Probability Theory of Phase Transitions

Final report, EPSRC Grant GR/L15425
1 October 1996–6 April 2000, PI: Geoffrey Grimmett

"The overall target of the proposal is to further the understanding by mathematicians of phase transitions in disordered systems."[Quoted from the Case for Support.]

Achievements in brief

The grant has enabled work which has been reported in 19 articles and 1 book (12 items already published, 6 accepted or submitted, and 2 in preparation). Of these 20 items, 9 have as author one of the two research assistants employed under the contract. Two further publications, listed as numbers [21, 22] on the attached list, have been written by a PhD student under the supervision of the PI during the period of the grant. The topics of these two papers formed part of the original proposal, the support for which was scaled down by the EPSRC from that sought.

Training of personnel

The first research assistant employed under this grant left for the Courant Institute, New York University. The second was appointed as GCHQ Research Fellow at Cambridge University. Part of the work was done by a PhD student under the supervision of the PI, supported in part by an EPSRC Research Studentship; he is now a fixed-term Hedricks Assistant Professor at UCLA.

General Report

This report comprises the abstracts of the papers produced under the grant, together with extra comments. Many of the papers are available via `www.statslab.cam.ac.uk/~grg/`

Percolation

1. *G. R. Grimmett*, Percolation and Disordered Systems, *Ecole d'Eté de Probabilités de Saint Flour* XXVI–1996, ed. P. Bernard, Lecture Notes in Mathematics no. 1665 (1997), 153–300, Springer-Verlag, Berlin. *See E1, E4, E5 of the Case for Support, and Questions 4.1, 4.2, 4.3, 5.1, 5.2.*

Comments. These notes are intended primarily to facilitate access by students and young researchers to a dynamic and important field. Some original research is reported in these notes.

5. C. Bezuidenhout, *G. R. Grimmett*, A. Löffler, Percolation and minimal spanning trees, *Journal of Statistical Physics* 92 (1998) 1–34. *See E1 of the Case for Support.*

Abstract. Consider a random set $\mathcal{V}_n$ of points in the box $[n, -n)^d$, generated either by a Poisson process with density p or by a site percolation process with parameter p. We analyse the empirical distribution function F_n of the lengths of edges in a minimal (Euclidean) spanning tree $\mathcal{T}_n$ on $\mathcal{V}_n$. We express the limit of F_n, as $n \to \infty$, in terms of the free energies of a family of percolation processes derived from $\mathcal{V}_n$ by declaring two points to be adjacent whenever they are closer than a prescribed distance. By exploring the singularities of such free energies, we show that the large-n limits of the moments of F_n are infinitely differentiable functions of p except possibly at values belonging to a certain infinite sequence $(p_{\mathrm{c}}(k) : k \geq 1)$ of critical percolation probabilities. It is believed that these limiting moments are twice differentiable at these singular values, but not thrice differentiable. This analysis provides a rigorous framework for the numerical experimentation of Dussert, Rasigni, Rasigni, Palmari, and Llebaria, who have proposed novel Monte Carlo methods for estimating the numerical values of critical percolation probabilities.

6. *G. R. Grimmett*, *A. M. Stacey*, Critical probabilities for site and bond percolation models, *Annals of Probability* 26 (1998) 1788–1812. *See E1 of the Case for Support, particularly Question 1.3.*

Abstract. Any infinite graph $G = (V, E)$ has a site percolation critical probability $p_{\mathrm{c}}^{\mathrm{site}}$ and a bond percolation critical probability $p_{\mathrm{c}}^{\mathrm{bond}}$. The well known weak inequality $p_{\mathrm{c}}^{\mathrm{site}} \geq p_{\mathrm{c}}^{\mathrm{bond}}$ is strengthened to strict inequality for a broad category of graphs G, including all the usual finite-dimensional lattices in two and more dimensions. The complementary inequality $p_{\mathrm{c}}^{\mathrm{site}} \leq 1 - (1 - p_{\mathrm{c}}^{\mathrm{bond}})^{\Delta - 1}$ is proved also, where Δ denotes the supremum of the vertex degrees of G.

Comments. This work has provoked further research by L. Chayes and R. Schonmann of UCLA (Mixed percolation as a bridge between site and bond percolation, *Annals of Applied Probability*, to appear).

10. *G. R. Grimmett*, *Percolation*, 2nd edn, 'Grundlehren der Mathematischen Wissenschaften' no. 321, Springer-Verlag, Heidelberg, 1999, 444 pp. *See E1 of the Case for Support.*

Comments. This monograph is a substantial revision and extension of the first edition. It is a compact account of the main properties of the percolation model and some of its fields of application. Some of the work therein has not appeared previously in print.

