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Harry Kesten's Publications

A Personal Perspective

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Writing about Harry Kesten's life work is a daunting task. At of the writing of this paper, he has published almost 150 papers totaling more than 5000 pages. The topics range from refined results for the classical topics of random walks, renewal theory, Lévy processes, and branching processes to questions of interest in statistical mechanics: first passage percolation, percolation, DLA, and the models named after Ising, Potts, and Heisenberg. In most cases Harry has solved other people's problems, so his publication list makes excursions into dozens of other topics from the local times for Markov processes [25, 43] to existence and uniqueness of Markov random fields [54]; from the speed of convergence of martingales [59] and properties of positive harmonic functions [61] to Chung-type laws of the iterated logarithm [143].

When I was born (i.e., emerged from graduate school) in 1976, the definitive work of the dynamic duo of Kesten and Spitzer on random walks [17, 20, 26] was legendary and, together with Kesten's individual work on random walks [21, 32, 34, 35] and his joint work with Stigum [27, 28, 33] and with Ney and Spitzer [31] on branching processes, was an important part of one's graduate education. In particular, the Kesten and Stigum result that $E(Z_1 \log^+ Z_1) < \infty$ is necessary and sufficient for the convergence of mean normalized branching processes Z_n/m^n to a limit with mean EZ_0 is something that is usually mentioned (but not proved) when branching processes are discussed. These and other results of that era [41, 45, 49, 51, 58] showed us that Kesten was someone who could prove results under a minimal number of assumptions and who could disentangle the mysteries of random walks with infinite mean.

However, most of all we knew Kesten as a problem solver. A classical and well-known example is Kesten's impressive work calculating "Hitting probabilities of single points for a process with independent increments" which appeared as Memoirs of the AMS, No. 93 [37] and was the subject of his 1970 address at the International Congress of Mathematicians in Nice [39]. A more personal bit of data is the following testimonial from G.R.Grimmett@statslab.cam.ac.uk: "Rick: I was perhaps understandably impressed by 'Supercritical branching processes with countably many types and the size of random Cantor sets' [106] which was inspired by Dekking/Grimmett. Harry was just able to blast through a version of the problem doing substantially better than anyone else. I understood then how well he grasped branching processes." Indeed, one does not really understand Kesten's prowess until you have seen him demolish a problem that you have worked on.

The first two decades of Kesten's work, while containing important contributions, are difficult for me to properly put in context since the work was already complete when I started learning probability. Because of this I have decided to tell the story of his work as I experienced it. Thus, following the style of some paperback novelists, I will begin in the middle of the story with some exciting events to grab the reader's attention. Then after the story line is established, I will go back and fill in earlier developments.

I spent the 1980–81 academic year at Cornell. At this time, Dynkin's Russian style seminar, held Wednesdays 7–9 PM, was a lively affair, with Avi Mandelbaum, Bob Vanderbei and Patrick Sheppard as students. Kesten had recently proved that "The critical probability for bond percolation on the square lattice equals 1/2" [67] and followed this up with "power estimates of functions in percolation theory" [71]. The title of the last paper is a double entendre. The results concern the power law behavior of functions near the critical value but introduced powerful new rigorous renormalization arguments. To facilitate writing his book [72], Kesten taught a graduate seminar on percolation and first passage percolation. Harry prepared for class while swimming laps in the pool at Teagle Hall, so in his lectures you got to see how he thought. He would start with the main idea of the proof, but then often would have to go back and insert a technicality at an angle on the margin of the board. This made it difficult for the students to get good notes, but for me it provided valuable insights about why things are true and how he went about solving problems.

In this brief article there is not enough space to discuss why things are true, so we will only discuss what Kesten (and others) have done. Our first two topics, percolation and first passage percolation (which we will interpret very broadly) are those of Harry's course in 1980 and of his Wald Lectures in 1986 (see [95]). In each case we will take the subject from the 80's up to today. For the third section of the paper we will go back to Kesten's Ph.D. thesis and follow his work on random walks and related topics up to the present. In these three forays we will touch on much, but by no means all, of Kesten's best work. Like a one week bus tour of Europe, there is only time to drive past the outside of some of the most important landmarks. We apologize in advance for the fact that in order to say things quickly, we will not always be able to say things carefully. We will never intentionally lie about what is true, but sometimes we will not take the time to sort out all the details of who did what when.

1.1 Percolation

Broadbent and Hammersley (1957), and Hammersley (1959) introduced percolation as a model for the spread of a fluid or gas through a random medium. To formulate the bond percolation model in d dimensions, we make the d-dimensional

integer lattice $\mathbb{Z}^d$ into a graph by drawing edges connecting adjacent sites. We imagine that the edges are channels and that fluid will move through a channel if and only if the channel is wide enough. We declare that the edges are independently designated as open (wide enough) or closed with probabilities p and $1-p$ respectively, and let P_p denote the resulting probability measure on the configurations of open and closed edges. We will also sometimes consider site percolation in which the sites are independently open with probability p or closed with probability $1-p$, but for this article the default process is bond percolation.

With that set-up it is natural to ask about the set of sites C_0 that can be reached from the origin by a path of open edges. The first papers mentioned above showed that if p is small, then the number of points in C_0, $|C_0|$, is always finite, while if p is close enough to 1, then

$$\theta(p) = P_p(|C_0| = \infty) > 0. \tag{1.1.1}$$

This and an obvious monotonicity establishes the existence of a critical value $p_c = \inf\{p : \theta(p) > 0\}$ but does not give much information about its value. The first step in that direction for two dimensional bond percolation was taken by Harris (1960). He noticed that when $p = 1/2$, symmetry dictates that the probability of a left to right crossing of a "sponge," an $n \times (n+1)$ piece of the square lattice, is 1/2, and used this observation to show that at $p = 1/2$ the origin is surrounded by infinitely many cut sets of vacant edges, so $p_c \geq 1/2$.

The next step in this direction was taken by Sykes and Essam (1964) who introduced a quantity they called the *free energy*:

$$\Delta(p) = E_p(1/|C_0|; |C_0| > 0). \tag{1.1.2}$$

Probabilistically, this is the limiting value of the number of clusters per unit volume. By analogy with the Ising model, Sykes and Essam argued that the phase transition in percolation must be manifest in a "singularity" at p_c. Their calculations showed that the square lattice, $\Delta(p) - \Delta(1-p)$ is a polynomial in p, so assuming that such a singularity was unique, they arrived at $p_c = 1/2$ for the square lattice. In addition, Sykes and Essam used variations of this argument to show that the critical value of site percolation on the triangular lattice is 1/2 and supplemented this with the star-triangle transformation to show that the critical values for bond percolation on the triangular and hexagonal lattice are ρ and $1-\rho$ where $\rho = 2\sin(\pi/18)$ is the unique root of $3\rho - \rho^3 = 1$ in $(0, 1)$.

Not much beyond Harris' result was rigorously proved about percolation until 1978, when Russo (1978) and Seymour and Welsh (1978) provided two valuable steps. The first is that if sponge crossing probabilities are large enough, then percolation occurs. To state the second, we need to define the dual of a planar graph, which is constructed by putting sites in each component of the complement of the graph and connecting two sites by an edge if the boundaries of their associated components share an edge. Denoting the dual by a star and defining

$$p_T = \sup\{p : E_p(|C_0|) = \infty\}, \tag{1.1.3}$$

the second fact is $p_c + p_T^* = 1$. The third crucial ingredient was provided by Kesten, who showed that if n is large, then the sponge crossing probability for two dimensional bond percolation is a very steep function of p near $p = 1/2$. Since small sponge crossing probabilities imply that the cluster size is finite, this completed the proof of $p_c = 1/2$.

