LATTICE EMBEDDINGS IN PERCOLATION

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ABSTRACT. Does there exist a Lipschitz injection of $\mathbb{Z}^d$ into the open set of a site percolation process on $\mathbb{Z}^D$, if the percolation parameter p is sufficiently close to 1? We prove a negative answer when $d = D$, and also when $d \geq 2$ in the case when the Lipschitz constant M is required to be 1. Combined with a recent result of Dirr, Dondl, Holroyd, Grimmett and Scheutzow, this suffices to answer the question for all d, D and M (in particular, the answer is positive for $d < D$ and $M = 2$). Our proof in the case $d = D$ uses Tucker's lemma from topological combinatorics. One application is an affirmative answer to a question of Peled concerning embeddings of random words in two and more dimensions.

1. INTRODUCTION

Let $\mathbb{Z}^d$ denote the d-dimensional integer lattice. Elements of $\mathbb{Z}^d$ are called **sites**. Let $\|\cdot\|_r$ denote the ℓ^r-norm on $\mathbb{Z}^d$, and abbreviate $\|\cdot\|_1$ to $\|\cdot\|$. We say that a map $f : \mathbb{Z}^d \to \mathbb{Z}^D$ is M-**Lipschitz**, or simply M-Lip, if $\|f(x) - f(y)\| \leq M$ for all $x, y \in \mathbb{Z}^d$ with $\|x - y\| = 1$.

For $p \in [0, 1]$, consider the site percolation model on $\mathbb{Z}^d$. That is, we declare each site to be **open** with probability p, and otherwise **closed**, with different sites receiving independent designations. Let $W_p(\mathbb{Z}^d)$ denote the random set of open sites, and write $\mathbb{P}_p$ and $\mathbb{E}_p$ for the associated probability measure and expectation operator.

We are interested primarily in the probability

$$(1) \quad L(d, D, M, p) := \mathbb{P}_p\Big(\exists \text{ an } M\text{-Lip injection from } \mathbb{Z}^d \text{ to } W_p(\mathbb{Z}^D)\Big).$$

Clearly L is increasing in D, M, and p, and decreasing in d. Furthermore, L is $\{0, 1\}$-valued, since $\mathbb{P}_p$ is a product measure and the event of (1) is invariant under translations of $\mathbb{Z}^D$. We define the critical probability

$$p_c(d, D, M) := \inf\{p : L(d, D, M, p) = 1\},$$

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$d \diagdown D$	1	2	3	4	5	$\dots$
1	∞	1	1	1	1	$\dots$
2	∞	∞	2	2	2	$\dots$
3	∞	∞	∞	2	2	$\dots$
4	∞	∞	∞	∞	2	$\dots$
5	∞	∞	∞	∞	∞	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1. The values of $M_c(d,D)$ for $d,D \geq 1$.

and furthermore

$$M_c(d,D) := \min\{M \geq 1 : p_c(d,D,M) < 1\},$$

(where $\min\varnothing := \infty$). That is, $M_c(d,D)$ is the smallest M such that, for some $p<1$, there exists, $\mathbb{P}_p$-a.s., an injective M-Lip map from $\mathbb{Z}^d$ to the open sites of $\mathbb{Z}^D$.

It is easily seen that $p_c(1,D,M)$ is simply the critical probability for site percolation on the lattice with vertex-set $\mathbb{Z}^D$ and an edge connecting every pair of sites at ℓ^1-distance at most M (see, for example, [4]). Therefore, for $M \geq 1$, we have that

$$p_c(1,D,M) \begin{cases} = 1 & \text{if } D = 1, \\ \in (0,1) & \text{if } D \geq 2. \end{cases}$$

This implies in particular that $p_c(d,D,M) > 0$ for all integers $d,D,M \geq 1$. The problem of interest is to determine for which d, D, M it is the case that $p_c(d,D,M)=1$.

Theorem 1. *Let d,D,M be positive integers.*

(a) *For all d, we have $p_c(d,d+1,2) < 1$, and hence $M_c(d,d+1) \leq 2$.*
(b) *For all $D \geq 2$, we have $p_c(2,D,1) = 1$, and hence $M_c(2,D) > 1$.*
(c) *For all $d \geq 2$ and all M, we have $p_c(d,d,M) = 1$, and hence $M_c(d,d) = \infty$.*

It is an elementary observation that if $d > D$ then $L(d,D,M,1) = 0$ for all M, and hence $M_c(d,D) = \infty$. (To check this, suppose that $f : \mathbb{Z}^d \to \mathbb{Z}^D$ is a M-Lip injection, and let $S_n := \{x \in \mathbb{Z}^d : \|x\| \leq n\}$. Then $|S_n|$ has order n^d, but $|f(S_n)|$ has order n^D, in contradiction to the injectivity of f.) Therefore, the above results suffice to determine the values of M_c for all d,D, as summarized in Table 1.

Theorem 1(a) is an immediate consequence of a substantially stronger statement proved in [2], which we state next. For $x =$

$(x_1, \dots, x_{d-1}) \in \mathbb{Z}^{d-1}$ and $z \in \mathbb{Z}$ we write $(x, z) := (x_1, \dots, x_{d-1}, z) \in \mathbb{Z}^d$. Write $\mathbb{Z}_+ := \mathbb{Z} \cap (0, \infty)$.

Theorem 2 (Lipschitz percolation; [2])**.** *Let $d \geq 2$ and suppose $p > 1-(2d)^{-2}$. There exists $\mathbb{P}_p$-a.s. a (random) 1-Lip function $F : \mathbb{Z}^{d-1} \to \mathbb{Z}_+$ such that for every $x \in \mathbb{Z}^{d-1}$, the site $(x, F(x)) \in \mathbb{Z}^d$ is open.*

For F as in Theorem 2, the map $x \mapsto (x, F(x))$ is evidently a 2-Lip injection, thus establishing Theorem 1(a). Other applications of Theorem 2 appear in [3, 6]. The proof of Theorem 1(b) is relatively straightforward (the proof involves showing that any 1-Lip injection from $\mathbb{Z}^2$ to $\mathbb{Z}^D$ is tightly constrained). The main contribution of this paper is the proof of Theorem 1(c). Interestingly, our proof of this non-existence result will use the existence result, Theorem 2. Another essential ingredient will be Tucker's Lemma from topological combinatorics.

