CUBIC GRAPHS AND THE GOLDEN MEAN

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Abstract. The connective constant $\mu(G)$ of a graph G is the exponential growth rate of the number of self-avoiding walks starting at a given vertex. We investigate the validity of the inequality $\mu \geq \phi$ for infinite, transitive, simple, cubic graphs, where $\phi := \frac{1}{2}(1+\sqrt{5})$ is the golden mean. The inequality is proved for several families of graphs including: (i) Cayley graphs of infinite groups with three generators and strictly positive first Betti number, and (ii) infinite, transitive, topologically locally finite (TLF) planar, cubic graphs. Bounds for μ are presented for transitive cubic graphs with girth either 3 or 4, and for certain quasi-transitive cubic graphs.

1. Introduction

Let G be an infinite, transitive, simple, rooted graph, and let σ_n be the number of n-step self-avoiding walks (SAWs) starting from the root. It was proved by Hammersley [13] in 1957 that the limit $\mu = \mu(G) := \lim_{n\to\infty} \sigma_n^{1/n}$ exists, and he called it the 'connective constant' of G. A great deal of attention has been devoted to counting SAWs since that introductory mathematics paper, and survey accounts of many of the main features of the theory may be found at [1, 17].

A graph is called *cubic* if every vertex has degree 3, and *transitive* if it is vertex-transitive (further definitions will be given in Section 2). Let $\mathcal{G}_d$ be the set of infinite, transitive, simple graphs with degree d, and let $\mu(G)$ denote the connective constant of $G \in \mathcal{G}_d$. The letter ϕ is used throughout this paper to denote the golden mean $\phi := \frac{1}{2}(1+\sqrt{5})$, with numerical value $1.618\cdots$. The basic question to be investigated here is as follows.

Question 1.1 ([10]). *Is it the case that $\mu(G) \geq \phi$ for $G \in \mathcal{G}_3$?*

This question has arisen within the study by the current authors of the properties of connective constants of transitive graphs, see [12] and the references therein. The question is answered affirmatively here for certain subsets of $\mathcal{G}_3$, but we have no

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complete answer to Question 1.1. Note that $\mu(G) \geq \sqrt{d-1} > \phi$ for $G \in \mathcal{G}_d$ with $d \geq 4$, by [10, Thm 1.1].

Here is some motivation for the inequality $\mu(G) \geq \phi$ for $G \in \mathcal{G}_3$. It well known and easily proved that the ladder graph $\mathbb{L}$ (see Figure 5.1) has connective constant ϕ. Moreover, the number of n-step SAWs can be expressed in terms of the Fibonacci sequence (an explicit such formula is given in [22]). It follows that $\mu(G) \geq \phi$ whenever there exists an injection from the set of (rooted) n-step SAWs on $\mathbb{L}$ to the corresponding set on G. One of the principal techniques of this article is to construct such injections for certain families of cubic graphs G, including (i) graphs supporting harmonic functions with certain properties, (ii) graphs supporting transitive graph height functions (this holds for many Cayley graphs), (iii) infinite, transitive, topologically locally finite (TLF) planar graphs with degree 3.

There are many infinite, transitive, cubic graphs, and we are unaware of a complete taxonomy. Various examples and constructions are described in Section 4, and the inequality $\mu \geq \phi$ is discussed in each case. In our search for cubic graphs, no counterexample has been knowingly revealed. However, there exist cubic graphs for which the inequality is neither proved nor disproved in this work, and a good example of this is the Cayley graph G of the Grigorchuk group. Our best lower bound in the last case is $\mu(G) \geq 12^{1/6} \approx 1.513$; see Example 7.3.

A substantial family of cubic graphs arises through the application of the so-called 'Fisher transformation' to a d-regular graph. We make explicit mention of the Fisher transformation here since it provides a useful technique in the study of connective constants.