The computation of the critical value for the square lattice was soon generalized to the other graphs Sykes and Essam considered, see [72, Chapters 1–3], and a flood of new results followed. Some of the results were proved for general d initially and most are known in that generality now, but to make the storytelling simple we will restrict our attention here to $d = 2$ and add the disclaimer that many other people's work was important in reaching the following conclusions. If you want the whole story and to have it told correctly, you should buy a copy of Grimmett's (1999) book.

Soon after Kesten's breakthrough, it was shown that $P_p(|C_0| \geq n)$ decayed exponentially fast for $p < p_c$ while for $p > p_c$ large finite clusters were very unlikely:

$$P_p(n \leq |C_0| < \infty) \leq C \exp(-\gamma n^{(d-1)/d}) \qquad \text{for } p > p_c.$$

Here one cannot do better than the power $n^{(d-1)/d}$ since a cube of vacant edges of radius r has probability $\exp(-cr^{d-1})$ and cuts off a volume of r^d. For a more recent look at estimates for the probability of a large cluster in supercritical percolation, see [105].

A second important consequence of Kesten's work is that it was possible for the first time to prove results about the behavior of various quantities near p_c. Taking $f(p) \approx |p - p_c|^\alpha$ to mean that $\log f(p)/\log|p-p_c| \to \alpha$, physicists tell us that near p_c we have

$$\theta(p) = P_p(|C_0| = \infty) \approx (p - p_c)^\beta \quad \text{as } p \downarrow p_c,$$
$$\chi(p) = E_p(|C_0|; |C_0| < \infty) \approx |p - p_c|^{-\gamma} \quad \text{as } p \to p_c$$

while if we use P_{cr} to denote P_p with $p = p_c$, then we can define three more critical exponents

$$P_{cr}(|C_0| \geq n) \approx n^{-1/\delta}$$
$$P_{cr}(\text{radius}(C_0) \geq n) \approx n^{-1/\delta_r}$$
$$P_{cr}((n,0) \in C_0) \approx n^{-(d-2+\eta)}.$$

Kesten was the first [71] to prove bounds which show that in two dimensions if β, γ, and δ exist, then they are positive and finite. These insights were deepened in [81] when he gave a rigorous definition in two dimensions of physicists' "incipient infinite cluster at criticality" (where the probability of an infinite cluster is 0) by conditioning on the event that the origin is connected to the boundary of the box of radius n and letting $n \to \infty$.

The incipient infinite cluster is a fractal of dimension $(2-\eta)/(1-1/\delta)$. Simulations show that the cluster consists of many dangling ends and very little backbone, i.e., the part that would carry electricity if the origin was electrified and the boundary of the box was grounded. In order to prove a result that captures this mental picture of the structure of the cluster, Kesten considered in [82] and [83] random walk on various random graphs. In the case of a graph that is the family tree of a critical branching process conditioned on non-extinction, Kesten was able to show that the normalized height of the walker at time n, $n^{-1/3}h(X_n)$, converge to a limit. His results were less complete for walks on the "incipient infinite cluster" but he was able to establish subdiffusive behavior, i.e., show that $|X_n|/n^{0.5-\epsilon}$ was tight for some $\epsilon > 0$. For recent related work on the geometry of critical percolation clusters, in particular results about lowest crossing, see Kesten and Zhang [123].

The incipient infinite cluster is not only a mathematical curiosity but also a useful technical device. It allowed Kesten to show [89] that the three exponents that we introduced for P_{cr} were simply related in $d = 2$ (assuming they exist):

$$\eta = 2/\delta_r, \qquad \delta = 2\delta_r - 1. \tag{1.1.4}$$

The last two equalities are two of many scaling relationships that relate the behavior of various quantities at and near p_c. However, before we can state more of these relations we need to introduce a quantity that is more subtle but equally important as those introduced above.

Taking the simplest of several possible definitions we can define the *correlation length* by

$$\xi(p) = \left(\frac{1}{\chi(p)} \cdot \sum_y |y|^2 P_p(y \in C_0, |C_0| < \infty)\right)^{1/2}$$

and the corresponding exponent by $\xi(p) \approx |p - p_c|^{-\nu}$ as $p \to p_c$. (See [97] for a definition in terms of the exponential decay of various connection probabilities.) Intuitively, the correlation length gives the radius of a "typical" finite cluster. Using the definition above, Kesten was able to show [90] that the critical exponents of two dimensional percolation satisfy

$$\beta = \frac{2\nu}{\delta + 1} \qquad \gamma = 2\nu \cdot \frac{\delta - 1}{\delta + 1}. \tag{1.1.5}$$

These equalities, which can be guessed by back-of-the-envelope calculations, were widely accepted by physicists, but it required a fair amount of ingenuity for Kesten to prove them in the two dimensional case.

Another of the "obvious" facts about percolation that needed mathematical proof was the fact that the infinite cluster, when it existed, was unique. A first step in this direction was taken by Newman and Schulman (1981) who showed that, with probability one, there were 0, 1, or ∞ infinite clusters. The last possibility occurs for trivial reasons for percolation on trees. Not many people believed that infinitely many clusters was a reasonable possibility on $\mathbf{Z}^d$, but it took another half decade before Aizenman, Kesten, and Newman (see [88] and [91]) could prove this by relating uniqueness to the differentiability of the free energy, $\Delta(p)$. Though the proof was slow to be found, it did not last long as the best argument around. In

(1988) Burton and Keane discovered a very beautiful geometric proof that worked for a number of dependent models as well.

There are many reasons for being interested in percolation. When generalized to the oriented case and then to "continuous time," these results have led to a wealth of information about the contact process, one of the most basic interacting particle systems. See Durrett (1984) and Bezuidenhout and Grimmett (1990), (1991). On a different level, comparison with oriented percolation can be used to prove the existence of interesting behavior for particle systems with long range or fast stirring. See Durrett (1995a).

Physicists view percolation as a prototypical example of a system with phase transitions. In the case of the Potts model (a multicolor version of the Ising model) the connection is more than an analogy. Suppose we use the independent percolation measure P_p with $0 \le p \le 1$ to define a new measure for $q > 0$ by

$$dQ_{q,p}/dP_p = q^{C(\omega,\Lambda)} \cdot \frac{1}{Z(q, p, \Lambda)} \tag{1.1.6}$$

where $C(\omega, \Lambda)$ is the number of connected components in the box Λ (for some specified boundary conditions) and $Z(q, p, \Lambda)$ is the normalizing constant to make the $Q_{q,p}$ a probability measure. Then the infinite volume limit $Q_{q,p}$ gives the distribution of the q-state Potts model, and taking $q = 2$ we have the Ising model. This great idea came from Fortuin and Kastelyn (1972). For more on this see Grimmett (1995).

This connection was used by Aizenman, J. Chayes, L. Chayes, and Newman (1988) to prove the discontinuity of magnetization in the one dimensional $1/|x-y|^2$ Ising and Potts models by using earlier results of Aizenman and Newman (1986) for the analogous percolation process. Later [99, 100, 102], and [103], in joint work with various subsets of {Bricmont, Lebowitz, Schonmann}, Kesten used these ideas to study the asymptotic behavior of Ising, Potts, and Heisenberg models as the dimension gets large. More recently in [120] with Bezuidenhout and Grimmett, Kesten used this connection to show that the critical value β_c of the ferromagnetic Potts model is a *strictly* decreasing function of the strengths of the interactions in the process.

Turning away from critical values, our next topic is critical exponents, specifically "mean field bounds" on them. To explain this term, we note that percolation on a tree in which each node has degree $k + 1$ is essentially a Galton-Watson process in which each individual has a binomial(k, p) number of children. Calculations for branching processes (exercise for the reader) show that the critical value $p_c = 1/k$ and critical exponents $\beta = 1$, $\gamma = 1$, $\delta_r = 1$, and $\delta = 2$.

