HEIGHT FUNCTIONS FOR AMENABLE GROUPS

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Abstract. The connective constant $\mu(G)$ of an infinite transitive graph G is the exponential growth rate of the number of self-avoiding walks from a given origin. Various properties of connective constants depend on the existence of so-called 'graph height functions', namely: (i) whether $\mu(G)$ is a local function on certain graphs derived from G, (ii) the equality of $\mu(G)$ and the asymptotic growth rate of bridges, and (iii) whether there exists a terminating algorithm for approximating $\mu(G)$ to a given degree of accuracy.

Graph height functions are explored here on Cayley graphs of infinite, finitely presented groups Γ, in which context they are related to integer-valued surjective homomorphisms on the finite-index subgroups of Γ. We prove that the Cayley graphs of infinite, finitely generated, elementary amenable groups support graph height functions, which are in addition harmonic. In contrast, we show that the Cayley graph of the first Grichorchuk group, which is amenable but not elementary amenable, does not have a graph height function.

Examples are given of non-amenable groups without graph height functions, of which one is the Higman group.

This work extends the set of groups for which graph height functions are known to exist, and resolves in the negative an open question concerning the existence of height functions on general transitive graphs.

1. Introduction

A *self-avoiding walk* on a graph $G = (V, E)$ is a path that visits no vertex more than once. The study of the number σ_n of self-avoiding walks of length n from a given initial vertex was initiated by Flory, [8], in his work on polymerization, and this topic has acquired an iconic status in the mathematics and physics associated with lattice-graphs. Hammersley and Morton, [18], proved in 1954 that, if G is vertex-transitive, there exists a constant $\mu = \mu(G)$, called the *connective constant* of G, such that $\sigma_n = \mu^{n(1+o(1))}$ as $n \to \infty$. This result is important not only for its intrinsic value, but also because its proof contained the introduction of subadditivity to the

Date: 12 October 2015.

2010 *Mathematics Subject Classification.* 20F65, 05C30, 60K35, 82B20.

Key words and phrases. Self-avoiding walk, connective constant, Cayley graph, amenable group, elementary amenable group, Grigorchuk group, Higman group, Baumslag–Solitar group, graph height function, group height function, harmonic function.

theory of interacting systems. Subsequent work has concentrated on understanding polynomial corrections in the above asymptotic for σ_n (see, for example, [2, 26]), and on finding exact values and inequalities for connective constants (for example, [7, 13]).

There are several natural questions about connective constants whose answers depend on whether or not the underlying graph admits a so-called graph height function. The first of these is whether $\mu(G)$ is a continuous function of the graph G (see [3, 14]). This so-called *locality* question has received serious attention also in the context of percolation and other disordered systems (see [4, 27, 29]), and has been studied in recent work of the current authors on general transitive graphs, [14], and also on Cayley graphs of finitely generated groups, [16]. Secondly, when G has a graph height function, one may define *bridge* self-avoiding walks on G, and show that their numbers grow asymptotically in the same manner as σ_n (see [14]). The third such question is whether there exists a terminating algorithm to approximate $\mu(G)$ within any given (non-zero) margin of accuracy (see [14, 15]).

Roughly speaking, a graph height function on $G = (V, E)$ is a non-constant function $h : V \to \mathbb{Z}$ whose increments are invariant under the action of a finite-index subgroup of automorphisms (a formal definition may be found at Definition 3.1). It is, therefore, useful to know which transitive graphs support graph height functions.

A method for constructing graph height functions on a certain class of transitive graphs is described in [14], and the question is posed there of deciding whether all transitive graphs support graph height functions. A rich source of interesting examples of transitive graphs is provided by Cayley graphs of finitely generated groups, as studied in [16]. It is proved there that the Cayley graphs of finitely generated, virtually solvable groups support graph height functions, which are in addition harmonic. The question is posed of determining whether or not the Cayley graph of the Grigorchuk group possesses a graph height function.

The current work has two principal results, one positive and the other negative. Firstly, it is proved that every Cayley graph of an infinite, finitely generated, elementary amenable group supports a graph height function. This extends [16, Thm 5.1] beyond the class of virtually solvable groups. Secondly, it is proved that the Cayley graph of the Grigorchuk group does not support a graph height function. This answers in the negative the above question of [14] (see also [16, Sect. 5]). Since the Grigorchuk group is amenable (but not elementary amenable), possession of a graph height function is not a characteristic of amenable groups. This is in contrast with work of Lee and Peres, [23], who have studied the existence of non-constant, Hilbert space valued, equivariant harmonic maps on amenable graphs.

An ancillary result is the non-existence of a graph height function for the Cayley graph of two non-amenable groups, namely the Higman group, [21], together with a variant with similar properties.

This paper is organized as follows. Relevant notation for groups and graphs is summarized in Section 2, and three different types of height functions are explained in Section 3. The class EG of elementary amenable groups is introduced in Section 4. The existence of graph height functions on infinite, finitely generated members of EG is stated in Theorem 4.1. The Grigorchuk group is defined in Section 5 and the non-existence of graph height functions thereon is given in Theorem 5.1. Similarly, the Higman group, together with another 'Higman-type' group, is presented and discussed in Section 6. Theorem 4.1 is proved in Section 7, and Theorems 4.1, 5.1, 6.1, and 6.2 in Sections 8–11.

2. Groups and graphs

The graphs $G = (V, E)$ in this paper are *simple*, in that they have neither loops nor multiple edges. The *degree* $\deg(v)$ of vertex $v \in V$ is the number of edges incident to v. We write $u \sim v$ for neighbours u and v, ∂v for the neighbour set of v, and $\partial_{\mathrm{e}} v$ for set of edges incident to v. The graph is *locally finite* if $|\partial v| < \infty$ for $v \in V$. An edge from u to v is denoted $\langle u, v\rangle$ when undirected, and $[u, v\rangle$ when directed from u to v.

The automorphism group of G is denoted $\mathrm{Aut}(G)$. The subgroup $\Gamma \le \mathrm{Aut}(G)$ is said to act *transitively* on G if, for $u, v \in V$, there exists $\alpha \in \mathrm{Aut}(G)$ with $\alpha(u) = v$. It acts *quasi-transitively* if there exists a finite subset $W \subseteq V$ such that, for $v \in V$, there exists $\alpha \in \Gamma$ and $w \in W$ such that $\alpha(v) = w$. The graph G is said to be *(vertex-)transitive* if $\mathrm{Aut}(G)$ acts transitively on V.

Let Γ be a group with generator set S satisfying $|S| < \infty$ and $\mathbf{1} \notin S$, where $\mathbf{1} = \mathbf{1}_\Gamma$ is the identity element. We shall assume that $S^{-1} = S$, while noting that this was not assumed in [16]. We write $\Gamma = \langle S \mid R\rangle$ with R a set of relators (or relations, when convenient). Such a group is called *finitely generated*, and is called *finitely presented* if, in addition, $|R| < \infty$.

The *Cayley graph* of the presentation $\Gamma = \langle S \mid R\rangle$ is the simple graph $G = G(\Gamma, S)$ with vertex-set Γ, and an (undirected) edge $\langle \gamma_1, \gamma_2\rangle$ if and only if $\gamma_2 = \gamma_1 s$ for some $s \in S$. Thus, our Cayley graphs are simple graphs. See [1, 24] for accounts of Cayley graphs, and [19] of geometric group theory.

3. Height functions

We review the definitions of the two types of height functions, and introduce a third type.