The Lipschitz injections discussed above act on $\mathbb{Z}^d$, with range the random set $W_p(\mathbb{Z}^D)$. Only little extra generality is achieved by considering injections that preserve values indexed by $\mathbb{Z}^d$, as follows. Let $\Omega = \Omega_d := \{0, 1\}^{\mathbb{Z}^d}$ be the space of percolation configurations, in which the value 1 (respectively, 0) is identified with the state 'open' (respectively, 'closed'). An **embedding** of a configuration $\eta \in \Omega_d$ into a configuration $\omega \in \Omega_D$ is an injection $f : \mathbb{Z}^d \to \mathbb{Z}^D$ such that $\omega(x) = f(\mu(x))$ for all $x \in \mathbb{Z}^d$.

Proposition 3. *For positive integers d, D, the following are equivalent:*

- (a) *$d < D$ (equivalently, $M_c(d, D) < \infty$),*
- (b) *$p_c(d, D, M) \to 0$ as $M \to \infty$,*
- (c) *for every $p \in (0, 1)$, there exists $M \geq 1$ such that: for every $\eta \in \Omega_d$ and for $\mathbb{P}_p$-a.e. $\omega \in \Omega_D$, there exists an M-Lip embedding of η into ω.*

Proposition 3 answers affirmatively a question posed by Ron Peled concerning the existence of M-Lip embeddings of d-dimensional random words into spaces of higher dimension, see [5, Sect. 5].

There is a close connection between the existence of embeddings and of quasi-isometries. A **quasi-isometry** between two metric spaces (X, δ) and (Y, ρ) is a map $f : X \to Y$ such that: there exist constants $c_i \in (0, \infty)$ with

- (i) $\forall\, x, x' \in X, \quad c_1\delta(x, x') - c_2 \leq \rho(f(x), f(x')) \leq c_3\delta(x, x') + c_4,$
- (ii) $\forall\, y \in Y,\ \exists\, x \in X$ such that $\quad \rho(f(x), y) \leq c_5.$

We call such f a **c-quasi-isometry** when we wish to emphasize the role of $\mathbf{c} = (c_1, \dots, c_5)$. It is not difficult to see that the existence of a quasi-isometry is a symmetric relation on metric spaces. Quasi-isometries of *random* metric spaces are discussed in [9]. A **subspace** of a metric space (X, δ) is a metric space (U, δ) with $U \subseteq X$.

Proposition 4. *For positive integers d, D, the following are equivalent.*

(a) *We have $d < D$ (equivalently, $M_c(d, D) < \infty$).*
(b) *For some $p \in (0, 1)$, with positive probability there exists a quasi-isometry between $(\mathbb{Z}^d, \ell^1)$ and some subspace of $(W_p(\mathbb{Z}^D), \ell^1)$.*
(c) *For all $p \in (0, 1)$, $\mathbb{P}_p$-a.s., there exists a quasi-isometry between $(\mathbb{Z}^d, \ell^1)$ and some subspace of $(W_p(\mathbb{Z}^D), \ell^1)$.*

The proofs of Theorem 1(b,c) appear respectively in Sections 2 and 3. Propositions 3 and 4 are proved in Section 4.

2. Nearest-neighbour maps

In this section we prove Theorem 1(b). A (self-avoiding) **path** in $\mathbb{Z}^d$ is a finite or infinite sequence of distinct sites, each consecutive pair of which is at ℓ^1-distance 1. Let $e_1, \dots, e_d \in \mathbb{Z}^d$ be the standard basis vectors of $\mathbb{Z}^d$, and let 0 denote the origin.

Lemma 5. *Let $x_1, \dots, x_k \in \mathbb{Z}^D$ be distinct, and let $A = A(x_1, \dots, x_k)$ be the event that there exists a singly-infinite path $0 = y_0, y_1, \dots$ in $\mathbb{Z}^D$ such that the sites $(x_i + y_j : i = 1, \dots, k,\ j = 0, 1, \dots)$, are distinct and open. If $p < (2D)^{-1/k}$ then $\mathbb{P}_p(A) = 0$.*

Proof. Let A_n be the event that there exists a path $0 = y_0, y_1, \dots, y_n$ of length n in $\mathbb{Z}^D$ such that the sites $(x_i + y_j : i = 1, \dots, k,\ j = 0, \dots, n)$ are distinct and open. Note that A is the decreasing limit of A_n as $n \to \infty$. Let N_n be the number of paths $0 = y_0, \dots, y_n$ with the properties required for A_n. Then

$$\mathbb{P}_p(A_n) \le \mathbb{E}_p N_n \le (2D)^n p^{nk} \xrightarrow{n\to\infty} 0, \qquad \text{if } 2Dp^k < 1.$$

Here, $(2D)^n$ is an upper bound for the number of n-step self-avoiding paths (y_j) starting from 0, while for those paths for which the sites $x_i + y_j$ are distinct, p^{nk} is the probability they are all open. □

Proof of Theorem 1(b). We must prove that, for any fixed $p < 1$ and $D \ge 2$, a.s. there exists no 1-Lip injection from $\mathbb{Z}^2$ to $W_p(\mathbb{Z}^D)$.