There are a number of results which show that the mean field values in general provide bounds for those of ordinary percolation. Aizenman and Newman (1984) showed that $\gamma \ge 1$. J.T. Chayes and L. Chayes (1986) showed $\beta \le 1$. Aizenman and Barsky (1987) showed $\delta \ge 2$. Going further in this direction, there are results which show that values of critical exponents do not take their mean field values in low dimensions. Reversing historical order we note that for $d = 2$, Kesten and Zhang [92] showed $\beta < 1$, while Kesten and van den Berg [79] showed that $\delta_r \ge 2 > 1$. The latter result is based on their famous inequality that the probability for two increasing events to occur disjointly is smaller than the product of their probabilities. For an up to date account of this inequality and its remarkable generalization to arbitrary events, see the article by Borgs, Chayes, and Randall in this volume.

Somewhat more surprising than the strict inequalities in low dimensions is the statement that above the critical dimension ($d_c = 6$ for percolation) all critical exponents take their mean field values. For a long time this statement was a claim that physicists made and mathematicians couldn't prove. The first steps toward a mathematical proof were taken by Aizenman and Newman (1984) who showed that if

$$\nabla \equiv \sum_{x,y} P_{cr}(0 \to x) P_{cr}(x \to y) P_{cr}(y \to 0) < \infty \tag{1.1.7}$$

then $\gamma = 1$. Barsky and Aizenman (1988) showed that if the "triangle condition" was satisfied, then we also have $\beta = 1$ and $\delta = 2$. The final step was taken by Hara and Slade (1989, 1990) who showed that the triangle condition held (i) for the nearest neighbor case in $d \ge d_o$ (where $d_o \le 19$, see Hara and Slade (1994)) or (ii) in $d > 6$ for a sufficiently spread out model. To be precise, they generalized the percolation model so that connections from x to y have probability $pL^{-d} g((y - x)/L)$, where g is a nice function, and showed that $\nabla < \infty$ if $L \ge L_o(d)$. Since there is no reason to believe that percolation with range 1 is different from range 10 or 100, (ii) gives a convincing demonstration that the critical dimension is ≤ 6.

Hara and Slade proved their results with the "lace expansion" which has turned out to be a powerful technique for understanding phase transitions in other systems. Recent applications of this method to lattice trees and the incipient infinite cluster are discussed in Slade's article in this volume. The limit in Slade's article involves a functional of super-Brownian motion, a process that is the subject of articles by LeGall and Cox, Durrett, and Perkins.

Results on percolation have continued to this day to be an important part of Kesten's work. Inspired by a seminar talk Larry Shepp gave at Cornell, Kesten and I solved a problem Shepp posed about long range percolation in one dimension, see [104]. In [119] he studied with Grimmett and Zhang random walk on the infinite cluster of bond percolation on $\mathbb{Z}^d$, showing that in the supercritical regime when $d \ge 3$ this random walk is a.s. transient. This conclusion was proved by considering the infinite percolation cluster as a random electrical network in which each open edge has unit resistance and showing that the effective resistance between a nominated point and points at infinity is almost surely finite.

Recently, Kesten has with Benjamini [130] and with Sidoravicius and Zhang [149], proved some fascinating results about the question: when can one with positive probability see every infinite word of 0's and 1's from a given site in a lattice of independent {0, 1} valued random variables? To be precise, a word is a binary sequence $(v_1, v_2, \ldots) \in \Xi = \{0, 1\}^N$ where $N = \{1, 2, 3, \ldots\}$. We say that the word is seen from x if there is a self-avoiding path starting from a neighbor of x along which we see the word. (We start at a neighbor for the trivial reason that there can be only one value at a given site.) Let $S(v)$ be the words seen from a given vertex v and $S_\infty = \cup_v S(v)$ be the words seen from some vertex.

To relate this to our previous discussion, note that the classic question of percolation can be phrased as: "Is $(1, 1, 1, \ldots) \in S(v)$?" Another variant that has been investigated is AB percolation (see Wierman and Appel (1997) and references therein). In the current setting the question may be phrased as: "Is $(1, 0, 1, 0, \ldots) \in S(v)$?" Benjamini and Kesten [130] studied the case in which 0's and 1's each had probability 1/2 in the original product measure. They showed that in $d \geq 10$, $P(S_\infty = \Xi) = 1$ while for $d \geq 40$, $P(S(v) = \Xi \text{ for some } v) = 1$.

The dimensions 10 and 40 came from considering analogous questions for oriented percolation and hence are not sharp. To approach the question from the other end, consider the triangular lattice in two dimensions. In this case, due to the lack of percolation at $p = 1/2$, $(1, 1, 1, \ldots)$ cannot be seen, but Wierman and Appel (1987) have shown $(1, 0, 1, 0, \ldots)$ can be seen. To ask how many words can be seen, we can take a comprehensive look by introducing product measures with density $0 < \beta < 1$, ν_β on Ξ and let $\rho(\xi)$ be the probability the word ξ is seen from some starting point. In [149], Kesten, Sidoravicius, and Zhang showed that $\rho(\xi) = 1$ for ν_β almost every word ξ.

Finally, while I have been writing this article, Kesten has done his best to create new results faster than I can digest what he has done. In [151] he and Zhonggen Su investigated ρ-percolation. Letting $X(e) = 1$ if the bond is open and 0 if it is closed, they asked if there is an infinite oriented path $v_0 = 0, v_1, v_2, \ldots$ starting at the origin so that $\liminf_{n\to\infty}(1/n)\sum_{i=1}^n X(v_{i-1}, v_i) \geq \rho$. Defining the critical value in the obvious way, they considered $D_1 = \lim_{d\to\infty} d^{1/\rho} p_c(d)$ and an analogous limit, D_2, for site percolation showing that $D_1 < D_2$ and that neither of these values is the equal to the corresponding limit for regular d-ary trees.

1.2 First Passage Percolation

Again our subject originates in England, but this time more than a half decade later in the work of Hammersley and Welsh (1965). To formulate the model in d dimensions, we again imagine that the edges connecting neighboring sites in $\mathbb{Z}^d$ are channels. However, we now assume that the fluid can flow through any channel but will take an amount of time t_e to flow through edge e, where the $t_e \in [0, \infty]$ are independent and identically distributed random times.

With this set-up it is natural to ask: at what time $\tau(x, y)$ will fluid first appear at y if we turn on a source at x at time 0? If we let $e_1 = (1, 0, \ldots, 0)$, then the passage times $a_{m,n} = \tau(me_1, ne_1)$ do not have independent increments, but by exploiting an obvious subadditivity property $a_{0,m} + a_{m,n} \geq a_{0,n}$ one can conclude (see Kingman (1968) or Chapter 5 of Smythe and Wierman (1978)) that if $Et_e < \infty$

then

$$a_{0,n}/n \to \mu \quad \text{a.s.} \quad \text{where } \mu = \inf_{m\geq 1} Ea_{0,m}/m. \tag{1.2.1}$$

Like many other people, Harry Kesten was attracted to the subject by Smythe and Wierman's (1978) monograph. Not surprisingly, his first result in this direction in [63] was closely related to his work on percolation. He showed that the time constant μ is 0 if and only if the atom at 0 in the passage time distribution is $\leq p_T$, the critical value for infinite mean cluster size. To put this result in context, we should mention that this was proved before it was known that $p_c = p_T$.