Let $\mathcal{G}$ be the set of all infinite, connected, transitive, locally finite, simple graphs, and let $G = (V, E) \in \mathcal{G}$. Let $\mathcal{H} \le \mathrm{Aut}(G)$. A function $F : V \to \mathbb{R}$ is said to be *$\mathcal{H}$-difference-invariant* if

$$F(v) - F(w) = F(\gamma v) - F(\gamma w), \qquad v, w \in V,\ \gamma \in \mathcal{H}. \tag{3.1}$$

Definition 3.1 ([14])**.** *A* graph height function *on G is a pair $(h, \mathcal{H})$, where $\mathcal{H} \le \mathrm{Aut}(G)$ acts quasi-transitively on G and $h : V \to \mathbb{Z}$, such that:*

(a) *$h(\mathbf{1}) = 0$,*
(b) *h is $\mathcal{H}$-difference-invariant,*
(c) *for $v \in V$, there exist $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$.*

Remark 3.2. *By Poincaré's Theorem for subgroups (see* [20, p. 48, Exercise 20]*), it is immaterial whether or not we require $\mathcal{H}$ to be a* normal *subgroup of* $\mathrm{Aut}(G)$ *in Definition 3.1.*

We turn to Cayley graphs of finitely generated groups. Let Γ be a finitely generated group with presentation $\langle S \mid R \rangle$. As in Section 2, we assume $S^{-1} = S$ and $\mathbf{1} \notin S$.

Definition 3.3. *A* group height function *on Γ (or on a Cayley graph of Γ) is a function $h : \Gamma \to \mathbb{Z}$ such that:*

(a) *$h(\mathbf{1}) = 0$, and h is not identically zero,*
(b) *if $\gamma = s_1 s_2 \cdots s_m$ with $s_i \in S$, then $h(\gamma) = \sum_{i=1}^m h(s_i)$,*
(c) *the values $(h(s) : s \in S)$ are such that, if $s_1 s_2 \cdots s_n = \mathbf{1}$ is a representation of the identity with $s_i \in S$, then $\sum_{i=1}^n h(s_i) = 0$.*

A necessary and sufficient condition for the existence of a group height function is given in [16, Thm 4.1]. In the language of group theory, this condition amounts to requiring that the first Betti number is strictly positive. It was pointed out in [16, Remark 4.2] that (when the non-zero $h(s)$, $s \in S$, are coprime) a group height function is simply a surjective homomorphism from Γ to $\mathbb{Z}$.

We introduce a third type of height function, which may be viewed as an intermediary between a graph height function and group height function.

Definition 3.4. *For a Cayley graph G of a finitely generated group Γ, we say that the pair $(h, \mathcal{H})$ is a* strong graph height function *of the pair (Γ, G) if*

(i) *$\mathcal{H} \trianglelefteq \Gamma$ acts on Γ by left multiplication, and $[\Gamma : \mathcal{H}] < \infty$,*
(ii) *$(h, \mathcal{H})$ is a graph height function.*

It is evident that a group height function h (of Γ) is a strong graph height function of the form (h, Γ), and a strong graph height function is a graph height function. The assumption in (i) above of the normality of $\mathcal{H}$ is benign, as in Remark 3.2.

We recall the definition of a harmonic function. A function $h : V \to \mathbb{R}$ is called *harmonic* on the graph $G = (V, E)$ if

$$h(v) = \frac{1}{\deg(v)} \sum_{u \sim v} h(u), \qquad v \in V.$$

It is an exercise to show that any group height function is harmonic.

4. Elementary amenable groups

The class EG of elementary amenable groups was introduced by Day in 1957, [6], as the smallest class of groups that contains the set EG_0 of all finite and abelian groups, and is closed under the operations of taking subgroups, and of forming quotients, extensions, and directed unions. Day noted that every group in EG is amenable (see also von Neumann [28]). An important example of an amenable but not elementary amenable group was described by Grigorchuk in 1984, [10]. Grigorchuk's group is important in the study of height functions, and we return to this in Section 5.

Let EFG be the set of infinite, finitely generated members of EG.

Theorem 4.1. *Let $\Gamma \in$ EFG. Any locally finite Cayley graph G of Γ admits a harmonic, strong graph height function.*

We prove a slightly stronger version of this at Theorem 7.1 in Section 7, using transfinite induction.

The class EFG includes all virtually solvable groups, and thus Theorem 4.1 extends [16, Thm 5.1]. Since any finitely generated group with polynomial growth is virtually nilpotent, [17], and hence lies in EFG, its locally finite Cayley graphs admit harmonic graph height functions.

5. The first Grigorchuk group

The (first) Grigorchuk group is an infinite, finitely generated, amenable group that is not elementary amenable. We show in Theorem 5.1 that there exists a locally finite Cayley graph of the Grigorchuk group with *no* graph height function. This answers in the negative Question 3.3 of [14] (see also [16, Sect. 3]).

Here is the definition of the group in question (see [9, 10, 12]). Let T be the rooted binary tree with root vertex $\varnothing$. The vertex-set of T can be identified with the set of finite strings u having entries 0, 1, where the empty string corresponds to the root $\varnothing$. Let T_u be the subtree of all vertices with root labelled u.

Let $\mathrm{Aut}(T)$ be the automorphism group of T, and let $a \in \mathrm{Aut}(T)$ be the automorphism that, for each string u, interchanges the two vertices $0u$ and $1u$.

Any $\gamma \in \mathrm{Aut}(G)$ may be applied in a natural way to either subtree T_i, $i = 0, 1$. Given two elements $\gamma_0, \gamma_1 \in \mathrm{Aut}(T)$, we define $\gamma = (\gamma_0, \gamma_1)$ to be the automorphism on T obtained by applying γ_0 to T_0 and γ_1 to T_1. Define automorphisms b, c, d of T recursively as follows:

$$b = (a, c), \quad c = (a, d), \quad d = (e, b), \tag{5.1}$$

where e is the identity automorphism. The Grigorchuk group is defined as the subgroup of $\mathrm{Aut}(T)$ generated by the set $\{a, b, c\}$.

Theorem 5.1. *The Cayley graph $G = (V, E)$ of the Grigorchuk group with generator set $\{a, b, c\}$ satisfies:*

(a) *G admits no graph height function,*
(b) *for $\mathcal{H} \trianglelefteq \mathrm{Aut}(G)$ with finite index, any $\mathcal{H}$-difference-invariant function on V is constant on each orbit of $\mathcal{H}$.*

The proof of Theorem 5.1 is given in Section 9. In the preceding Section 8, two approaches are developed for showing the absence of a graph height function within particular classes of Cayley graph. In the case of the Grigorchuk group, two reasons combine to forbid graph height functions, namely, the Cayley group has no automorphisms beyond the action of the group itself, and the group is a torsion group in that every element has finite order.

Since the Grigorchuk group is amenable, Theorems 4.1 and 5.1 yield that: within the class of infinite, finitely generated groups, every elementary amenable group has a graph height function, but there exists an amenable group without a graph height function. The Grigorchuk group is finitely *generated* but not finitely *presented*, [10, Thm 6.2]. Two Cayley graphs of finitely presented, *non-amenable* groups are considered in Section 6, one of which is shown to possess no graph height function.

We ask if there exists an infinite, finitely *presented*, amenable group with a Cayley graph having no graph height function. A natural candidate might be the group $\Gamma = \langle S \mid R \rangle$ of [11, Thm 1], with

$$S = \{a, c, d, t\},$$
$$R = \big\{a^2 = c^2 = d^2 = (ad)^4 = (adacac)^4 = \mathbf{1}, t^{-1}at = aca, t^{-1}ct = dc, t^{-1}dt = c\big\}.$$

This finitely presented, amenable HNN-extension of the Grigorchuk group is not elementary amenable. However, since it contains the free group generated by the stable letter t, it possesses a group height function. More precisely, the function

$$h(\mathbf{1}) = 0, \quad h(t) = 1, \quad h(t^{-1}) = -1, \qquad h(s) = 0 \text{ for } s \in S,\ s \neq t^{\pm 1}$$

defines a group height function.

