CUBIC GRAPHS AND THE GOLDEN MEAN

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Abstract. The connective constant $\mu(G)$ of a graph G is the exponential growth rate of the number of self-avoiding walks starting at a given vertex. We investigate the validity of the inequality $\mu \geq \phi$ for infinite, transitive, simple, cubic graphs, where $\phi := \frac{1}{2}(1+\sqrt{5})$ is the golden mean. The inequality is proved for several families of graphs including (i) Cayley graphs of infinite groups with three generators and strictly positive first Betti number, (ii) infinite, transitive, topologically locally finite (TLF) planar, cubic graphs, and (iii) cubic Cayley graphs with two ends. Bounds for μ are presented for transitive cubic graphs with girth either 3 or 4, and for certain quasi-transitive cubic graphs.

1. Introduction

Let G be an infinite, transitive, simple, rooted graph, and let σ_n be the number of n-step self-avoiding walks (SAWs) starting from the root. It was proved by Hammersley [22] in 1957 that the limit $\mu = \mu(G) := \lim_{n\to\infty} \sigma_n^{1/n}$ exists, and he called it the 'connective constant' of G. A great deal of attention has been devoted to counting SAWs since that introductory mathematics paper, and survey accounts of many of the main features of the theory may be found at [1, 21, 29].

A graph is called *cubic* if every vertex has degree 3, and *transitive* if it is vertex-transitive (further definitions will be given in Section 2). Let $\mathcal{G}_d$ be the set of infinite, transitive, simple graphs with degree d, and let $\mu(G)$ denote the connective constant of $G \in \mathcal{G}_d$. The letter ϕ is used throughout this paper to denote the golden mean $\phi := \frac{1}{2}(1+\sqrt{5})$, with numerical value $1.618\cdots$. The basic question to be investigated here is as follows.

Question 1.1 ([18]). *Is it the case that $\mu(G) \geq \phi$ for $G \in \mathcal{G}_3$?*

This question has arisen within the study by the current authors of the properties of connective constants of transitive graphs, see [21] and the references therein. The

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question is answered affirmatively here for certain subsets of $\mathcal{G}_3$, but we have no complete answer to Question 1.1. Note that $\mu(G) \ge \sqrt{d-1} > \phi$ for $G \in \mathcal{G}_d$ with $d \ge 4$, by [18, Thm 1.1].

Here is some motivation for the inequality $\mu(G) \ge \phi$ for $G \in \mathcal{G}_3$. It well known and easily proved that the ladder $\mathbb{L}$ (see Figure 5.1) has connective constant ϕ. Moreover, the number of n-step SAWs can be expressed in terms of the Fibonacci sequence (an explicit such formula is given in [41]). It follows that $\mu(G) \ge \phi$ whenever there exists an injection from a sufficiently large set of rooted n-step SAWs on $\mathbb{L}$ to the corresponding set on G. As domain for such injections, we take the set $\mathbb{W}_n$ of n-step 'eastward' SAWs on the singly infinite ladder $\mathbb{L}_+$ of Figure 5.1 (see Section 5). One of the principal techniques of this article is to construct such injections for certain families of cubic graphs G. We state some of our main results next, and refer the reader to the appropriate sections for the precise terminology in use.

Theorem 1.2. *Let $\mathcal{G}_3$ be the set of connected, infinite, transitive, cubic graphs.*

A. *The connective constant $\mu = \mu(G)$ satisfies $\mu \ge \phi$ whenever one or more of the following holds:*
   (a) *$G \in \mathcal{G}_3$ has a transitive graph height function, (Theorem 3.1(b)),*
   (b) *G is the Cayley graph of the Grigorchuk group with three generators, (Theorem 8.1),*
   (c) *$G \in \mathcal{G}_3$ is topologically locally finite, (Theorem 9.1),*
   (d) *$G \in \mathcal{G}_3$ is the Cayley graph of a finitely presented group with two ends, (Theorem 10.1).*

   Further to the last item, if Γ is a finitely presented group with infinitely many ends, there exists a symmetric set of generators such that the corresponding Cayley graph G satisfies $\mu(G) \ge \phi$, (Theorem 10.2).

B. *$G \in \mathcal{G}_3$ satisfies*
$$\mu \begin{cases} \in (1.529, 1.770) & \text{if } G \text{ has girth } 3, \\ \in (1.513, 1.900) & \text{if } G \text{ has girth } 4. \end{cases}$$
   (Theorems 7.1 and 7.2.)

There are many infinite, transitive, cubic graphs, and we are unaware of a complete taxonomy. Various examples and constructions are described in Section 4 (including the illustrious case of the hexagonal lattice, see [6]), and the inequality $\mu \ge \phi$ is discussed in each case. In our search for cubic graphs, no counterexample has been knowingly revealed. Our arguments can frequently be refined to obtain stronger lower bounds for connective constants than ϕ, but we do not explore that here.

A substantial family of cubic graphs arises through the application of the so-called 'Fisher transformation' to a d-regular graph (see Section 7). We make explicit mention of the Fisher transformation here since it provides a useful technique in the study of connective constants.

The family of Cayley graphs provides a set of transitive graphs of special interest and structure. The Cayley graph of the Grigorchuk group is studied by a tailored argument in Section 8. The case of 2-ended Cayley graphs is handled in Section 10, and also certain ∞-ended Cayley graphs.

This paper is structured as follows. General criteria that imply $\mu \geq \phi$ are presented in Section 3 and proved in Section 5. In Section 4 is given a list of cubic graphs known to satisfy $\mu \geq \phi$ (for some such graphs, the inequality follows from earlier results as noted, and for others by the results of the current article). *Transitive* graph height functions are discussed in Section 6, including sufficient conditions for their existence. Upper and lower bounds for connective constants for cubic graphs with girth 3 or 4 are stated and proved in Section 7. The Grigorchuk group is considered in Section 8. In Section 9, it is proved that $\mu \geq \phi$ for all transitive, topologically locally finite (TLF) planar, cubic graphs. The final Section 10 is devoted to 2- and ∞-ended Cayley graphs. (Theorem 8.1 and much of the proof of Theorem 10.1 are due to Anton Malyshev (personal communication).)

2. Preliminaries

The graphs $G = (V, E)$ of this paper will be assumed to be connected, infinite, and simple. We write $u \sim v$ if $\langle u, v\rangle \in E$, and say that u and v are *neighbours*. The set of neighbours of $v \in V$ is denoted ∂v. The *degree* $\deg(v)$ of vertex v is the number of edges incident to v, and G is called *cubic* if $\deg(v) = 3$ for $v \in V$.

The automorphism group of G is written $\mathrm{Aut}(G)$. A subgroup $\Gamma \leq \mathrm{Aut}(G)$ is said to *act transitively* if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma v = w$. It *acts quasi-transitively* if there is a finite subset $W \subseteq V$ such that, for $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $\gamma v = w$. The graph is called *(vertex-)transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively).

A *walk* w on the (simple) graph G is a sequence $(w_0, w_1, \dots, w_n)$ of vertices w_i such that $n \geq 0$ and $e_i = \langle w_i, w_{i+1}\rangle \in E$ for $i \geq 0$. Its *length* $|w|$ is the number of its edges, and it is called *closed* if $w_0 = w_n$. The distance $d_G(v, w)$ between vertices v, w is the length of the shortest walk of G between them.

An *n-step self-avoiding walk* (SAW) on G is a walk $(w_0, w_1, \dots, w_n)$ of length $n \geq 0$ with no repeated vertices. The walk w is called *non-backtracking* if $w_{i+1} \neq w_{i-1}$ for $i \geq 1$. A *cycle* is a walk $(w_0, w_1, \dots, w_n)$ with $n \geq 3$ such that $w_i \neq w_j$ for $0 \leq i < j < n$ and $w_0 = w_n$. Note that a cycle has a specified orientation. The *girth* of G is the length of its shortest cycle. A *triangle* (respectively, *quadrilateral*) is a cycle of length 3 (respectively, 4).

We denote by $\mathcal{G}$ the set of infinite, rooted, connected, transitive, simple graphs with finite vertex-degrees, and by $\mathcal{Q}$ the set of such graphs with 'transitive' replaced by 'quasi-transitive'. The subset of $\mathcal{G}$ containing graphs with degree d is denoted $\mathcal{G}_d$, and the subset of $\mathcal{G}_d$ containing graphs with girth g is denoted $\mathcal{G}_{d,g}$. A similar notation is valid for $\mathcal{Q}_d$ and $\mathcal{Q}_{d,g}$. The root of such graphs is denoted 0 (or $\mathbf{1}$ when the graph is a Cayley graph of a group with identity $\mathbf{1}$).

Let $\Sigma_n(v)$ be the set of n-step SAWs starting at $v \in V$, and $\sigma_n(v) := |\Sigma_n(v)|$ its cardinality. *Assume that G is connected, infinite, and quasi-transitive.* It is proved in [22, 23] that the limit

$$\mu = \mu(G) := \lim_{n\to\infty} \sigma_n(v)^{1/n}, \qquad v \in V, \tag{2.1}$$

exists, and $\mu(G)$ is called the *connective constant* of G. We shall have use for the SAW *generating function*

$$Z_v(\zeta) = \sum_{\substack{\pi \text{ a SAW} \\ \text{from } v}} \zeta^{|\pi|} = \sum_{n=0}^{\infty} \sigma_n(v)\zeta^n, \qquad v \in V,\ \zeta \in \mathbb{R}. \tag{2.2}$$

By (2.1), each Z_v has radius of convergence $1/\mu(G)$. We shall sometimes consider SAWs joining midpoints of edges of G (in the manner of [6, 15]).

There are two (related) types of graph functions relevant to this work. We recall first the definition of a 'graph height function', as introduced in [16] in the context of connective constants.

Definition 2.1 ([16])**.** *Let $G \in \mathcal{Q}$. A* graph height function *on G is a pair $(h, \mathcal{H})$ such that:*

(a) *$h : V \to \mathbb{Z}$ and $h(0) = 0$,*
(b) *$\mathcal{H}$ is a subgroup of* $\mathrm{Aut}(G)$ *acting quasi-transitively on G such that h is $\mathcal{H}$-*difference-invariant *in the sense that*

$$h(\alpha v) - h(\alpha u) = h(v) - h(u), \qquad \alpha \in \mathcal{H},\ u, v \in V,$$

(c) *for $v \in V$, there exist $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$.*

A graph height function $(h, \mathcal{H})$ of G is called transitive *if $\mathcal{H}$ acts transitively on G.*

The properties of normality and unimodularity of the group $\mathcal{H}$ are discussed in [16], but do not appear to be especially relevant to the current work.

Secondly we remind the reader of the definition of a harmonic function on a graph $G = (V, E)$. A function $h : V \to \mathbb{R}$ is called *harmonic* if

$$h(v) = \frac{1}{\deg(v)} \sum_{u \sim v} h(u), \qquad v \in V.$$

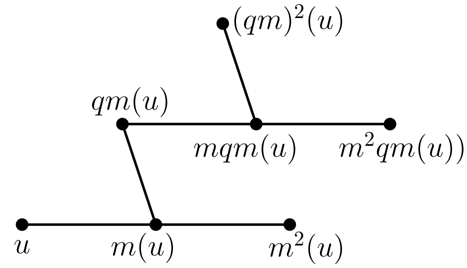

FIGURE 3.1. An illustration of the notation of equations (3.2)–(3.3).

Cayley graphs of finitely generated groups make appearances in this paper, and the reader is referred to [19, 20] for background material on such graphs. We denote by $\mathbf{1}$ the identity of any group Γ under consideration.

3. General results

Let $G = (V, E)$ be an infinite, connected graph. For $h : V \to \mathbb{R}$, we define two functions $m : V \to V$ and $M : V \to \mathbb{R}$, depending on h, by

$$m(u) \in \operatorname{argmax}\{h(x) - h(u) : x \sim u\}, \quad M_u = h(m(u)) - h(u), \qquad u \in V. \tag{3.1}$$

There may be more than one candidate vertex for $m(u)$, and hence more than one possible value for the term $M_{m(u)}$. If so, we make a choice for the value $m(u)$, and we fix $m(u)$ thereafter. Let $q(v)$ denote the unique neighbour of $v := m(u)$ other than u and $m(v)$. We shall apply the functions m and q repeatedly, and shall omit parentheses in that, for example, $mqm(u)$ denotes the vertex $m(q(m(u)))$, and $(qm)^2(u)$ denotes $qmqm(u)$. This notation is illustrated in Figure 3.1.

Let $\mathcal{Q}_{\rm harm} \subseteq \mathcal{Q}_3$ be the subset of graphs G with the following properties: there exists $h : V \to \mathbb{R}$ such that h is harmonic and, for $u \in V$,

$$M_{m(u)} - M_u < \min\{M_u, M_{qm(u)}\}, \tag{3.2}$$

$$2M_{qm(u)} > M_{m(u)} - M_u + M_{mqm(u)}. \tag{3.3}$$

Although inequalities (3.2) and (3.3) lack obvious motivation, they turn out to be useful (see Theorem 3.1) in establishing certain cases of the inequality $\mu(G) \ge \phi$. We note two consequences of (3.2) and (3.3):

(a) since h is assumed harmonic, we have $M_u \ge 0$ for $u \in V$, and hence $M_u > 0$ by (3.2),
(b) it is proved at (5.6) that, subject to (3.2) and (3.3),
$$h(qm(u)) > h(u),\ h(mqm(u)) > h(m(u)),\ h((qm)^2(u)) > h(m(u)),$$
whence $qm(u) \neq u$, $mqm(u) \neq m(u)$, $(qm)^2(u) \neq m(u)$.

Conditions (3.2)–(3.3) will be used in the proof of part (a) of the following theorem. Less obscure but still sufficient conditions are contained in Remark 3.2, following.

Theorem 3.1. *We have that $\mu(G) \ge \phi$ if any of the following hold.*

(a) $G \in \mathcal{Q}_{\mathrm{harm}}$.
(b) $G \in \mathcal{G}_3$ *has a transitive graph height function.*
(c) $G \in \mathcal{Q}_{3,g}$ *where* $g \ge 3$, *and there exists a function* $h : V(G) \to \mathbb{R}$ *such that, for* $u \in V$,

$$h(qm(u)) > h(u), \quad h(mqm(u)) > h(m(u)), \tag{3.4}$$

$$h((qm)^\gamma q(u)) > h(u), \tag{3.5}$$

where $\gamma = \lceil \frac{1}{2}(g-1) \rceil$.

(d) $G \in \mathcal{Q}_{3,g}$ *where* $g \ge 3$, *and there exists a harmonic function* h *on* G *satisfying* (3.2) *and* (3.5).

Remark 3.2. *Condition* (3.2) *holds whenever there exists* $A > 0$ *and a harmonic function* $h : V \to \mathbb{R}$ *such that, for* $u \in V$, $A < M_u \le 2A$. *Similarly, both* (3.2) *and* (3.3) *hold whenever there exists* $A > 0$ *such that, for* $u \in V$, $2A < M_u \le 3A$.

Let Γ be an infinite, finitely presented group, and let G be a locally finite Cayley graph of Γ. If there exists a surjective homomorphism F from Γ to $\mathbb{Z}$, then F is a transitive graph height function on G (see [19]). Such a graph height function is called a *group height function*.

Example 3.3. *Here are some examples of Theorem 3.1 in action.*

(a) *The hexagonal lattice* $\mathbb{H}$ *supports a harmonic function* h *with* $M_u \equiv 1$, *so that part (a) of the theorem applies (see Remark 3.2). To see this, we embed* $\mathbb{H}$ *into the plane as in the dashed lines of Figure 3.2. Let each edge have length* 1, *and let* $h(u)$ *be the horizontal coordinate of the vertex* u.
(b) *The Cayley graph of a finitely presented group* $\Gamma = \langle S \mid R \rangle$ *with* $|S| = 3$ *has a transitive graph height function whenever it has a group height function, and hence part (b) applies. See Theorem 6.3 for a sufficient condition on a transitive cubic graph to possess a transitive graph height function.*
(c) *The Archimedean lattice* $\mathbb{A} = [\![4, 6, 12]\!]$ *lies in* $\mathcal{Q}_{3,4}$ *and possesses a harmonic function satisfying* (3.2) *and* (3.3). *The harmonic function in question is illustrated in Figure 6.1, and the claimed inequalities may be checked from the figure. See also Remark 9.8.*
(d) *A further way of proving* $\mu(\mathbb{A}) \ge \phi$ *is as follows. The lattice* $\mathbb{A}$ *can be embedded into the plane as in the solid lines of Figure 3.2. As in (a) above, let* $h(u)$ *be the horizontal coordinate of* u. *By Theorem 3.1(c), the connective constant is at least* ϕ.

The proof of Theorem 3.1 is found in Section 5.

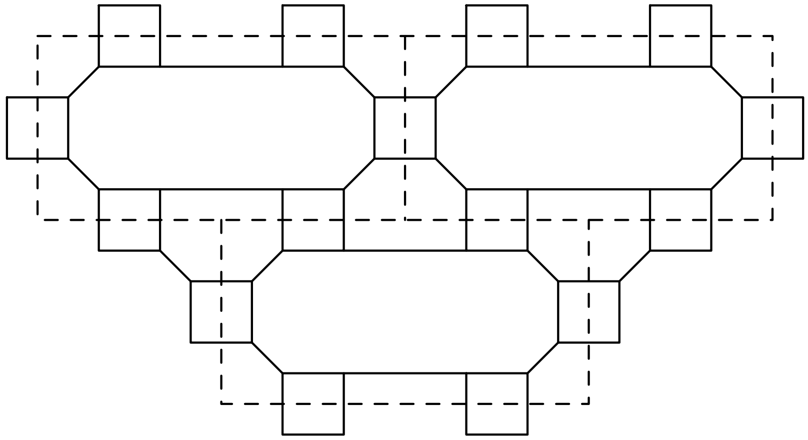

FIGURE 3.2. The dashed lines form the hexagonal lattice $\mathbb{H}$, and the solid lines the $[\![4,6,12]\!]$ lattice

4. Examples of infinite, transitive, cubic graphs

4.1. **Cubic graphs with $\mu \geq \phi$.** We list here examples of infinite, cubic graphs G with $\mu(G) \geq \phi$. As mentioned earlier, we have no example that violates the inequality (however, see Section 4.2). A number of these examples are well known, and others have been studied by other authors. In some cases, Theorem 3.1 may be applied, and such cases are prefixed by the part of the theorem that applies. Most of these examples are transitive, and all are quasi-transitive.