Having mentioned the possibility of $\mu = 0$, we will now define it out of existence by supposing for the rest of our discussion that the passage time distribution has $F(0) = 0$. There is nothing special about the direction e_1 in the limit theorem quoted above. Taking inspiration from Richardson (1973), one can define a propagation speed for each direction and then patch them together to get a shape theorem for the *wet region*

$$W_t = \{x \in \mathbb{Z}^d : \tau(0, x) \leq t\}.$$

To do this let $Q = [-1/2, 1/2]^d$ be the cube of side 1 and convert the wet region at time t into a solid blob by letting $\bar{W}_t$ be the union of $x + Q$ over all $x \in W_t$. Cox and Durrett (1980) showed that for any distribution F with $\lim_{x\to\infty} F(x) = 1$ (e.g, we do not have to assume the existence of a mean) there is a limiting convex set G so that for any $\epsilon > 0$

$$P(\bar{W}_t \subset (1+\epsilon)tG,\ |\bar{W}_t/t - G| \leq \epsilon) \to 1. \tag{1.2.2}$$

Here $|A|$ denotes the Lebesgue measure of A.

An easy consequence of this result is that if we let $b(0, n) = \min\{\tau(0, x) : x_1 = n\}$ be the *point to hyperplane passage time*, then for any distribution F we have

$$b_{0,n}/n \to \mu \quad \text{a.s.} \quad \text{where } \mu = \inf_{m\geq 1} Ea_{0,m}/m. \tag{1.2.3}$$

In contrast, some moment condition is needed to have almost sure convergence of $a_{0,n}/n$ in (1.2.1). For, otherwise, the minimum of the $2d$ bonds ending at ne_1 may be $\geq \epsilon n$ infinitely often, spoiling the convergence. For $a_{0,n}/n$, Cox and Durrett (1980) showed that the necessary condition implicit in the previous sentence is sufficient for convergence to μ almost surely. The stubborn points are responsible for little holes in the limiting shape which force the complicated formulation. I would like to thank Harry Kesten for explaining this aspect of Cox and Durrett's work to me.

The time constant $\mu = \inf_{m\geq 1} Ea_{0,m}/m$ is a mysterious object. One can, with considerable pain, get upper bounds on μ by estimating $Ea_{0,n}/n$ for $n = 1$ or 2. However, to my knowledge it cannot be computed exactly in any case in which $\mu > \inf\{x : F(x) > 0\}$. Cox (1980) was the first to investigate continuity properties of μ as a function of the underlying distribution F. He proved that if the F_n were dominated by a single distribution G with finite mean and $F_n \Rightarrow F$ in the sense of weak convergence, then $\mu(F_n) \to \mu(F)$. As the "no moment" result in (1.2.3) might suggest, the domination condition is not needed. It was removed by Cox and Kesten [70] who showed that if $F_n \Rightarrow F$, then $\mu(F_n) \to \mu(F)$.

Eden (1961) was interested in the Markovian case of first passage percolation in which each edge had a mean one exponential distribution. Early simulations, and a little wishful thinking, suggested that the limit shape might be a ball. However in the mid 1980's a super-computer solved the problem by showing there was roughly a 2% difference between the speeds along the axis and on the line at 45 degrees. (See Zabolitsky and Stauffer (1986a,b).) In words, the fact that the L^1 distance to $(n/\sqrt{2}, n/\sqrt{2})$ is $\sqrt{2}n$ rather than n to the point $(n, 0)$ is almost exactly compensated by the fact that there are many more paths of minimum length to the first point. Kesten showed in Section 8 of [78] that in high dimensions the balance between these forces breaks down. If we suppose that the underlying distribution F has density function $F'(0) = 1$, then the time constant is of order $(\log d)/d$ while the passage time in the direction $(1, 1, \dots, 1)/\sqrt{d}$ is of order $1/d$.

There is only one special case in which we have some rigorous concrete information about the limiting shape G in (1.2.2). Consider two dimensions for simplicity and suppose that $P(t_e \geq 1) = 1$ and $P(t_e = 1) = p$. Since the fluid can move at most one unit per time, the limiting set must be contained in the diamond $\{(x, y) : |x|+|y| \leq 1\}$. Durrett and Liggett (1981) showed that if p was larger than the critical value for oriented percolation in two dimensions, then the boundary of the limiting shape contained an interval in $x + y = 1$.

Closely related to the topic of first passage percolation is the notion of random resistor networks. To formulate the model in two dimensions, we imagine that the edges connecting adjacent sites in the square lattice are resistors with random resistances r_e that are independent and identically distributed random values $\in [0, \infty]$. Grimmett and Kesten [74] investigated the bulk properties of random resistor networks consisting of $n \times n$ chunks of the square lattice, with a special interest in the case in which the values 1 and ∞ had probabilities p and $1 - p$. They also looked at the flow through networks where edges have random capacities. This study involved a look at large deviations probabilities for the various passage times. In a second study [75] they examined properties of random electrical networks on complete graphs. The reader should not be surprised to hear that the results are more precise and detailed in this context.

Kesten's work on percolation and first passage percolation earned him his second invitation for a 45 minute lecture at the International Congress of Mathematicians, which met in Warsaw in 1983. The odd numbered year is not a typo. The Congress was delayed for a year due to political unrest in Poland. Like many mathematicians, Kesten showed his solidarity with Solidarity by not going to the Congress.

In 1984 Kesten lectured (with René Carmona and John Walsh) at École d'Été de Probabilités de Saint Flour XIV. Kesten's lecture notes cover many of the topics we have referred to above and in many cases present new refinements. We have already mentioned the asymptotics for large dimensions that were given in Section 8 of his notes. The work of Grimmett and Kesten [74] is taken further in Section 5 by giving large deviations results and rates of convergence of $Ea_{0,n}/n$ to the passage time, and in Section 7 with a look at convergence rates in the case $\mu = 0$ in $d = 2$. For more recent results on the last topic, see Kesten and Zhang [142].

The rate of convergence to the time constant given in (5.16) of the St. Flour notes was crude: $O((\log n)^{-1/(9d+3)})$ but it took several years before Kesten [117] and Alexander (1993) could improve that bound to the very respectable $O(n^{-1/2} \log n)$. Kesten showed in [117] that the fluctuations $a_{0,n} - Ea_{0,n}$ are at most diffusive, i.e., the variance of $a_{0,n}$ is $\leq Cn$. Novice readers might expect to hear next of a central limit theorem being proved. However, physicists tell us (see Kardar, Parisi, and Zhang (1986), Zabolitsky and Stauffer (1986a,b), and Krug and Spohn (1991)) that in two dimensions the standard deviation of the first passage time $t(0, (n, 0))$ is of order $n^{1/3}$.

The fluctuations in the passage times to ne_1 or, more geometrically, of the boundary of the wet region at time n, $\bar{W}_n$, can be used to define a critical exponent, χ, by declaring that they are $O(n^{\chi})$. In this new notation, Kesten's result is that $\chi \leq 1/2$, while physicists claim that $\chi = 1/3$. Lower bounds on the fluctuations have turned out to be more difficult. Pemantle and Peres (1994) and Newman and Piza (1995) have shown that, in $d = 2$, fluctuations diverge at least logarithmically fast. This result can be improved if one is willing to introduce hypotheses that seem reasonable, but that cannot at the moment be proved. Wehr and Aizenman (1990) studied an exponent ξ, defined so that the point the wet region first touches the hyperplane $x_1 = n$ is $O(n^{\xi})$, and proved that

$$\chi \geq \frac{1 - (d-1)\xi}{2}. \tag{1.2.4}$$

Combining this with Newman and Piza's (1995) result $\xi \leq 3/4$ in $d = 2$ gives $\chi \geq 1/8$. Weaker versions of the last conclusion can be proved without invoking any unverified hypotheses.