12. P. Balister, B. Bollobas, *A. M. Stacey*, A model for dependent percolation, *Probability Theory and Related Fields* 17 (2000) 495–514. *See E1 of the Case for Support.*

Abstract. For a natural number k, define an oriented site percolation on $\mathbb{Z}^2$ as follows. Let x_i, y_j be independent random variables with values uniformly distributed in $\{1, \ldots, k\}$. Declare a site $(i,j) \in \mathbb{Z}^2$ *closed* if $x_i = y_j$, and *open* otherwise. Peter Winkler conjectured some years ago that if $k \geq 4$ then with positive probability there is an infinite oriented path $P = (i_0, j_0)(i_1, j_1)\ldots$ such that $0 = i_0 \leq i_1 \leq \ldots$, $0 = j_0 \leq j_1 \leq \ldots$, and each site (i_n, j_n) is open. Rather surprisingly, this conjecture is still open; in fact, it is not known whether the conjecture holds for *any* value of k. In this note, we shall prove the weaker result that the corresponding assertion holds in the *unoriented* case: if $k \geq 4$ then the probability that there is an infinite path that starts at the origin and consists only of open sites is positive. Furthermore, we shall show that our method can be applied to a wide variety of distributions of (x_i) and (y_j). Independently, Peter Winkler has recently proved a variety of similar assertions by different methods.

Comments. This is one of the few rigorous works of its type, studying percolation with heavily correlated edges. There is a variety of such two-dimensional problems for which individuals worldwide have tried unsuccessfully to prove the existence of percolation for some large parameter.

15. *G. R. Grimmett*, Percolation, in *Development of Mathematics 1950–2000*, Birkhäuser, 2000. *See E1 of the Case for Support.*

Comments. This historical account of percolation and applications will be a chapter in a book which reviews the principal components of mathematics during the last 50 years.

16. *G. R. Grimmett*, *A. E. Holroyd*, Entanglement in percolation, *Proceedings of the London Mathematical Society. See E1 of the Case for Support. The work arises from Question 1.4.*

Abstract. We study finite and infinite entangled graphs in the bond percolation process in three dimensions with density p. After a discussion of the relevant definitions, the entanglement critical probabilities are defined. The size of the maximal entangled graph at the origin is studied for small p, and it is shown that this graph has radius whose tail decays at least as fast as $\exp(-\alpha n/\log n)$; indeed, the logarithm may be replaced by any iterate of logarithm for an appropriate positive constant α. We explore the question of almost sure uniqueness of the infinite maximal open entangled graph when p is large, and we establish two relevant theorems. We make several conjectures concerning the properties of entangled graphs in percolation.

Comments. This is the first fully rigorous account of entanglement in random systems, and has potential applications to polymer physics. It is closely related to the work reported in [22], and is related to [21]. It has provoked later work of O. Häggström to appear in *Mathematical Proceedings of the Cambridge Philosophical Society*.

17. *G. R. Grimmett*, Conformal invariance, droplets, and entanglement, 2000, van Zwet Festschrift. *See E1 of the Case for Support, particularly Question 1.2.*

Abstract. Very brief surveys are presented of three topics of importance for interacting random systems, namely conformal invariance, droplets, and entanglement. For ease of description, the emphasis throughout is upon progress and open problems for the percolation model, rather than for the more general random-cluster model. Substantial recent progress has been made on each of these topics, as summarised here. Detailed bibliographies of recent work are included.

18. *G. R. Grimmett*, Randomly oriented lattices, *Random Structures and Algorithms. See E1 of the Case for Support.*

Abstract. The square lattice is used to generate an oriented graph in which a rightward or upward arrow is present on each edge with probability a, and a leftward or downward arrow with probability b. Independence between different edges of the square lattice is assumed, but nothing is assumed concerning the dependence between the two possible orientations at any given edge. A property of self-duality is exploited to show that, when $a + b = 1$, the process is, in a sense to be made precise, either critical or supercritical but not subcritical. This observation enables progress with the percolation problem in which each horizontal edge is oriented rightwards with probability p and otherwise leftwards, and each vertical edge is oriented upwards with probability p and otherwise downwards.

Comments. This marks partial progress on a 'quite old chestnut' of two-dimensional percolation.