First, suppose f is a 1-Lip injection from $\mathbb{Z}^2$ to the full lattice $\mathbb{Z}^D$, and consider the image of a unit square. Specifically, take $(i, j) \in \mathbb{Z}^2$ and let

$$r_1 = f(i+1, j) - f(i, j), \qquad r_1' = f(i+1, j+1) - f(i, j+1),$$
$$r_2 = f(i, j+1) - f(i, j), \qquad r_2' = f(i+1, j+1) - f(i+1, j).$$

Note that: the four vectors r_1, r_2, r_1', r_2' are elements of $\{\pm e_j : j = 1, \dots, D\}$ (by the 1-Lip property); they satisfy $r_1 + r_2' = r_1' + r_2$ (by definition); the pair r_1, r_2 are neither equal to nor negatives of each other; and similarly for r_1, r_2' (a consequence of injectivity). It follows that $r_1 = r_1'$ and $r_2 = r_2'$. Since this holds for every unit square, for any distinct $i, i' \in \mathbb{Z}$, the images under f of the two paths $\{(i, j) : j \in \mathbb{Z}\}$ and $\{(i', j) : j \in \mathbb{Z}\}$ are two disjoint self-avoiding paths that are translates of each other. (Another consequence, which we shall not need, is that there exists $\Delta \subset \{1, \dots, D\}$ such that all horizontal edges have images in $\{\pm e_j : j \in \Delta\}$, and all vertical edges have images in the complement $\{\pm e_j : j \notin \Delta\}$).

Let B be the event that there exist $x_1, x_2, \dots \in \mathbb{Z}^D$ and a self-avoiding path $0 = y_0, y_1, \dots$ in $\mathbb{Z}^D$ such that the sites $(x_i + y_j : i \geq 1, j \geq 0)$ are distinct and open. The above argument implies that, if there exists a 1-Lip injection $f : \mathbb{Z}^2 \to W_p(\mathbb{Z}^D)$, then B occurs. We shall now show that $\mathbb{P}_p(B) = 0$ for all $p < 1$ and $d \geq 1$. Let k be large enough that $p < (2D)^{-1/k}$. Define B_k similarly to B, except in that it requiring the existence of only k sites $x_1, \dots, x_k$. Lemma 5 implies that $\mathbb{P}_p(B_k) = 0$, because B_k is the countable union over all possible $x_1, \dots, x_k$ of the events $A(x_1, \dots, x_k)$. Finally, we have $B \subseteq B_k$. $\square$

3. The case of equal dimensions

In this section we prove Theorem 1(c). We denote integer intervals by $(\!|a, b]\!] := (a, b] \cap \mathbb{Z}$, etc. Fix any $d \geq 2$, $M \geq 1$ and $p \in (0, 1)$. We will prove that a.s. there does not exist an M-Lip injection from $\mathbb{Z}^d$ to $W_p(\mathbb{Z}^d)$.

The idea behind the proof is as follows. Suppose that f is such an injection. By a *hole* we mean a cube of side length M in $\mathbb{Z}^d$ all of whose sites are closed (actually, a slightly different definition will be convenient in the formal proof, but this suffices for the current informal sketch). Holes are rare (if p is close to 1), but the typical spacing between them is a fixed function of d, M, and p. We will consider the image under f of a cuboid $[\![1, n]\!]^{d-1} \times [\![1, m]\!] \subset \mathbb{Z}^d$, where $m \gg n \gg 1$. We will arrange that the images of the two opposite faces $[\![1, n]\!]^{d-1} \times \{1\}$ and $[\![1, n]\!]^{d-1} \times \{m\}$ are far apart, and separated by a $(d-1)$-dimensional 'surface of holes' (at the typical spacing). This implies that image of the interior of the cuboid must pass through this surface, avoiding all the holes. To do so, the image must be in some sense be folded up so as to be locally $(d-1)$-dimensional, and this will give a contradiction to the injectivity of f if n is chosen large enough compared with the spacing of the holes.

In the case $d = 2$ (and perhaps for other small values of d), the above ideas can be formalized using ad-hoc geometric methods, but for general d we need a more systematic approach. The surface of holes will be constructed using Theorem 2, and we will augment it with a colouring of the nearby open sites using $d-1$ colours, in such a way that the coloured sites separate the two sides of the surface from each other, but regions of any given colour have bounded size. Via the map f, this colouring will induce a colouring of the cuboid that contradicts a certain topological fact. The following notation will be used extensively. A **colouring** of a set of sites $U \subseteq \mathbb{Z}^d$ is a map χ from U to a finite set Q. A site $u \in U$ is said to have **colour** $\chi(u) \in Q$. We introduce the graph $G = G(U, \ell^r, k)$ having vertex set U and an edge between $u, v \in U$ if and only if $0 < \|u - v\|_r \leq k$. An important special case is the **star-lattice** $G^* = G^*_d := G(\mathbb{Z}^d, \ell^\infty, 1)$. Given a graph G and a colouring χ as above, a q-**cluster** (of χ with respect to G) is a connected component in the subgraph of G induced by the set of vertices of colour q. The **volume** of a cluster is defined to be the number of its sites.

We next state the two main ingredients of the proof: a topological result on colouring a cuboid, and a result on existence of random coloured surfaces in the percolation model.

Proposition 6 (Colour blocking)**.** *Let d, n, m be positive integers, and consider a colouring*

$$\chi : [\![1, n]\!]^{d-1} \times [\![1, m]\!] \to \{-\infty, +\infty, 1, 2, \ldots, d-1\}.$$

If χ satisfies:

(i) *all sites in $[\![1, n]\!]^{d-1} \times \{1\}$ have colour $-\infty$;*
(ii) *all sites in $[\![1, n]\!]^{d-1} \times \{m\}$ have colour $+\infty$; and*
(iii) *no site of colour $+\infty$ is adjacent in G^* to a site of colour $-\infty$,*

then χ has a j-cluster with respect to G^* of volume at least n, for some $j \in \{1, 2, \ldots, d-1\}$.

Proposition 7 (Coloured surfaces)**.** *Fix $d \geq 2$, $J \geq 1$, and $p \in (0, 1)$. There exist constants $K, C < \infty$ (depending on d, J, and p) such that $\mathbb{P}_p$-a.s. there is a (random) colouring*

$$\lambda : W_p(\mathbb{Z}^d) \to \{-\infty, +\infty, 1, 2, \ldots, d-1\}$$

of the open sites of $\mathbb{Z}^d$ with the following properties.