6. The Higman group

The *Higman group* Γ of [21] is the infinite, finitely presented group with presentation $\Gamma = \langle S \mid R \rangle$ where

$$\begin{aligned} S &= \{a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}\}, \\ R &= \{a^{-1}ba = b^2, b^{-1}cb = c^2, c^{-1}dc = d^2, d^{-1}ad = a^2\}. \end{aligned} \tag{6.1}$$

This group is interesting since it has no proper normal subgroup with finite index, and the quotient of Γ by its maximal proper normal subgroup is an infinite, finitely generated, simple group. By [16, Thm 4.1(b)], Γ has no group height function. The above two reasons conspire to forbid graph height functions.

Theorem 6.1. *The Cayley graph $G = (V, E)$ of the Higman group $\Gamma = \langle S \mid R \rangle$ has no graph height function.*

A further group of Higman type is given as follows. Let S be as above, and let $\Gamma' = \langle S \mid R' \rangle$ be the finitely presented group with

$$R' = \{a^{-1}ba = b^2, b^{-2}cb^2 = c^2, c^{-3}dc^3 = d^2, d^{-4}ad^4 = a^2\}.$$

Note that Γ' is infinite and non-amenable, since the subgroup generated by the set $\{a, c, a^{-1}, c^{-1}\}$ is a free group (as in the corresponding step for the Higman group at [21, pp. 62–63]).

Theorem 6.2. *The Cayley graph $G = (V, E)$ of the above group $\Gamma' = \langle S \mid R' \rangle$ has no graph height function.*

The proofs of the above theorems are given in Sections 10 and 11, respectively.

7. Proof of Theorem 4.1

We shall prove the following stronger form of Theorem 4.1.

Theorem 7.1. *Let $\Gamma \in$ EFG. There exists a normal subgroup $\mathcal{H} \trianglelefteq \Gamma$ with $1 < [\Gamma : \mathcal{H}] < \infty$ such that any locally finite Cayley graph G of Γ possesses a harmonic, strong graph height function of the form $(h, \mathcal{H})$.*

Whereas every member of EFG has a proper, normal subgroup with finite index, it is proved in [22] that there exist amenable *simple* groups.

We review next the structure of EG. Let EG_0 be the class of all groups that are either finite or abelian (or both), and let $\mathcal{O}$ be the class of all ordinals. Let $\alpha \in \mathcal{O}$, $\alpha \neq 0$, and assume we have defined EG_β for each $\beta \in \mathcal{O}$, $\beta < \alpha$. Each $\alpha \in \mathcal{O}$ is either a limit ordinal or a successor ordinal. If α is a limit ordinal, we set

$$\mathrm{EG}_\alpha = \bigcup_{\beta<\alpha} \mathrm{EG}_\beta. \tag{7.1}$$

If α is a successor ordinal, let EG_α be the class of groups which can be obtained from members of $\mathrm{EG}_{\alpha-1}$ by no more than one operation of extension or directed union.

Theorem 7.2 ([5])**.** *We have that $\mathrm{EG} = \bigcup_{\alpha\in\mathcal{O}} \mathrm{EG}_\alpha$.*

Proof of Theorem 7.1. Let $\mathrm{EFG}_\alpha = \mathrm{EFG} \cap \mathrm{EG}_\alpha$. For $\alpha \in \mathcal{O}$, let H_α be the following statement:

H_α : for $\beta \in \mathcal{O}$, $\beta \leq \alpha$, and $\Gamma \in \mathrm{EFG}_\beta$, there exists $\mathcal{H} \trianglelefteq \Gamma$ such that every locally finite Cayley graph of Γ admits a harmonic, strong graph height function of the form $(h, \mathcal{H})$.

Now, EFG_0 is the set of infinite, finitely generated, abelian groups. By [16, Prop. 4.3, Thm 5.2(b)], any locally finite Cayley graph of Γ has a group height function, and hence a harmonic, strong graph height function of the form (h, Γ). Therefore, H_0 holds, and we turn to the induction step.

Let $\alpha \in \mathcal{O}$, $\alpha \neq 0$, and assume H_β holds for all $\beta < \alpha$. Let $\Gamma \in \mathrm{EFG}_\alpha$ with α the smallest such ordinal. There are two cases to consider, depending on whether or not α is a limit ordinal. *If α is a limit ordinal*, by (7.1), there exists $\beta \in \mathcal{O}$, $\beta < \alpha$, such that $\Gamma \in \mathrm{EFG}_\beta$. The claim now follows by H_β.

We assume for the remainder of this proof that α is a successor ordinal. By Theorem 7.2, the group $\Gamma \in \mathrm{EFG}_\alpha$ is obtained from groups in $\mathrm{EFG}_{\alpha-1}$ by exactly one operation of either extension or directed union. That is, there are two sub-cases to consider.

(a) There exist $\mathcal{N}', \mathcal{Q}' \in \mathrm{EFG}_{\alpha-1}$ such that $\mathcal{N}'$ is isomorphic to a normal subgroup $\mathcal{N}$ of Γ, and $\mathcal{Q}' \simeq \mathcal{Q} := \Gamma/\mathcal{N}$.

(b) There exist a directed set Λ and a family $(S_\lambda : \lambda \in \Lambda)$ satisfying
 (i) $S_\lambda \in \mathrm{EFG}_{\alpha-1}$,
 (ii) $S_{\lambda_1} \subseteq S_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$,
 (iii) $\Gamma = \bigcup_{\lambda\in\Lambda} S_\lambda$.

Assume (a) holds. Since Γ is finitely generated, so is $\mathcal{Q}$.

Suppose $\mathcal{Q}$ is infinite. We shall use the fact that $\mathcal{Q} \in \mathrm{EFG}_{\alpha-1}$. Let S be a finite set of generators of Γ with $S = S^{-1}$ and $\mathbf{1} \notin S$, and let $G = G(\Gamma, S)$ be the corresponding Cayley graph of Γ. A locally finite Cayley graph $G_\mathcal{Q}$ of $\mathcal{Q}$ may be constructed as follows. Let

$$\overline{S} = \{\overline{s} = s\mathcal{N} : s \in S\},$$

be the (finite) generator set of $\mathcal{Q}$ derived from S. The vertex-set of $G_\mathcal{Q} = G_\mathcal{Q}(Q, \overline{S})$ is the set of cosets $\{\overline{v} := v\mathcal{N} : v \in \Gamma\}$, and two such vertices $\overline{v}$, $\overline{w}$ are connected by an edge of $G_\mathcal{Q}$ if and only if there exist $v \in \overline{v}$, $w \in \overline{w}$ such that v and w are connected by an edge in G.

By $\mathrm{H}_{\alpha-1}$, there exists $\overline{\mathcal{H}} \trianglelefteq \mathcal{Q}$, not depending on the choice of S, such that $G_\mathcal{Q}$ admits a harmonic, strong graph height function $(h_\mathcal{Q}, \overline{\mathcal{H}})$. Let $h : \Gamma \to \mathbb{Z}$ and $\mathcal{H} \subseteq \Gamma$ be given by

$$h(v) = h_\mathcal{Q}(\overline{v}), \qquad \mathcal{H} = \bigcup_{\gamma\mathcal{N}\in\overline{\mathcal{H}}} \gamma\mathcal{N}. \tag{7.2}$$

The following lemma completes the proof of this case.