A. (b) The 3-regular tree has connective constant 2.
B. (a) The 'ladder' $\mathbb{L}$ (see Figure 5.1) has $\mu = \phi$. This exact value is elementary and well known; see, for example, [18, p. 284].
C. The 'twisted ladder' $\mathbb{TL}$ (see Figure 5.2) has $\mu = \sqrt{1+\sqrt{3}}) \approx 1.653 > \phi$. To see this, observe that the generating function of SAWs from 0 (see (2.2)) that move only rightwards or within quadrilaterals is $Z(\zeta) = \sum_{m=0}^{\infty} f(\zeta)^m$, where $f(\zeta) = 2\zeta^2 + 2\zeta^4$. The radius of convergence, $1/\mu(\mathbb{TL})$, of Z is the root of the equation $f(\zeta) = 1$.
D. (a) The hexagonal lattice $\mathbb{H}$ satisfies $\mu(\mathbb{H}) \geq \phi$, by Example 3.3(a). It has been proved in [6] that $\mu = \sqrt{2+\sqrt{2}}$.
E. (a) It is explained in [17, Ex. 4.2] that the square/octagon lattice $[\![4,8,8]\!]$ satisfies $\mu > \phi$.
F. (a, c) The Archimedean $[\![4,6,12]\!]$ lattice has connective constant at least ϕ. See Example 3.3(c, d) and Remark 9.8.

G. (b) The Cayley graph of the lamplighter group has a so-called group height function, and hence a transitive graph height function. See Example 3.3(b) and [19, Ex. 5.3].

H. The following examples concern so-called Fisher graphs (see [15] and Section 7). For $G \in \mathcal{G}_3$, the *Fisher graph* G_{F} ($\in \mathcal{Q}_3$) is obtained by replacing each vertex by a triangle. It is shown at [15, Thm 1] that the value of $\mu(G_{\mathrm{F}})$ may be deduced from that of $\mu(G)$, and furthermore that $\mu(G_{\mathrm{F}}) > \phi$ whenever $\mu(G) > \phi$.

I. In particular, the Fisher graph $\mathbb{H}_{\mathrm{F}}$ of $\mathbb{H}$ satisfies $\mu(\mathbb{H}_{\mathrm{F}}) > \phi$.

J. The Archimedean lattices mentioned above are the hexagonal lattice $\mathbb{H} = [\![6, 6, 6]\!]$, the square/octagon lattice $[\![4, 8, 8]\!]$, together with $[\![4, 6, 12]\!]$, and $\mathbb{H}_{\mathrm{F}} = [\![3, 12, 12]\!]$. To this list we may add the ladder $\mathbb{L} = [\![4, 4, \infty]\!]$.

   These are examples of so-called transitive, TLF-planar graphs [33], and all such graphs are shown in Section 9 to satisfy $\mu \geq \phi$.

K. More generally, if $G \in \mathcal{G}_d$ where $d \geq 3$, and

$$\frac{1}{\mu(G)} \leq \begin{cases} \dfrac{1}{\phi^r} & \text{if } d = 2r + 1, \\ \dfrac{2}{\phi^{r+1}} & \text{if } d = 2r, \end{cases}$$

   then its (generalized) Fisher graph satisfies $\mu(G_{\mathrm{F}}) \geq \phi$. See Proposition 7.4.

   Since $\mu \leq d - 1$, the above display can be satisfied only if $d \leq 10$.

L. The Cayley graph G of the group $\Gamma = \langle S \mid R \rangle$, where $S = \{a, b, c\}$ and $R = \{c^2, ab, a^3\}$, is the Fisher graph of the 3-regular tree, and hence $\mu(G) > \phi$. The exact value of $\mu(G)$ may be calculated by [15, Thm 1] (see also Proposition 7.4(a) and [8, Ex. 5.1]).

   We note that the $[\![3, 12, 12]\!]$ lattice is a quotient graph of G by adding the further relator $(ac)^6$. Since the last lattice has connective constant at least ϕ, so does G (see [17, Cor. 4.1]).

M. The Cayley graph G of the group $\Gamma = \langle S \mid R \rangle$, where $S = \{a, b, c\}$ and $R = \{a^2, b^2, c^2, (ac)^2\}$, is the generalized Fisher graph of the 4-regular tree. The connective constant $\mu(G)$ may be calculated exactly, as in Theorem 7.3, and satisfies $\mu > \phi$.

   Since the ladder $\mathbb{L}$ is the quotient graph of G obtained by adding the further relator $(bc)^2$, we have by [17, Cor. 4.1] that $\mu(G) > \phi$. (see [17]).

N. The Cayley graph of the Grigorchuk group with three generators has $\mu \geq \phi$. The proof uses a special construction due to Malyshev based on the orbital Schreier graphs, and is presented in Section 8.

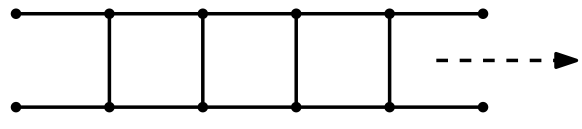

FIGURE 5.1. The singly infinite ladder $\mathbb{L}_+$. The doubly infinite ladder $\mathbb{L}$ extends to infinity both leftwards and rightwards.

O. (b) A group height function of a Cayley graph is also a transitive graph height function (see [19]). Therefore, any cubic Cayley graph with a group height function satisfies $\mu \geq \phi$.

P. (b) Let $G \in \mathcal{G}_3$ be such that: there exists $\mathcal{H} \leq \mathrm{Aut}(G)$ that acts transitively but is not unimodular. By [19, Thm 3.5], T has a transitive graph height function, whence $\mu \geq \phi$.

4.2. **Open question.** We mention a general situation in which we are unable to show that $\mu \geq \phi$. Let G be the Cayley graph of an infinite, finitely generated, virtually abelian group $\Gamma = \langle S \mid R \rangle$ with $|S| = 3$. Is it generally true that $\mu(G) \geq \phi$? Whereas such groups are abelian-by-finite, the finite-by-abelian case is fairly immediate (see Theorem 6.6).

A method for constructing such graphs was described by Biggs [2, Sect. 19] and developed by Seifter [34, Thm 2.2]. Cayley graphs with two or more ends are considered in Section 10.

5. Proof of Theorem 3.1

We begin with some notation that will be used throughout this article. Let $\mathbb{L}_+$ be the singly-infinite ladder of Figure 5.1. An *eastward* SAW on $\mathbb{L}_+$ is a SAW starting at 0 that, at each stage, steps either to the right (that is, *horizontally*, denoted H) or between layers (that is, *vertically*, denoted V). Note that the first step of an eastward walk is necessarily H, and every V step is followed by an H step. Let $\mathbb{W}_n$ be the set of n-step eastward SAWs on $\mathbb{L}_+$.

It is clear that $\mathbb{W}_n$ is in one–one correspondence with the set of n-letter words w in the alphabet $\{\mathrm{H}, \mathrm{V}\}$ that start with the letter H, and having no pair of consecutive appearances of the letter V. We shall frequently consider $\mathbb{W}_n$ as this set of words, and we shall make use of the set $\mathbb{W}_n$ throughout this paper.

It is elementary, by considering the first two steps, that $\eta_n = |\mathbb{W}_n|$ satisfies the recursion

$$\eta_n = \eta_{n-1} + \eta_{n-2}, \qquad n \geq 2,$$

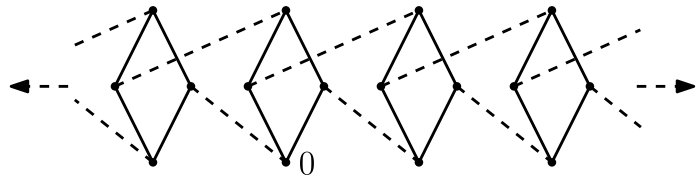

FIGURE 5.2. The doubly infinite 'twisted ladder' $\mathbb{TL}$ is obtained from the ladder by twisting every other quadrilateral.

with $\eta_0 = \eta_1 = 1$. Therefore,

$$\lim_{n\to\infty} \eta_n^{1/n} = \phi. \tag{5.1}$$

Let $G = (V, E) \in \mathcal{Q}_3$, and let $\mathcal{W}_n$ denote the set of n-step walks starting at the root 0. Let $h : V \to \mathbb{R}$. We shall first construct an injection $f : \mathbb{W}_n \to \mathcal{W}_n$, and then we will show that, subject to appropriate conditions, each $f(w)$ is a SAW. In advance of giving the formal definition of f, we explain it informally. When thinking of an element of $\mathbb{W}_n$ as a word of length n, we apply the function m at every appearance of H, and q at every appearance of V; for example, the word HVHH corresponds to the vertex $m^2qm(0)$.

Definition 5.1. *For $w = (w_1w_2\cdots w_n) \in \mathbb{W}_n$, we let $f(w) = (f_0, f_1, \dots, f_n)$ be the n-step walk on G given as follows.*

1. *$f_0 = 0$, $f_1 = m(f_0)$.*
2. *Assume $k \ge 1$ and $(f_0, f_1, \dots, f_k)$ have been defined.*
   (a) *If $w_{k+1} = \mathrm{H}$, then $f_{k+1} = m(f_k)$.*
   (b) *If $w_{k+1} = \mathrm{V}$, then $f_{k+1} = q(f_k)$.*

Lemma 5.2. *The function f is an injection from $\mathbb{W}_n$ to $\mathcal{W}_n$.*

Proof. Let $w, w' \in \mathbb{W}_n$ satisfy $w \ne w'$, and let l be such that $w_i = w'_i$ for $1 \le i < l$, and $w_l = \mathrm{H}$, $w'_l = \mathrm{V}$. It is necessarily the case that $l \ge 2$ and $w_{l-1} = w'_{l-1} = \mathrm{H}$. We have that $f_i(w) = f_i(w')$ for $1 \le i < l$, and

$$f_l(w) = m^2(u), \qquad f_l(w') = qm(u),$$

where $u = f_{l-2}(w)$. Since $m^2(u) \ne qm(u)$, we have $f(w) \ne f(w')$ as required. □

Proof of Theorem 3.1(a). Let $G = (V, E) \in \mathcal{Q}_{\mathrm{harm}}$, and let $h : V \to \mathbb{R}$ be harmonic such that (3.2)–(3.3) hold.

Lemma 5.3. *The function f is an injection from $\mathbb{W}_n$ to $\Sigma_n(0)$.*

Proof. In the light of Lemma 5.2, it suffices to show that each $f(w)$ is a SAW.

Let $u \in V$. The three neighbours of $m(u)$ are u, $qm(u)$, $m^2(u)$ (see Figure 3.1). Since h is harmonic,

$$3h(m(u)) = h(u) + h(qm(u)) + h(m^2(u)), \qquad u \in V,$$

so that

$$h(qm(u)) - h(m(u)) = M_u - M_{m(u)}. \tag{5.2}$$

Therefore, by (3.2),

$$\begin{aligned} h(qm(u)) - h(u) &= M_u - M_{m(u)} + \big[h(m(u)) - h(u)\big] \\ &= 2M_u - M_{m(u)} > 0, \end{aligned} \tag{5.3}$$

$$\begin{aligned} h(mqm(u)) - h(m(u)) &= M_u - M_{m(u)} + \big[h(mqm(u)) - h(qm(u)\big] \\ &= M_u - M_{m(u)} + M_{qm(u))} > 0, \end{aligned} \tag{5.4}$$

and, by (5.2) with $m(u)$ replaced by $mqm(u)$, and (3.3),

$$\begin{aligned} h((qm)^2(u)) - h(m(u)) &= M_u - M_{m(u)} + M_{qm(u)} + \big(M_{qm(u)} - M_{mqm(u)}\big) \\ &> 0. \end{aligned} \tag{5.5}$$

See Figure 3.1 again. By (5.3)–(5.5),

$$qm(u) \neq u, \qquad mqm(u) \neq m(u), \qquad (qm)^2(u) \neq m(u). \tag{5.6}$$

Let $w \in \mathbb{W}_n$. Let S_k be the statement that

(a) $f_0, f_1, \dots, f_k$ are distinct, and
(b) if $w_k = \mathrm{H}$, then $h(f_k) > h(f_i)$ for $0 \le i \le k-1$, and
(c) if $w_k = \mathrm{V}$, then $h(f_k) > h(f_i)$ for $0 \le i \le k-2$.

If S_k holds for every k, then the f_k are distinct, whence $f(w)$ is a SAW. We shall prove the S_k by induction.

Evidently, S_0 and S_1 hold. Let $K \ge 2$ be such that S_k holds for $k < K$, and consider S_K.

1. Suppose first that $w_K = \mathrm{V}$, so that $w_{K-1} = \mathrm{H}$. By (5.3) (or (5.6)) with $u = f_{K-2}$ and $v = m(f_{K-2}) = f_{K-1}$, we have that $h(f_K) > h(f_{K-2})$.
   (a) If $w_{K-2} = \mathrm{H}$, the claim follows by S_{K-2}.
   (b) Assume $w_{K-2} = \mathrm{V}$ (so that, in particular, $K \ge 4$). We need also to show that $h(f_K) > h(f_{K-3})$. In this case, we take $u = f_{K-4}$ so that $m(u) = m(f_{K-4}) = f_{K-3}$, and $mqm(u) = f_{K-1}$ in (5.5), thereby obtaining that $h(f_K) > h(f_{K-3})$ as required.
2. Assume next that $w_K = \mathrm{H}$.
   (a) If $w_{K-1} = \mathrm{H}$, the relevant claims of S_K follow by S_{K-1} and the fact that $f_K = m(f_{K-1})$.

(b) If $w_{K-1} = \mathrm{V}$, then $w_{K-2} = \mathrm{H}$. By (5.4), $h(f_K) > h(f_{K-2})$, and the claim follows by S_{K-1} and S_{K-2}.

This completes the induction, and the lemma is proved. $\square$

By Lemma 5.3, $|\Sigma_n(0)| \geq |\mathbb{W}_n|$, and part (a) follows by (5.1). $\square$

Proof of Theorem 3.1(b). Let $G \in \mathcal{G}_3$ and let $(h, \mathcal{H})$ be a transitive graph height function. For $u \in V$, let $M = \max\{h(v) - h(u) : v \sim u\}$ as in (3.1). We have that $M > 0$ and, by transitivity, M does not depend on the choice of u. Since h is $\mathcal{H}$-difference-invariant, the neighbours of any $v \in V$ may be listed as v_1, v_2, v_3 where

$$h(v_i) - h(v) = \begin{cases} M & \text{if } i = 1, \\ -M & \text{if } i = 2, \\ \eta & \text{if } i = 3, \end{cases}$$

where η is a constant satisfying $|\eta| \leq M$. By the transitive action of $\mathcal{H}$, we have that $-\eta \in \{-M, \eta, M\}$, whence $\eta \in \{-M, 0, M\}$.

If $\eta = 0$, h is harmonic and satisfies (3.2)–(3.3), and the claim follows by part (a). If $\eta = M$, it is easily seen that the construction of Definition 5.1 results in an injection from $\mathbb{W}_n$ to $\Sigma_n(v)$. If $\eta = -M$, we replace h by $-h$ to obtain the same conclusion. $\square$

Proof of Theorem 3.1(c). This is a variant of the proof of part (a). With $\gamma = \lceil \frac{1}{2}(g-1) \rceil$ as in the theorem, let T_k be the statement that

(a) $f_0, f_1, \dots, f_k$ are distinct, and
(b) if $w_k = \mathrm{H}$, then $h(f_k) > h(f_i)$ for $0 \leq i \leq k-1$, and
(c) if $w_k = \mathrm{V}$, then $h(f_k) > h(f_i)$ if i satisfies either
  (i) $i = k - 2s \geq 0$ for $s \in \mathbb{N}$, or
  (ii) $i = k - (2t+1) \geq 0$ for $t \in \mathbb{N}$, $t \geq \gamma$.

Lemma 5.4. *Assume that T_k holds for every k. The f_k are distinct, so that each $f(w)$ is a SAW.*

Part (c) follows from this by Lemma 5.2, as in the proof of part (a).

Proof of Lemma 5.4. If $w_k = \mathrm{H}$ then, by T_k, $f_k \neq f_0, f_1, \dots, f_{k-1}$. Assume that $w_k = \mathrm{V}$. By T_k, we have that $f_k \neq f_i$ for $0 \leq i < k$ except possibly for the values $i \in I := \{k-1, k-3, \dots, k-(2\gamma-1)\}$. If $f_k = f_i$ with $i \in I$, then G has girth not exceeding $2\gamma - 1$, a contradiction. $\square$

We next prove the T_k by induction. Evidently, T_0 and T_1 hold. Let $K \geq 2$ be such that T_k holds for $k < K$, and consider T_K.

Suppose first that $w_K = \mathrm{V}$, so that $w_{K-1} = \mathrm{H}$. By (3.4) with $u = f_{K-2}$,

$$h(f_K) > h(f_{K-2}). \tag{5.7}$$

(a) Assume $w_{K-2} = \mathrm{H}$. By (5.7) and T_{K-2}, we have that $h(f_K) > h(f_i)$ for $i \le K-2$.

(b) Assume $w_{K-2} = \mathrm{V}$ (so that, in particular, $K \ge 4$). By T_{K-2},

$$\begin{aligned} &h(f_{K-2}) > h(f_{K-2-2s}), &&\text{for } s \in \mathbb{N},\ K-2-2s \ge 0,\\ &h(f_{K-2}) > h(f_{K-2-(2t+1)}), &&\text{for } t \ge \gamma,\ K-2-(2t+1) \ge 0. \end{aligned}$$

Hence, by (5.7),

$$\begin{aligned} &h(f_K) > h(f_{K-2s}), &&\text{for } s \in \mathbb{N},\ K-2s \ge 0,\\ &h(f_K) > h(f_{K-(2t+3)}), &&\text{for } t \ge \gamma,\ K-2-(2t+1) \ge 0. \end{aligned} \tag{5.8}$$

It remains to show that

$$h(f_K) > h(f_{K-(2\gamma+1)}). \tag{5.9}$$

Exactly one of the following two cases occurs.

(i) There are two (or more) consecutive appearances of H in $w_K, \dots, w_{K-2\gamma}$. In this case there exists $1 \le t \le \gamma$ such that $w_{K-2t} = \mathrm{H}$, implying by T_{K-2t} that

$$h(f_{K-2t}) > h(f_i), \qquad 0 \le i \le K-2t-1.$$

Inequality (5.9) follows by (5.8).

(ii) We have that $(w_K, \dots, w_{K-2\gamma}) = (\mathrm{V,H,V,H,\dots,V})$, in which case (5.9) follows from (3.5).

Assume next that $w_K = \mathrm{H}$.

(a) If $w_{K-1} = \mathrm{H}$, the relevant claims of T_K follow by T_{K-1} and the fact that $f_K = m(f_{K-1})$.