(i) *No site of colour $+\infty$ is adjacent to a site of colour $-\infty$ in $G(W_p(\mathbb{Z}^d), \ell^\infty, J)$.*
(ii) *For each $j \in \{1, 2, \ldots, d-1\}$, every j-cluster with respect to $G(W_p(\mathbb{Z}^d), \ell^\infty, J)$ has volume at most K.*

(iii) *There exists a (random) non-negative real-valued function* $g : \mathbb{Z}^{d-1} \to [0, \infty)$*, with the Lipschitz property that* $|g(u) - g(v)| \le \frac{1}{d}\|u - v\|_1$ *for all* $u, v \in \mathbb{Z}^{d-1}$*, such that all open sites in*

$$S^- := \{(u, z) : u \in \mathbb{Z}^{d-1},\ z < g(u)\}$$

are coloured $-\infty$*, while all open sites in*

$$S^+ := \{(u, z) : u \in \mathbb{Z}^{d-1},\ z > g(u) + C\}$$

are coloured $+\infty$*.*

(In Proposition 7(iii), note in particular that all open sites in the half-space $\mathbb{Z}^{d-1} \times (\!|-\infty, 0|\!)$ are colored $-\infty$).

The proof of Theorem 1(c) will proceed by playing Propositions 6 and 7 against each other to obtain a contradiction. The number of allowed colours is crucial – if one colour more were added to $1, \dots, d-1$ then the conclusion of Proposition 6 would no longer hold, while with one colour fewer, the conclusion of Proposition 7 would not hold. It should also be noted that the use of the star-lattice G^* is essential in Proposition 6 – the statement does not hold for the nearest-neighbour lattice $G(\mathbb{Z}^d, \ell^1, 1)$.

Our proof of Proposition 6 will use Tucker's Lemma, a beautiful result of topological combinatorics. The general version of [10, 7] applies to triangulations of a ball, and is a close relative of the Borsuk-Ulam fixed-point theorem; see [8] for background. We need only a special case, for the cuboid, which is also proved in [1].

For $t \in [\![1, \infty)\!)^d$, consider the cuboid $T = T(t) := [\![0, t_1]\!] \times \cdots \times [\![0, t_d]\!] \subset \mathbb{Z}^d$ with opposite corners 0 and t, and define the boundary $\partial T := T \setminus (\!(0, t_1)\!) \times \cdots \times (\!(0, t_d)\!)$. We say that boundary sites $x, y \in \partial T$ are **antipodal** if $x + y = t$.

Lemma 8 (Tucker's Lemma for the cuboid; [1])**.** *Let* $T \subset \mathbb{Z}^d$ *be a cuboid as above, and suppose* $\beta : T \to \{\pm 1, \dots, \pm d\}$ *is a colouring such that for each antipodal pair* $x, y \in \partial T$ *we have* $\beta(x) = -\beta(y)$*. Then there exist* $u, v \in T$ *that are adjacent in* G^* *(and, in fact, that satisfy* $u_i \le v_i \le u_i + 1\ \forall i$*) such that* $\beta(u) = -\beta(v)$*.*

Proof of Proposition 6. Throughout the proof, adjacency and clusters refer to G^*. The (ℓ^∞-)**diameter** of a cluster is the maximum ℓ^∞-distance between two of its sites. It suffices to show that for a coloring χ satisfying the given conditions, there is a j-cluster of diameter at least n for some $j \ne \pm\infty$. Suppose that this is false. We will construct a modified coloring that leads to a contradiction.

First define a coloring χ' of the larger cuboid $T := [\![0, n+1]\!]^{d-1} \times [\![0, m+1]\!]$ as follows. Let χ' agree with χ on $T \setminus \partial T$, except with colour ∞ everywhere changed to d, and $-\infty$ changed to $-d$. Colour ∂T as follows. For each $i = 1, \dots, d$, let χ' assign colour $-i$ to the face $\{x \in T : x_i = 0\}$, and colour $+i$ to the antipodal face (this rule creates conflicts at the intersections of faces; for definiteness assign such sites the candidate colour of smallest absolute value). Thus χ' satisfies the condition of Lemma 8 on the boundary.

Now let β be the colouring of T obtained by modifying χ' as follows. For each $i = 1, \dots, d-1$, recolour with colour $-i$ all i-clusters that are adjacent to the face coloured $-i$. Since there were no i-clusters of diameter as large as n in χ, this does not affect the colours on ∂T. Hence Lemma 8 applies, so there are adjacent sites $u, v \in T$ with $\beta(u) = -\beta(v)$, which is impossible. □

The proof of Proposition 7 relies on Theorem 2 concerning Lipschitz surfaces in percolation, together with the following deterministic fact.

Lemma 9 (Periodic colouring)**.** *For any integers $d \geq 1$ and $R \geq 2d$, there exists a colouring $\alpha : \mathbb{Z}^d \to \{0, 1, \dots, d\}$ with the following properties.*

(i) *The colouring is periodic with period R in each dimension; that is, $\alpha(x + Ry) = \alpha(x)$ for all $x, y \in \mathbb{Z}^d$.*
(ii) *For each $j \in \{0, 1, \dots, d\}$, every j-cluster with respect to G^* has volume at most R^d.*
(iii) *The 0-clusters with respect to G^* are precisely the cubes $Rx + [\![-(d-1), (d-1)]\!]^d$, for $x \in \mathbb{Z}^d$.*

Proof. The construction is illustrated in Figure 1. Define a **slice** to be any set of sites of the form $Y = Rx + (I_1 \times \cdots \times I_d)$, where each I_i is either $\{0\}$ or $[\![1, R-1]\!]$. If $[\![1, R-1]\!]$ appears k times in this product then we call Y a ***k*-slice**. The set of all slices forms a partition of $\mathbb{Z}^d$. Let $a_k := d - 1 - k$. For a k-slice Y, define the associated ***k*-slab** to be the set obtained from Y by replacing each occurrence of $\{0\}$ in the product $I_1 \times \cdots \times I_d$ with $[\![-a_k, a_k]\!]$ (thus 'thickening' the slice by distance a_k). We now define the colouring: for each site x, let $\alpha(x)$ be the smallest k for which x lies in some k-slab.

