Probability Theory of Phase Transitions

Summary of achievements, EPSRC Grant GR/L15425
1 October 1996–6 April 2000, PI: Geoffrey Grimmett

The overarching target of the proposal was to further the understanding by mathematicians of phase transitions in disordered systems. This target has been pursued on several fronts, namely percolation theory, random-cluster models, contact and epidemic models, scattering models and random labyrinths, and random processes on graphs. The work is reported in 21 papers and 1 book, of which 14 items are already published, 6 are accepted or submitted, and 2 are in preparation. Several of the research papers have provoked subsequent work by the authors and by other mathematicians. Two research assistants have been employed in sequence, and a PhD student supported in part by the EPSRC through a Research Studentship has participated in some of the research under the supervision of the PI. The first RA left for the Courant Institute, New York University. The second was appointed as GCHQ Research Fellow at Cambridge University. The then PhD student is now a fixed-term Hedricks Assistant Professor at UCLA.

The research achievements are extremely close to the specific objectives of the Case for Support, a few possible lines of enquiry having been passed over in favour of more promising directions. The principal achievements are as follows.

Percolation. Inequalities relating the site and bond critical probabilities have been proved, for lattices and for more general graphs.

The existence of infinite 'rigid' components in the percolation model has been studied, together with inequalities for the rigidity critical probability, and the uniqueness of the infinite rigid cluster. (This was done by the PhD student under the supervision of the PI.) Similar progress has been made for infinite 'entangled' subsets in the percolation model. The phase transition was explored, together with the uniqueness of the infinite entanglement, and the tail of the entanglement at the origin for small p. Many open questions remain.

There is a famous open problem concerning the possible existence of *oriented* infinite paths in a certain correlated percolation model. The question was answered affirmatively in the *unoriented* case.

Continuum percolation was studied, and questions from the physics literature answered concerning the empirical distribution function of the edge lengths of a minimum spanning tree of a Poisson process.

It was proved for a certain randomly oriented model in two dimensions that infinite paths exist when the process dominates (strictly) the corresponding self-dual process. This is progress towards a resolution of 'quite an old chestnut'.

Random-cluster models. The exponential decay of the two-point connectivity function has been proved under the assumption that it decays at least polynomially. This is a generalization of a well known result of Simon (1980) for the Ising model.

The strict inequality of the critical point was proved for random-cluster models with random interactions.

Contact and epidemic models. We have studied contact models with recovery, proving exponential decay, and exploring the phase diagram. We have explored the structure of a contact model in which the infection rate depends on whether the site in question has or has not been infected previously.

Further work relates to the contact model on a tree, and particularly how the properties of the process on a *finite* subtree are related to the weak/strong survival of the process on the *infinite* tree.

Scattering models and random labyrinths. We have made progress in understanding the transience/recurrence of the motion of a light ray through a random field of reflecting bodies, subject to a stochastic relaxation. A result of significance in this stochastic version of the Lorenz/Ehrenfest problem is a functional central limit theorem.

Random processes on graphs. For a low-temperature model for directed polymers in a random environment, we have established inequalities for certain critical exponents.

Search algorithms on randomly-weighted trees have been studied, and a conjecture of Aldous disproved.

Phase transitions for branching random walks on graphs have been established, and related to the property of amenability, as well as compared with the phase structure of the corresponding contact model.