Seeking to understand the spread of first passage percolation, Newman (1995) introduced a graph that consists of the union of the time minimizing paths from one fixed point, say the origin, to all of the other points. It is easy to see that this graph must be a spanning tree, and that the spanning tree must have at least one infinite path, which Newman called a one sided geodesic. Proving that one sided geodesics exist going in all directions, or the more mysterious claim by physicists that two sided geodesics do not exist, has proved to be difficult. See Licea and Newman (1996). In this volume, Howard and Newman report on recent progress for models that take place on $\mathbf{R}^d$. Here rotational invariance can be used to great advantage once one pays the price of generalizing the lattice results to the new setting.

The exact solution for the critical value of two dimensional percolation rests on a duality between planar graphs. If one considers bond percolation in the three dimensions, then a natural dual two-dimensional object is a family of two dimensional plaquettes, i.e., the squares with side 1 perpendicular to the mid-point of the segments from x to $x + z$ where z is one of the six nearest neighbors of the origin: $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, and $(0, 0, -1)$. Aizenman, Chayes, Chayes, Fröhlich, and Russo (1983) showed that if we make plaquettes occupied or vacant with probabilities p and $1 - p$ and consider the event that there is a surface of occupied plaquettes with boundary exactly equal to a given rectangle, then the probability is of order exp(−area) if $p < p_c$, while it is of order exp(−surface) when $p > p_c$.

Inspired by this, and using analogies with the max-flow/min-cut theorem, Kesten considered in [87] a first passage percolation theory for random surfaces, i.e., the minimal cost surface that can be constructed with boundary equal to a given rectangle of side n. In three dimensions the asymptotic cost of such a loop is proportional to the area, and dividing by n^2 leads to a limit. This may all sound very straightforward, but it definitely is not since the topology of surfaces in $\mathbf{R}^3$ rears its ugly head. Kesten's paper was an important first step, but much remains to be done. A proper understanding of the issue involved would probably help us sort out the asymptotic behavior of the contact process on $[0, L]^d$. Specifically, the problem for the contact process is to show that if τ_L is the time the process dies out (i.e., reaches all sites vacant) starting from all sites occupied, then

$$(1/L^2) \log \tau_L \to \gamma \tag{1.2.5}$$

in probability as $L \to \infty$. This is known to be true in $d = 1$, see Durrett and Schonmann (1988); but only partial results exist in $d > 1$, see Mountford (1993).

In an opposite direction from the concept of surfaces with minimal weights is the notion of greedy lattice animals. To set up the problem, suppose we have i.i.d. positive random variables for the sites in the d-dimensional integer lattice, $\{X_v : v \in \mathbf{Z}^d\}$. Let M_n be the largest sum that we can get from a self-avoiding path of length n containing the origin, and let N_n be the largest sum for an animal (connected subset) of size n containing 0. In joint work with Cox, Gandolfi, and Griffin [121] and with Gandolfi [122], Kesten showed that if $EX_i^{d+a} < \infty$ for some $a > 0$, then $M_n/n \to \mu$ and $N_n/n \to \nu$ almost surely.

In an opposite direction from the notion of greedy lattice animals is that of minimal spanning trees. Let $X_1, X_2, \ldots$ be independent and identically distributed with common distribution μ that has support in $[0, 1]^d$, and choose a spanning tree T to minimize $M_\alpha = \sum_{e \in T} |e|^\alpha$ where $|e|$ is the length of the edge. Steele (1988) has shown that if $0 < \alpha < d$, then the minimum length satisfies

$$n^{-(d-\alpha)/d} M_\alpha \to c(\alpha; d) \int f(x)^{(d-\alpha)/d}\, dx \quad \text{a.s.} \tag{1.2.6}$$

where f is the density of the absolutely continuous part of μ and $C(\alpha, d)$ is a constant that only depends on α and d.

Aldous and Steele (1992) complemented this result by showing that for the uniform distribution when $\alpha = d$, $M_d \to c(d, d)$ in L^2. The central limit theorem had to wait for a while until Alexander (1996) and Kesten and Lee [137] independently showed that if μ is the uniform distribution, then for any $\alpha > 0$

$$n^{-(d-2\alpha)/2d}(M_\alpha - EM_\alpha) \Rightarrow \text{normal}(0, \sigma^2_{\alpha,d}) \tag{1.2.7}$$

where again $\sigma^2_{\alpha,d}$ is a constant that only depends on α and d.

Reversing direction yet again, we will motivate the last two topics in this section, by noting that (i) Eden's growth model can be thought of as a continuous time process in which vacant sites become occupied at a rate equal to the number of occupied neighbors, and (ii) if we assign the passage times to the sites instead and look at the embedded discrete time chain, then Eden's model becomes a process in which at each step a randomly chosen vacant site on the boundary of the wet region becomes occupied.

Taking (ii) first, we consider a new, more complicated model, called diffusion limited aggregation or DLA, in which boundary sites are added according to a non-uniform rule. Let $A_0 = \{0\}$, i.e., just the origin. Having defined A_n for $n \geq 0$, A_{n+1} is formed from A_n by releasing a particle at ∞ and letting it perform a nearest neighbor symmetric random walk on $\mathbf{Z}^d$ until it reaches a site on the boundary of A_n. Simulations of this process produce starfish like creatures. The intuition is easy to see: once arms with narrow valleys between them form, random walks are more likely to attach near the tips rather than traversing the fjords to get stuck in the interior.

It is a very difficult unsolved problem to show that arms form and the diameter of DLA grows at rate $n^{0.5+\epsilon}$. Kesten proved a result in the opposite direction in [93] showing that the arms of DLA can't grow any faster than $Cn^{2/3}$. In hindsight the answer is easy to see: the enhancement of adding at the tip is maximized if the configuration is always an interval (or a plus sign), and in this case the growth rate for the radius is 2/3. The proof of this result involves interesting estimates for hitting probabilities of random walks on $\mathbf{Z}^d$ [94] and has led to relationships between solutions to discrete and continuous Dirichlet problems [112]. Physicists, in their quest for new and exciting pictures generalized the original model to include versions where the probability of attachment is a power η of the "harmonic measure," i.e., the hitting distribution for the random walk. Kesten [113] kept up with them as best as he could, proving results to explain their simulations.

Returning now to (i), Kesten and Schonmann [133] considered a variant of Eden's growth model in which each site on $\mathbf{Z}^d$ becomes occupied at rate 1 if the site has at least θ occupied neighbors, at rate ϵ if at least one but $< \theta$ occupied neighbors, and at rate 0 if it has no occupied neighbor. In the case $\theta = 2$ they were able to show that the asymptotic growth rate for the model was $O(\epsilon^{1/d})$, i.e., was bounded above and below by constant multiples of this quantity, and that the limiting shape after rescaling is a cube as $\epsilon \to 0$. The model with $\theta = 3$ is not interesting in two dimensions, but in $d = 3$ presents an intriguing open problem that seems to have connections to so-called bootstrap percolation. See Aizenman and Lebowitz (1988).

1.3 Random Walks

Having taken two trips through the most recent twenty years of Kesten's work, we now go back to a time when symmetric random walk transition probabilities were better known as Toeplitz matrices: [11, 12, 18]. The natural place to start is with

Kesten's Ph.D. thesis in which he considered symmetric random walks on groups. Let G be a countable group and let $p = (p_x)_{x \in G}$ be a symmetric probability distribution whose support generates G. Consider the random walk on G in which every step corresponds to right multiplication by x with probability p_x. Kesten's thesis explored connections between the spectrum of the transition probability (as an operator on $L^2(G)$) and the structure of the group G. The operator is self-adjoint (since the random walk is symmetric) and has norm ≤ 1, so its spectrum is a subset of $[-1, 1]$. Kesten showed that the spectral radius $\lambda(G, p)$ is the maximal value in the spectrum and

$$\lambda(G, p) = \lim_{n\to\infty} a_{2n}^{1/2n}$$

where a_{2n} is the probability that the walk is back at the origin after $2n$ steps. Thus $\lambda(G, p) < 1$ if and only if $a_{2n} \to 0$ exponentially fast.