The required properties (i) and (iii) are immediate (the cubes in (iii) are precisely the 0-slabs). For (ii), note that any k-cluster is contained within one k-slab; indeed, it is straightforward to check that any connection in G^* between two different k-slabs is prevented by sites of smaller colours (here it is important that a_k is strictly decreasing in k). The volume of a k-slab is $(R-1)^k (2a_k + 1)^{d-k} < R^d$. □

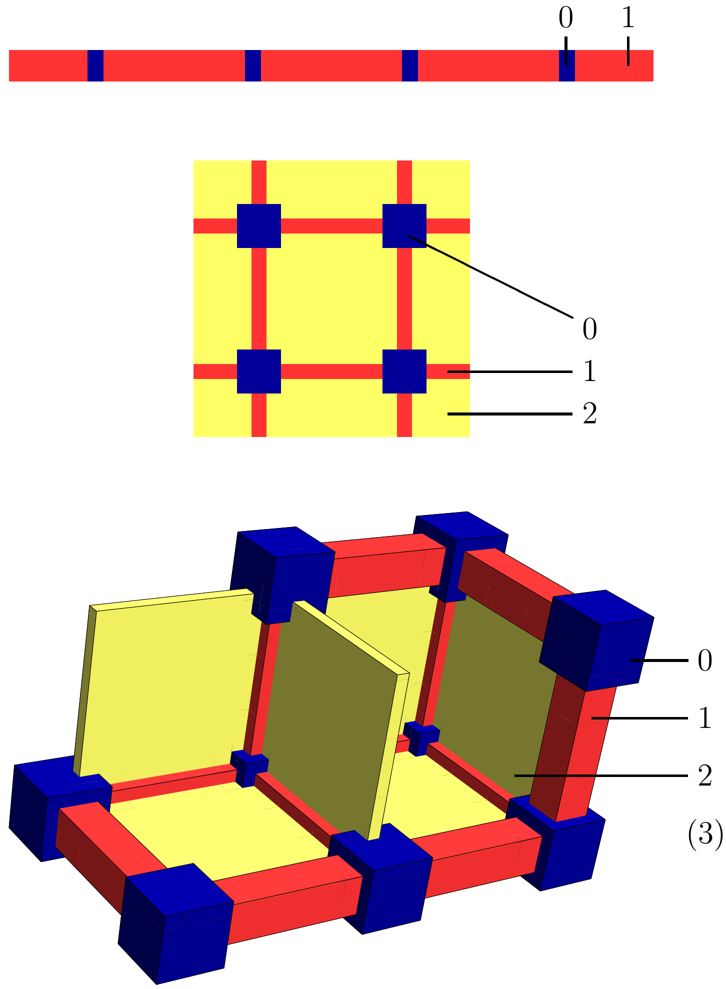

FIGURE 1. Part of the coloring α of Lemma 9, for $d = 1$ (top), $d = 2$ (middle), $d = 3$ (bottom; colour 3 is shown transparent, and only selected slabs are shown).

In what follows we sometimes refer to the d coordinate at vertical, with positive and negative senses being up and down respectively, and the other coordinate directions as horizontal.

Proof of Proposition 7. See Figure 2 for an illustration of the construction. Let L be a large constant, a multiple of J, to be determined later, and let α be the colouring from Lemma 9 with parameters $d-1$ and $R := L/J$. Let α' be the colouring obtaining by dilating α by a factor J, i.e. for all $u \in \mathbb{Z}^{d-1}$, let $\alpha'(u) = \alpha([u/J])$ (where $[v]$ denotes v with each co-ordinate rounded to the nearest integer, rounding up in case of ties). From property (i) in Lemma 9 note that α' has period

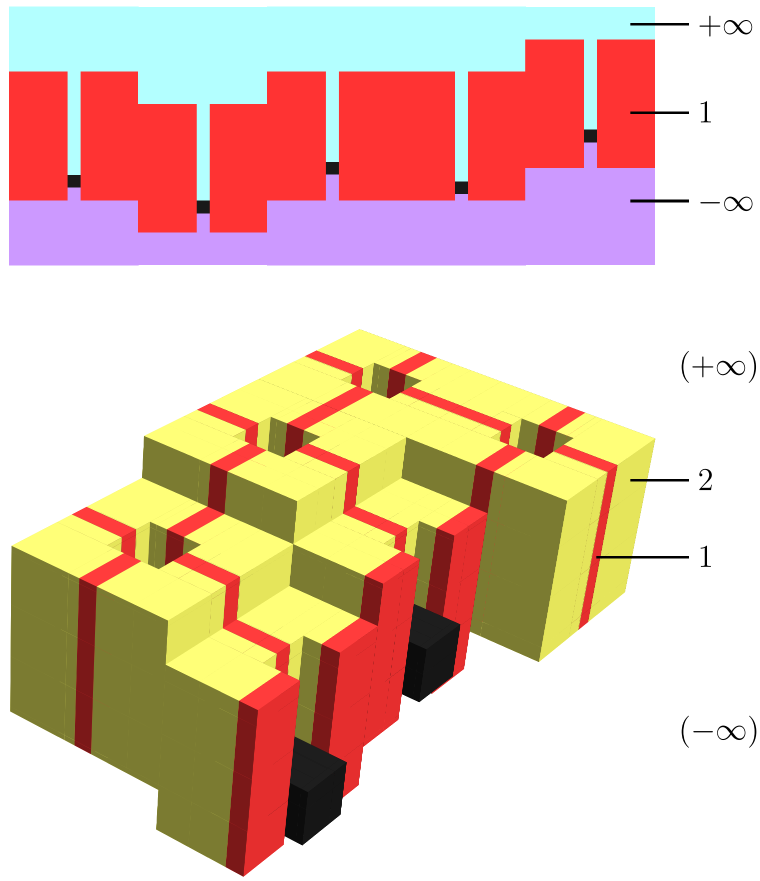

FIGURE 2. Part of the random coloring λ of Proposition 7, for $d = 2$ (top), and $d = 3$ (bottom, with colours $\pm\infty$ shown transparent). Holes are shown black.