Lemma 7.3. *We have that:*

(i) *$\mathcal{H} \trianglelefteq \Gamma$, and $\mathcal{H}$ acts quasi-transitively on G by left-multiplication,*

(ii) *the pair $(h, \mathcal{H})$ is a harmonic, strong graph height function of G.*

Proof. (i) Since $\overline{\mathcal{H}} \trianglelefteq \mathcal{Q}$, we have that $(a\mathcal{N})\overline{\mathcal{H}}(a\mathcal{N})^{-1} = \overline{\mathcal{H}}$ for $a \in \Gamma$, whence

$$(a\gamma a^{-1})\mathcal{N} \in \overline{\mathcal{H}} \quad \text{whenever} \quad a,\gamma \in \Gamma,\ \gamma\mathcal{N} \in \overline{\mathcal{H}}. \tag{7.3}$$

It is elementary that, for $a \in \Gamma$ and $\gamma_1\mathcal{N}, \gamma_2\mathcal{N} \in \overline{\mathcal{H}}$,

$$(a\gamma_1 a^{-1})\mathcal{N} = (a\gamma_2 a^{-1})\mathcal{N} \quad \text{if and only if} \quad \gamma_1\mathcal{N} = \gamma_2\mathcal{N}. \tag{7.4}$$

Since $\overline{\mathcal{H}}$ is a group, so is $\mathcal{H}$. For $a \in \Gamma$, by (7.2)–(7.4),

$$\begin{aligned} a\mathcal{H}a^{-1} &= \bigcup_{\gamma\mathcal{N}\in\overline{\mathcal{H}}} a(\gamma\mathcal{N})a^{-1} = \bigcup_{\gamma\mathcal{N}\in\overline{\mathcal{H}}} (a\gamma a^{-1})\mathcal{N} \\ &= \bigcup_{\gamma\mathcal{N}\in\overline{\mathcal{H}}} \gamma\mathcal{N} = \mathcal{H}. \end{aligned}$$

Therefore, $\mathcal{H} \trianglelefteq \Gamma$. We prove next that $[\Gamma : \mathcal{H}] < \infty$.

Since $(h_{\mathcal{Q}}, \overline{\mathcal{H}})$ is a graph height function, we have that $[\mathcal{Q} : \overline{\mathcal{H}}] < \infty$. Let $\overline{W}_1, \overline{W}_2, \dots, \overline{W}_k$ be the cosets of $\overline{\mathcal{H}}$ in $\mathcal{Q}$, and let

$$W_i = \bigcup_{\gamma\mathcal{N}\in\overline{W}_i} \gamma\mathcal{N}.$$

We show next that each W_i is contained in an orbit of $\mathcal{H}$ acting on Γ. (Actually the W_i *are* the orbits.) It follows that $\mathcal{H}$ acts quasi-transitively on G.

Without loss of generality, let $u, v \in W_1$. We shall show that there exists $\nu \in \mathcal{H}$ such that $v = \nu u$. Suppose $u \in a\mathcal{N}$, $v \in b\mathcal{N}$ where $a\mathcal{N}, b\mathcal{N} \in \overline{W}_1$. There exists $\gamma\mathcal{N} \in \overline{\mathcal{H}}$ such that $\gamma\mathcal{N}a\mathcal{N} = b\mathcal{N}$, which is to say that $a\mathcal{N}b^{-1} \in \overline{\mathcal{H}}$.

There exist n_i such that $u = an_1$, $v = bn_2$. Then, $u = (an_1n_2^{-1}b^{-1})v$, and $\nu := a(n_1n_2^{-1})b^{-1} \in \mathcal{H}$ by (7.2).

(ii) It is trivial that $h(\mathbf{1}) = h_{\mathcal{Q}}(\overline{\mathbf{1}}) = 0$. For $\gamma \in \mathcal{H}$ and $u, v \in \Gamma$, we have

$$\begin{aligned} h(\gamma u) - h(\gamma v) &= h_{\mathcal{Q}}(\overline{\gamma u}) - h_{\mathcal{Q}}(\overline{\gamma v}) \\ &= h_{\mathcal{Q}}(\overline{\gamma}\,\overline{u}) - h_{\mathcal{Q}}(\overline{\gamma}\,\overline{v}) \quad \text{since } \mathcal{N} \text{ is normal} \\ &= h_{\mathcal{Q}}(\overline{u}) - h_{\mathcal{Q}}(\overline{v}) \qquad \text{since } h_{\mathcal{Q}} \text{ is } \overline{\mathcal{H}}\text{-difference invariant, } \overline{\gamma} \in \overline{\mathcal{H}} \\ &= h(u) - h(v). \end{aligned}$$

Therefore, h is $\mathcal{H}$-difference-invariant.

For $v \in \Gamma$, there exist $\overline{s}_1, \overline{s}_2 \in \overline{S}$ such that

$$h_{\mathcal{Q}}(\overline{v}\,\overline{s}_1) < h_{\mathcal{Q}}(\overline{v}) < h_{\mathcal{Q}}(\overline{v}\,\overline{s}_2),$$

whence, since $\mathcal{N}$ is a normal subgroup of Γ,

$$h(vs_1) < h(v) < h(vs_2).$$

In conclusion, $(h, \mathcal{H})$ is a strong graph height function of G.

We show finally that h is harmonic on the Cayley graph $G = (V, E)$. The edges incident to the vertex labelled $\gamma \in \Gamma$ have the form $\langle \gamma, \gamma s \rangle$ for $s \in S$. Since $h_{\mathcal{Q}}$ is harmonic on the quotient graph, it suffices to show that the cardinality $N_s := |\partial\gamma \cap (\gamma \mathcal{N} s)|$ does not depend on the choice of $s \in S \setminus \mathcal{N}$. For $s \in S \setminus \mathcal{N}$ and $n \in \mathcal{N}$, $\gamma \sim \gamma n s$ if and only if $ns \in S$, which is to say that $n \in s^{-1}S$, whence $N_s = |S|$. $\square$

Suppose $\mathcal{Q}$ is finite. Since $\mathcal{N} \simeq \mathcal{N}' \in \mathrm{EFG}_{\alpha-1}$, we have that $\mathcal{N} \in \mathrm{EFG}_{\alpha-1}$ and $1 < [\Gamma : \mathcal{N}] < \infty$. By $\mathrm{H}_{\alpha-1}$, there exists $\mathcal{H}' \trianglelefteq \mathcal{N}$ with $[\mathcal{N} : \mathcal{H}'] < \infty$ such that any locally finite Cayley graph $G_{\mathcal{N}}$ of $\mathcal{N}$ admits a strong graph height function of the form $(h_{\mathcal{N}}, \mathcal{H}')$.

Since $|\Gamma/\mathcal{H}'| = |\Gamma/\mathcal{N}| \cdot |\mathcal{N}/\mathcal{H}'| < \infty$, there exists (by Poincaré's Theorem for subgroups) a subgroup $\mathcal{H} \le \mathcal{H}'$ that is normal in Γ with finite index, that is, $\mathcal{H} \trianglelefteq \Gamma$ and $1 < [\Gamma : \mathcal{H}] < \infty$. Choose a locally finite Cayley graph $G_{\mathcal{N}}$ of $\mathcal{N}$, and find a strong graph height function of the form $(h_{\mathcal{N}}, \mathcal{H}')$. Let $F : \mathcal{H} \to \mathbb{Z}$ be the restriction of $h_{\mathcal{N}}$ to $\mathcal{H}$.

Lemma 7.4. *The function F is a group height function on the group $\mathcal{H}$.*

Proof. As noted in [16, Remark 4.2], a group height function is a homomorphism from $\mathcal{H}$ to $\mathbb{Z}$ that is not identically zero. For $\gamma_1, \gamma_2 \in \mathcal{H}$,

$$\begin{aligned} F(\gamma_1\gamma_2) - F(\gamma_1) &= h_{\mathcal{N}}(\gamma_1\gamma_2) - h_{\mathcal{N}}(\gamma_1) \\ &= h_{\mathcal{N}}(\gamma_2) - h_{\mathcal{N}}(\mathbf{1}) = F(\gamma_2), \end{aligned}$$

since $\gamma_1 \in \mathcal{H}'$ and $h_{\mathcal{N}}$ is $\mathcal{H}'$-difference invariant. Therefore, F is a homomorphism.