(b) If $w_{K-1} = \mathrm{V}$, then $w_{K-2} = \mathrm{H}$. By (3.4) and T_{K-2}, $h(f_K) > h(f_{K-2}) > h(f_i)$ for $0 \le i \le K-3$. Finally, $h(f_K) > h(f_{K-1})$ since $f_K = m(f_{K-1})$.

This completes the induction. □

Proof of Theorem 3.1(d). It suffices by part (c) to show that the harmonic function h satisfies (3.4). This holds as in (5.3) and (5.4). □

6. Transitive graph height functions

By Theorem 3.1(b), the possession of a transitive graph height function suffices for the inequality $\mu(G) \ge \phi$. It is not currently known exactly which $G \in \mathcal{G}_3$ possess graph height functions, and it is shown in [20, Thm 5.1] that the Cayley graph of the Grigorchuk group has no graph height function at all. We pose a weaker question here. Suppose $G \in \mathcal{G}_3$ possesses a graph height function $(h, \mathcal{H})$. Under what further condition does G possess a *transitive* graph height function? A natural candidate function $g : V \to \mathbb{Z}$ is obtained as follows.

Proposition 6.1. *Let Γ act transitively on $G = (V, E) \in \mathcal{G}_d$ where $d \geq 3$. Assume that $(h, \mathcal{H})$ is a graph height function of G, where $\mathcal{H} \trianglelefteq \Gamma$ and $[\Gamma : \mathcal{H}] < \infty$. Let $\kappa_i \in \Gamma$ be representatives of the cosets, so that $\Gamma/\mathcal{H} = \{\kappa_i \mathcal{H} : i \in I\}$, and let*

$$g(v) = \sum_{i \in I} h(\kappa_i v), \qquad v \in V. \tag{6.1}$$

The function $g : V \to \mathbb{Z}$ is Γ-difference-invariant.

A variant of the above will be useful in the proof of Theorem 10.1.

Proof. The function g is given in terms of the representatives κ_i of the cosets, but its differences $g(v) - g(u)$ do not depend on the choice of the κ_i. To see this, suppose κ_1 is replaced in (6.1) by some $\kappa_1' \in \kappa_1 \mathcal{H}$. Since $\mathcal{H}$ is a normal subgroup, $\kappa_1' = \eta\kappa_1$ for some $\eta \in \mathcal{H}$. The new function g' satisfies

$$g'(v) - g(v) = h(\kappa_1' v) - h(\kappa_1 v) = h(\eta\kappa_1 v) - h(\kappa_1 v),$$

so that

$$[g'(v) - g'(u)] - [g(v) - g(u)] = [h(\eta\kappa_1 v) - h(\kappa_1 v)] - [h(\eta\kappa_1 u) - h(\kappa_1 u)] = 0,$$

since $\eta \in \mathcal{H}$ and h is $\mathcal{H}$-difference-invariant.

We show as follows that g is Γ-difference-invariant. Let $\alpha \in \Gamma$, and write $\alpha = \kappa_j \eta$ for some $j \in I$ and $\eta \in \mathcal{H}$. Since $\Gamma/\mathcal{H}$ can be written in the form $\{\kappa_i \kappa_j \mathcal{H} : i \in I\}$,

$$\begin{aligned} g(\alpha v) - g(\alpha u) &= \sum_i \Big[h(\kappa_i \kappa_j \eta v) - h(\kappa_i \kappa_j \eta u) \Big] \\ &= g(\eta v) - g(\eta u) \\ &= g(v) - g(u), \end{aligned}$$

since g is $\mathcal{H}$-difference-invariant. □

If the function g of (6.1) is non-constant, it follows that $(g - g(0), \Gamma)$ is a transitive graph height function, implying by Theorem 3.1(b) that $\mu(G) \geq \phi$. This is not invariably the case, as the following example indicates.

Example 6.2. *Consider the Archimedean lattice $\mathbb{A} = [\![4, 6, 12]\!]$ of Figure 6.1. Then $\mathbb{A}$ is transitive and cubic, but it has no transitive graph height function. This is seen by examining the structure of $\mathbb{A}$. There are a variety of ways of showing $\mu(\mathbb{A}) \geq \phi$, and we refer the reader to Theorem 3.1 and the stronger inequality of Remark 9.8.*

Theorem 6.3. *Let Γ act transitively on $G = (V, E) \in \mathcal{G}_3$. Let $(h, \mathcal{H})$ be a graph height function of G, where $\mathcal{H} \trianglelefteq \Gamma$ and $[\Gamma : \mathcal{H}] < \infty$. Pick $\kappa_i \in \Gamma$ such that $\Gamma/\mathcal{H} = \{\kappa_i \mathcal{H} : i \in I\}$, and let $g : V \to \mathbb{Z}$ be given by (6.1). If there exists a constant $C < \infty$ such that*

$$d_G(v, \kappa_i v) \leq C, \qquad v \in V,\ i \in I, \tag{6.2}$$

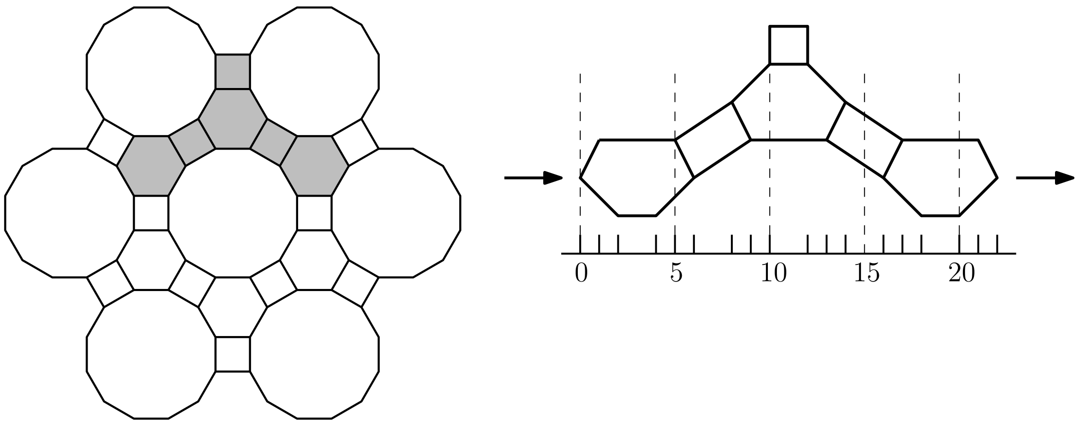

FIGURE 6.1. The left figure depicts part of the Archimedean lattice $\mathbb{A} = [\![4, 6, 12]\!]$. Potentials may be assigned to the vertices as illustrated in the right figure, and the potential differences are duplicated by translation, and by reflection in a horizontal axis. The resulting harmonic function satisfies (3.2).

then $(g - g(0), \Gamma)$ *is a transitive graph height function.*

Proof. By the comment prior to Example 6.2, we need to show that g is non-constant. Since $(h, \mathcal{H})$ is a graph height function, we may pick $v \in V$ such that $h(v) > 2C\delta$, where

$$\delta := \max\{|h(v) - h(u)| : u \sim v\}.$$

By (6.2),

$$[h(\kappa_i v) - h(\kappa_i)] \in [h(v) - h(\mathbf{1})] + [-2C\delta, 2C\delta].$$

Therefore, by (6.1),

$$[g(v) - g(\mathbf{1})] \in |I|h(v) + \big[-2C\delta|I|, 2C\delta|I|\big],$$

so that $g(v) > g(\mathbf{1})$ as required. □

Corollary 6.4. *Let* $\Gamma = \langle S \mid R \rangle$ *be an infinite, finitely-generated group. Let* $\mathcal{H} \trianglelefteq \Gamma$ *be a finite-index normal subgroup, and let* $(h, \mathcal{H})$ *be a graph height function of the Cayley graph* G *(so that it is a 'strong' graph height function, see [19]). Pick* $\kappa_i \in \Gamma$ *such that* $\Gamma/\mathcal{H} = \{\kappa_i \mathcal{H} : i \in I\}$*, and let* $g : V \to \mathbb{Z}$ *be given by (6.1). If*

$$\max_{1 \le i \le k} \big|[\kappa_i]\big| < \infty, \tag{6.3}$$

where $[\kappa_i] = \{g^{-1}\kappa_i g : g \in \Gamma\}$ *is the conjugacy class of* κ_i*, then* $(g - g(0), \Gamma)$ *is a transitive graph height function.*

Proof. Since $d_G(g, \kappa_i g) = d_G(0, g^{-1}\kappa_i g)$, condition (6.2) holds by (6.3). □

Example 6.5. *An* FC-group *is a group all of whose conjugacy classes are finite (see, for example,* [37]*). Clearly,* (6.3) *holds for FC-groups.*

We note a further situation in which there exists a transitive graph height function.

Theorem 6.6. *Let Γ act transitively on $G = (V, E) \in \mathcal{G}_d$ where $d \geq 3$, and let $(h, \mathcal{H})$ be a graph height function on G. If there exists a short exact sequence $\mathbf{1} \to K \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} \mathcal{H} \to \mathbf{1}$ with $|K| < \infty$, then G has a transitive graph height function.*

Proof. Suppose such an exact sequence exists. Fix a root $v_0 \in V$, find $\gamma \in \Gamma$ such that $v = \gamma v_0$, and define $g(v) := h(\beta_\gamma v_0)$.

Certainly $g(v_0) = 0$ and g is non-constant. It therefore suffices to show that g is Γ-difference-invariant. Let $u, v \in V$ and find $\gamma' \in \Gamma$ such that $u = \gamma' v_0$. For $\rho \in \Gamma$,

$$\begin{aligned} g(\rho v) - g(\rho u) &= h(\beta_{\rho\gamma} v_0) - h(\beta_{\rho\gamma'} v_0) \\ &= h(\beta_\rho \beta_\gamma v_0) - h(\beta_\rho \beta_{\gamma'} v_0) \\ &= h(\beta_\gamma v_0) - h(\beta_{\gamma'} v_0) \qquad \text{since } \beta_\rho \in \mathcal{H} \\ &= g(v) - g(u), \end{aligned}$$

and the proof is complete. □

7. Graphs with girth 3 or 4

We recall the subset $\mathcal{G}_{d,g}$ of $\mathcal{G}$ containing graphs with degree d and girth g. Our next theorem is concerned with $\mathcal{G}_{3,3}$, and the following (Theorem 7.2) with $\mathcal{G}_{3,4}$.

Theorem 7.1. *For $G \in \mathcal{G}_{3,3}$, we have that*

$$x_1 \leq \mu(G) \leq x_2, \tag{7.1}$$

where $x_1, x_2 \in (1, 2)$ satisfy

$$\frac{1}{x_1^2} + \frac{1}{x_1^3} = \frac{1}{\sqrt{2}}, \tag{7.2}$$

$$\frac{1}{x_2^2} + \frac{1}{x_2^3} = \frac{1}{2}. \tag{7.3}$$

Moreover, the upper bound x_2 is sharp.

The bounds of (7.2)–(7.3) satisfy $x_1 \approx 1.529 < 1.618 \approx \phi$ and $x_2 \approx 1.769$, so that $\phi \in (x_1, x_2)$. The upper bound x_2 is achieved by the Fisher graph of the 3-regular tree (see Proposition 7.4 and [8, 15]).

Theorem 7.2. *For $G \in \mathcal{G}_{3,4}$, we have that*

$$y_1 \leq \mu(G) \leq y_2, \tag{7.4}$$

where

$$y_1 = 12^{1/6}, \tag{7.5}$$

and $y_2 = 1/x$ *where* x *is the largest real root of the equation*

$$2x(x + x^2 + x^3) = 1. \tag{7.6}$$

Moreover, the upper bound y_2 *of* (7.4) *is sharp.*

The lower bound of (7.5) satisfies $12^{1/6} \approx 1.513 < 1.618 \approx \phi$. The upper bound is approximately $y_2 \approx 1.900$, and is achieved by the Fisher graph of the 4-regular tree (see Proposition 7.4). The proofs of Theorems 7.1 and 7.2 are given later in this section.

The emphasis of the current paper is upon lower bounds for connective constants of cubic graphs. The upper bounds of Theorems 7.1–7.2 are included as evidence of the accuracy of the lower bounds, and in support of the unproven possibility that $\mu \geq \phi$ in each case. We note a more general result (derived from results of [8, 40]) for upper bounds of connective constants as follows.

Theorem 7.3. *For* $G \in \mathcal{G}_{d,g}$ *where* $d, g \geq 3$*, we have that* $\mu(G) \leq y$ *where* $\zeta := 1/y$ *is the smallest positive real root of the equation*

$$(d-2)\frac{M_1(\zeta)}{1 + M_1(\zeta)} + \frac{M_2(\zeta)}{1 + M_2(\zeta)} = 1, \tag{7.7}$$

where

$$M_1(\zeta) = \zeta, \qquad M_2(\zeta) = 2(\zeta + \zeta^2 + \cdots + \zeta^{g-1}). \tag{7.8}$$

The upper bound y *is sharp, and is achieved by the free product graph* $F := K_2 * K_2 * \cdots * K_2 * \mathbb{Z}_g$*, with* $d-2$ *copies of the complete graph* K_2 *on two vertices and one copy of the cycle* $\mathbb{Z}_g$ *of length* g*.*

The extremal graph of this theorem is the (simple) Cayley graph F of the free product group $\langle S \mid R \rangle$ with $S = \{a_1, a_2, \ldots, a_{d-2}, b\}$ and $R = \{a_1^2, a_2^2, \ldots, a_{d-2}^2, b^g\}$.

The proofs follow. Let $G = (V, E) \in \mathcal{G}_d$ where $d \geq 3$. The (generalized) *Fisher graph* G_{F} is obtained from G by replacing each vertex by a d-cycle, called a *Fisher cycle*, as illustrated in Figure 7.1. The Fisher transformation originated in the work of Fisher [7] on the Ising model. We shall study the relationship between $\mu(G)$ and $\mu(G_{\mathrm{F}})$, and to that end we need G_{F} to be quasi-transitive (see (2.1)). When $d = 3$, G_{F} is invariably quasi-transitive but, when $d \geq 4$, one needs to be specific about the choice of the Fisher cycles. Let $v \in V$, and order the neighbours of v in a fixed but arbitrary manner as $(u_1, u_2, \ldots, u_d)$. We replace v by a Fisher cycle, denoted F_v, with ordered vertex-set in one–one correspondence with the edges $\langle v, u_i \rangle$, $i = 1, 2, \ldots, d$, in that order. For $x \in V$, find $\alpha_x \in \mathrm{Aut}(G)$ such that $\alpha_x(v) = x$, and replace x by

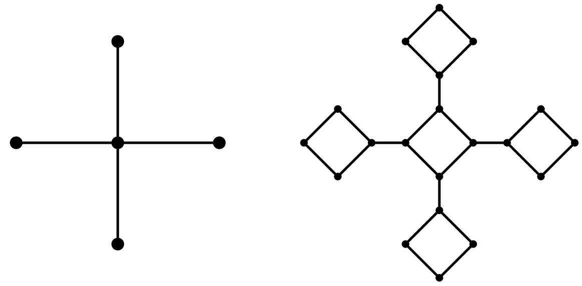

FIGURE 7.1. Each vertex of G is replaced in the Fisher graph G_{F} by a cycle.

the Fisher cycle $\alpha_x(F_v)$. The family $\{\alpha_x : x \in V\}$ acts quasi-transitively on G_{F}, as required.

The connective constants of G and G_{F} are related as follows.

Proposition 7.4. *Let $G \in \mathcal{G}_d$ where $d \geq 3$.*

(a) [15, Thm 1] *If $d = 3$,*

$$\frac{1}{\mu(G_{\mathrm{F}})^2} + \frac{1}{\mu(G_{\mathrm{F}})^3} = \frac{1}{\mu(G)}. \tag{7.9}$$

(b) *If $d = 2r \geq 4$ is even,*

$$\frac{2}{\mu(G_{\mathrm{F}})^{r+1}} \leq \frac{1}{\mu(G)}. \tag{7.10}$$

(c) *If $d = 2r + 1 \geq 5$ is odd,*

$$\frac{1}{\mu(G_{\mathrm{F}})^{r+1}} + \frac{1}{\mu(G_{\mathrm{F}})^{r+2}} \leq \frac{1}{\mu(G)}. \tag{7.11}$$

Proof of Proposition 7.4. We use the methods of [15], where a proof of part (a) appears at Theorem 1. Consider SAWs on G and G_{F} that start and end at midpoints of edges. Let π be such a SAW on G. When π reaches a vertex v of G, it can be directed around the corresponding d-cycle C of G_{F}. There are $d-1$ possible exit points for C relative to the entry point. For each, the SAW may be redirected around C either clockwise or anticlockwise (as illustrated in Figure 7.2). If the exit lies s ($\leq d/2$) edges along C from the entry, a single step of π becomes a walk of length either $s+1$ or $d-s+1$. Such a substitution is made at each vertex of π. It is easily checked that (i) the outcome is a SAW π' on G_{F}, and (ii) by observation of π', one may recover the choices made at each v.

We formalize the above by following the arguments of [15]. Let $d = 2r \geq 4$ (the case of odd d is similar). Write $G = (V, E)$ and $G_F = (V_F, E_F)$. The set E may be

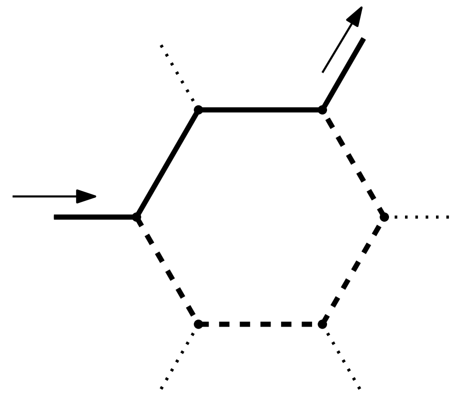

FIGURE 7.2. The entry and exit of a SAW at a Fisher 6-cycle. It follows either two edges clockwise, or 4 edges anticlockwise.

considered as a subset of E_F. By the argument leading to [15, eqn (15)], it suffices to consider SAWs on G_F that begin and end at midpoints of edges of E.