L in each dimension, while from (ii), for $j \in \{0, 1, \ldots, d-1\}$, each j-cluster of α' with respect to $G(\mathbb{Z}^{d-1}, \ell^\infty, J)$ has volume at most L^{d-1}. Write $r := J(2d - 3)$. From Lemma 9 (iii), the 0-clusters of α' are $(d-1)$-dimensional cubes of side length r centered (approximately) at the elements of the lattice $L\mathbb{Z}^{d-1}$.

For $u \in \mathbb{Z}^{d-1}$ we will use $\alpha'(u)$ to determine the colours (other than $\pm\infty$) in λ of sites in the vertical column containing $(u, 0)$. Colours $1, \ldots, d-1$ will be used as-is, while colour 0 will be treated in a different way.

We will introduce a renormalized percolation process; we start by defining some sets associated with it. For a site $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, write $\underline{x} := (x_1, \ldots, x_{d-1}) \in \mathbb{Z}^{d-1}$ and $\overline{x} := x_d$, so that $x = (\underline{x}, \overline{x})$. Let $\underline{C}_{\underline{x}} \subset \mathbb{Z}^{d-1}$ be the 0-cluster of α' centred at $L\underline{x}$. Let $s := \lfloor L/d \rfloor$ (where $\lfloor \cdot \rfloor$ denotes the integer part). Let $\overline{C}_{\overline{x}}$ be the interval $[\![s\overline{x}, s(\overline{x}+1))\!) \subset \mathbb{Z}$. Define the **cell** corresponding to $x \in \mathbb{Z}^d$ to be the set of sites $C_x := \underline{C}_{\underline{x}} \times \overline{C}_{\overline{x}}$. Thus, each cell is a cuboid of height s, and side length r in each horizontal dimension. The centres of the cells are spaced at distance s vertically (so that they abut each other), and at distance L horizontally.

Define a **hole** to be any cube of the form $z + [\![1, r]\!]^d$, where $z \in \mathbb{Z}^d$, all of whose sites are closed in the percolation configuration. We say that the cell C_x is **holey** if it contains some hole as a subset. Now we return to the issue of choosing L. Since a hole has volume r^d (a function of J and d), and a cell has height $s = \lfloor L/d \rfloor$, we may choose L a large enough multiple of J (depending on J, d, and p) so that the probability that a cell is holey exceeds $1 - (2d)^{-2}$. For later purposes, ensure also that L is large enough that $s > J$ and $\lfloor (L-r)/2 \rfloor > J$. By Proposition 2, there exists a.s. a function $F : \mathbb{Z}^{d-1} \to [1, \infty)$, satisfying the Lipschitz condition (ii) of that proposition, such that all the cells $C_{(u,F(u))}$ for $u \in \mathbb{Z}^{d-1}$ are holey.

We will next specify a set of sites surrounding each of the holey cells considered above, to be coloured according to α'. For any $\underline{x} \in \mathbb{Z}^{d-1}$, let $\underline{B}_{\underline{x}}$ be the cube $\{v \in \mathbb{Z}^{d-1} : [v/L] = \underline{x}\}$ (so that these cubes partition $\mathbb{Z}^{d-1}$). Let $\overline{B}_{\overline{x}}$ be the interval $[\![s\overline{x}, s\overline{x} + L)\!)$. Define the **block** corresponding to $x \in \mathbb{Z}^d$ to be the set of sites $B_x := \underline{B}_{\underline{x}} \times \overline{B}_{\overline{x}}$. Thus B_x is a cube of side L which contains the cell C_x (at its bottom-centre).

Now we define the colouring λ. For each $u \in \mathbb{Z}^{d-1}$, call the block $B_{(u,F(u))}$ **active**. To each open site $y \in B_{(u,F(u))}$, assign the colour $\alpha'(\underline{y})$, provided this is one of the colours $1, 2, \ldots, d-1$. For the remaining sites y in the active block (those satisfying $\alpha'(\underline{y}) = 0$), we proceed as follows. Since the cell is holey, choose one hole $H_u \subset C_{(u,F(u))}$. Since the sites in H_u are closed they do not receive any colours. Assign colour ∞ to all open sites in the block that lie above the hole H_u, and assign colour $-\infty$ to those that lie below H_u. (We say that a site x lies **above** a set S if for all $y \in S$ with $\underline{x} = \underline{y}$ we have $\overline{x} > \overline{y}$, and **below** is defined analogously). We have assigned colours to all open sites lying in active blocks. Finally, assign colour ∞ to all open sites that lie above some active block, and colour $-\infty$ to all those that lie below some active block.

Now we must check that the colouring λ has all the claimed properties. For property (ii), note first that if the function F were constant, then each j-cluster for $j = 1, \ldots, d-1$ would have volume at most

$L^{d-1} \times L = L^d$, since the colouring α' has merely been 'thickened' vertically to thickness L. However, the effect of taking a non-constant F is to displace the active blocks in the vertical direction, and this clearly cannot make these clusters any larger, so we can take $K = L^d$.

Property (iii) follows easily from the Lipschitz property of F. The constant $\frac{1}{d}$ arises because for $u, v \in \mathbb{Z}^{d-1}$ with $\|u-v\|_1 = 1$, the centres of the corresponding blocks are at horizontal displacement L from each other, and vertical displacement at most $s \le L/d$. Once the function g is determined for the centres of the blocks, it can be defined elsewhere by linear interpolation.

To check property (i), suppose on the contrary that there exist two sites x, y with respective colours $\infty, -\infty$ within ℓ^∞ distance J of each other. If there is a single active block such that both x and y lie above, below or within it, this contradicts the presence of a hole (which has side length $r > J$) in the corresponding cell. Also, if one of x, y lies within an active block then the other cannot lie above, below or within a different active block, since $\lfloor (L-r)/2 \rfloor > J$. Therefore the only other case to consider is that x and y lie respectively above and below two different active blocks, say $B_{(u,F(u))}$ and $B_{(v,F(v))}$. In this case we must have $\|u-v\|_\infty = 1$ and therefore $|F(u)-F(v)| \le \|u-v\|_1 \le d-1$, so the height intervals $\overline{B}_{F(u)}$ and $\overline{B}_{F(v)}$ overlap by at least $L-(d-1)s \ge s > J$, so such x, y cannot exist. □

To complete the proof of Theorem 1(c) we will need the following a simple geometric fact, in order to find a separating surface. For a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, write $\widehat{x}_r$ for the $(d-1)$-vector obtained by dropping the r-coordinate.