Kesten was especially interested in determining when $\lambda(G, p) = 1$. He showed that this is a property of G alone and does not depend on the choice of p. He studied what happened for finite direct products and proved comparison results between groups and their normal subgroups and quotients. Using that machinery he showed that if G is an Abelian group, then $\lambda(G, p) = 1$, while if $\lambda(G, p) = 1$, then G has no free subgroups on more than one generator. He calculated $\lambda(G, p)$ explicitly if G is free and p assigns equal probability to the generators and their inverses. Later in [5] he showed that $\lambda(G, p) = 1$ if and only if G is amenable. This famous result is now known as Kesten's criterion for amenability.

Kesten returned to random walks on groups in [32] where he considered, among other things, the question of recurrence. Kesten's results and questions inspired a generation of workers, culminating in Varopoulos's beautiful solution to "Kesten's conjecture": Simple random walk on a finitely generated group G is recurrent if and only if G is virtually Abelian of rank ≤ 2.

From the first few entries in his publication list, you can see that even as a young man Kesten was already hard at work solving other people's problems. An interesting thread in his early work is what one might call ergodic number theory. Kac and Kesten [3] considered the transformation $Tx = 1/x - [1/x]$ of the unit interval, where $[1/x]$ is the integer part of $1/x$ and $[1/T^{n-1}x]$ gives the digits in the continued fraction representation of x. Using Lévy's result that T is rapidly mixing, they were able to show that the number of times a specified digit occurs among the first n digits in a continued fraction representation is asymptotically normally distributed. This was published in the Bulletin of the American Mathematical Society, although they later learned that the result was due to Doeblin (1940).

A second set of results in this direction concern uniform distribution mod 1. Let $f(\xi)$ be the indicator function of the interval $[0, t]$ extended to be periodic with period 1, that is, $f(\xi + 1) = f(\xi)$. In [8] and [13] Kesten showed that if X and Y are independent and uniform on $[0, 1]$, then

$$(\log n)^{-1} \sum_{k=1}^{n} \{f(Y + kX) - t\} \tag{1.3.1}$$

has a limiting Cauchy distribution. This contrasts sharply with the behavior Kac (1946) observed for lacunary series, where $n^{-1/2} \sum_{k=1}^{n} \{f(2^k X) - t\}$ converges to a normal distribution. Fine (1954) sharpened Kac's result by proving convergence of finite dimensional distributions for the process indexed by t, while Ciesielski and Kesten [15] established tightness to complete the proof of weak convergence to a limiting Gaussian process. Related results and refinements were proved by Kesten in [14, 24, 29], and [30].

Like a good mystery story, parts of Kesten's early work foreshadow later developments. Inspired by work of Bellman (1954) for i.i.d. sequences, Furstenberg and Kesten [19] considered products ${}^nY^1 = X^n X^{n-1} \cdots X^1$ where the X^i are an ergodic stationary sequence of $k \times k$ matrices, and showed that if $E(\log^+ \|X^1\|) < \infty$, then with probability one

$$\lim_{n\to\infty} n^{-1} \log \|{}^nY^1\| = \lim_{n\to\infty} n^{-1} E \log \|{}^nY^1\|. \tag{1.3.2}$$

Nowadays this is a textbook application of Kingman's (1968) subadditive ergodic theorem. See e.g., pages 367–369 of Durrett (1995b). While the above law of large numbers for the norm of the matrix, $\|{}^nY^1\|$ and related results for the entries ${}^nY^1_{ij}$ are widely known and definitive, the corresponding central limit question has not been much investigated (see [9] and Ishitani (1977)) and still has room for improvement. For more recent work on products of random matrices see Kesten and Spitzer [76], Cohen and Newman (1984), and the collection of papers from a 1984 AMS Summer Research Conference edited by Cohen, Kesten, and Newman (1984).

A second harbinger of the future, again related to subadditivity, is Kesten's work on self-avoiding walks in [22] and [23]. Let χ_n be the number of self-avoiding walks on the integer lattice in d dimensions that start at the origin. The obvious inequality $\chi_m \chi_n \le \chi_{m+n}$ leads one easily to the conclusion that

$$(\chi_n)^{1/n} \to \beta_d \equiv \inf_{m\ge1} (\chi_m)^{1/m}. \tag{1.3.3}$$

It has long been conjectured (see Hammersley (1961) and references therein) that the ratio $\chi_{n+1}/\chi_n \to \beta_d$. Kesten [22] proved more and less than this when he showed that

$$|\chi_{n+2}/\chi_n - \beta_d^2| \le Cn^{-1/3}. \tag{1.3.4}$$

The result in (1.3.3) clearly allows one to compute upper bounds on β_d by calculating χ_m for small values of m. Results about the limit β_d are much harder to come by. Kesten [23] attacked this question by considering $\chi_{n,2r}(d)$ = the number of n-step walks on the integer lattice in d- dimensions with no loops of $2r$ steps or less. He showed that $\beta_{d,2r} = \lim_{n\to\infty}(\chi_{n,2r}(d))^{1/n}$ existed and satisfied $\beta_{d,2r} - \beta_d = O(d^{-r})$ as $d \to \infty$. Taking $r = 2$ leads to an asymptotic expansion

$$\beta_d = 2d - 1 - \frac{1}{2d} + O(1/d^2). \tag{1.3.5}$$

In the three and a half decades since Kesten's paper there has been an explosion of results on self-avoiding random walks and related processes. See the books by

Madras and Slade (1993) and Lawler (1991), and the paper by Lawler on "loop-erased walk" in this volume.

Returning to ordinary random walks on the integer lattice, let $p^k(x, y)$ be the probability of going from x to y in k steps, let T be the time of the first visit to the origin, 0, and let $r_n = P_0(T > n)$. Kesten, Ornstein, and Spitzer [17] proved that for all $x \neq 0$,

$$\lim_{n\to\infty} \frac{P_x(T > n)}{P_0(T > n)} = a(x), \tag{1.3.6}$$

where $a(x) = \sum_{k=0}^{\infty} p^k(0,0) - p^k(x,0)$, which exists by earlier work of Spitzer (1962). Note that the last result holds for ANY random walk. This was the first of many ratio limit theorems for arbitrary random walks. See Kesten and Spitzer [20], and Kesten [21, 38]. These three papers cover more than 100 pages in *Journal d'Analyse Mathématique*, so it would be difficult to even sketch their contents here.

Inspired by the pioneering work of Hunt (1957–1958) developing a potential theory for transient Markov process, the 60's were the golden age of potential theory of random walks. Using the notation of the previous paragraph, this can be defined as the study of the potential kernels $\sum_{k=0}^{\infty} p^k(0,x)$ and $\sum_{k=0}^{\infty} p^k(0,0) - p^k(x,0)$, the former appropriate for the transient and the latter for the recurrent case. Spitzer's beautiful (1964) book contained definitive results for random walks on the d-dimensional integer lattice, which Kesten and Spitzer [26] generalized to countably infinite Abelian groups. Ornstein (1969) and Port and Stone (1969) generalized this work to R^d and to locally compact Abelian groups.

The work of Kesten and Spitzer [26] referred to in the previous paragraph led to Kesten's paper on "The Martin boundary of recurrent random walks on countable groups" which appeared in the Proceedings of the 5th Berkeley Symposium. The breadth and depth of talent at that meeting can be illustrated by noting that Part II of Volume II featured papers by Blumenthal and Getoor, Breiman, Dynkin, Kakutani, Karlin and McGregor, Kesten, Kendall, Kunita and Watanabe, Lamperti, Neveu, Ornstein, Ray, Rosenblatt, Smith, and Spitzer.