The generating functions of SAWs beginning at a given midpoint e of E on the graph G is given as

$$Z(\zeta) = \sum_{\pi \in \Sigma(G)} \zeta^{|\pi|}, \tag{7.12}$$

where $\Sigma(G)$ is the set of such SAWs. Let Z_F be the generating function of SAWs on G_F starting at e and ending in the set of midpoints of E. The function Z_F is obtained from Z as follows. For $\pi \in \Sigma(G)$, let $e_0, e_1, \dots, e_n$ be the midpoints visited by π, and let C_i be the Fisher cycle of G_F touching e_i and e_{i+1}. Considering the e_i as midpoints of E_F, let k_i be the length of the shorter of the two routes from e_i to e_{i+1} around C_i. We replace the product $\zeta^{|\pi|}$ in (7.12) by

$$P_\pi(\zeta) := \prod_{i=0}^{n-1} (\zeta^{k_i+1} + \zeta^{d-k_i+1})$$

to obtain

$$Z_F(\zeta) = \sum_{\pi \in \Sigma(G)} P_\pi(\zeta).$$

Since $1 \le k_i \le r$, we deduce that

$$Z_F(\zeta) \ge Z\big(\min\{\zeta^2 + \zeta^d, \zeta^3 + \zeta^{d-1}, \dots, 2\zeta^{r+1}\}\big), \qquad \zeta \ge 0. \tag{7.13}$$

The radius of convergence of Z_{F} is $1/\mu(G_{\mathrm{F}})$, and (7.10) follows from (7.13) on letting $\zeta \uparrow 1/\mu(G_{\mathrm{F}})$ and noting that the minimum in (7.13) is achieved by $2\zeta^{r+1}$. $\square$

Lemma 7.5. *Let* $G = (V, E) \in \mathcal{G}_{3,3}$.

(a) *For $v \in V$, there exists exactly one triangle passing through v.*
(b) *If each such triangle of G is contracted to a single vertex, the ensuing graph G' satisfies $G' \in \mathcal{G}_3$.*

Proof. (a) Assume the contrary: each $u \in V$ lies in two or more triangles. Since $\deg(u) = 3$, there exists $v \in V$ such that $\langle u, v\rangle$ lies in two distinct triangles, and we write w_1, w_2 for the vertices of these triangles other than u, v. Since each w_i has degree 3, we have than $w_1 \sim w_2$. This implies that G is finite, which is a contradiction.

(b) Let $\mathcal{T}$ be the set of triangles in G, so that the elements of $\mathcal{T}$ are vertex-disjoint. We contract each $T \in \mathcal{T}$ to a vertex, thus obtaining the graph $G' = (V', E')$. Since each vertex of G' arises from a triangle of G, the graph G' is cubic, and G is the Fisher graph of G'. Since G is infinite, so is G'.

We show next that G' is transitive. Let $v_1', v_2' \in V'$, and write $T_i = \{a_i, b_i, c_i\}$, $i = 1, 2$, for the corresponding triangles of G. Since G is transitive, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a_1) = a_2$. By part (a), $\alpha(T_1) = T_2$. Since $\alpha \in \mathrm{Aut}(G)$, it induces an automorphism $\alpha' \in \mathrm{Aut}(G')$ such that $\alpha'(v_1') = v_2'$, as required.

Finally, we show that G' is simple. If not, there exist two vertex-disjoint triangles of G, T_1 and T_2 say, with two edges between their vertex-sets. Each vertex in these two edges belongs to two faces of size 3 and 4. By transitivity, every vertex has this property. By consideration of the various possible cases, one arrives at a contradiction. $\square$

Proof of Theorem 7.1. Since G is the Fisher graph of $G' \in \mathcal{G}_3$, by Proposition 7.4(a),

$$\frac{1}{\mu(G)^2} + \frac{1}{\mu(G)^3} = \frac{1}{\mu(G')}.$$

By [18, Thm 4.1],

$$\sqrt{2} \le \mu(G') \le 2,$$

and (7.1) follows. When G' is the 3-regular tree T_3, we have $\mu(G') = 2$, and the upper bound is achieved. $\square$

The following lemma is preliminary to the proof of Theorem 7.2.

Lemma 7.6. *Let $G = (V, E) \in \mathcal{G}_{3,4}$. If G is not the doubly infinite ladder $\mathbb{L}$, each $v \in V$ belongs to exactly one quadrilateral.*

Proof. Let $G = (V, E) \in \mathcal{G}_{3,4}$ and $v \in V$. Assume v belongs to two or more quadrilaterals. We will deduce that $G = \mathbb{L}$.

By transitivity, there exist two (or more) quadrilaterals passing through every vertex v, and we pick two of these, denoted $C_{v,1}$, $C_{v,2}$. Since v has degree 3, exactly one of the following occurs (as illustrated in Figure 7.3).

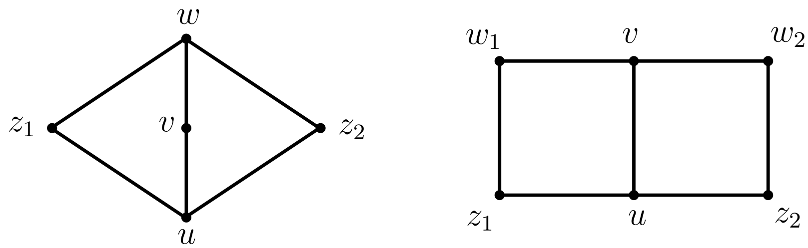

FIGURE 7.3. The two situations in the proof of Lemma 7.6.

(a) $C_{v,1}$ and $C_{v,2}$ share two edges incident to v.
(b) $C_{v,1}$ and $C_{v,2}$ share exactly one edge incident to v.

Assume first that Case (a) occurs, and let Π_x be the property that $x \in V$ belongs to three quadrilaterals, any two of which share exactly one incident edge of x, these $\binom{3}{2} = 3$ edges being distinct.

Let $\langle u, v\rangle$ and $\langle w, v\rangle$ be the two edges shared by $C_{v,1}$ and $C_{v,2}$, and write $C_{v,i} = (u, v, w, z_i)$, $i = 1, 2$. Observe that u lies in the quadrilaterals $C_{v,1}$, $C_{v,2}$, and (u, z_1, w, z_2). Since u has degree 3, if u lies in any fourth quadrilateral Q, then Q has two edges incident to u in common with one of the first three quadrilaterals above. Therefore, Π_u occurs, so that Π_x occurs for every x by transitivity.

Let x be the adjacent vertex of v other than u and w. Note that $x \notin \{z_1, z_2\}$ and $x \not\sim u, w$, since otherwise G would have girth 3. By Π_v, either $x \sim z_1$ or $x \sim z_2$. Assume without loss of generality that $x \sim z_1$. If $x \sim z_2$ in addition, then G is finite, which is a contradiction. Therefore, $x \not\sim z_2$.

Let y be the incident vertex of z_2 other than u and w, and note that $y \notin \{u, v, w, x, z_1, z_2\}$. By Π_{z_2}, there exists a quadrilateral containing both $\langle y, z_2\rangle$ and $\langle z_2, u\rangle$. Since u has degree 3, either $y \sim z_1$ or $y \sim v$. However, neither is possible since both z_2 and v have degree 3. Therefore, Case (a) does not occur.

Assume Case (b) occurs, and write $C_{v,i} = (u, v, w_i, z_i)$, $i = 1, 2$, for the above two quadrilaterals passing through v. Let Π_x^2 (respectively, Π_x^3) be the property that $x \in V$ belongs to two quadrilaterals (respectively, three quadrilaterals), and each incident edge of x lies in at least one of these quadrilaterals (respectively, every pair of incident edges of x lie in at least one of these quadrilaterals). Since Π_v^2 occurs, by transitivity Π_x^2 occurs for every $x \in V$.

Since G is infinite, there exists a 'new' edge incident to the union of $C_{v,1}$ and $C_{v,2}$. Without loss of generality, we take this as $\langle z_1, x\rangle$ with $x \notin \{u, v, w_1, w_2, z_1, z_2\}$. By $\Pi_{z_1}^2$, there exists a quadrilateral of the form (z_1, x, y, z). Since G is simple with degree 3 and girth 4, and $d_G(y, z_1) = 2$, $y \notin \{z_1, u, v, w_1, w_2\}$.

We claim that $y \neq z_2$, as follows. If $y = z_2$, then Π_u^3 occurs, whence $\Pi_{z_1}^3$ occurs by transitivity. Therefore, there exists a quadrilateral passing through the

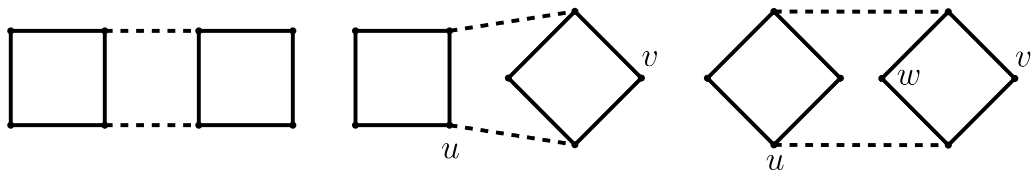

FIGURE 7.4. Three ways in which two quadrilaterals may be joined by two edges.

two edges $\langle x, z_1\rangle$, $\langle z_1, w_1\rangle$, and we denote this (x, z_1, w_1, y'). It is immediate that $y' \notin \{u, v, w_2, z_2\}$ since G is simple with degree 3 and girth 4, and therefore y' is a 'new' vertex. By $\Pi^3_{w_1}$, $y' \sim w_2$, and G is finite, a contradiction. Therefore, $y \neq z_2$, and hence y is a 'new' vertex, and $z = w_1$.

We iterate the above procedure, adding at each stage a new quadrilateral to the graph already obtained. It could be that the resulting graph is a singly infinite ladder with two 'terminal' vertices of degree 2 (as in Figure 5.1). If so, we then turn attention to these terminal vertices, and use the fact that, by transitivity, there exists $D < \infty$ such that $d_{G\setminus e}(a, b) \leq D$ for every edge $e = \langle a, b\rangle \in E$. $\square$

Proof of Theorem 7.2. There are two special cases. If $G = \mathbb{L}$, then $\mu = \phi$, which satisfies (7.4). If G is the 'twisted ladder' $\mathbb{TL}$ of Figure 5.2, then $\mu(G)$ is the root of the equation $2\mu^{-2} + 2\mu^{-4} = 1$, which satisfies $\mu = \sqrt{1 + \sqrt{3}} \approx 1.653 > \phi$ (see item C of Section 4.1).

Assume that $G \neq \mathbb{L}, \mathbb{TL}$. Let $\mathcal{T}$ be the set of quadrilaterals of G, and recall Lemma 7.6. We contract each element of $\mathcal{T}$ to a degree-4 vertex, thus obtaining a graph G'. We claim that

$$G' \in \mathcal{G}_4, \text{ and } G \text{ is the Fisher graph of } G'. \tag{7.14}$$

Suppose for the moment that (7.14) is proved. By [18, Thm 4.1], $\mu(G') \geq \sqrt{3}$, and, by Proposition 7.4(b),

$$\frac{2}{\mu(G)^3} \leq \frac{1}{\mu(G')} \leq \frac{1}{\sqrt{3}},$$

which implies $\mu(G) \geq 12^{1/6}$.

We prove (7.14) next. It suffices that $G' = (V', E') \in \mathcal{G}_4$, and G is then automatically the required Fisher graph. Evidently, G' has degree 4. We show next that G' is transitive. Let $v'_1, v'_2 \in V'$, and write $Q_i = (a_i, b_i, c_i, d_i)$, $i = 1, 2$, for the corresponding quadrilaterals of G. Since G is transitive, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a_1) = a_2$. By Lemma 7.6, $\alpha(Q_1) = Q_2$. Since $\alpha \in \mathrm{Aut}(G)$, it induces an automorphism $\alpha' \in \mathrm{Aut}(G')$ such that $\alpha'(v'_1) = v'_2$, as required.

Suppose that G' is not simple, in that it has parallel edges. Then there exist two quadrilaterals joined by two edges, which can occur in any of the three ways drawn in Figure 7.4. The first is impossible by Lemma 7.6. In the second, u lies in both a 4-cycle and a 5-cycle having exactly one edge in common; on the other hand, v cannot have this property since it would require some vertex in the diagram to have degree 4 or more. In the third, u is in a 4-cycle and four 6-cycles, and each 6-cycle has a path of length 2 in common with the 4-cycle. By transitivity, v has the same property, which implies that there is a path of length 4 from v to w which is disjoint from the rest of the diagram. By iteration, we find that $G = \mathbb{TL}$, a contradiction. In summary, G' is infinite, transitive, cubic, and simple, whence $G' \in \mathcal{G}_4$.

For the sharpness of the upper bound, we refer to the proof of the more general Theorem 7.3, following. □

Proof of Theorem 7.3. Let $G \in \mathcal{G}_{d,g}$ where $d, g \geq 3$, and let F ($\in \mathcal{G}_{d,g}$) be the given free product graph. By [40, Thm 11.6], F covers G. Therefore, there is an injection from SAWs on G with a given root to a corresponding set on F, whence $\mu(G) \leq \mu(F)$.

By [8, Thm 3.3], $\mu(F) = 1/\zeta$ where ζ is the smallest positive real root of (7.7). □

8. The Grigorchuk group

The Grigorchuk group Γ is defined as follows (see [9, 10, 11]). Let T be the rooted binary tree with root vertex $\varnothing$. The vertex-set of T can be identified with the set of finite strings u having entries 0, 1, where the empty string corresponds to the root $\varnothing$. Let T_u be the subtree of all vertices with root labelled u.

Let $\mathrm{Aut}(T)$ be the automorphism group of T, and let $a \in \mathrm{Aut}(T)$ be the automorphism that, for each string u, interchanges the two vertices $0u$ and $1u$ together with their subtrees.

Any $\gamma \in \mathrm{Aut}(G)$ may be applied in a natural way to either subtree T_i, $i = 0, 1$. Given two elements $\gamma_0, \gamma_1 \in \mathrm{Aut}(T)$, we define $\gamma = (\gamma_0, \gamma_1)$ to be the automorphism on T obtained by applying γ_0 to T_0 and γ_1 to T_1. Define automorphisms b, c, d of T recursively as follows:

$$b = (a, c), \quad c = (a, d), \quad d = (\mathbf{1}, b), \tag{8.1}$$

where $\mathbf{1}$ is the identity automorphism. The Grigorchuk group Γ is defined as the subgroup of $\mathrm{Aut}(T)$ generated by the set $\{a, b, c\}$.

The 2-neighbourhood of $\mathbf{1}$ in the Cayley graph G of Γ is drawn in Figure 8.1. Since $G \in \mathcal{G}_{3,4}$, we have by Theorem 7.2 that $y_1 \leq \mu(G) \leq y_2$. The lower bound may be improved as follows.

Theorem 8.1. *The Cayley graph G of the Grigorchuk group Γ satisfies $\mu(G) \geq \phi$.*

Proof. This theorem is due to Anton Malyshev, who has kindly given permission for it to be included here. A *ray* of T is a SAW on T starting at $\varnothing$. The collection of all infinite rays is called the *boundary* of T and denoted ∂T. Since each $\gamma \in \Gamma$ preserves

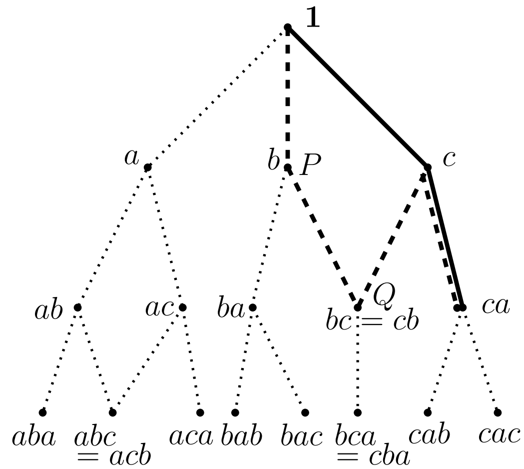

FIGURE 8.1. The 3-neighbourhood of the identity. The walk $(\mathbf{1}, c, ca)$ may be re-routed as $(\mathbf{1}, b, bc, c, ca)$.

the root $\varnothing$, the orbit of any $v \in T$ is a subset of the generation of T containing v. Since $\gamma \in \Gamma$ preserves adjacency, γ maps ∂T into ∂T.

The *orbit* $\Gamma\rho$ of $\rho \in \partial T$ gives rise to a graph, called the *orbital Schreier graph* of ρ, and denoted here by $S(\rho)$. The vertex-set of $S(\rho)$ is $\Gamma\rho$. For $\rho_1, \rho_2 \in \Gamma\rho$, $S(\rho)$ has an edge between ρ_1 and ρ_2 if and only if $\rho_2 = x\rho_1$ for some $x \in \{a, b, c\}$; we label this edge with the generator x and call it an x-edge. (Recall that $x^2 = \mathbf{1}$ for $x \in \{a, b, c\}$.) Such orbital Schreier graphs have been studied in [12, 13, 38] and the references therein.

Let 1^∞ denote the rightmost infinite ray of T, with orbital Schreier graph $S := S(1^\infty)$ illustrated in Figure 8.2. It is standard (see the above references) that, if $\rho \in \Gamma 1^\infty$, $S(\rho)$ is graph-isomorphic to the singly infinite graph S (the edge-labels are generally different). Otherwise, $S(\rho)$ is graph-isomorphic to a certain doubly infinite chain.

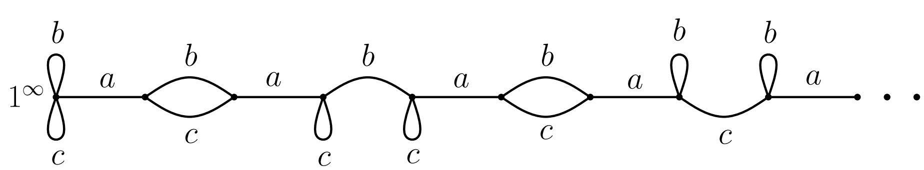

FIGURE 8.2. The one-ended orbital Schreier graph of the ray 1^∞.

Let $\mathcal{W}$ be the set of labelled walks on S starting at the root 1^∞ that, at each step, either move one step rightwards (in the sense of Figure 8.2), or pass around a loop (no loop may be traversed more than once). Members of $\mathcal{W}$ may be considered as words without consecutive repetitions in the alphabet $\aleph = \{a, b, c\}$. Walks in $\mathcal{W}$ are not generally self-avoiding, but we shall see next that they lift to a set $\overline{\mathcal{W}}$ of self-avoiding walks on G starting at its root $\mathbf{1}$. In order to obtain the required inequality, we shall need to augment $\mathcal{W}$ to a certain set $\mathcal{W}'$ constructed as follows.

Each $w \in \mathcal{W}$ lifts to a distinct walk $\overline{w}$ on G. Furthermore,

$$\text{each } w \in \mathcal{W} \text{ lifts to a SAW } \overline{w} \text{ on } G. \tag{8.2}$$

To see the last, if $\overline{w}$ is not a SAW, then w contains some shortest subword s of length 3 or more satisfying $s = \mathbf{1}$. On considering the action of Γ on S, we deduce that S contains a cycle of length 3 or more, a contradiction.