Lemma 10. *Let $a_{\pm 1}, \ldots, a_{\pm d}$ be positive constants and define for $i = 1, \ldots, d$ the sets*

$$A_i := \left\{ x \in \mathbb{R}^d : x_i \le \tfrac{1}{d} \|\widehat{x}_i\|_1 + a_i \right\};$$
$$A_{-i} := \left\{ x \in \mathbb{R}^d : x_i \ge -\tfrac{1}{d} \|\widehat{x}_i\|_1 - a_{-i} \right\}.$$

Then $\bigcap_{i=\pm 1,\ldots,\pm d} A_i$ is bounded.

Proof. We may assume without loss of generality that the a_i are all equal, to a say. For $x \in A_i \cap A_{-i}$ we have $|x_i| \le \frac{1}{d}\|\widehat{x}_i\|_1 + a$, hence for x in the given intersection, summing the last inequality over i gives $\|x\|_1 \le \frac{d-1}{d}\|x\|_1 + da$, hence $\|x\|_1 \le d^2 a$. □

Proof of Theorem 1(c). Fix $d \ge 2$, $M \ge 1$ and $p \in (0,1)$, and suppose that f is an M-Lip injection from $\mathbb{Z}^d$ to $W_p(\mathbb{Z}^d)$. Let K be the constant from Proposition 7 for the given values of p, d, and with $J := dM$. Let

$n := K+1$. Let N be large enough so that the image $f(\llbracket 1,n\rrbracket^{d-1}\times\{1\})$ is a subset of $\llbracket -N,N\rrbracket^d$.

Now apply Proposition 7, but to the translated lattice having its origin at $(N+1)e_d$, to obtain (a.s.) a colouring of $W_p(\mathbb{Z}^d)$ in which all open sites in $\mathbb{Z}^{d-1}\times(\!|-\infty,N\rrbracket$ have colour $-\infty$. Call this colouring λ_d, and let S_d^+ be the set corresponding to S^+ in Proposition 7(iii) (all of whose open sites are coloured ∞). Similarly, for each of the two senses of the d coordinate directions, apply Proposition 7 to the lattice rotated and translated so that the part of the half-axis at distance greater than N from the origin is mapped to the positive d-axis. Thus we obtain $2d$ colorings λ_i of $W_p(\mathbb{Z}^d)$, with associated sets S_i^+, for $i=\pm1,\dots,\pm d$, such that all the colourings assign colour $-\infty$ to $\llbracket -N,N\rrbracket^d$, and λ_i assigns colour ∞ to sites sufficiently far along the i coordinate half-axis.

For each i as above, let S_i^{++} be the set of sites y such that every site within ℓ^1-distance dMn of y lies in S_i^+. We claim that

$$Z := \mathbb{Z}^d \setminus \bigcup_{i=\pm1,\dots,\pm d} S_i^{++}$$

is a finite set. This follows from Lemma 10, because $\mathbb{Z}^d\setminus S_i^{++}$ lies in a set of the form A_i in the lemma (here it is important the the Lipschitz constant in Proposition 7(iii) is $\frac{1}{d}$). Since f is injective, it follows that for some $m>1$, the site $f((1,\dots,1,m))$ lies outside Z, and hence lies in S_I^{++} for some I. Since $f(\llbracket 1,n\rrbracket^{d-1}\times\{m\})$ has ℓ^1-diameter at most dMn, this implies that $f(\llbracket 1,n\rrbracket^{d-1}\times\{m\})$ is a subset of S_I^+, and is therefore coloured ∞ in λ_I.

Now define a colouring

$$\chi : \llbracket 1,n\rrbracket^{d-1}\times\llbracket 1,m\rrbracket \to \{\infty,-\infty,1,2,\dots,d-1\}$$

via $\chi := f\circ\lambda_I$. By the construction, χ satisfies properties (i) and (ii) of Proposition 6. Now, if x,y are adjacent sites in G^* then $\|x-y\|_1\le d$, and therefore the M-Lip property gives

$$\|f(x)-f(y)\|_\infty \le \|f(x)-f(y)\|_1 \le dM = J,$$

so $f(x),f(y)$ are adjacent in $G(W_p(\mathbb{Z}^d),\ell^\infty,J)$. Hence, property (i) in Proposition 7 implies that χ has no two adjacent sites in G^* coloured $\pm\infty$, which is property (iii) of Proposition 6. Therefore by Proposition 6, for some $j\neq\pm\infty$, χ has a j-cluster of volume at least n with respect to G^*. Let A be such a cluster. Since f is injective, $f(A)$ also has volume at least n. But by the above observation on adjacency, $f(A)$ is a subset of some j-cluster of λ_I with respect to $G(W_p(\mathbb{Z}^d),\ell^\infty,J)$. This contradicts property (ii) in Proposition 7 because $n>K$. □

4. Embeddings and quasi-isometries

We will use the following simple renormalization construction. Fix an integer $r \geq 1$. For a site $x = (x_1, \ldots, x_D) \in \mathbb{Z}^D$ define the corresponding **block** (or r-**block**) to be the set of r sites given by:

$$B_x := \big\{(x_1, \ldots x_{D-1}, r x_D + i) : i \in [\![0, r-1]\!]\big\}.$$

The blocks $(B_x : x \in \mathbb{Z}^D)$ form a partition of $\mathbb{Z}^D$, with the geometry of $\mathbb{Z}^D$ stretched by a factor r in the Dth coordinate. If $\|x - y\| = k$ then, for all $u \in B_x$ and $v \in B_y$, $\|u - v\| \leq (2r-1)k$.

Proof of Proposition 3. We will prove that (a) is equivalent to each of (b) and (c).