It suffices now to show that $F \not\equiv 0$ on $\mathcal{H}$. Assume the converse, that $F \equiv 0$ on $\mathcal{H}$. For $\gamma \in \Gamma$, there exists $a_\gamma \in \mathcal{N}$ such that $\gamma \in a_\gamma \mathcal{H}$, so that $\gamma = a_\gamma \nu$ for $\nu \in \mathcal{H}$. Since $h_{\mathcal{N}}$ is $\mathcal{H}$-difference-invariant,

$$h_{\mathcal{N}}(\gamma) = h_{\mathcal{N}}(a_\gamma) + F(\nu) = h_{\mathcal{N}}(a_\gamma). \tag{7.5}$$

Now $|\mathcal{N}/\mathcal{H}| < \infty$, so we may restrict consideration to only finitely many a_γ. Therefore, $h_{\mathcal{N}}(\gamma)$ is bounded, which is impossible since $h_{\mathcal{N}}$ is a graph height function. We deduce that $F \not\equiv 0$ on $\mathcal{H}$. $\square$

Let $G = G(\Gamma, S)$ be a locally finite Cayley graph of Γ. The triple $(\Gamma, \mathcal{H}, F)$ satisfies the conditions of [16, Thm 3.5] with $\mathcal{H}$ acting by left multiplication, and it follows that G possesses a harmonic graph height function of the form $(h, \mathcal{H})$.

Assume (b) holds. Let Γ be finitely generated with finite generator set $S = \{s_1, s_2, \ldots, s_k\}$. Since $\Gamma = \bigcup_{\lambda \in \Lambda} S_\lambda$, there exists $\lambda_i \in \Lambda$ such that $s_i \in S_{\lambda_i}$. Let $L = \max\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, so that $S_L = \Gamma$. Then $\Gamma \in \mathrm{EFG}_{\alpha-1}$, which contradicts the minimality of α. $\square$

8. Criteria for the absence of height functions

This section contains some observations relevant to proofs of the non-existence of graph height functions.

Let $\Gamma = \langle S \mid R\rangle$ where $|S| < \infty$, and let $G = (V, E)$ be the corresponding Cayley graph. Let Π be the set of permutations of S that preserve Γ up to isomorphism, and write $e \in \Pi$ for the identity. Thus, $\pi \in \Pi$ acts on Γ by: for $w = s_1 s_2 \cdots s_m$ with $s_i \in S$, we have $\pi(w) = \pi(s_1)\pi(s_2)\cdots\pi(s_m)$. It follows that $\Pi \subseteq \mathrm{Aut}(G)$. For $\gamma = g_1 g_2 \cdots g_n \in \Gamma$ with $g_i \in S$, and $\pi \in \Pi$, we define $\gamma\pi \in \mathrm{Aut}(G)$ by $\gamma\pi(w) = g_1 g_2 \cdots g_n \pi(w)$, $w \in V$. Write $\Gamma\Pi \subseteq \mathrm{Aut}(G)$ for the subgroup containing all such $\gamma\pi$, and note that γe operates on G in the manner of γ with left-multiplication.

The *stabilizer* Stab_v of $v \in V$ is the set of automorphisms of G that preserve v, that is,

$$\mathrm{Stab}_v = \{\eta \in \mathrm{Aut}(G) : \eta(v) = v\}.$$

Proposition 8.1. *Suppose* $\mathrm{Stab}_{\mathbf{1}} = \Pi$.

(a) $\mathrm{Aut}(G) = \Gamma\Pi$.
(b) *If* $\mathcal{M} \trianglelefteq \mathrm{Aut}(G)$ *has finite index, the subgroup* $\mathcal{N} = \mathcal{M} \cap \Gamma$ *satisfies* $\mathcal{N} \trianglelefteq \Gamma$ *and* $[\Gamma : \mathcal{N}] < \infty$.
(c) *If G has a graph height function, then it has a strong graph height function.*

Proof. Assume $\mathrm{Stab}_{\mathbf{1}} = \Pi$.

(a) Let $\eta \in \mathrm{Aut}(G)$, and write $\gamma = \eta(\mathbf{1})$. Then $\gamma^{-1}\eta \in \mathrm{Stab}_{\mathbf{1}}$, which is to say that $\gamma^{-1}\eta = \pi \in \Pi$, and thus $\eta = \gamma\pi \in \Gamma\Pi$ so that $\mathrm{Aut}(G) = \Gamma\Pi$. Note for future use that

$$[\mathrm{Aut}(G) : \Gamma] = |\Pi| < \infty.$$

(b) Let $\mathcal{M} \trianglelefteq \mathrm{Aut}(G)$ be a finite-index normal subgroup, and let $\mathcal{N} = \{\gamma e : \gamma e \in \mathcal{M}\}$. Viewed as automorphisms, we have that $\gamma e = \gamma$, and hence $\mathcal{N} \le \Gamma \le \mathrm{Aut}(G)$. For $\alpha \in \Gamma$, $\nu \in \mathcal{N}$, we have that $(\alpha^{-1}\nu\alpha)e = \alpha^{-1}(\nu e)\alpha \in \mathcal{M}$, since $\mathcal{M} \trianglelefteq \mathrm{Aut}(G)$. Therefore, $\mathcal{N} \trianglelefteq \Gamma$.

Since $\Gamma, \mathcal{M} \le \mathrm{Aut}(G)$ and $\mathcal{N} = \Gamma \cap \mathcal{M}$, we have that

$$[\mathrm{Aut}(G) : \mathcal{N}] \le [\mathrm{Aut}(G) : \Gamma] \cdot [\mathrm{Aut}(G) : \mathcal{M}] < \infty,$$

which implies $[\Gamma : \mathcal{N}] < \infty$, as required.

(c) Let $(h, \mathcal{H})$ be a graph height function of G. Since $\mathcal{H}$ is a finite-index normal subgroup of $\mathrm{Aut}(G)$, by part (b), there exists $\mathcal{N} \le \mathcal{H}$ that is a finite-index normal subgroup of Γ. Since $\mathcal{N} \le \mathcal{H}$, h acts on Γ and is $\mathcal{N}$-difference invariant, whence $(h, \mathcal{N})$ is a strong graph height function. □

Corollary 8.2. *Let* $\Gamma = \langle S \mid R\rangle$ *have Cayley graph G satisfying* $\mathrm{Stab}_{\mathbf{1}} = \Pi$.

(a) *If Γ has no proper, normal subgroup with finite index, any graph height function of G is also a group height function of Γ.*

(b) *If every element in* Γ *has finite order, then* G *has no graph height function.*

Proof. (a) Let $(h, \mathcal{M})$ be a graph height function of G. If Γ satisfies the given condition then, by Proposition 8.1(b), $\mathcal{M} \supseteq \Gamma$. Therefore, (h, Γ) is a graph height function and hence a group height function.

(b) If G has a graph height function, by Proposition 8.1(c), G has a strong graph height function $(h, \mathcal{N})$. Assume every element of Γ has finite order. For $\gamma \in \mathcal{N}$ with $\gamma^n = \mathbf{1}$, we have that $h(\gamma^n) = nh(\gamma) = 0$, whence $h \equiv 0$ on $\mathcal{N}$.

We now use the argument around (7.5). For $\gamma \in \Gamma$, find α_γ such that $\gamma \in \alpha_\gamma \mathcal{N}$. Since h is $\mathcal{N}$-difference-invariant, there exists $\nu \in \mathcal{N}$ such that

$$h(\gamma) = h(\alpha_\gamma) + h(\nu) = h(\alpha_\gamma). \tag{8.1}$$

Now $[\Gamma : \mathcal{N}] < \infty$, so we may consider only finitely many choices for α_γ. Therefore, h is bounded on Γ, in contradiction of the assumption that it is a graph height function. $\square$

9. Proof of Theorem 5.1

The main step is to show that

$$\mathrm{Stab}_{\mathbf{1}} = \{e\}, \tag{9.1}$$

where e is the identity of $\mathrm{Aut}(G)$. Once this is shown, claim (a) follows from Corollary 8.2(b) and the fact that every element of the Grigorchuk group has finite order, [19]. It therefore suffices for (a) to show (9.1), and to this end we study the structure of the Cayley graph $G = (V, E)$.