21. *A. E. Holroyd*, Existence and uniqueness of infinite components in generic rigidity percolation, *Annals of Applied Probability* 8 (1998) 944–973. *See E1 of the Case for Support, particularly Question 1.4.*

Abstract. We consider a percolation configuration on a general lattice in which edges are included independently with probability p. We study the rigidity properties of the resulting configuration, in the sense of generic rigidity in d dimensions. We give a mathematically rigorous treatment of the problem, starting with a definition of an infinite rigid component. We prove that, for a broad class of lattices, there exists an infinite rigid component for some p strictly below unity. For the particular case of two-dimensional rigidity on the two-dimensional triangular lattice, we prove first that the critical probability for rigidity percolation lies strictly above that for connectivity percolation and second that the infinite rigid component (when it exists) is unique for all but countably many values of p. We comjecture that this uniqueness in fact holds for all p. Some of our arguments could be applied to two-dimensional lattices in more generality.

Comments. This first rigorous work on rigidity percolation was provoked by work of Jacobs and Thorpe (1996) in the physics literature. The basic mathematics of the problem are explored, and various results of importance are identified and established. This has led to further work by Holroyd to appear in the *Annals of Applied Probability*, and to work of O. Häggström (to appear).

22. *A. E. Holroyd*, Existence of a phase transition for entanglement percolation, *Mathematical Proceedings of the Cambridge Philosophical Socety* 129 (2000) 231–251. *See E1 of the Case for Support. This work is related to Question 1.4.*

Abstract. We consider the bond percolation model on the three-dimensional cubic lattice, in which individual edges are retained independently with probability p. We shall describe a subgraph of the lattice as 'entangled' if, roughly speaking, it cannot be 'pulled apart' in three dimensions. We shall discuss possible ways of turning this into a rigorous definition of entanglement. For a broad class of such definitions, we shall prove that for p sufficiently close to zero, the graph of retained edges has no infinite entangled subgraph almost surely, thereby establishing that there is a phase transition for entanglement at some value of p strictly between zero and unity.

Comments. The problem of proving that the critical point for entanglement is *strictly* positive had apparently been overlooked. Such questions are usually straightforward to answer, using established methods, but this one turns out to require new and geometrically complex ideas. See also [16].

Random-cluster models

2. *G. R. Grimmett, M. Piza*, Decay of correlations in subcritical Potts and random-cluster models, *Communications in Mathematical Physics* 189 (1997) 465–480. *See E4 of the Case for Support, Question 4.1.*

Abstract. We prove exponential decay for the tail of the radius R of the cluster at the origin, for subcritical random-cluster models, under an assumption slightly weaker than that $E(R^{d-1}) < \infty$ (here, d is the number of dimensions). Specifically, if $E(R^{d-1}) < \infty$ throughout the subcritical phase, then $P(R \geq n) \leq \exp(-\alpha n)$ for some $\alpha > 0$. This implies the exponential decay of the two-point correlation function of subcritical Potts models, subject to a hypothesis of (at least) polynomial decay of this function. Similar results are known already for percolation and Ising models, and for Potts models when the number q of available states is sufficiently large; indeed the hypothesis of polynomial decay has been proved rigorously for these cases. In two dimensions, the hypothesis that $E(R) < \infty$ is

weaker than requiring that the susceptibility be finite, i.e., that the two-point function be summable. The principal new technique is a form of Russo's formula for random-cluster models reported by Bezuidenhout, Grimmett, and Kesten. For the current application, this leads to an analysis of a first-passage problem for random-cluster models, and a proof that the associated time constant is strictly positive if and only if the tail of R decays exponentially.

Comments. The main point is to establish an inequality of the type of the Hammersley–Simon–Lieb inequality for percolation/Ising systems, in order to obtain exponential decay in generality for random-cluster models. This is achieved via a novel inequality for the derivative of the probability of an increasing event. The technology has applications elsewhere also, see [16]. It is interesting to note that the condition obtained is very slightly weaker than that of Simon–Lieb (1980) when $d = 2$.

8. *G. R. Grimmett*, Inequalities and entanglements for percolation and random-cluster models, in *Perplexing problems in Probability, Festschrift in Honor of Harry Kesten*, ed. M. Bramson and R. Durrett, 1999, 91–106, Birkhäuser. *See E4 of the Case for Support.*

Abstract. We discuss inequalities and applications for percolation and random-cluster models. The relevant areas of methodology concern the following two types of inequality: inequalities involving the probability of a general increasing event, and certain differential inequalities involving the percolation probability. We summarise three areas of application of such inequalities, namely strict inequality between the bond and site critical percolation probabilities of a general graph, the general study of entanglements in percolation, and strict inequalities for critical points of disordered random-cluster models.