It is immediate that (b) implies (a). Conversely, suppose (a) holds. Any given block contains one or more open sites with probability $1 - (1-p)^r$. If this probability exceeds $p_c(d, D, M)$, there exists a.s. a M-Lip injection $f : \mathbb{Z}^d \to \mathbb{Z}^D$ such that, for each $y \in f(\mathbb{Z}^d)$, the block B_y contains some open site. By choosing one representative open site in each such block, we obtain a $(2r-1)M$-Lip injection from $\mathbb{Z}^d$ to $W_p(\mathbb{Z}^D)$. Hence,

$$p_c\big(d, D, (2r-1)M\big) \leq 1 - \big(1 - p_c(d, D, M)\big)^{1/r}.$$

By the monotonicity of p_c in M, (b) follows.

It is immediate that (c) implies (a) by taking $\eta \equiv 1$, the all-1 configuration, and using the monotonicity of p_c in M. Conversely, suppose (a) holds, and write $m = M_c(d, D)$. Given $p \in (0, 1)$, choose r sufficiently large that any given block contains both an open and a closed site with probability exceeding $p_c(d, D, m)$. There exists a.s. an m-Lip injection $f : \mathbb{Z}^d \to \mathbb{Z}^D$ such that, for each $y \in f(\mathbb{Z}^d)$, the block B_y contains both an open and a closed site. Hence, for any configuration η, by choosing the open or the closed site as appropriate in each block, we obtain a $(2r-1)m$-Lip embedding of η into ω. $\square$

Proof of Proposition 4. Evidently (c) implies (b). We will prove that (a) implies (c), and (b) implies (a).

Assuming that (a) holds, we will deduce (c). We assume without loss of generality that $D = d + 1$. Given $p < 1$, take r sufficiently large that a given block in $\mathbb{Z}^D$ contains some open site with probability exceeding $1 - (2D)^{-2}$. By Theorem 2 there exists a 1-Lip map $F : \mathbb{Z}^{D-1} \to \mathbb{Z}_+$ such that for all $u \in \mathbb{Z}^{D-1}$, the block $B_{(u, F(u))}$ contains some open site. By choosing an arbitrary open site to represent each such block, we obtain the required quasi-isometry.

Now assume (b), i.e. with positive probability there exists a quasi-isometry from $(\mathbb{Z}^d, \ell^1)$ to some subset of $(W_p(\mathbb{Z}^D), \ell^1)$. We will prove that, for some $p' \in (0,1)$ and $M \geq 1$, there exists a M-Lip injection $g : \mathbb{Z}^d \to W_{p'}(\mathbb{Z}^D)$, which will imply (a).

Recall the parameters $\mathbf{c} = (c_1, c_2, \ldots, c_5)$ in the definition of a $\mathbf{c}$-quasi-isometry, and let $Q_{\mathbf{c}}$ be the event that there exists a $\mathbf{c}$-quasi-isometry from $(\mathbb{Z}^d, \ell^1)$ to some subset of $(W_p(\mathbb{Z}^D), \ell^1)$. Since $Q_{\mathbf{c}}$ is invariant under the action of translations of $\mathbb{Z}^D$, it has probability 0 or 1. Under the above assumption, the event $\bigcup_{\mathbf{c}} Q_{\mathbf{c}}$ has positive probability. By the obvious monotonicities in the parameters c_i, this union is equal to the union $\bigcup_{\mathbf{c}\in(\mathbb{Q}\cap(0,\infty))^5} Q_{\mathbf{c}}$ over rational parameters, and hence there exists a *deterministic* $\mathbf{c}$ such that $Q_{\mathbf{c}}$ has probability 1. We choose $\mathbf{c}$ accordingly, and let $\mathcal{F}_{\mathbf{c}}$ be the (random) set of quasi-isometries of the required type.

A quasi-isometry $f \in \mathcal{F}_{\mathbf{c}}$ is not necessarily an injection, but, by the properties of a $\mathbf{c}$-quasi-isometry, there exists $C = C(d, D, \mathbf{c})$ such that, for all $y \in W_p(\mathbb{Z}^D)$ we have $|f^{-1}(y)| \leq C$. Let $r = C$, and take $p' \in (0,1)$ sufficiently large that, with probability at least p, every site in any given block is p'-open. Let $f \in \mathcal{F}_{\mathbf{c}}$ be such that: for $y \in f(\mathbb{Z}^d)$, every site in B_y is p'-open. Since the pre-image under f of any $y \in \mathbb{Z}^D$ has cardinality C or less, we may construct an injection $g : \mathbb{Z}^d \to W_{p'}(\mathbb{Z}^D)$ such that for $y \in \mathbb{Z}^D$, every $x \in f^{-1}(y)$ has $g(x) \in B_y$, and furthermore distinct elements $x \in f^{-1}(y)$ have distinct images $g(x)$. It is easily seen that g is a M-Lip injection for some $M = M(d, D, \mathbf{c})$. □

Open Questions

(i) Derive quantitative versions of Theorem 1. For example, fix d, M, p, and let $N = N(n)$ be the smallest integer such that there exists an M-Lip injection from the cube $[\![1, n]\!]^d$ to the open sites of $[\![1, N]\!]^d$ with probability at least $\frac{1}{2}$. How does N behave as $n \to \infty$?

(ii) For which graphs G and which M is it the case that for p sufficiently close to 1 there exists an M-Lipschitz injection from $V(G)$ to the open sites of $V(G)$ (where M-Lipschitz refers to the graph metric of G)? Theorem 1(c) shows that for $\mathbb{Z}^d$, no $M < \infty$ suffices. On the other hand, for the 3-regular tree, $M = 2$ suffices, by the well-known fact that percolation on a ternary tree contains a binary tree for p sufficiently close to 1.

(iii) We may interpolate between 1-Lipschitz and 2-Lipschitz maps as follows. Let S be a subset of $\{-1, 0, 1\}^d$, and let G be the graph with vertex set $\mathbb{Z}^d$ and an edge between u, v whenever

$u - v$ or $v - u$ belongs to S. For which S does there exist an injection from $\mathbb{Z}^{d-1}$ to the open sites of $\mathbb{Z}^d$ that maps neighbours in $\mathbb{Z}^d$ to neighbours in G?

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