It was shown in [25] (see also [12, eqn (4.7)]) that $\Gamma = \langle S \mid R \rangle$ where $S = \{a, b, c, d\}$, R is the following set of relations

$$\begin{aligned} \mathbf{1} &= a^2 = b^2 = c^2 = d^2 = bcd \\ &= \sigma^k((ad)^4) = \sigma^k((adacac)^4), \qquad k = 0, 1, 2, \dots, \end{aligned} \tag{9.2}$$

and σ is the substitution

$$\sigma : \begin{cases} a \mapsto aca, \\ b \mapsto d, \\ c \mapsto b, \\ d \mapsto c. \end{cases}$$

It follows that the following, written in terms of the reduced generator set $\{a, b, c\}$ after elimination of d, are valid relations:

$$\mathbf{1} = a^2 = b^2 = c^2 = (bc)^2 = (abc)^4 = (ac)^8 = (abcacac)^4 = (acab)^8 = (ab)^{16}, \tag{9.3}$$

(see also [12, Sect. 1]). Note the asymmetry between b and c in that ab (respectively, ac) has order 16 (respectively, 8).

Let

$$V_n = \{v \in \Gamma : \mathrm{dist}(v, \mathbf{1}) = n\},$$

where dist denotes graph-distance on G. Since G is locally finite, $|V_n| < \infty$. For $\eta \in \mathrm{Stab}_{\mathbf{1}}$, η restricted to V_n is a permutation of V_n. As illustrated in Figure 9.1,

$$V_0 = \{\mathbf{1}\}, \qquad V_1 = \{a, b, c\}, \qquad V_2 = \{ab, ac, ba, bc = cb, ca\}.$$

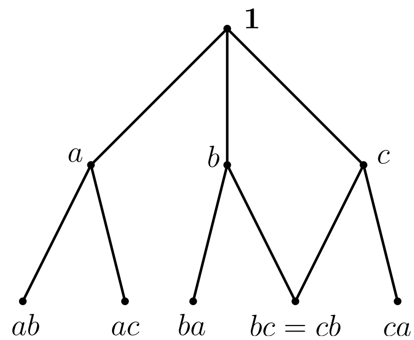

FIGURE 9.1. The subgraph of G on $V_0 \cup V_1 \cup V_2$.

Let $\eta \in \mathrm{Stab}_{\mathbf{1}}$, so that $\eta(a) \in V_1$. Since the shortest cycles using the edges $\langle \mathbf{1}, b\rangle$ and $\langle \mathbf{1}, c\rangle$ have length 4, and using $\langle \mathbf{1}, a\rangle$ greater than 4 (see Figure 9.1), we have that $\eta(a) = a$. By a similar argument, we obtain that, for $n \geq 1$,

$$\eta(va) = \eta(v)a, \qquad v \in V_n,\ va \in V_{n+1}, \tag{9.4}$$

which we express by saying that η maps a-type edges to a-type edges.

We show next that

$$\eta(vc) = \eta(v)c, \qquad v \in V,\ \eta \in \mathrm{Stab}_{\mathbf{1}}, \tag{9.5}$$

which is to say that η maps c-type edges to c-type edges. By (9.4)–(9.5), $\eta \in \mathrm{Stab}_{\mathbf{1}}$ maps b-type edges to b-type edges also, whence $\eta = e$ as required. It remains to prove (9.5).

Assume, in contradiction of (9.5), that there exists $v \in V$, $\eta \in \mathrm{Stab}_{\mathbf{1}}$ such that $\eta(vc) = \eta(v)b$. Since ac has order 8, we have that $(ca)^8 = \mathbf{1}$. Let C be the directed cycle corresponding to the word $v(ca)^8$; thus, C includes the edge $[v, vc\rangle$. Then $\eta(C)$ is a cycle of length 16 including the edge $[\eta(v), \eta(v)b\rangle$. Since C contains exactly 8 a-type edges at alternating positions, by (9.4), so does $\eta(C)$. Therefore, $\eta(C)$ has the form $\eta(v)ba\prod_{i=2}^{8}(x_i a)$, where $x_i \in \{b, c\}$ for $i = 2, 3, \ldots, 8$. In particular,

$$ba\prod_{i=2}^{8}(x_i a) = \mathbf{1}, \qquad x_i \in \{b, c\},\ i = 2, 3, \ldots, 8. \tag{9.6}$$

The word problem of the Grigorchuk group is solvable (see [10] and [12, Sect. 4]), in that there exists an algorithm to determine whether or not $w = \mathbf{1}$ for any given word $w \in \{a, b, c\}^*$. By applying this algorithm (see below), we deduce that (9.6) has no solution. Equation (9.5) follows, and the proof of part (a) is complete.

Finally, here is a short amplification of the analysis of (9.6). The word in (9.6) has the form $b(ay_1a)z_1(ay_2a)z_2(ay_3a)z_3(ay_4a)$, where $y_i, z_j \in \{b, c\}$. By (5.1), the effect of such a word on the right sub-tree T_1 is $\gamma_1 := ca(c/d)a(c/d)a(c/d)a$, where each term of the form (y/z) is to be interpreted as 'either y or z'. The effect of γ_1 on the left sub-tree T_{10} of T_1 is $\gamma_{10} := a(d/b)(a/e)(d/b)$. If there is an odd number of appearances of a in γ_{10}, then γ_{10} is not the identity, and thus we may assume $\gamma_{10} := a(d/b)a(d/b)$. It is immediate that none of the four possibilities is the identity, and the claim follows.

Part (b) holds as follows. Suppose there exists $\mathcal{H} \trianglelefteq \Gamma$, $\gamma \in \Gamma$, and a non-constant $\mathcal{H}$-difference-invariant function $F : \gamma\mathcal{H} \to \mathbb{Z}$. It is elementary that $\mathcal{H}$ is unimodular and symmetric (see, for example, [15, Sect. 4]). By [16, Thm 3.5] and the comment near the beginning of [16, Sect. 8], G has a graph height function, in contradiction of part (a).

10. Proof of Theorem 6.1

We shall prove three statements:

(i) Γ has no group height function,

(ii) Π is the cyclic group generated by the permutation $(abcd)$, with the convention that $\eta(x^{-1}) = \eta(x)^{-1}$, for $\eta \in \Pi$, $x \in \{a, b, c, d\}$,

(iii) $\mathrm{Stab}_{\mathbf{1}} = \Pi$.

It is proved in [21] that the Higman group has no proper, finite-index, normal subgroup, and the result follows from the above statements by Corollary 8.2(a).

(i) The absence of a group height function is immediate by [16, Example 6.3].

(ii) Evidently, Π contains the given cyclic group, and we turn to the converse. Since elements of Π preserve Γ up to isomorphism,

$$\eta(x^{-1}) = \eta(x)^{-1}, \qquad x \in S. \tag{10.1}$$

We next rule out the possibility that $\eta(x) = y^{-1}$ for some $x, y \in \{a, b, c, d\}$. Suppose, for illustration, that $\eta(a) = b^{-1}$. By (10.1), the relation $a^{-1}ba = b^2$ becomes $b\beta b^{-1} = \beta^2$ where $\beta = \eta(b)$. The Higman group has no such relation with $\beta \in S$. In summary,

$$\eta(x) \in \{a, b, c, d\}, \quad \eta(x^{-1}) = \eta(x)^{-1}, \qquad x \in \{a, b, c, d\}. \tag{10.2}$$

The shortest cycles containing the edge $\langle \mathbf{1}, a\rangle$, modulo rotation and reversal, arise from the relations $ab^2a^{-1}b^{-1} = \mathbf{1}$ and $ada^{-2}d^{-1} = \mathbf{1}$ (see Figure 10.2). The first uses $a^{\pm 1}$ twice and $b^{\pm 1}$ thrice, and the second uses $a^{\pm 1}$ thrice and $d^{\pm 1}$ twice. Let $\eta \in \Pi$, and suppose for illustration that $\eta(a) = b$ (the same argument is valid for any $\eta(x)$,

$x \in \{a, b, c, d\}$). By considering the cycles starting $\langle \mathbf{1}, b\rangle$, $\langle \mathbf{1}, c\rangle$, $\langle \mathbf{1}, d\rangle$, and using (10.2), we deduce that

$$\eta(b) = c, \quad \eta(c) = d, \quad \eta(d) = a,$$

and the claim is proved.