Comments. The principal progress in this paper is the proof of the strict monotonicity of the critical point for a random-cluster model with random interactions. We understand that this result has applications in the theory of spin glasses.

Contact and epidemic models

4. J. van den Berg, *G. R. Grimmett*, R. Schinazi, Dependent random graphs and spatial epidemics, *Annals of Applied Probability* 8 (1998) 317–336. *See E3 of the Case for Support, particularly Questions 3.1, 3.2, 3.3.*

Abstract. We extend certain exponential decay results of subcritical percolation to a class of locally dependent random graphs, introduced by Kuulasmaa as models for spatial epidemics on $\mathbb{Z}^d$. In these models, infected individuals eventually die (are removed) and are not replaced. We combine these results with certain continuity and re-scaling arguments in order to improve our knowledge of the phase diagram of a modified epidemic model in which new susceptibles are born at some positive rate. In particular, we show that, throughout an intermediate phase where the infection rate lies between a certain two critical values, no coexistence is possible for sufficiently small positive values of the recovery rate. This result provides a converse to results of Durrett–Neuhauser and Andjel–Schinazi. We show also that such an intermediate phase indeed exists for every $d \geq 1$ (i.e., that the two critical values mentioned above are distinct). An important technique is the general version of the BK inequality for disjoint occurrence, proved in 1994 by Reimer.

Comments. The difficulty is that this contact process with recovery is not monotone. This causes serious difficulty in the analysis, and one proceeds instead by making comparisons with other processes which have better understood structure. In following this project, the authors have obtained a general exponential-decay type theorem for oriented percolation with certain correlations. This appears to be one of the very few 'essential' applications of Reimer's inequality found to date.

13. R. Pemantle, *A. M. Stacey*, The branching random walk and contact process on Galton–Watson and non-homogeneous trees. *See E4 of the Case for Support.*

Abstract. We show that branching random walk on a Galton–Watson tree may have one or two phase transitions, depending on the relative sizes of the mean degree and the maximum degree. We show that there are some Galton–Watson trees on which the branching random walk has one phase transition while the contact process has two; this contradicts a conjecture of Madras and Schinazi. We show that the contact process has only one phase transition on some trees of uniformly exponential growth and bounded degree, contradicting a conjecture of Pemantle.

14. *A. M. Stacey*, The contact process on finite homogeneous trees, *Probability Theory and Related Fields*. *See E3 of the Case for Support.*

Abstract. Let $\mathbb{T}_d$ be a homogeneous tree in which every vertex has $d+1$ neighbours, where $d \geq 2$. The contact process on such a tree is known to have three distinct phases. We consider the process on a finite subtree, namely the rooted tree of depth h and branching factor d, and relate the behaviour of the process on the infinite tree to its behaviour on the finite tree for large h. In the phase of strong survival, we show that probability ϵ independent of h, the process on the subtree starting from a single infection survives for a time which is doubly exponential in h and almost exponential in the number of vertices of the finite tree. In the phase of weak survival on the infinite tree, the survival time on the finite tree is approximately linear in h. In the phase of no survival, the survival time on the finite tree is linear in h if one starts with all vertices initially infected, and bounded by a random variable (independent of h) with an exponential tail if one starts from a single infection.

Comments. This is part of an ongoing project to understand multiphase properties of stochastic processes on trees, initiated by Grimmett–Newman and Pemantle.

19. *A. M. Stacey*, Partial immunization processes. *See E3 of the Case for Support.*

Comments. A variant of the contact process is considered, which was introduced independently by Grassberger, Chaté and Rousseau (1997), and by Durrett and Schinazi. A site has three possible states corresponding to never infected, healthy but previously infected, and currently infected. There are two parameters l_{n} and l_{o} (the suffices standing for new and old). Never infected sites become infected at a rate equal to l_{n} times the number of infected neighbours. Previously infected sites become infected at a rate equal to l_{o} times the number of infected neighbours. All infected sites recover (i.e., flip to 'previously infected') at rate 1. One typically starts with one site infected and all other sites 'never infected'. It is shown that on $\mathbb{Z}^d$ one cannot have strong survival (in the usual sense of repeated infection at the origin) when l_{o} is less than the critical value for the standard contact process. This strengthens results of Grassberger et al., and establishes a conjecture of Durrett–Schinazi. This implies that one can have a phase of weak survival. Some results are also obtained which show that, on other graphs, the phase diagram can look qualitatively different.