Let w_0 be a SAW on S starting at $\mathbf{1}$ and ending immediately after an a-edge, and let $F(w_0)$ be the set of final endpoints of the a-edges in w_0, ordered by the order they are encountered by w_0 (thus w_0 traverses rightwards in Figure 8.2 using no loops). We think of w_0 as a word in the alphabet $\aleph$. Then w_0 can be broken into sections spanning two consecutive members of $F(w_0)$, and each such section may be replaced by any of a certain set of words. Let σ be such a section, with first endpoint $z := z(\sigma)$. There are three possible cases (see Figure 8.2).

(a) If both b and c are rightward edges from z, we may replace σ by the word labelled by any element of $\{ba, ca\}$.
(b) If b is rightward from z, and c is a loop at z, we may replace σ by any element of $\{ba, cba, bca, cbca\}$.
(c) If c is rightward from z, and b is a loop at z, we may replace σ by any element of $\{ca, bca, cba, bcba\}$.

The resulting set of words corresponds to a subset $\mathcal{W}_a \subseteq \mathcal{W}$ of walks on S, and we use the symbol $\mathcal{W}_a$ to denote both the word-set and the walk-set.

The augmented set $\mathcal{W}'_a$ of words is obtained as above but with (a) replaced as follows.

(a′) If both b and c are rightward edges from z, we may replace σ by the word labelled by any element of $\{ba, ca, bcba, cbca\}$.

Let $\overline{\mathcal{W}}'_a$ be the corresponding set of lifted walks on G. Once again, each $w' \in \mathcal{W}'_a$ lifts to a distinct walk $\overline{w}' \in \overline{\mathcal{W}}'_a$, and we claim that

$$\text{each } w' \in \mathcal{W}'_a \text{ lifts to a SAW } \overline{w}' \text{ on } G. \tag{8.3}$$

We argue as follows to check the last statement.

Consider first a single instance of the 'additional' substitution $bcba$ in (a′), which we view as a substitute for ca (the same argument applies to $cbca$ viewed as a substitute for ba). Let $w = x(ca)y \in \mathcal{W}_a$ where x, y are words terminating with the letter a, and let $w' = x(bcba)y$ be obtained from w by replacing the instance of ca by $bcba$. This amounts to rerouting $\overline{w}$ around an image (under the action of x) of the quadrilateral

$(\mathbf{1}, b, bc = cb, c)$ as indicated in Figure 8.1. The new lifted walk $\overline{w}'$ fails to be a SAW only if $\overline{w}$ visits either $xP := xb$ or $xQ := xbc$ (where P and Q are indicated Figure 8.1) with its final step across the relevant a-type edge. Consider the first case, the second being similar. Let $w_1 = x(ca)y'$ be the subword of w that equals xP, noting that the final character of y' is a. Then $w_1 b \in \mathcal{W}$, but $\overline{w_1 b}$ contains a cycle, and is therefore not a SAW. This contradicts (8.2), and we deduce that $\overline{w}'$ is a SAW.

Suppose next that the word w' is obtained from $w \in \mathcal{W}_a$ by making several such additional substitutions, which necessarily involve disjoint subwords of w. We consider these substitutions one by one, in the natural order of w. If $\overline{w}'$ is not a SAW, there is an earliest substitution which creates a cycle. The above argument may be applied to that substitution to obtain a contradiction. In conclusion, (8.3) holds.

The generating function Z of $\overline{\mathcal{W}}_a'$ (see (2.2)) may be expressed in the form

$$Z(\zeta) = A_0 \sum_{n=0}^{\infty} A_1 A_2 \cdots A_n,$$

where $A_0 = 2\zeta^2$ and each A_k, for $k \geq 1$, is either

$$Z_1(\zeta) = 2\zeta^2 + 2\zeta^4 \quad \text{or} \quad Z_2 = \zeta^2 + 2\zeta^3 + \zeta^4.$$

Furthermore, Z_1 appears infinitely often in the sequence $(A_k : k = 1, 2, \dots)$. Since $Z_1(1/\phi) > 1$ and $Z_2(1/\phi) = 1$, we have that $Z(\zeta) = \infty$ for $\zeta > 1/\phi$. The claim of the theorem follows. $\square$

9. Transitive TLF-planar graphs

9.1. **Background and main theorem.** There are only few infinite, transitive, cubic graphs that are planar, and each has $\mu \geq \phi$. These graphs belong to the larger class of so-called TLF-planar graphs, and we study such graphs in this section. The basic properties of such graphs were presented in [33], to which the reader is referred for further information. In particular, the class of TLF-planar graphs includes the one-ended planar Cayley graphs and the normal transitive tilings.

We use the word *plane* to mean a simply connected Riemann surface without boundaries. An *embedding* of a graph $G = (V, E)$ in a plane $\mathcal{P}$ is a function $\eta : V \cup E \to \mathcal{P}$ such that η restricted to V is an injection and, for $e = \langle u, v \rangle \in E$, $\eta(e)$ is a C^1 image of $[0, 1]$. An embedding is ($\mathcal{P}$-)*planar* if the images of distinct edges are disjoint except possibly at their endpoints, and a graph is ($\mathcal{P}$-)*planar* if it possesses a ($\mathcal{P}$-)planar embedding. An embedding is *topologically locally finite* (*TLF*) if the images of the vertices have no accumulation point, and a connected graph is called *TLF-planar* if it possesses a planar TLF embedding. Let $\mathcal{T}_d$ denote the class of transitive, TLF-planar graphs with vertex-degree d. We shall sometimes confuse a

TLF-planar graph with its TLF embedding. The *boundary* of $S \subseteq \mathcal{P}$ is given as $\partial S := \overline{S} \cap \overline{\mathcal{P} \setminus S}$, where $\overline{T}$ is the closure of T.

The principal theorem of this section is as follows.

Theorem 9.1. *Let $G \in \mathcal{T}_3$ be infinite. Then $\mu(G) \geq \phi$.*

The principal methods of the proof are as follows: (i) the construction of an injection from eastward SAWs on $\mathbb{L}_+$ to SAWs on G, (ii) a method for verifying that certain paths on G are indeed SAWs, and (iii) a generalization of the Fisher transformation of [15].

A *face* of a TLF-planar graph (or, more accurately, of its embedding) is an arc-connected component of the (topological) complement of the graph. The *size* $k(F)$ of a face F is the number of vertices in its topological boundary, if bounded; an unbounded face has size ∞. Let $G = (V, E) \in \mathcal{T}_d$ and $v \in V$. The *type-vector* $[k_1, k_2, \dots, k_d]$ of v is the sequence of sizes of the d faces incident to v, taken in cyclic order around v. Since G is transitive, the type-vector is independent of choice of v modulo permutation of its elements, and furthermore each entry satisfies $k_i \geq 3$. We may therefore refer to the type-vector $[\![k_1, k_2, \dots, k_d]\!]$ of G, and we define

$$f(G) = \sum_{i=1}^{d} \left(1 - \frac{2}{k_i}\right),$$

with the convention that $1/\infty = 0$. We shall use the following two propositions.

Proposition 9.2 ([33, p. 2827])**.** *Let $G = (V, E) \in \mathcal{T}_3$.*

(a) *If $f(G) < 2$, G is finite and has a planar TLF embedding in the sphere.*
(b) *If $f(G) = 2$, G is infinite and has a planar TLF embedding in the Euclidean plane.*
(c) *If $f(G) > 2$, G is infinite and has a planar TLF embedding in the hyperbolic plane (the Poincaré disk).*

Moreover, all faces of the above embeddings are regular polygons.

There is a moderately extensive literature concerning the function f and the Gauss–Bonnet theorem for graphs. See, for example, [4, 25, 27].

Proposition 9.3. *The type-vector of an infinite graph $G \in \mathcal{T}_3$ is one of the following:*

A. $[\![m, m, m]\!]$ *with $m \geq 6$,*
B. $[\![m, 2n, 2n]\!]$ *with $m \geq 3$ odd, and $m^{-1} + n^{-1} \leq \frac{1}{2}$,*
C. $[\![2m, 2n, 2p]\!]$ *with $m, n, p \geq 2$ and $m^{-1} + n^{-1} + p^{-1} \leq 1$.*

Recall that the elements of a type-vector lie in $\{3, 4, \dots\} \cup \{\infty\}$.

Proof. See [33, p. 2828] for an identification of the type-vectors in $\mathcal{T}_3$. The inequalities on m, n, p arise from the condition $f(G) \geq 2$. ∎

9.2. **Proof of Theorem 9.1.** Let $G \in \mathcal{T}_3$ be infinite. By Proposition 9.2, $f(G) \geq 2$. If $f(G) = 2$ then, by Proposition 9.3, the possible type-vectors are precisely those with type-vectors $[\![6,6,6]\!]$, $[\![3,12,12]\!]$, $[\![4,8,8]\!]$, $[\![4,6,12]\!]$, $[\![4,4,\infty]\!]$, that is, the hexagonal lattice [6] and its Fisher graph [15, Thm 1], the square/octagon lattice [17, Example 4.2], the Archimedean lattice $[\![4,6,12]\!]$ of Examples 3.3(c), 6.2 and Remark 9.8, and the doubly infinite ladder of Figure 5.1. It is explained in the above references that each of these has $\mu \geq \phi$.

It remains to prove Theorem 9.1 when $G \in \mathcal{T}_3$ is infinite with $f(G) > 2$. By Proposition 9.3, the cases to be considered are:

A. $[\![m,m,m]\!]$ where $m > 6$,
B. $[\![m,2n,2n]\!]$ where $m \geq 3$ is odd and $m^{-1} + n^{-1} < \frac{1}{2}$,
C. $[\![2m,2n,2p]\!]$ where $m,n,p \geq 2$ and $m^{-1} + n^{-1} + p^{-1} < 1$.

These cases are covered in the following order, as indexed by section number.

§9.3. $\min\{k_i\} \geq 5$, $[\![k_1,k_2,k_3]\!] \neq [\![5,8,8]\!]$,
§9.4. $\min\{k_i\} = 3$,
§9.5. $[\![4,2n,2p]\!]$ where $p \geq n \geq 4$ and $n^{-1} + p^{-1} < \frac{1}{2}$,
§9.6. $[\![4,6,2p]\!]$ where $p \geq 6$,
§9.7. $[\![5,8,8]\!]$.

Note that Section 9.6 includes the case of the Archimedean lattice $\mathbb{A} = [\![4,6,12]\!]$ with $f(\mathbb{A}) = 2$ (see also Example 3.3(c)).

We identify G with a specific planar, TLF embedding in the hyperbolic plane every face of which is a regular polygon. The proof is similar in overall approach to that of Theorem 3.1, as follows. Let $\mathbb{W}_n$ be the set of eastward n-step SAWs from 0 on the singly-infinite ladder $\mathbb{L}_+$ of Figure 5.1. Fix a root $v \in V$, and let $\Sigma_n(v)$ be the set of n-step SAWs on G starting at v. We shall construct an injection from $\mathbb{W}_n$ to $\Sigma_n(v)$, and the claim will follow by (5.1).

We construct next an injection from $\mathbb{W}_n$ to $\Sigma_n(v)$. Let $w \in \mathbb{W}_n$. We shall explain how the word w encodes an element of $\Sigma_n(v)$. In building an element of $\Sigma_n(v)$ sequentially, at each stage there is a choice between two new edges, which, in the sense of the embedding, we may call 'right' and 'left'. The key step is to show that the ensuing paths on G are indeed SAWs so long as the cumulative differences between the aggregate numbers of right and left steps remain sufficiently small.

Some preliminary lemmas follow. Let $G \in \mathcal{T}_d$ be infinite, where $d \geq 3$. A cycle C of G is called *clockwise* if its orientation after embedding is clockwise. Let C be traversed clockwise, and consider the changes of direction at each turn. Since the vertex-degree is d, each turn is along one of $d-1$ possible non-backtracking edges, exactly one of which may be designated *rightwards* and another *leftwards* (the other $d-3$ are neither rightwards nor leftwards). Let $r = r(C)$ (respectively, $l = l(C)$) be the number of right (respectively, left) turns encountered when traversing

C clockwise, and let

$$\rho(C) = r(C) - l(C). \tag{9.1}$$

Lemma 9.4. *Let $G \in \mathcal{T}_d$ be infinite with $d \geq 3$. Let C be a cycle of G, and let $\mathcal{F} := \{F_1, F_2, \dots, F_s\}$ be the set of faces enclosed by C. There exists $F \in \mathcal{F}$ such that the boundary of $\mathcal{F} \setminus F$ is a cycle of G. The set of edges lying in $\partial F \setminus C$ forms a path.*

Proof. Let C be a cycle of G, and let $\mathcal{F}' \subseteq \mathcal{F}$ be the subset of faces that share an edge with C. Let I be the (connected) subgraph of G comprising the edges and vertices of the faces in $\mathcal{F}'$, and let I_{d} be its dual graph (with the infinite face omitted). Then I_{d} is finite and connected, and thus has some spanning tree T which is non-empty. Pick a vertex t of T with degree 1, and let F be the corresponding face. The first claim follows since the removal of t from T results in a connected subtree. The second claim holds since, if not, the interior of C is disconnected, which is a contradiction. □

Lemma 9.5. *Let $G \in \mathcal{T}_d$ be infinite with $d \geq 3$. For any cycle $C = (c_0, c_1, \dots, c_n)$ of G,*

$$\rho(C) \begin{cases} = 6 + \displaystyle\sum_{i=1}^{s} \big[k(F_i) - 6\big] & \textit{if } d = 3, \\ \geq 4 + \displaystyle\sum_{i=1}^{s} \big[k(F_i) - 4\big] & \textit{if } d \geq 4, \end{cases} \tag{9.2}$$

where $\mathcal{F} = \{F_1, F_2, \dots, F_s\}$ is the set of faces enclosed by C.

Proof. The proof is by induction on the number $s = s(C)$ of faces enclosed by C. It is trivial when $s = 1$ that $r(C) = k(F_1)$ and $l(C) = 0$, and (9.2) follows in that case.

Let $S \geq 2$ and assume that (9.2) holds for all C with $s(C) < S$. Let $C = (c_0, c_1, \dots, c_n)$ be such that $s(C) = S$, and pick $F \in \mathcal{F}$ as in Lemma 9.4. Let π be the path of edges in $\partial F \setminus C$.

Let C_F (respectively, C') be the boundary cycle of F (respectively, $\mathcal{F} \setminus F$), each viewed clockwise. We write π in the form $\pi = (c_a, \psi_1, \psi_2, \dots, \psi_m, c_b)$ where $a \neq b$, $\psi_i \notin C$. We claim that

$$\rho(C) \begin{cases} = \rho(C') + \rho(C_F) - 6 & \text{if } d = 3, \\ \geq \rho(C') + \rho(C_F) - 4 & \text{if } d \geq 4. \end{cases} \tag{9.3}$$

The induction step is proved by applying the induction hypothesis to C' and noting that $\rho(C_F) = k(F)$.

We prove (9.3) by considering the contributions to $\rho(D)$ from vertices in the cycles $D = C, C', C_F$. The contribution from any $y \in C \setminus \{c_a, c_b\}$ occurs exactly once on the left and the right sides of (9.3). We turn, therefore, to vertices in the remaining path π.

1. The cycle C_F (respectively, C') takes a right (respectively, left) turn at each vertex ψ_i. The net contribution from ψ_i to the right (respectively, left) side of (9.3) is $1-1=0$ (respectively, 0).
2. Consider the turn at a vertex $x \in \{c_a, c_b\}$.
   (a) Suppose $d = 3$. At x, C_F takes a right turn, C' takes a right turn, and C takes a left turn. The net contribution from x to the right (respectively, left) side of (9.3) is $1+1=2$ (respectively, -1).
   (b) Suppose $d \geq 4$. At x, C_F takes a right turn. Furthermore, if C' takes a right turn, then C does not take a left turn. The net contribution from x to $[\rho(C') + \rho(C_F)] - \rho(C)$ is at most 2.

We sum the above contributions, noting that case 2 applies for exactly two values of x, to obtain (9.3). The proof is complete. □

Lemma 9.6. *Let $G \in \mathcal{T}_d$ be infinite with type-vector $[\![k_1, k_2, \ldots, k_d]\!]$, and let C be a cycle of G.*

(a) *If $d = 3$ and $\min\{k_i\} \geq 6$, then $\rho(C) \geq 6$.*
(b) *If $d = 3$ and $[\![k_1, k_2, k_3]\!] = [\![5, 2n, 2n]\!]$ with $n \geq 5$, then $\rho(C) \geq 5$.*
(c) *If $d \geq 4$ and $\min\{k_i\} \geq 4$, then $\rho(C) \geq 4$.*

Proof. (a, c) These are immediate consequences of (9.2).
(b) Suppose $[\![k_1, k_2, k_3]\!] = [\![5, 2n, 2n]\!]$ with $n \geq 5$, and let $M = M(C)$ be the number of size-$2n$ faces enclosed by a cycle C. We shall prove $\rho(C) \geq 5$ by induction on $M(C)$. If $M = 0$, then C encloses exactly one size-5 face, and $\rho(C) = 5$. Let $S \geq 1$, and assume $\rho(C) \geq 5$ for any cycle C with $M(C) < S$.

Let C be a cycle with $M(C) = S$. Since every vertex of C is incident to no more than one size-5 face inside C, C contains some size-$2n$ face F with at least one edge in common with C. Let C' be the boundary of the set obtained by removing F from the inside of C; that is, C' may be viewed as the sum of the cycles C and ∂F with addition modulo 2. Then C' may be expressed as the edge-disjoint union of cycles $C_1, C_2, \ldots, C_m$ satisfying $M(C_i) < S$ for $i = 1, 2, \ldots, m$.

By (9.2) and the induction hypothesis,

$$\begin{aligned}\rho(C) &= 6 + [2n-6] + \sum_{i=1}^{m} [\rho(C_i) - 6] \\ &\geq 2n - r.\end{aligned}$$

Each C_i shares an edge with ∂F, and no two such edges have a common vertex. Therefore, $r \leq n$, and the induction step is complete since $n \geq 5$. □

9.3. **Proof that $\mu \geq \phi$ when** $\min\{k_i\} \geq 5$ **and** $[\![k_1, k_2, k_3]\!] \neq [\![5, 8, 8]\!]$**.** This case covers the largest number of instances. It is followed by consideration of certain other special families of type-vectors. By Proposition 9.3, it suffices to assume

$$\text{either } \min\{k_i\} \geq 6, \text{ or } [\![k_1, k_2, k_3]\!] = [\![5, 2n, 2n]\!] \text{ with } n \geq 5. \tag{9.4}$$

We shall construct an injection from the set $\mathbb{W}_n$ to the set $\Sigma_n(v)$ of SAWs on G starting at $v \in V$. For $w \in \mathbb{W}_n$, we shall define an n-step SAW $\pi(w)$ on G, and the map $\pi : \mathbb{W}_n \to \Sigma_n(v)$ will be an injection. The idea is as follows. With G embedded in the plane, one may think of the steps of a SAW on G (after its first edge) as taking a sequence of right and left turns. For given $w \in \mathbb{W}_n$, we will explain how the letters H and V in w are mapped to the directions right/left.