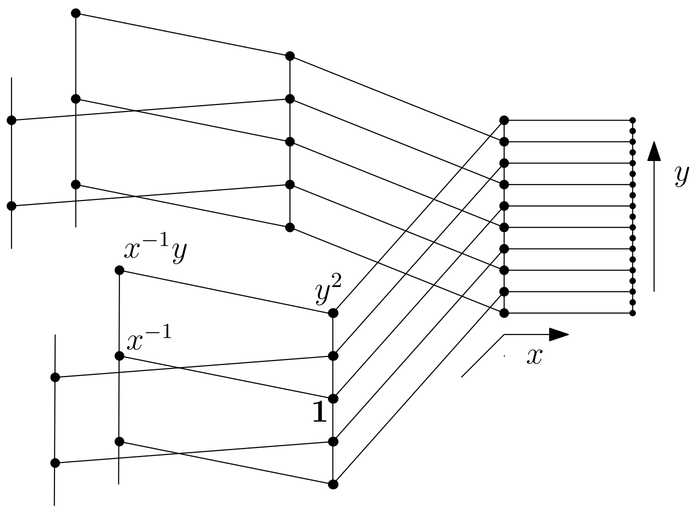

FIGURE 10.1. Part of the Cayley graph of the Baumslag–Solitar group $\mathrm{BS}(x, y)$.

(iii) We begin with some observations concerning the Baumslag–Solitar (BS) group $\mathrm{BS}(x, y)$ with presentation $\langle x, y, x^{-1}, y^{-1} \mid x^{-1}yx = y^2\rangle$, of which the Cayley graph is sketched in Figure 10.1. Edges of the form $\langle \gamma, \gamma x^{\pm 1}\rangle$ have *type* x, and of the form $\langle \gamma, \gamma y^{\pm 1}\rangle$ *type* y. By inspection, the shortest cycles have length 5 (see Figure 10.2), and, for $\gamma \in \mathrm{BS}(x, y)$,

(10.3) for $p, q = \pm 1$, the edges $\langle \gamma, \gamma x^p\rangle$ and $\langle \gamma, \gamma y^q\rangle$ lie in a common 5-cycle,

(10.4) the third edge of any directed 5-cycle beginning $[\gamma, \gamma x\rangle$ has type y,

(10.5) the third edge of any directed 5-cycle beginning $[\gamma, \gamma x^{-1}\rangle$ has type x,

(10.6) every 5-cycle contains two consecutive edges of type y, and not of type x,

(10.7) a type x (respectively, type y) edge lies in 2 (respectively, 3) 5-cycles.

Returning to the Higman group, for convenience, we relabel the vector (a, b, c, d) as (s_0, s_1, s_2, s_3), with addition and subtraction of indices modulo 4. Let G be the Cayley graph of the Higman group $\Gamma = \langle S \mid R\rangle$, rooted at $\mathbf{1}$. An edge of G is said to be of *type* s_i if it has the form $\langle \gamma, \gamma s_i^{\pm 1}\rangle$ with $\gamma \in \Gamma$. We explain next how to obtain information about the types of the edges of G, by examination of G only, and without further information about the vertex-labellings as elements of Γ.

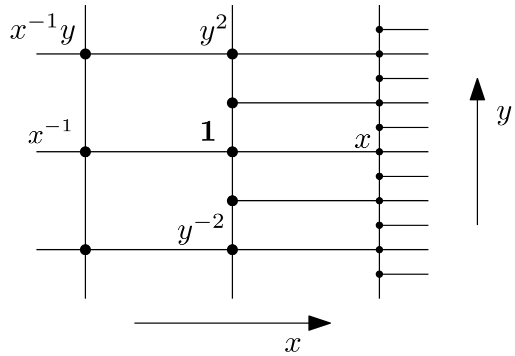

FIGURE 10.2. Part of one 'sheet' of the Cayley graph of $\mathrm{BS}(x, y)$.

We consider first the set $\partial_{\mathrm{e}}\mathbf{1}$ of edges of G incident to $\mathbf{1}$. Let $e_1 = \langle \mathbf{1}, v\rangle$, $t \in \{0, 1, 2, 3\}$, and $p \in \{-1, 1\}$. Assume that

$$v = s_t^p, \tag{10.8}$$

so that, in particular, e_1 has type s_t. By (10.3), for $j = \pm 1$, e_1 lies in a 5-cycle of $\mathrm{BS}(s_{t-1}, s_t)$ (respectively, $\mathrm{BS}(s_t, s_{t+1})$) containing $\langle \mathbf{1}, s_{t-1}^j\rangle$ (respectively, $\langle \mathbf{1}, s_{t+1}^j\rangle$). On the other hand, by consideration of the relator set R, e_1 lies in no 5-cycle including an edge of type s_{t+2}. Therefore, the edges of the form $\langle \mathbf{1}, s_{t+2}^{\pm 1}\rangle$ may be identified by examination of G, and we denote these as g_1, g_2. There is exactly one further edge of $\partial_{\mathrm{e}}\mathbf{1}$ that lies in no 5-cycle containing either g_1 or g_2, and we denote this edge as e_2. In summary,

$$\{e_1, e_2\} = \big\{\langle \mathbf{1}, s_t^{-1}\rangle, \langle \mathbf{1}, s_t\rangle\big\}, \qquad \{g_1, g_2\} = \big\{\langle \mathbf{1}, s_{t+2}^{-1}\rangle, \langle \mathbf{1}, s_{t+2}\rangle\big\}.$$

Having identified the edges of $\partial_{\mathrm{e}}\mathbf{1}$ with types s_t and s_{t+2}, we move to the other endpoint $v = s_t^p$ of e_1, and apply the same argument. Let e_1, e_1' be the two type-s_t edges incident to v.

We turn next to the remaining four edges of $\partial_{\mathrm{e}}\mathbf{1}$. Let k be such an edge, and consider the property: *k lies in a 5-cycle of G containing both e_1 and e_1'*. By (10.6) and examination of the Cayley graphs of the four groups $\mathrm{BS}(s_i, s_{i+1})$, $0 \le i < 4$, we see that k has this property if it has type $t-1$, and not if it has type $t+1$. Thus we may identify the types of the four remaining edges of $\partial_{\mathrm{e}}\mathbf{1}$, which we write as

$$\{f_1, f_2\} = \big\{\langle \mathbf{1}, s_{t+1}^{-1}\rangle, \langle \mathbf{1}, s_{t+1}\rangle\big\}, \qquad \{h_1, h_2\} = \big\{\langle \mathbf{1}, s_{t+3}^{-1}\rangle, \langle \mathbf{1}, s_{t+3}\rangle\big\}.$$

Having determined the types of edges in $\partial_{\mathrm{e}}\mathbf{1}$ (relative to the type t of the initial edge e_1), we move to an endpoint of such an edge other than $\mathbf{1}$, and apply the same argument. By iteration, we deduce the types of all edges of G. Let $T(k)$ denote the type of edge k. It follows from the above that

$$T(k) - T(e_1) \text{ is independent of } t = T(e_1), \tag{10.9}$$

with arithmetic on indices, modulo 4.

