polynomial lower bound is an indicator of the criticality of the model.

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