Kesten gave a 45 minute talk at the 1970 International Congress in Nice [39] on "Hitting of sets by processes with stationary independent increments." The highlight of that talk was his result announced earlier in the Bulletin of the AMS [36] and presented in detail in a 129 page volume of the Memoirs of the AMS [37] giving necessary and sufficient conditions for processes with independent increments to hit points with positive probability. As Kesten [36] explains in his announcement (see that paper for precise references) he was motivated by earlier work of Lévy, Erdös, Kac, and Port, who resolved the question for symmetric stable processes, and by a convolution equation of Chung that P.A. Meyer had shown was related to the probabilities of hitting points. From the list of people who had worked on the problem and the fact that Neveu and McKean had already published false solutions of Chung's problem, you can see that its solution was an impressive achievement. With his characteristic modesty, Kesten told me when I was quizzing him about some of the details of his early work, that "I was very happy when I was able to solve that problem." Kesten did other work in this general

area concerning "positivity intervals for stable processes" [19], finding results that generalized the arcsine law, and on "Lévy processes with a nowhere dense range" which considered related questions concerning the complement of the range.

Continuing to track Kesten's career through his prestigious lectures, our next stop is his 1971 Rietz Lecture given at the annual meeting of the IMS in Fort Collins Colorado, September 20–23, 1971. As the associated paper [45] indicates, his intent was to survey generalizations or analogues of classical limit theorems which do not make a priori moment or smoothness assumptions on the underlying distribution, e.g., results that hold for any random walk. Three topics were discussed: (i) ratio limit theorems, which we have touched on above; (ii) a concentration function inequality that gives an upper bound on the probability a random walk lies in an interval of length L, and (iii) the set of accumulation points of normalized random walk, i.e, given a normalizing sequence γ_n,

$$A(F, \gamma_n) = \cap_{m=1}^{\infty} \overline{\{S_n/\gamma_n : n \geq m\}} \tag{1.3.7}$$

where the bar denotes closure, and F is the distribution function for a single step.

The main result in category (ii) is the one proved in [35]. Its name is a mouthful: "A sharper form of the Doeblin-Lévy-Kolmogorov-Rogozin inequality" but I have fond memories of it, since it was very useful in Bramson, Durrett, and Swindle (1989) by providing uniform upper bounds for the sequence of random walks we considered. Restricting our attention to the simplest case of one dimensional random walks, the results in (iii) start from the simple observation that if F has finite mean μ, then the law of large numbers implies $A(F, n) = \{\mu\}$. The single points $+\infty$ and $-\infty$ are obviously possible limits. Kesten showed in [34] that if $A(F, n)$ contains at least two points, then it must contain $+\infty$ and $-\infty$. Conversely, any closed set A of $[-\infty, \infty]$ that contains $+\infty$ and $-\infty$ is $A(F, n)$ for some distribution F.

Turning to smaller normalizations, we note that if F has mean 0 and finite variance σ^2, then Strassen's (1964) version of the law of the iterated logarithm implies

$$A(F, (2n \log\log n)^{1/2}) = [-\sigma, \sigma]. \tag{1.3.8}$$

In [41] Kesten solved his own open problem by showing that (assuming F is not a point mass at 0) (a) if $\alpha < 1/2$ and $A(F, n^{-\alpha})$ has a finite limit point, then it contains all real numbers, while (b) if $A(F, n^{-1/2})$ has a finite limit point, then it contains a half line of values $(-\infty; b]$ or $[b, \infty)$. It is natural to conjecture, as Kesten did in [41], that in case (b) $A(F, n^{-1/2}) = [-\infty, \infty]$, but this seems to be an open problem.

Erickson and Kesten [49] introduced the notion of strong limit points of random walks as the set of values $B(F, n^{-\alpha})$ so that $n_k^{-\alpha} S_{n_k} \to b$ for some deterministic sequence n_k. If $\alpha \leq 1/2$ and F is not a point mass at 0, $B(F, n^{-\alpha}) = \emptyset$. Depending upon F and the value of α, $B(F, n^{-\alpha})$ may be $\emptyset$, $\{\infty\}$, $\{-\infty\}$, $[0, \infty]$, $[-\infty, 0]$, or $\mathbf{R}$. Four theorems and five examples in [49] carefully described the possible behaviors, though in words we have heard more than once in Kesten's work "we do not give the proof of Theorem 4 since it is rather lengthy." For more on strong limit points, see Kesten and Maller [144].

Our next topic, Random Walks in Random Environments, are not random walks at all. In the formulation of Solomon's Ph.D. thesis (1975) they are discrete time birth and death chains X_m on the integers in which $x \to x+1$ has probability α_x and $x \to x-1$ has probability $1-\alpha_x$, where the environment α_x is a sequence of independent and identically distributed random variables, and we suppose for simplicity here that $0 < \epsilon \le \alpha_x \le 1-\epsilon < 1$. Let $\sigma = (1-\alpha_0)/\alpha_0$. It is a straightforward exercise in the theory of birth and death chains to show that X_n is recurrent if and only if $E \ln \sigma = 0$. However, this simple problem and some related questions about branching processes in random environments inspired Kesten [48] to produce some very nice results on "random difference equations."

Returning to the original problem, things become very interesting when one considers limit theorems. Solomon (1975) showed that

$$\text{If } E\sigma < 1 \text{ then } \lim_{n\to\infty} X_n/n = (1-E\sigma)/(1+E\sigma).$$

$$\text{If } E(\sigma^{-1}) < 1 \text{ then } \lim_{n\to\infty} X_n/n = -(1-E(\sigma^{-1})/(1+E(\sigma^{-1}).$$

$$\text{If } (E\sigma)^{-1} \le 1 \le E(\sigma^{-1}) \text{ then } \lim_{n\to\infty} X_n/n = 0.$$

Kesten, Kozlov and Spitzer [53] probed the middle ground where $E \ln \sigma < 0$ but $E\sigma \ge 1$. In this case if one defines κ by $E\sigma^\kappa = 1$, then

$$\lim_{t\to\infty} P(t^{-\kappa} X_t \le x) = 1 - L_\kappa(x^{-1/\kappa}), \tag{1.3.9}$$

where L_κ is the stable law with index κ. Other non-normal limit theorems were found for $1 \le \kappa \le 2$ in [53] and generalized by Kesten and Kawazu [77]. Ritter (1976) proved some results about the critical case $E \ln \sigma = 0$ in his thesis, but a complete solution had to wait until Sinai (1982) showed that $(\log n)^{-2} X_n$ converges in distribution to a nondegenerate limit defined in terms of a functional of a Brownian motion associated with the environment. The distribution of the limit was later calculated by Kesten [84].

If one drops the assumption of nearest neighbor jumps in $d=1$, the problem becomes technically more difficult. Key's (1984) thesis gives results for the finite range model in $d=1$. The answers are not as explicit as in the nearest neighbor case since they are most naturally framed in terms of Lyapunov exponents of random matrices, which typically cannot be computed explicitly. While the finite range case in $d=1$ is hard, the nearest neighbor model in $d>1$ proved to be almost impossible. Some remarkably clever arguments were used by Kalikow in his (1981) thesis to prove transience of some lopsided models in $d=2$. However, little was known in $d \ge 2$ until Bricmont and Kupiainen (1991), (1992) used rigorous renormalization group methods to show that the critical dimension was 2, i.e., one has central limit theorem behavior in $d>2$. For an overview of this and related work see the text of Kupiainen's (1990) talk at the International Congress of Mathematicians in Kyoto.