We explain next how to identify the value of $p = p(v)$ in (10.8) from the graphical structure of G. Let S_i be the subgraph of G containing all edges with type either s_i or s_{i+1}, so that each component of S_i is isomorphic to the Cayley graph of $\mathrm{BS}(s_i, s_{i+1})$. By (10.4)–(10.5), every directed 5-cycle of $\mathrm{BS}(s_t, s_{t+1})$ starting with the edge $[\mathbf{1}, s_t\rangle$ has third edge with type s_{t+1}, whereas every directed 5-cycle starting with $[\mathbf{1}, s_t^{-1}\rangle$ has third edge with type s_t. We examine S_t to determine which of these two cases holds, and the outcome determines the value of $p = p(v)$.

The above argument is applied to each directed edge $[\gamma, \gamma s_i^{\pm 1}\rangle$ of G, and the power of s_i is thus determined from the graphical structure of G.

Let $\eta \in \mathrm{Stab}_{\mathbf{1}}$. By (10.9), the effect of η is to change the edge-types by

$$T(k) \mapsto T(k) + T(\eta(e_1)) - t.$$

Now, $\eta(v)$ is adjacent to $\mathbf{1}$ and, by the above, once $\eta(v)$ is known, the action of η on the rest of G is determined. Since $\eta \in \mathrm{Aut}(G)$, $\eta(v)$ may be any neighbour w of $\mathbf{1}$ with the property that $p(w) = p(v)$. There are exactly four such neighbours (including v) and we deduce from (10.9) that η lies in the cyclic group generated by the permutation $(s_0 s_1 s_2 s_3)$.

11. Proof of Theorem 6.2

We shall prove three statements:

(i) Γ has no group height function,
(ii) $\mathrm{Stab}_{\mathbf{1}} = \Pi$ where $\Pi = \{e\}$,
(iii) Γ has no proper normal subgroup with finite index.

The result follows from these statements by Corollary 8.2(a), and we turn to their proofs.

(i) The absence of a group height function is immediate by [16, Thm 4.1(b)].

(ii) Let $\eta \in \mathrm{Stab}_{\mathbf{1}}$ and $\gamma \in \Gamma$. We consider the action of η on directed edges of G. By inspection of the set R' of relations, an edge of the type $\langle \gamma, \gamma x\rangle$ lies in shortest cycles of length

$$\begin{cases} 5,\ 8 & \text{if } x = a^{\pm 1}, \\ 5,\ 7 & \text{if } x = b^{\pm 1}, \\ 7,\ 8 & \text{if } x = c^{\pm 1}, \\ 9,\ 11 & \text{if } x = d^{\pm 1}. \end{cases}$$

Since the four combinations are distinct, it must be that

$$\eta([\gamma, \gamma x\rangle) = [\gamma', \gamma' x^{\pm 1}\rangle, \qquad \gamma \in \Gamma,\ x \in S, \tag{11.1}$$

where $\gamma' = \eta(\gamma)$. We show next that

$$\eta([\gamma, \gamma x\rangle) \neq [\gamma', \gamma' x^{-1}\rangle, \qquad \gamma \in \Gamma,\ x \in S, \tag{11.2}$$

which combines with (11.1) to imply $\eta = e$ as required.

It suffices to consider the case $x = a$ in (11.2), since a similar proof holds in the other cases. Suppose $\eta([\gamma, \gamma a\rangle) = [\gamma', \gamma' a^{-1}\rangle$, and consider the cycle corresponding to $\gamma ab^{-2}a^{-1}b^{-1}$, that is $(\gamma, \gamma a, \gamma ab^{-1}, \gamma ab^{-2}, \gamma ab^{-2}a^{-1}, \gamma ab^{-2}a^{-1}b^{-1} = \gamma)$. By (11.1), this is mapped under η to the cycle corresponding to $\gamma' a^{-1}b^{\pm 2}a^{\pm 1}b^{\pm 1}$. By examining the relation set R', the only cycles beginning $\gamma' a^{-1}b^{\pm 1}$ with length not exceeding 5 are $\gamma' a^{-1}bab^{-2}$ and $\gamma' a^{-1}b^{-1}ab^2$, in contradiction of the above (since the third step of these two cycles is a rather than the required $b^{\pm 1}$).

(iii) Suppose $\mathcal{N}$ is a proper normal subgroup of Γ with finite index. The quotient group $\Gamma/\mathcal{N}$ is non-trivial and finite with generators $\overline{s} = s\mathcal{N}$, $s \in S$, satisfying

$$\begin{aligned} \overline{a}^{-1}\overline{b}\overline{a} &= \overline{b}^2, \qquad & \overline{b}^{-2}\overline{c}\overline{b}^2 &= \overline{c}^2, \\ \overline{c}^{-3}\overline{d}\overline{c}^3 &= \overline{d}^2, \qquad & \overline{d}^{-4}\overline{a}\overline{d}^4 &= \overline{a}^2. \end{aligned} \tag{11.3}$$

Since $\Gamma/\mathcal{N}$ is finite, each $\overline{s}$ has finite order, denoted $\mathrm{ord}(\overline{s})$. It follows from (11.3) that

$$\mathrm{ord}(\overline{s}) > 1, \qquad s = a, b, c, d. \tag{11.4}$$

To see this, suppose for illustration that $\mathrm{ord}(\overline{c}) = 1$, so that $\overline{c} = \overline{\mathbf{1}}$. By the third equation of (11.3), $\mathrm{ord}(\overline{d}) = 1$, so that $\overline{d} = \overline{\mathbf{1}}$, and similarly for $\overline{a}$ and $\overline{b}$, implying that $\Gamma/\mathcal{N}$ is trivial, a contradiction.

By induction, for $n \geq 1$,

$$\begin{aligned} \overline{a}^{-n}\overline{b}\overline{a}^n &= \overline{b}^{2^n}, \qquad & \overline{b}^{-2n}\overline{c}\overline{b}^{2n} &= \overline{c}^{2^n}, \\ \overline{c}^{-3n}\overline{d}\overline{c}^{3n} &= \overline{d}^{2^n}, \qquad & \overline{d}^{-4n}\overline{a}\overline{d}^{4n} &= \overline{a}^{2^n}, \end{aligned}$$

whence, by setting $n = \mathrm{ord}(\overline{a})$, etc,

$$\begin{aligned} \mathrm{ord}(\overline{b}) \,\big|\, (2^{\mathrm{ord}(\overline{a})} - 1), \qquad & \mathrm{ord}(\overline{c}) \,\big|\, (2^{\mathrm{ord}(\overline{b})} - 1), \\ \mathrm{ord}(\overline{d}) \,\big|\, (2^{\mathrm{ord}(\overline{c})} - 1), \qquad & \mathrm{ord}(\overline{a}) \,\big|\, (2^{\mathrm{ord}(\overline{d})} - 1), \end{aligned} \tag{11.5}$$

where $u \mid v$ means that v is a multiple of u. We shall deduce a contradiction from (11.4) and (11.5). This is done as in [21], of which we reproduce the proof for completeness.

Let p be the least prime factor of the four integers $\mathrm{ord}(\overline{s})$, $s \in \{a, b, c, d\}$. By (11.4), $p > 1$. Suppose that $p \mid \mathrm{ord}(\overline{a})$ (with a similar argument if $p \mid \mathrm{ord}(\overline{s})$ for some other parameter s). Then $p \mid 2^{\mathrm{ord}(\overline{d})} - 1$ by (11.5), and in particular p is odd and therefore coprime with 2. Let r be the multiplicative order of 2 mod p, that is, the least positive integer r such that $p \mid 2^r - 1$. In particular, $r > 1$, so that r has a prime factor q. By Fermat's little theorem, $r \mid p-1$ so that $q < p$. Furthermore, $r \mid \mathrm{ord}(\overline{d})$ so that $q \mid \mathrm{ord}(\overline{d})$, in contradiction of the minimality of p. We deduce that Γ has no proper, normal subgroup with finite index.

Acknowledgements

GRG was supported in part by the Engineering and Physical Sciences Research Council under grant EP/103372X/1. ZL acknowledges support from the Simons Foundation under grant #351813.

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