Let $n \geq 1$ and $w = (w_1 w_2 \cdots w_n) \in \mathbb{W}_n$, so that in particular $w_1 = \text{H}$. The path $\pi = \pi(w)$ is constructed iteratively as follows. In order to fix an initial direction, we choose a 2-step SAW (v', v, v'') of G starting at some neighbour v' of v, and we assume in the following that the turn in the path (v', v, v'') is rightwards (the other case is similar). We set $\pi'(w) = (v', v, v'')$ if $n = 1$. The first letter of w is $w_1 = \text{H}$, and the second is either H or V, and the latter determines whether the next turn is the same as or opposite to the previous turn. We adopt the rule that:

$$\begin{aligned} &\text{if } (w_1 w_2) = (\text{HV}), && \text{the next turn is the same as the previous,} \\ &\text{if } (w_1 w_1) = (\text{HH}), && \text{the next turn is opposite to the previous.} \end{aligned} \tag{9.5}$$

For $k \geq 3$, the kth turn of π' is either to the right or the left, and is either the same or opposite to the $(k-1)$th turn. Whether it is the same or opposite is determined as follows:

$$\begin{aligned} &\text{when } (w_{k-2} w_{k-1} w_k) = (\text{HHH}), && \text{it is opposite,} \\ &\text{when } (w_{k-2} w_{k-1} w_k) = (\text{HHV}), && \text{it is the same,} \\ &\text{when } (w_{k-2} w_{k-1} w_k) = (\text{HVH}), && \text{it is opposite,} \\ &\text{when } (w_{k-2} w_{k-1} w_k) = (\text{VHH}), && \text{it is the same,} \\ &\text{when } (w_{k-2} w_{k-1} w_k) = (\text{VHV}), && \text{it is opposite.} \end{aligned} \tag{9.6}$$

When the iterative construction is complete, a path $\pi' = (v' = \pi'_{-1}, v = \pi'_0, v'' = \pi'_1, \dots, \pi'_n)$ on G ensues. Since π' proceeds by right or left turns, it is non-backtracking. The following claim will be useful in showing it is also self-avoiding.

Lemma 9.7. *Let $i \in \{0, 1, \dots, n\}$. For any subpath of π' beginning at π'_i, the numbers of right turns and left turns differ by at most* 3.

Proof. A subpath of π' corresponds to some subword w_1 of w. If this subpath is a cycle, then the length m of w' is necessarily at least 3. A *block* of w' is a subword B of w_1 of the form VH^kV, where H^k denotes k (≥ 1) consecutive appearances of H.

A block B generates $k+1$ turns in π' corresponding to the letters $\mathrm{H}^k\mathrm{V}$, and B is called *even* (respectively, *odd*) according to the parity of k. These $k+1$ turns are determined by the $k+1$ triples HVH, HHH, $\dots$, HHV. By inspection of (9.6), the corresponding turns are related to their predecessors by the sequence osoo $\cdots$ os, where o (respectively, s) means 'opposite' (respectively, 'same'), that is, with $k-1$ opposites and 2 sames. Suppose, for illustration, that the turn immediately prior to the block was R, where R (respectively, L) denotes right (respectively, left). Then the corresponding sequence of turns begins LLRL $\cdots$

(a) If B is odd, then, in the corresponding $k+1$ turns made by π', the numbers of right and left turns are equal. Moreover, if the first turn is to the right (respectively, left), then the last turn is to the left (respectively, right).
(b) If B is even, the numbers of right and left turns differ by 3. Moreover, the first turn is to the right if and only if the last turn is to the right, and in that case there are 3 more right turns than left turns.

Let B be an odd block. By (a), B makes no contribution to the aggregate difference between the number of right and left turns. Furthermore, the first turn of B equals the first turn following B (since the last turn of B is opposite to the first, and the following subword HVH results in a turn equal to the first). We may therefore consider w_1 with all odd blocks removed, and we assume henceforth that w_1 has no odd blocks.

Using a similar argument for even blocks based on (b) above, the effects of two even blocks cancel each other, and we may therefore remove any even number of even blocks from w without altering the aggregate difference. After performing these reductions, we obtain from w_1 a reduced word w_2 with the form either $\mathrm{H}^a\mathrm{V}\mathrm{H}^b$ or $\mathrm{H}^a\mathrm{V}\mathrm{H}^{2r}\mathrm{V}\mathrm{H}^b$ where $a \ge 0$, $r \ge 1$, $b \ge 0$. Each of these cases may be considered separately to obtain the lemma. ☐

Write $\pi'(w) = (v', v = x_0, v'' = x_1, \dots, x_n)$, and remove the first step to obtain a SAW $\pi(w) = (v = x_0, x_1, \dots, x_n)$. By Lemmas 9.6(a, b) and 9.7, subject to (9.4), $\pi(w)$ contains no cycle and is thus a SAW. This is seen as follows. Suppose $\nu = (x_i, x_{i+1}, \dots, x_j = x_i)$ is a cycle. The cycle has one more turn than the path, and hence, by Lemma 9.7, $|\rho(\nu)| \le 4$, in contradiction of Lemma 9.6(a, b). Therefore, π maps $\mathbb{W}_n$ to $\Sigma_n(v)$.

We conclude that π maps $\mathbb{W}_n$ into $\Sigma_n(v)$. It is an injection since, by examination of (9.5)–(9.6), $\pi(w) \ne \pi(w')$ if $w \ne w'$. We deduce by (5.1) that $\mu(G) \ge \phi$.

The above difference between counting turns on *paths* and *cycles* can be overcome by considering SAWs between midpoints of edges (as in Section 9.7).

9.4. **Proof that $\mu \ge \phi$ when $\min\{k_i\} = 3$.** Assume $\min\{k_i\} = 3$. By Proposition 9.3 and the assumption $f(G) > 2$, the type-vector is $[\![3, 2n, 2n]\!]$ for some $n \ge 7$.

On contracting each triangle to a single vertex, we obtain the graph $G' = [\![n, n, n]\!]$; therefore, G is a Fisher graph of G'. By Proposition 7.4(a),

$$\frac{1}{\mu(G)^2} + \frac{1}{\mu(G)^3} = \frac{1}{\mu(G')}.$$

It is proved in Section 9.3 that $\mu(G') \geq \phi$, and the inequality $\mu(G) \geq \phi$ follows (see also [15]).

9.5. **Proof that $\mu \geq \phi$ for $[\![4, 2n, 2p]\!]$ with $p \geq n \geq 4$ and $n^{-1} + p^{-1} < \frac{1}{2}$.** Let $G = (V, E) \in \mathcal{T}_3$ be infinite with type-vector $[\![4, 2n, 2p]\!]$ where $p \geq n \geq 4$ and $n^{-1} + p^{-1} < \frac{1}{2}$. Note that G has girth 4, and every vertex is incident to exactly one size-4 face.

From G, we obtain a new graph G' by contracting each size-4 face to a vertex. For $u, v \in V$ lying in different size-4 faces F_u, F_v of G, there exists $\alpha \in \mathrm{Aut}(G)$ that maps v to v', and hence maps F_u to F_v. Therefore, α induces an automorphism of G', so that G' is transitive. We deduce that $G' \in \mathcal{T}_4$, and in addition G' is infinite with girth $n \geq 4$ and type-vector $[n, p, n, p]$. Recall Lemma 9.6(c). We shall make use of G' later in the proof.

Let $v \in V$. We will construct an injection from $\mathbb{W}_n$ to $\Sigma_n(v)$ in a manner similar to the argument following (9.4). An edge of G is called *square* if it lies in a size-4 face, and *non-square* otherwise. Let $w = (w_1 w_2 \cdots w_n) \in \mathbb{W}_n$. We shall construct a *non-backtracking* n-step path $\pi = \pi(w)$ on G from v, and then show it is a SAW. For $k = 1$, set $\pi(w) = (v, v')$ for some neighbour v' of v chosen arbitrarily. We perform the following construction for $k = 2, 3, \ldots, n$, in which the edges of π are denoted $e_1, e_2, \ldots, e_n$ in order.

1. Suppose $(w_{k-1} w_k) = (\mathrm{HV})$. The edge e_k is always square.
   - (a) If the edge e_{k-1} of π corresponding to w_{k-1} is square, then the next edge e_k of π is square. That is, e_{k-1} and e_k form a length-2 path on the same size-4 face of G.
   - (b) Suppose e_{k-1} is non-square. Then the next edge e_k is one of the two possible square edges, chosen as follows. In contracting G to G', the path $(\pi_0, \pi_1, \ldots, \pi_{k-1})$ contracts to a non-backtracking path π' on G'. Find the most recent turn at which π' turns either right or left. If, at that turn, π' turns left (respectively, right), the non-backtracking path π on G turns left (respectively, right). If no turn of π' is rightwards or leftwards, then π turns left.
2. Suppose $(w_{k-1} w_k) = (\mathrm{HH})$.
   - (a) If the edge e_{k-1} of π corresponding to w_{k-1} is square, then the next edge e_k of π is the unique possible non-square edge.

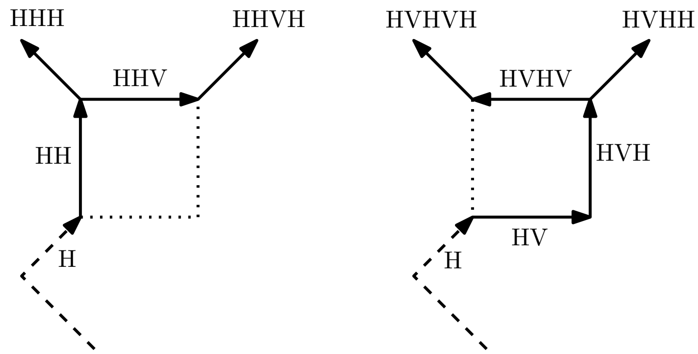

FIGURE 9.1. The dashed line is the projected SAW on G'. After a right (respectively, left) turn, the projection either moves straight or turns left (respectively, right).

(b) Suppose e_{k-1} is non-square. Then e_k is one of the two possible square edges, chosen as follows. In the notation of 1(b) above, find the most recent turn at which π' turns either right or left. If at that turn, π' turns left (respectively, right), the non-backtracking path π on G turns right (respectively, left). If π' has no such turn, then π turns right.

3. Suppose $(w_{k-1}w_k) = (\text{VH})$, so that, in particular, $k \geq 3$. The edge e_{k-1} of G corresponding to w_{k-1} must be square. If e_{k-2} is square (respectively, non-square), then e_k is the unique possible non-square (respectively, square) edge.

We claim that the mapping $\pi : \mathbb{W}_n \to \Sigma_n(v)$ is an injection. By construction, $\pi(w) = \pi(w')$ if and only if $w = w'$, and, furthermore, $\pi(w)$ is non-backtracking. It remains to show that each $\pi(w)$ is a SAW. In order to show this, we consider the projected walk on the graph G'.

We next give some motivation for the above construction, with reference to Figure 9.1. Each non-square edge is followed by a square edge of some size-4 face F. Having touched a size-4 face F, the path π proceeds around F before departing along the currently unique non-square edge. The above rules are determined in such a way that π never traverses consecutively more then three edges of any F (see Figure 9.1).

At steps 1(b) and 2(b), certain choices are made in order that π be self-avoiding. The outcomes of these choices depend on the projections π' of the path so far onto G', and they are chosen in such a way that the numbers of right and left turns of π' have difference at most 1 (each left turn of π' is followed by a right turn, and vice versa). Therefore, in any subwalk ν of $\pi'(w)$, the numbers of right and left turns differ by at most 1. By Lemma 9.6(c) or directly, ν cannot form a cycle. Hence $\pi'(w)$ is a SAW, and the proof is complete.

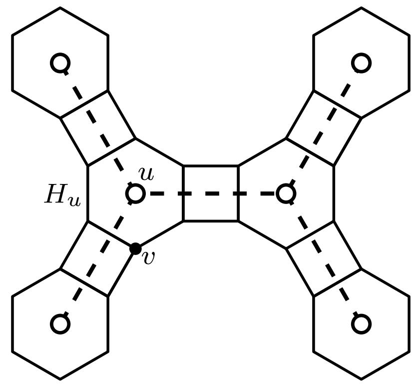

FIGURE 9.2. The graph G with an embedded copy of the graph $P = [\![p, p, p]\!]$.

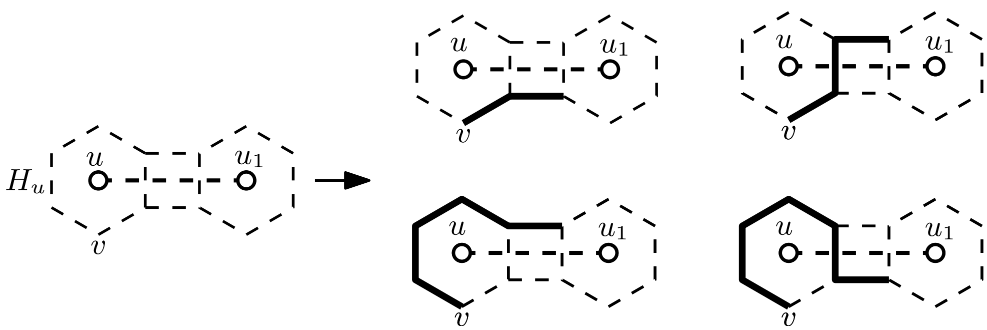

FIGURE 9.3. The step (u, u_1) on P may be mapped to any of the four SAWs on G from v, as drawn on the right.

9.6. **Proof that $\mu \geq \phi$ for $[\![4, 6, 2p]\!]$ with $p \geq 6$.** Let $G \in \mathcal{T}_3$ be infinite with type-vector $[4, 6, 2p]$ where $p \geq 6$. (We include the case $p = 6$, being the Archimedean lattice of Figure 6.1.) Associated with G is the graph $P := [\![p, p, p]\!]$ as drawn in Figure 9.2. As illustrated in the figure, each vertex u of P lies in the interior of some size-6 face of G denoted H_u. Let u be a vertex of P and let v be a vertex of H_u. Let $\pi = (u_0 = u, u_1, \dots, u_n)$ be a SAW on P from u. We shall explain how to associate with π a family of SAWs on G from v. The argument is similar to that of the proof of Proposition 7.4.

A hexagon H of G has six edges, which we denote according to approximate compass bearing. For example, $p_{\mathrm{w}}(H)$ is the edge on the west side of H, and similarly p_{nw}, p_{ne}, p_{e}, p_{se}, p_{sw}. For definiteness, we assume that H_u has orientation as in Figure 9.2, and $v \in p_{\mathrm{sw}}(H_u)$, as in Figure 9.3.

Let $\Sigma_n(u)$ be the set of n-step SAWs on P from u, the first edge of which is either north-westwards or eastwards (that is, away from $p_{\mathrm{sw}}(H_u)$). *Suppose the first step of* *the SAW $\pi \in \Sigma_n(u)$ is to the neighbour u_1 that lies eastwards of u* (the other cases are similar). With the step (u, u_1), we may associate any of four SAWs on G from v to $p_{\mathrm{w}}(H_{u_1})$, namely those illustrated in Figure 9.3. These paths have lengths 2, 3, 5, 6. If u_1 lies to the north-west of u, the corresponding four paths have lengths 3, 4, 4, 5.

We now iterate the above construction. At each step of π, we construct a family of 4 SAWs on G that extend the walk on G to a new hexagon. When this process is complete, the ensuing paths on G are all SAWs, and they are distinct.

Let $Z_P(\zeta)$ (respectively, $Z_G(\zeta)$) be the generating function of SAWs on P from u (respectively, on G from v), subject to above italicized assumption. In the above construction, each step of π is replaced by one of four paths, with lengths lying in either $(2, 3, 5, 6)$ or $(3, 4, 4, 5)$, depending on the initial vertex of the segment. Since

$$\zeta^2 + \zeta^3 + \zeta^5 + \zeta^6 \ge \zeta^3 + 2\zeta^4 + \zeta^5 \quad [= \zeta^3(1+\zeta)^2], \qquad \zeta \in \mathbb{R},$$

we have, by the argument that led to (7.13), that

$$Z_P(\zeta^3(1+\zeta)^2) \le Z_G(\zeta), \qquad \zeta \ge 0. \tag{9.7}$$

Let $z > 0$ satisfy

$$z^3(1+z)^2 = \frac{1}{\mu(P)}. \tag{9.8}$$

Since $1/\mu(P)$ is the radius of convergence of Z_P, (9.7) implies $z \ge 1/\mu(G)$, which is to say that

$$\mu(G) \ge \frac{1}{z}. \tag{9.9}$$

As in Section 9.3, $\mu(P) \ge \phi$. It suffices for $\mu(G) \ge \phi$, therefore, to show that the (unique) root in $(0, \infty)$ of

$$x^3(1+x)^2 = \frac{1}{\phi}$$

satisfies $x \le 1/\phi$, and it is easily checked that, in fact, $x = 1/\phi$.

Remark 9.8 (Archimedean lattice $\mathbb{A} = [\![4, 6, 12]\!]$)**.** *The inequality $\mu(\mathbb{A}) \ge \phi$ may be strengthened. In the special case $p = 6$, we have that $\mu(P) = \sqrt{2+\sqrt{2}}$; see* [6]*. By* (9.8)–(9.9)*, $\mu(G) \ge 1.676$.*

9.7. **Proof that $\mu \ge \phi$ for $[\![5, 8, 8]\!]$.** Let $G \in \mathcal{T}_3$ be infinite with type-vector $[\![5, 8, 8]\!]$. The proof in this case is essentially the same as that of Section 9.5, but with squares replaced by pentagons.

Let G' be the simple graph obtained from G by contracting each size-5 face of G to a vertex. As in the corresponding step at the beginning of Section 9.5, we have that

$G' \in \mathcal{T}_5$ is infinite with type-vector $[4, 4, 4, 4, 4]$. Recall Lemma 9.6(c). A midpoint of G is called *pentagonal* if it belongs to a size-5 face, and *non-pentagonal* otherwise.