Scattering models and random labyrinths

7. C. Bezuidenhout, *G. R. Grimmett*, A central limit theorem for random walks in random labyrinths, *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* 35 (1999) 631–683. *See E5 of the Case for Support, particularly Questions 5.1, 5.2.*

Abstract. A beam of light shines through the lattice $\mathbb{Z}^d$, and is subjected to reflections determined by a random environment of mirrors at the vertices of $\mathbb{Z}^d$. The behaviour of the light ray is investigated under the hypothesis that the environment contains a strictly positive density of vertices at which the light behaves in the manner of a random walk. When $d \geq 2$ and the density of non-trivial reflectors is sufficiently small, the environment contains almost surely a unique infinite 'inter-illuminating' class of vertices. Furthermore, when the light beam originates within this class, then its trajectory obeys a functional central limit theorem with a strictly positive diffusion constant. These facts are obtained using percolation-type arguments, together with the invariance principle proposed by Kipnis and Varadhan.

Comments. This is a continuation of earlier work of the PI on the motion of an electron through a random environment of scatterers. Subject to stochastic relaxation, the authors prove the diffusivity of the motion. It is a hard and important open question of probability and physics to establish (or not) diffusivity in the absence of the assumption involved in the stochastic relaxation, i.e., for Lorenz/Ehrenfest models.

9. *G. R. Grimmett*, Stochastic pin-ball, in *Random Walks and Discrete Potential Theory*, ed. M. Picardello and W. Woess, Cambridge University Press, 1999, 205–213. *See E5 of the Case for Support.*

Abstract. A ball is propelled through a random environment of obstacles off which it rebounds with perfect reflection. What is the behaviour of the trajectory of the ball? We summarise known mathematical results concerning this model, which we call 'stochastic pin-ball', and which is known also as a 'Lorentz lattice gas' and a version of which is sometimes termed the 'Ehrenfest wind–tree

model'. The rigorous theory is more extensively developed if the environment is allowed to include a positive density of space in which the ball behaves in the manner of a random walk. For a lattice model of this type, one may employ arguments of percolation theory in order to prove theorems concerning non-localisation, transience, and asymptotic normality, under certain assumptions on the environment.

Random processes on graphs

3. *M. Piza*, Directed polymers in a random environment: some results on fluctuations, *Journal of Statistical Physics* 89 (1997) 581–603. *Related to E6 of the Case for Support.*

Abstract. We consider a polymer model on $\mathbb{Z}_+^d$ where to each edge is associated a random variable $v(e)$. A polymer configuration is represented by a directed path r and has a weight $\exp[-\beta \sum_{e\in r} v(e)]$, with $\beta = 1/T$ the inverse temperature. We extend some rigorous results that have been obtained for the ground state of this model to finite temperatures. In particular we obtain some upper and lower bounds on sample to sample free energy fluctuations, and also rigorous scaling inequalities between the exponents describing free energy fluctuations and transversal displacements of polymer configurations.

Comments. There have been enormous efforts towards an understanding of disordered system in which the interactions are themselves random variables (see [8] for an example of such work). The present paper extends certain work, related to that of Alexander, Kesten, Newman, Piza, Wehr, Aizenman, from the ground state of a directed polymer model to the regime of non-zero temperatures.

11. *A. M. Stacey*, Searches on a random tree with random edge-weights, *Combinatorics, Probability, and Computing* 8 (1999) 555–565. *Work in the vein of the project but not announced in the Case for Support.*

Abstract. Each edge of the standard rooted binary tree is equipped with a random weight; weights are independent and identically distibuted. The value of a vertex is the sum of the weights on the path from the root to the vertex. We wish to search the tree to find a vertex of large weight. A very natural conjecture of Aldous states that, in the sense of stochastic domination, an obvious greedy algorithm is best possible. We show that this conjecture is false. We prove, however, that in a weaker sense there is no significantly better algorithm.

20. *A. M. Stacey*, Modified branching random walks and amenability. *Work in the vein of the project but not announced in the Case for Support.*

Comments. A modified version of the branching random walk is considered and shown to have particularly good behaviour. On a connected graph of bounded degree this process has two phase transitions if and only if the graph is non-amenable. Once formulated, this result follows fairly easily from known results about the simple random walk. The standard branching random walk does not behave in such a simple way but has advantages over the modified one in other respects which are discussed. With a little more work it is shown that the result for the modified branching random walk can be extended to the standard branching random walk if, additionally, the graph is quasi-transitive.