Kesten's two papers with Papanicolaou, [60] and [64], studied a different type of motion in a random environment. The first studied turbulent diffusion, that is solutions to $dx(t)/dt = V(x(t))$ with $V(x) = v + \epsilon F(x)$ where $v \ne 0$, ϵ is small, and F is a mean zero stationary random vector field. When F satisfied suitable hypotheses they were able to show that as $\epsilon \to 0$, $x^\epsilon(t) = x(t/\epsilon^2) - vt/\epsilon^2$ converged to a diffusion process with constant coefficients that came from averaging F. The second paper, [64], used similar methods to study stochastic acceleration $d^2x(t)/dt^2 = \epsilon F(x(t))$.

Readers who recall that Kesten's result "$p_c = 1/2$" is in [67] realize that we have now almost reached the beginning of Kesten's work on percolation and first passage percolation. He still wrote beautiful papers on questions about random walks: [58] proves a conjecture of Erickson to the effect that any genuinely d-dimensional random walk S_n in $d \ge 3$ goes to infinity at least as fast as simple random walk. However, increasingly his work on random walk was motivated by ideas from physics. An example is his work with Spitzer [62] on random walk in random scenery. They studied the limiting behavior of sums of the form $W_n = \sum_{k=1}^n \xi(S_k)$ where S_k is a random walk on the integers and the $\xi(x)$ are i.i.d. and independent of S_k. They found that $n^{-3/4} W_{nt}$ converged weakly to a limit Δ_t that had stationary increments and was self-similar, i.e., Δ_{ct} has the same distribution as $c^{3/4}\Delta_t$.

Physics was not the only science to provide Kesten with problems. He had been for some time interested in models for population growth. See [40, 42, 46], and [55]. In [65] and [66] he studied the number of alleles in the stepwise mutation model. Simulations of Ohta and Kimura had suggested that the number of different alleles $\Lambda(N)$ found in a population of size N remained bounded in distribution as $N \to \infty$, but Kesten showed that it went to infinity very slowly. To state his result, we begin with the rapidly increasing sequence defined by $\gamma_0 = 0$ and $\gamma_{k+1} = \exp(\gamma_k)$ for $k \ge 0$. This begins

$$\gamma_1 = e, \quad \gamma_2 = 15.15, \quad \gamma_3 = 3,814,279, \quad \gamma_4 > 10^{1,656,620}$$

so the inverse function $\lambda(n) = \max\{k : \gamma_k \le n\}$ grows very slowly. Kesten's result says (in the symmetric nearest neighbor case) that

$$P(|\Lambda(N) - \lambda(N)| > \log \lambda(N)) \to 0$$

as $N \to \infty$. A second contact with biology can be seen in his work with Ogura [69] giving recurrence properties of Lotka-Volterra models with random fluctuations.

From biology we move next to the study of river networks. The problem studied in [107] came from a sabbatical visit to Cornell by Ed Waymire. The problem may be formulated in a purely mathematical way as follows. Consider the family tree of a branching process starting from a single progenitor and conditioned to have v edges (total progeny). To each edge e we associate a weight $W(e)$ which we think of as the length of the edge. Interest then focuses on the height of the tree, i.e., the maximum sum of weights that can be achieved by a self- avoiding path starting at the progenitor. Kesten refined the results in [126, 134, 135]. Some of this later work was inspired by connections with Aldous' (1993) continuum random tree and LeGall's (1991) random snake construction of super-processes.

Our next paper [110] involves a sabbatical visit to Cornell by Greg Lawler and a letter from Spataru to Spitzer. Spataru's question, after a little rewriting, asks: Suppose that a casino offers p fair games. Can we make money playing the games (a) according to a fixed schedule or (b) using a strategy that depends on our wealth? Somewhat surprisingly the answer to (a) is yes. Let $\alpha < 1/2$. If we choose p large enough and construct the fair games carefully then our fortune at time n will have $\liminf S_n/n^\alpha = \infty$. In the other direction if the fair games all have finite variance then the answer to (b) is No. A more detailed study of these questions was carried out in [112].

The phenomena in the last paragraph come from the fact that the behavior of the first n steps of a random walk is dictated primarily by the part of the distribution F between $F^{-1}(1/n)$ and $F^{-1}(1-1/n)$ and this truncated distribution may have a mean different from 0. This trimming of the distribution may be used for good rather than evil. In the 70's and 80's statisticians realized that the removal of outliers from a random sample led to robust estimators with reduced variability, and probabilists realized that trimming could produce central limit behavior from distributions with even heavy tails.

Given Kesten's expertise with random walks, it was natural for him to get involved in this area, where much of his work has been done in collaboration with Ross Maller. The eight papers cited in this paragraph total almost 300 journal pages, so we will just mention some random results to arouse the reader's interest. [114] and [115] concern conditions which guarantee that a sum of independent random variables is much larger than the largest summand. These results are of interest in relation to the law of large numbers, central limit behavior, and law of the iterated logarithm. In [115] a necessary and sufficient condition for $P(S_n \geq 0) \to 1$, answering a simple sounding problem first mentioned by Révész. [125] and [131] show that deleting a fixed finite number of terms cannot affect asymptotic normality of normed sums, that is, the trimmed sum has a limit if and only if the untrimmed one does. In the other direction, [128] shows that a fixed trimming can have a significant effect. The original random walk may be recurrent while the trimmed one is transient. [141] studies questions from the renewal theory for random walks with $S_n \to \infty$. [145] and [146] concern random walks crossing curved, e.g., power law, boundaries.

A somewhat less technical problem, at least in its formulation, is the question: can you distinguish sceneries by observing them along a random walk path? Given is a sequence of random variables $\xi_x, x \in Z$ taking values in a finite alphabet and a symmetric nearest neighbor random walk S_n on the integers starting at 0. A robot walks according to S_n and calls out the symbols she sees: $\xi(S_0), \xi(S_1), \ldots$. The question is: can we reconstruct the underlying scenery from this information? Partial results can be found in [138, 139], and [147]. However, it was Harry's student Henry Matzinger who was finally able to answer the question in the affirmative in his Ph.D. thesis. For three or more symbols this involves the pretty idea of using the observed sequence to define a self-intersecting path on a tree in order to disentangle the underlying sequence. We leave it to the reader to fill in the remaining details and to contemplate the more complicated case of an alphabet with two symbols.

In the other direction, if you can't do the case in which the ξ_x are i.i.d. uniform on (0,1), you should not operate a motorized vehicle.

The last stop on our random walk through the theory of random walks is Kesten's recent work [150] with van den Berg on the asymptotic density in a coalescing random walk model. Particles perform continuous time random walks on $\mathbf{Z}^d$ but interact only when a particle jumps onto a site at which there are j particles present, in which case the jumping particle is removed with probability p_j. If we start with at most one particle per site and have $p_1 = 1$, this is classical coalescing random walk. In this case asymptotics for the density are known from work of Sawyer (1979) and Bramson and Griffeath (1980). Kesten and van den Berg show (under some natural assumptions) that in $d \geq 6$ the density of particles $u(t) \sim C(d)/t$. The last sentence should be correct with 6 replaced by 3. For the reader who wants to study this question I have some good news and some bad news: it would even be interesting to extend the methods of [151] to prove the Bramson-Griffeath-Sawyer result in $d \geq 3$.

1.4 Denouement

At this point we have exhausted the author, but not Kesten's publication list. I would like to thank Maury Bramson, Ken Brown, Geoff Grimmett, Harry Kesten, Ross Maller, Chuck Newman, Yuval Peres, and Gordon Slade for reading various drafts and making numerous corrections. They are, of course, responsible for all errors that remain, even in the parts that they never read. I would like to express my appreciation to Harry Kesten, not only for the lessons he gave me in connection with rewriting this paper, but also for his insights and his friendship for the twenty years I have known him. He is not only a brilliant mathematician, but also one of the nicest people you could hope to meet.

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