We opt to consider SAWs that start and end at midpoints of edges. Let m be the midpoint of some non-pentagonal edge of G, and let $\Sigma_n(m)$ be the set of n-step SAWs on G from m. We will find an injection from $\mathbb{W}_n$ to $\Sigma_n(m)$. Let $w = (w_1 w_2 \cdots w_n) \in \mathbb{W}_n$. We construct as follows a non-backtracking path $\pi = \pi(w)$ on G starting from m. The first step of $\pi(w)$ is (v, v') where v' is an arbitrarily chosen midpoint adjacent to m. We write $\pi = (\pi_0, \pi_1, \dots, \pi_n)$.

For any path π' of G', let $\rho(\pi') = r(\pi') - l(\pi')$, where $r(\pi')$ (respectively, $l(\pi')$ is the number of right (respectively, left) turns of π'. Since paths move between midpoints, each step of π' involves a turn, and thus the terminology is consistent with its previous use.

We iterate the following for $k = 2, 3, \dots, n$ (cf. the construction of Section 9.5).

1. Suppose $(w_{k-1} w_k) = (\mathrm{HV})$. The midpoint π_k is always pentagonal.
   (a) If π_{k-1} is pentagonal, the next point π_k is also pentagonal.
   (b) Suppose π_{k-1} is non-pentagonal. On contracting G to G', the path on G, so far, gives rise to a non-backtracking path π' on G'. If $\rho(\pi') < 0$ (respectively, $\rho(\pi') \geq 0$), then the next turn of π is to the left (respectively, right).
2. Suppose $(w_{k-1} w_k) = (\mathrm{HH})$.
   (a) If π_{k-1} is pentagonal, then π_k is non-pentagonal.
   (b) Suppose π_{k-1} is non-pentagonal. In the notation of 1(b) above, if $\rho(\pi') < 0$ (respectively, $\rho(\pi') \geq 0$), then the next turn of π is to the right (respectively, left).
3. Suppose $(w_{k-1} w_k) = (\mathrm{VH})$, and note that π_{k-1} is necessarily pentagonal. If π_{k-2} is pentagonal (respectively, non-pentagonal), then π_k is non-pentagonal (respectively, pentagonal).

We claim that the mapping $\pi : \mathbb{W}_n \to \Sigma_n(m)$ is an injection, and this claim is justified very much as in the corresponding step of Section 9.5. It is straightforward that π is an injection from $\mathbb{W}_n$ to the set of n-step non-backtracking paths of G from m, and it suffices to show that any $\pi(w)$ is self-avoiding. For $w \in \mathbb{W}_n$, at most three consecutive edges of $\pi(w)$ are pentagonal. As in Section 9.5, we need to show that, after contracting each pentagon to a vertex, the ensuing non-backtracking walk $\pi'(w)$ is a SAW on G'. For any subwalk ν of $\pi'(w)$, it may be checked (as in the proof of Section 9.5) that its numbers of right and left turns differ by at most 1. By Lemma 9.6(c) or directly, ν cannot form a cycle. Hence $\pi'(w)$ is a SAW, and the proof is complete.

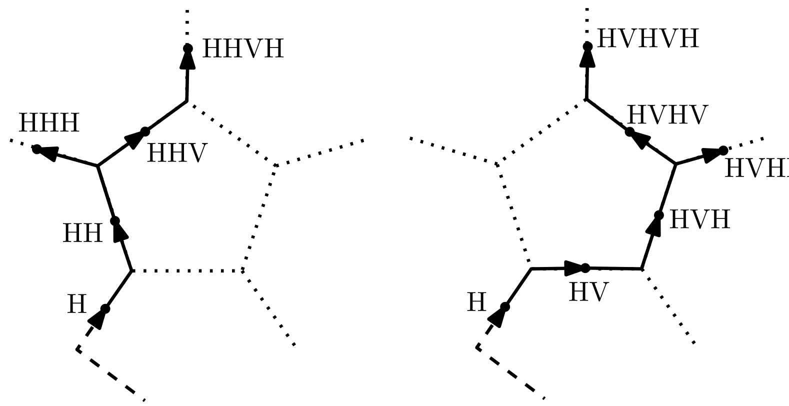

FIGURE 9.4. The dashed line is the projected SAW π' on G', assumed in the figure to satisfy $\rho(\pi') \geq 0$. When $\rho(\pi') \geq 0$ (respectively, $\rho(\pi') < 0$), the projection may move leftwards but not rightwards (respectively, rightwards but not leftwards) at its next pentagon.

10. Groups with two or more ends

10.1. **Groups with many ends.** The number of *ends* of a connected graph G is the supremum over its finite subgraphs H of the number of infinite components that remain after removal of H. We recall from [31, Prop. 6.2] that the number of ends of an infinite transitive graph is invariably 1, 2, or ∞. Moreover, a two-ended (respectively, ∞-ended) graph is necessarily amenable (respectively, non-amenable). The number of ends of a finitely presented group is the number of ends of any of its Cayley graphs.

We present two principal theorems in this section concerning Cayley graphs of multiply ended groups, and further results in Section 10.3. Theorems 10.1 and 10.2 are proved, respectively, in Sections 10.2 and 10.4. As in [17], all Cayley graphs in this paper are in their *simple* form, that is, multiple edges are allowed to coalesce.

Theorem 10.1. *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group with two ends. Any Cayley graph G of Γ with degree 3 or more satisfies $\mu(G) \geq \phi$.*

We turn to our main result for ∞-ended Cayley graphs of finitely generated groups $\Gamma = \langle S \mid R \rangle$. For clarity, we shall consider only finite generating sets S with $\mathbf{1} \notin S$ and which are *symmetric* in that $S = S^{-1}$. A symmetric set of generators S is called *minimal* if no proper subset is a symmetric set of generators.

Theorem 10.2. *Let Γ be a finitely generated group with infinitely many ends. There exists a minimal symmetric set of generators S such that the Cayley graph G of Γ with respect to S has connective constant satisfying $\mu(G) \geq \phi$.*

We do not know whether *every* Cayley graph of such Γ satisfies $\mu \geq \phi$. Certain partial results in this direction are presented in Section 10.3, and will be used in the proof of Theorem 10.2. Neither do we do know whether the above two results can be extended to multiply ended transitive *graphs*. Indeed, we have no example of a 2-ended, transitive, cubic graph that is not a Cayley graph (see [39]).

10.2. **Proof of Theorem 10.1.** We are grateful to Anton Malyshev for his permission to present his ideas in this proof. Let Γ be as in the statement of the theorem, and recall from [5, Thm 1.6] (see also [26, 32]) that there exists $\beta \in \Gamma$ with infinite order such that the infinite cyclic subgroup $\mathcal{H} := \langle \beta \rangle$ of Γ has finite index, and β preserves the ends of Γ. By Poincaré's theorem for subgroups, we may choose β such that $\mathcal{H} \trianglelefteq \Gamma$. We write ω_1 for the end of Γ containing the ray $\{\beta^k \mathbf{1} : k = 1, 2, \dots\}$, and ω_0 for its other end.

Let $F : \mathcal{H} \to \mathbb{Z}$ be given by $F(\beta^n) = n$, and let G be a locally finite Cayley graph of Γ. By [19, Thm 3.4(ii)], there exists a harmonic, $\mathcal{H}$-difference-invariant function $h : \Gamma \to \mathbb{R}$ that agrees with F on $\mathcal{H}$.

Let g be a harmonic function on G. For an edge $\vec{e} = [u, v\rangle$ of G endowed with an orientation, we write $\Delta g(\vec{e}) = g(v) - g(u)$. A *cut* of G is a finite set of edges that separates the two ends of G; a cut is *minimal* if no strict subset is a cut. The (g-)*size* of a cut C is given as the aggregate g-flow across C, that is,

$$s_C(g) = \sum_{\vec{e} \in C} \Delta g(\vec{e}),$$

where the sum is over all edges in C oriented such that initial vertex (respectively, final vertex) of each edge is connected in $G \setminus C$ to ω_0 (respectively, ω_1). Since g is harmonic, $s_C(g)$ is constant for all minimal cuts C (this elementary fact follows, for example, by the discussion after [14, Defn 1.14]); we write $s(g) := s_C(g)$ for the *size* of g. Turning to the above function h, we have that $s(h) > 0$. Here is one way of seeing the last statement. Let C be a cut, and choose n sufficiently large that some finite component X of $G \setminus (C_1 \cup C_2)$ contains the vertices $\mathbf{1}$ and β, where $C_1 = \beta^{-n} C$, $C_2 = \beta^n C$. If $s_C(h) = 0$ then, by $\mathcal{H}$-difference-invariance, $s_{C_i}(h) = 0$ for $i = 1, 2$. By the theory of electrical networks and flows (see [14, Chap. 1]), we have that h is constant on X, and in particular $h(\mathbf{1}) = h(\beta)$, a contradiction.

We now develop the argument of Proposition 6.1. Let $\{\kappa_i : i \in I\}$, be representatives of the cosets of $\mathcal{H}$, so that $\Gamma/\mathcal{H} = \{\kappa_i \mathcal{H} : i \in I\}$ and $|I| < \infty$. For $\kappa \in \Gamma$, we write $\operatorname{sign}(\kappa) = 1$ (respectively, $\operatorname{sign}(\kappa) = -1$) if the ends of Γ are κ-invariant (respectively, the ends are swapped under κ). Note that

$$s(\kappa h) = \operatorname{sign}(\kappa) s(h) \tag{10.1}$$

where $\kappa h(\alpha) := h(\kappa \alpha)$ for $\alpha \in \Gamma$.

Let $g : \Gamma \to \mathbb{R}$ be given by

$$g(\alpha) = \sum_{i \in I} \operatorname{sign}(\kappa_i) h(\kappa_i \alpha), \qquad \alpha \in \Gamma. \tag{10.2}$$

Since g is a linear combination of harmonic functions, it is harmonic. Furthermore, (as in the proof of Proposition 6.1), g is Γ-*skew*-difference-invariant in that

$$g(\alpha v) - g(\alpha u) = \operatorname{sign}(\alpha)[g(v) - g(u)], \qquad u, v \in \Gamma,\ \alpha \in \Gamma. \tag{10.3}$$

By (10.1) and (10.2), $s(g) = |I| s(h) > 0$, whence g is non-constant.

Let a, b, c denote the values of $g(v) - g(u)$ for $v \in \partial u$. By (10.3), a, b, c are independent of the choice of u up to negation, and, since g is harmonic, $a+b+c = 0$. By re-scaling and re-labelling where necessary, since g is non-constant we may assume $|a|, |b| \le c = 1$. The directed edge $\vec{e} = [u, v\rangle$ is *labelled* with the corresponding letter (with ambiguities handled as below), and is allocated weight $\Delta g(\vec{e})$. Thus, a directed edge labelled d has weight $\pm d$.

A SAW $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ is called *maximal* if $g(\pi_k) < g(\pi_n)$ for $k < n$. We shall construct a family of maximal SAWs π of sufficient cardinality to yield the claim. Choose (π_0, π_1) such that $g(\pi_1) = g(\pi_0) + 1$. There are three possibilities for the vector (a, b, c).

(a) Suppose $(a, b, c) = (0, -1, 1)$. For $n \ge 1$, a maximal SAW $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ can be extended to two distinct maximal SAWs by adding either (i) the directed edge $[\pi_n, w\rangle$ with weight 1, or (ii) the directed edge $[\pi_n, w\rangle$ with weight 0, followed by the edge $[w, x\rangle$ with weight 1. The number w_n of such walks of length n from a given starting point satisfies $w_n = w_{n-1} + w_{n-2}$, whence $\mu \ge \phi$.

(b) Suppose $(a, b, c) = (-\frac{1}{2}, -\frac{1}{2}, 1)$. Since there are no odd cycles comprising only edges with weight $\pm\frac{1}{2}$, the labels of such edges, $\langle u, v\rangle$ say, may be arranged in such a way that $[u, v\rangle$ and $[v, u\rangle$ receive the same label. A maximal SAW $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ that ends with a c-edge can be extended by following sequences of additional directed edges labelled one of ac, bc, $abac$, $babc$, thus creating a new walk denoted π'. Since π is maximal, we have that $h(\pi_n) - h(\pi_{n-1}) = 1$. By tracking the signs of the weights of any additional edge, we see that any such π' is both self-avoiding and maximal. The number w_n of such SAWs with length n from a given starting point satisfies $w_n = 2w_{n-2} + 2w_{n-4}$. Therefore, $\lim_{n\to\infty} w_n^{1/n}$ equals the root in $[1, \infty)$ of the equation $x^4 = 2(x^2 + 1)$, namely $x = \sqrt{1 + \sqrt{3}} > \phi$.

(c) Suppose $b < a < 0$, $-a - b = c = 1$. There are no cycles comprising only directed edges labelled either a or b. A maximal SAW $\pi = (\pi_0, \pi_1, \dots, \pi_n)$ that ends with a c-edge can be extended by following edges labelled either (i) ac, or (ii) bc, $babc$, $bababc$, and so on; any such extension results in a maximal

SAW, as in case (b) above. The number w_n of such SAWs with an odd length n from a given starting point satisfies $w_n = 2w_{n-2} + w_{n-4} + w_{n-6} + \cdots + w_1$. It is easily checked that $w_n \geq C\phi^n$, as required.

The proof is complete.

10.3. **Multiply ended graphs.** Let Γ be an infinite, finitely generated group. By Stalling's splitting theorem (see [35, 36]), Γ has two or more ends if and only if one of the following two properties holds.

(i) $\Gamma = \langle S, t \mid R, t^{-1}C_1 t = C_2\rangle$ is an HNN extension, where $H = \langle S \mid R\rangle$ is a presentation of the group H, C_1 and C_2 are isomorphic finite subgroups of H, and t is a new symbol.

(ii) $\Gamma = H *_C K$ is a free product with amalgamation, where H, K are groups, and $C \neq H, K$ is a finite group.

Part (i) above may be taken as the definition of an HNN extension, named after the authors of [24].

Readers are referred to [3, 28] for the definition and background of amalgamated products. Although the group C, in (ii), is not required to be a subgroup of either H or K, when we speak of C as such a subgroup we mean the image of C under the corresponding map of the amalgamated product. We next remind the reader of the normal form theorem for such groups, and then we summarise the results of this section in Theorem 10.4.

Theorem 10.3 (Normal form, [3, Sect. 2.2], [28, p. 187], [30, Cor. 4.4.1])**.**

(a) *Every $g \in H *_C K$ can be written in the* reduced form $g = cv_1 \cdots v_n$ *where $c \in C$, and the v_i lie in either $H \setminus C$ or $K \setminus C$ and they alternate between these two sets. The length $l(g) := n$ of g is uniquely determined, and $l(g) = 0$ if and only if $g \in C$. Two such expressions of the form $v_1 \cdots v_n$, $w_1 \dots w_n$ represent the same element in $H *_C K$ if and only if there exist c_0 $(= \mathbf{1}), c_2, \dots, c_n$ $(= \mathbf{1}) \in C$ such that $w_k = c_{k-1} v_k c_k^{-1}$.*

(b) *Let A (respectively, B) be a set of right coset representatives of (the image of) C in H (respectively, K), where the representatives of C are $\mathbf{1}$. Every $g \in H *_C K$ can be expressed uniquely in the* normal form $g = cx_1 \cdots x_n$ *where $c \in C$, and the x_i lie in either A or B, and they alternate between these two sets.*

Theorem 10.4.

(i) *Let Γ be an HNN extension as above. Any locally finite Cayley graph G of Γ admits a group height function (see [19]). If such G is cubic, then $\mu(G) \geq \phi$.*

(ii) *Let Γ be an amalgamated free product as above.*

(a) *Suppose* $C = \{\mathbf{1}\}$, *and let* S_H *(respectively,* S_K*) be a finite symmetric generator set of* H *(respectively,* K*). If the generator set* $S = S_H \cup S_K$ *of* Γ *satisfies* $|S| \geq 3$, *then it generates a Cayley graph* G *with* $\mu(G) \geq \phi$.

(b) *Suppose* $C \neq \{\mathbf{1}\}$, *Any symmetric generator set* S *satisfying both*
   1. $S \cap C \neq \varnothing$, $|S| \geq 3$, *and*
   2. *there exists* $s_1 \in S$ *(respectively,* $s_2 \in S$*) with a normal form beginning with an element of* $H \setminus C$ *(respectively, an element of* $K \setminus C$*),*

   generates a Cayley graph G *with* $\mu(G) \geq \phi$.

(c) *Suppose* $C \neq \{\mathbf{1}\}$ *and* C *is a normal subgroup of both* H *and* K. *Any symmetric generator set* S *satisfying* $S \cap C \neq \varnothing$ *generates a Cayley graph* G *with* $\mu(G) \geq \phi$.

Proof of Theorem 10.4(i). Let S_H be a finite set of generators of H, and define $h : \Gamma \to \mathbb{Z}$ by, for $v \in \Gamma$,

$$\begin{aligned} h(vt) - h(v) &= 1, \\ h(vs) - h(v) &= 0. \qquad s \in S_H. \end{aligned}$$

By [19, Thm 4.1(a)] applied to the unit vector in $\mathbb{Z}^{S_H}$ with 1 in the entry labelled t, h is a group height function (and hence a transitive graph height function) on any locally finite Cayley graph of Γ. When G is cubic, the inequality $\mu(G) \geq \phi$ follows by Theorem 3.1(b). $\square$

We turn to the proof of Theorem 10.4(ii). By [18, Thm 1], we have $\mu(G) \geq \sqrt{3} > \phi$ if the symmetric generator-set S satisfies $|S| \geq 4$. We may, therefore, assume henceforth that $|S| = 3$. It is straightforward to check the following lemma.

Lemma 10.5. *Let* Γ *be a finitely generated group with two or more ends. Let* $S = \{s, s_1, s_2\}$ *be a symmetric generator set whose Cayley graph* G *is cubic. Then, subject to permutation of the generators, exactly one of the following holds.*

A. $s^2 = s_1^2 = s_2^2 = \mathbf{1}$.
B. $s^2 = s_1 s_2 = \mathbf{1}$.

Proof of Theorem 10.4(a). When $C = \{\mathbf{1}\}$, Γ is a free product. Since S is symmetric, without loss of generality we may write $S = \{s, s_1, s_2\}$ with $s \in H$, $s_1, s_2 \in K$, and either A or B of Lemma 10.5 holds. It suffices to construct an injection from $\mathbb{W}_n$ into the set of n-step SAWs on G starting from $\mathbf{1}$.

Assume A of Lemma 10.5 holds. Let $w \in \mathbb{W}_n$, and construct a SAW $\pi = (\pi_0, \pi_1, \ldots, \pi_n)$ on G as follows. We set $\pi_0 = \mathbf{1}$, $\pi_1 = s$, and we iterate the following construction for $k = 2, 3, \ldots, n$.

1. If $w_k = \mathrm{V}$, the kth edge of π is that labelled s_2.

2. If $(w_{k-1}w_k) = (\mathrm{HH})$, the kth edge is that labelled s (respectively, s_1) if the $(k-1)$th edge is labelled s_1 (respectively, s).
3. If $(w_{k-1}w_k) = (\mathrm{VH})$, the kth edge is that labelled s.

The outcome may be expressed in the form $\pi = p_1 s_2 p_2 s_2 \cdots$ where each $p_i = s s_1 s s_1 \cdots$ is a word of alternating s and s_1. The letters in π alternate between the two sets $H \setminus \{\mathbf{1}\}$ and $K \setminus \{\mathbf{1}\}$ with the possible exception of isolated appearances of $s_1 s_2$, each of which is in $K \setminus \{\mathbf{1}\}$. Now $s_1 s_2 \neq \mathbf{1}$, whence $s_1 s_2 \in K \setminus \{\mathbf{1}\}$. The claim follows by Theorem 10.3(a).

Assume B of Lemma 10.5 holds. Let G_n be the Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_n$ for $3 \le n < \infty$. We have that $H \cong \mathbb{Z}_2$, and the Cayley graph of K is a cycle of length at least 4. Therefore, G is isomorphic to G_n for some $n \ge 4$. The exact value $\mu(G)$ may be deduced from [8, Thm 3.3], but it suffices here to note that

$$\mu(G) \ge \mu(G_3) > \phi. \tag{10.4}$$

The above strict inequality holds since G_3 is the graph obtained from the cubic tree by the Fisher transformation of [15] (see item H of Section 4.1). □

Proof of Theorem 10.4(b). Let Γ, G, S, s_1, s_2 be as given, and $s \in S \cap C$. We may write $S = \{s, s_1, s_2\}$ and $s^2 = \mathbf{1}$. Either A or B of Lemma 10.5 holds. Under A, the normal form of s_1 (respectively, s_2) begins and ends with elements of $H \setminus C$ (respectively, $K \setminus C$). Under B, the normal form of s_1 (respectively, s_2) ends with an element of $K \setminus C$ (respectively, $H \setminus C$).

Let $w \in \mathbb{W}_n$, and construct a SAW π on G as follows. We set $\pi_0 = \mathbf{1}$, $\pi_1 = s_1$, and we iterate the following for $k = 2, 3, \dots, n$.

1. Suppose $w_k = \mathrm{V}$. The kth edge of π is that labelled s.
2. Suppose $(w_{k-1}w_k) = (\mathrm{HH})$. The kth edge is that labelled s_1 (respectively, s_2) if the $(k-1)$th edge is labelled by the member of $\{s_1, s_2\}$ whose normal form ends with an element of $K \setminus C$ (respectively, $H \setminus C$).
3. Suppose $(w_{k-1}w_k) = (\mathrm{VH})$. The kth edge is that labelled s_1 (respectively, s_2) if the $(k-2)$th step is labelled by the member of $\{s_1, s_2\}$ whose normal form ends with an element of $K \setminus C$ (respectively, $H \setminus C$).

We claim that the resulting π is a SAW. If not, there exists a representation of the identity of the form

$$\mathbf{1} = p_1 s p_2 s \cdots s p_r, \tag{10.5}$$

where $r \ge 1$, and each p_i is a non-empty alternating product of elements of $H \setminus C$ and $K \setminus C$ such that $p_1 p_2 \cdots p_r$ is such a product also, with some aggregate length $L \ge 1$ (we allow also that p_1 and/or p_r may equal $\mathbf{1}$). We move the occurrences of s to the left in (10.5) by noting that a term of the form gs, with $g \in (H \setminus C) \cup (K \setminus C)$, lies in some right coset of C, say $gs = ca$ with $c \in C$ and $a \in A \cup B$ (in the notation of Theorem 10.3(b)). We iterate this procedure to obtain a normal form $g = c'v_1v_2\cdots v_L$, which cannot equal the identity since $L \geq 1$. This contradicts (10.5), and the claim of part (b) follows. □

Proof of Theorem 10.4(c). We may assume $|S| = 3$, and we write $S = \{s, s_1, s_2\}$. Clearly, $|S \cap C| \leq 2$, since C is a proper subgroup of both H and K.

Assume that $|S \cap C| = 2$, and let $\{s_1, s_2\} = S \cap C$ and $\{s\} = S \setminus C$, so that $s^2 = \mathbf{1}$. Since C is a normal subgroup of both H and K, we have by Theorem 10.3 that $\alpha C \alpha^{-1} = C$ for $\alpha \in \Gamma$. Since S generates Γ, every $g \in \Gamma$ may be expressed as a word in the alphabet $\{s, s_1, s_2\}$, and hence in the form $g = c_1 s c_2 s \cdots s c_r$ with $c_i \in C$. By the normality of C, $g = cs^k$ for some $c \in C$, $k \in \mathbb{N}$. However, $s^2 = \mathbf{1}$, so that there are only finitely many choices for g, a contradiction.

Therefore, we have $|S \cap C| = 1$, and we write $\{s\} = S \cap C$ and $\{s_1, s_2\} = S \setminus C$. Either A or B of Lemma 10.5 holds.

If one of $\{s_1, s_2\}$ has a normal form starting from an element in $H \setminus C$, and the other has a normal form starting from an element in $K \setminus C$, then the claim follows by Theorem 10.4(b). For the remaining case, we may assume without loss of generality that the normal forms of both s_1 and s_2 start in $H \setminus C$. It follows that, under either A and B, both normal forms end in $H \setminus C$.

Here is an intermediate lemma, proved later.

Lemma 10.6. *For $j \in \mathbb{N}$,*

$$\begin{aligned} &\text{if A holds,} \quad (s_1s_2)^j, (s_1s_2)^{j-1}s_1, (s_2s_1)^{j-1}s_2 \notin C, \\ &\text{if B holds,} \quad s_1^j \notin C. \end{aligned} \tag{10.6}$$

We shall construct an injection from the set $\mathbb{W}_n$ into the set of n-step SAWs on G from $\mathbf{1}$. For $w \in \mathbb{W}_n$, we construct a SAW π on G with $\pi_0 = \mathbf{1}$, $\pi_1 = s_1$ as follows.

1. Each letter V in w corresponds to an edge in π with label s.
2. Assume A holds. The letters H in w correspond to the elements of the sequence $(s_1, s_2, s_1, s_2, \dots)$, in order. That is, for $k \geq 1$, the $(2k-1)$th (respectively, $(2k)$th) occurrence of H corresponds to s_1 (respectively, s_2).
3. Assume B holds. The letters H in w correspond to edges labelled s_1.

We show next that the resulting walks are self-avoiding.

Assume B holds. If one of the corresponding walks fails to be self-avoiding, there exists a representation of the identity as

$$\mathbf{1} = s_1^{k_1} s s_1^{k_2} s \cdots s s_1^{k_r}, \tag{10.7}$$

where $r \geq 1$, $k_1, k_r \in \mathbb{N} \cup \{0\}$, $k_i \in \mathbb{N}$ for $2 \leq i < r$, and $K = k_1 + \cdots + k_r \geq 1$. Since C is normal, we have $\mathbf{1} = cs_1^K$ for some $c \in C$. This contradicts (10.6), and we deduce that each such π is self-avoiding.

Assume A holds. The above argument remains valid with adjusted (10.7), and yields that $\mathbf{1} = ct$ for some $c \in C$ and

$$t \in \big\{(s_1s_2)^j, (s_2s_1)^j, (s_1s_2)^{j-1}s_1, (s_2s_1)^{j-1}s_2 : j \in \mathbb{N}\big\}.$$

Note that $[(s_2s_1)^j]^{-1} = (s_1s_2)^j$. Therefore, $t = c^{-1} \in C$, in contradiction of (10.6). We deduce that each such π is self-avoiding. $\square$

Proof of Lemma 10.6. Let $t_1 \in H \setminus C$ and $t_2 \in K \setminus C$, so that

$$l([t_1t_2]^n) = 2n, \qquad n \in \mathbb{N}.$$

Since S generates $H *_C K$, we can express t_1t_2 as a word in the alphabet $\{s, s_1, s_2\}$, denoted $t(s, s_1, s_2)$. Let $\widetilde{t}$ be the word obtained from $t(s, s_1, s_2)$ by removing all occurrences of s and using the group relations on S to reduce the outcome to a minimal form. More precisely, since $s \in C$ and C is normal in H and K, every occurrence of s in $t(s, s_1, s_2)$ may be moved leftwards to obtain $t(s, s_1, s_2) = ct'(s_1, s_2)$ for some $c \in C$ and some word $t'(s_1, s_2)$. On reducing t' by the group relations on S, we obtain $\widetilde{t}$, and note that

$$\text{(10.8)}\qquad \begin{aligned} &\text{if A holds,} \quad \widetilde{t} \text{ is an alternating product of } s_1 \text{ and } s_2.\\ &\text{if B holds,} \quad \widetilde{t} \in \{s_1^k, s_2^k : k \in \mathbb{N}\}. \end{aligned}$$

Since $\widetilde{t} = c^{-1}t_1t_2$, we have $l(\widetilde{t}) = l(t_1t_2) = 2$. By the normality of C again, there exists $c' \in C$ such that $l(\widetilde{t}^n) = l(c'[t_1t_2]^n) = 2n$. In particular, by Theorem 10.3(a),

$$\text{(10.9)}\qquad \widetilde{t}^n \notin C, \qquad n \in \mathbb{N}.$$

Suppose A holds. By (10.8),

$$\text{(10.10)}\qquad \widetilde{t} \in \big\{(s_1s_2)^k, (s_2s_1)^k, (s_1s_2)^{k-1}s_1, (s_2s_1)^{k-1}s_2 : k \in \mathbb{N}\big\}.$$

If $\widetilde{t} \in \{(s_1s_2)^{k-1}s_1, (s_2s_1)^{k-1}s_2 : k \in \mathbb{N}\}$, we have $\widetilde{t}^2 = 1$, which contradicts (10.9). Therefore,

$$\widetilde{t} \in \big\{(s_1s_2)^k, (s_2s_1)^k : k \in \mathbb{N}\big\}.$$

If $(s_1s_2)^j \in C$ for some $j \in \mathbb{N}$, then

$$\widetilde{t}^j \in \big\{[(s_1s_2)^j]^k, [(s_2s_1)^j]^k : k \in \mathbb{N}\big\} \subseteq C,$$

which contradicts (10.9). Hence $(s_1s_2)^j \notin C$ for $j \in \mathbb{N}$, as required. Suppose next that $c := (s_1s_2)^{j-1}s_1 \in C$ for some $j \in \mathbb{N}$. Since C is a normal subgroup of both H and K, we have $s_2cs_2^{-1} = s_2(s_1s_2)^j \in C$. Therefore, $(s_1s_2)^{2j} \in C$, which contradicts (10.9) as above. A similar argument holds for the case $c := (s_2s_1)^{j-1}s_2$. The first statement of (10.6) is proved.

Suppose B holds. A similar argument is valid by (10.8), as follows. Suppose $\widetilde{t} = s_1^k$ (a similar argument holds in the other case, using the fact that $s_1s_2 = \mathbf{1}$). If $s_1^j \in C$

for some $j \in \mathbb{N}$, then $\widetilde{t}^j = (s_1^j)^k \in C$, in contradiction of (10.9). The second statement of (10.6) follows. □

10.4. **Proof of Theorem 10.2.** Since Γ has infinitely many ends, we have $|S| \geq 3$. The claim follows by Theorem 10.4(i) when Γ is an HNN extension, and we assume henceforth that $\Gamma = H *_C K$ is an amalgamated product as in Section 10.3. If Γ has a minimal symmetric generator set S satisfying $|S| \geq 4$, the corresponding Cayley graph G satisfies $\mu(G) \geq \sqrt{3} > \phi$. We may, therefore, assume that every minimal symmetric generator set of Γ has cardinality 3.

By Theorem 10.3, we may pick a symmetric generator set S satisfying $S \subseteq H \cup K$. We may assume S is minimal, as follows. If S is not minimal, there exists $s \in S$ that is expressible as a word in the alphabet $S \setminus \{s, s^{-1}\}$, and we may remove such s and its inverse to obtain a new symmetric generator set S'. By iteration, we obtain a minimal set S''.

Since C is a proper subset of both H and K, there exist $s_1 \in S \cap (H \setminus C)$ and $s_2 \in S \cap (K \setminus C)$. Let $s \in S \setminus \{s_1, s_2\}$ and, without loss of generality, assume $s \in H$. If $s \in C$, the claim follows by Theorem 10.4(b). Assume without loss of generality that $s \in H \setminus C$, so that

$$s, s_1 \in H \setminus C, \quad s_2 \in K \setminus C. \tag{10.11}$$

By Lemma 10.5, one of the following occurs.

A. $s^2 = s_1^2 = s_2^2 = \mathbf{1}$.
B. $s_2^2 = ss_1 = \mathbf{1}$.

Assume A occurs. If $ss_1 \in C$, consider the minimal generator set $\widetilde{S} = \{ss_1, s_1, s_2\}$. If $\widetilde{S}$ is not symmetric, there exists a minimal symmetric generator set of Γ with cardinality at least 4, a contradiction. On the other hand, if $\widetilde{S}$ is symmetric, then the claim follows by Theorem 10.4(b).

Suppose $ss_1 \in H \setminus C$. We construct an injection from $\mathbb{W}_n$ into the set of n-step SAWs on G from $\mathbf{1}$ as follows. Let $w \in \mathbb{W}_n$, and let π denote the following walk on G. Set $\pi_0 = \mathbf{1}$, $\pi_1 = s_2$.

1. At each occurrence of V in w, π traverses the edge labelled s_1.
2. Any run of the form H^r in w corresponds to a walk $s_2, s_2s, s_2ss_2, s_2ss_2s, \dots$ of length r in π.

The resulting π traverses the edges of G in a word of the form $\alpha = (a_1s_1a_2s_1 \cdots s_1a_r)$ where each a_i is a word starting with s_2 and alternating s and s_2 (we allow a_r to be empty). By (10.11), each a_i is in the reduced form of Theorem 10.3(a). At each occurrence of s_1 in α, there may be a consecutive appearance of generators in $H \setminus C$ taking the form ss_1. At each such instance, we may group ss_1 as a single element of $H \setminus C$, thus obtaining a normal form for α.

If π is not self-avoiding, some non-trivial subword of α equals the identity $\mathbf{1}$. By Theorem 10.3, this subword must have length 0, which cannot occur. Therefore, π is a SAW.

Assume B occurs, so that $s_1^{-1} = s$. If $C = \{\mathbf{1}\}$, the claim follows by Theorem 10.4(a). Assume $C \neq \{\mathbf{1}\}$. We construct an injection from a suitable set W_n into the set of n-step SAWs on the Cayley graph G from $\mathbf{1}$, as follows. Let $w \in W_n$ and write π for the corresponding walk on G.

(a) Suppose $s_1^2 \notin C$. Let Π_n be the set of n-step walks π on G with $\pi_0 = \mathbf{1}$ and satisfying: π may expressed as a word of the form $\alpha = (a_1 s_2 a_2 s_2 \cdots s_2 a_r)$ where each a_i lies in $T := \{s, s^2, s_1, s_1^2\}$ (we allow a_1 and a_r to be empty). Note that $T \subseteq H \setminus C$ and $s_2 \in K \setminus C$. Such π is self-avoiding since, if not, some non-trivial subword of α is a reduced form with length 0, which cannot occur.

Let G_3 be the Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_3$, and let W_n be the set of SAWs on G_3 from a given vertex. An edge of G_3 is called *triangular* if it lies in a triangle. We define an injection $f : W_n \to \Pi_n$ as follows. Let $w \in W_n$, and follow w from its start to its end. Each time w visits a triangular edge (respectively, non-triangular edge), $f(w)$ visits an edge of G labelled either s or s_1 (respectively, s_2). As in (10.4), we have that $\mu(G) \geq \mu(G_3) > \phi$.

(b) Suppose $s_1^2 \in C$, and consider the minimal symmetric set of generators $\widetilde{S} = \{s_2, u := s_2 s_1, v := s s_2\}$ with corresponding Cayley graph $\widetilde{G}$. We construct an injection from $\mathbb{W}_n$ into the set of n-step SAWs on $\widetilde{G}$ from $\mathbf{1}$, as follows. Set $\pi_0 = \mathbf{1}$, $\pi_1 = u$, and let $k \geq 2$.
   1. If $w_k = \mathrm{V}$, the kth edge of π is labelled s_2.
   2. If $w_k = \mathrm{H}$, the kth edge of π lies in $\{u, v\}$.
      (i) If $(w_{k-1} w_k) = (\mathrm{HH})$, the kth edge of π has the same label as the $(k-1)$th.
      (ii) If $(w_{k-1} w_k) = (\mathrm{VH})$, the kth edge of π is labelled as the inverse of that of the $(k-2)$th.

The resulting π has the form of the word

$$\alpha = (u^{k_1} s_2 v^{k_2} s_2 \cdots s_2 w^{k_r}) \tag{10.12}$$

where $r \geq 2$, $k_i \geq 1$ (we allow $k_r = 0$), the terms in u and v alternate, and $w \in \{u, v\}$ as appropriate. By considering the various possibilities, we obtain that every non-trivial subword of α has non-zero length, and hence π is a SAW.

We explain the last stage as follows. Suppose the walk π contains some cycle. Then the word α of (10.12) contains a subword of the form $\beta = t_1^{l_1} s_2 t_2^{l_2} s_2 \cdots s_2 w_1^{l_m}$ that satisfies $\beta = \mathbf{1}$, where $m \geq 2$, $l_i > 0$ (we allow $l_1 = 0$

and $l_m = 0$, but not both), $\{t_1, t_2\} = \{u, v\}$, and $w_1 \in \{u, v\}$ is chosen so that the powers of u and v alternate. The only cancellations that can arise in β from the group relations on S (under case B) are of the form $s_2 s_2 = \mathbf{1}$. Such a product appears only where either (i) β ends with the sequence vs_2 (so that $l_m = 0$), or (ii) some s_2 is preceded by v and followed by u, thus making the subsequence vs_2u. At each such occurrence, exactly one cancellation occurs. The resulting word, in this reduced form β', has strictly positive length. In addition, β' is an alternating product of terms in $H \setminus C$ and $K \setminus C$. By Theorem 10.3(a), $\beta' \neq \mathbf{1}$, a contradiction. We conclude thet π is a SAW.

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