This paper is structured as follows. General criteria that imply $\mu \geq \phi$ are presented in Section 3 and proved in Section 5. In Section 4 is given a list of cubic graphs known to satisfy $\mu \geq \phi$. *Transitive* graph height functions are discussed in Section 6, including sufficient conditions for their existence. Upper and lower bounds for connective constants for cubic graphs with girth 3 or 4 are stated and proved in Section 7, and these include our best results for the Cayley graph of the Grigorchuk group. In the final Section 8, it is proved that $\mu \geq \phi$ for all transitive, topologically locally finite (TLF) planar, cubic graphs.

2. Preliminaries

The graphs $G = (V, E)$ of this paper will be assumed to be connected, infinite, and simple. We write $u \sim v$ if $\langle u, v \rangle \in E$. The *degree* $\deg(v)$ of vertex v is the number of edges incident to v, and G is called *cubic* if $\deg(v) = 3$ for $v \in V$.

The automorphism group of G is written $\mathrm{Aut}(G)$. A subgroup $\Gamma \leq \mathrm{Aut}(G)$ is said to *act transitively* if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma v = w$. It *acts quasi-transitively* if there is a finite subset $W \subseteq V$ such that, for $v \in V$, there exist

$w \in W$ and $\gamma \in \Gamma$ with $\gamma v = w$. The graph is called *(vertex-)transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively).

A *walk* w on the (simple) graph G is a sequence $(w_0, w_1, \dots, w_n)$ of vertices w_i such that $n \ge 0$ and $e_i = \langle w_i, w_{i+1} \rangle \in E$ for $i \ge 0$. Its *length* $|w|$ is the number of its edges, and it is called *closed* if $w_0 = w_n$. The distance $d_G(v, w)$ between vertices v, w is the length of the shortest walk between them.

An *n-step self-avoiding walk* (SAW) on G is a walk $(w_0, w_1, \dots, w_n)$ of length $n \ge 0$ with no repeated vertices. The walk w is called *non-backtracking* if $w_{i+1} \ne w_{i-1}$ for $i \ge 1$. A *cycle* is a walk $(w_0, w_1, \dots, w_n)$ with $n \ge 3$ such that $w_i \ne w_j$ for $0 \le i < j < n$ and $w_0 = w_n$. Note that a cycle has a chosen orientation. The *girth* of G is the length of its shortest cycle. A *triangle* (respectively, *quadrilateral*) is a cycle of length 3 (respectively, 4).

We denote by $\mathcal{G}$ the set of infinite, connected, transitive, simple graphs with finite vertex-degrees, and by $\mathcal{Q}$ the set of such graphs with 'transitive' replaced by 'quasi-transitive'. The subset of $\mathcal{G}$ containing graphs with degree d is denoted $\mathcal{G}_d$, and the subset of $\mathcal{G}_d$ containing graphs with girth g is denoted $\mathcal{G}_{d,g}$. A similar notation is valid for $\mathcal{Q}_d$ and $\mathcal{Q}_{d,g}$.

Let $\Sigma_n(v)$ be the set of n-step SAWs starting at $v \in V$, and $\sigma_n(v) := |\Sigma_n(v)|$ its cardinality. *Assume that G is connected, infinite, and quasi-transitive.* It is proved in [13, 14] that the limit

$$\mu = \mu(G) := \lim_{n\to\infty} \sigma_n(v)^{1/n}, \qquad v \in V, \tag{2.1}$$

exists, and $\mu(G)$ is called the *connective constant* of G. We shall have use for the SAW *generating function*

$$Z_v(\zeta) = \sum_{\substack{\pi \text{ a SAW} \\ \text{from } v}} \zeta^{|\pi|} = \sum_{n=0}^{\infty} \sigma_n(v)\zeta^n, \qquad v \in V,\ \zeta \in \mathbb{R}.$$

We shall sometimes consider SAWs joining midpoints of edges of G.

There are two (related) types of graph functions relevant to this work. We recall first the definition of a 'graph height function', as introduced in [8] in the context of the study of connective constants.

Definition 2.1 ([8]). *Let $G \in \mathcal{Q}$. A* graph height function *on G is a pair $(h, \mathcal{H})$ such that:*

(a) *$h : V \to \mathbb{Z}$ and $h(\mathbf{1}) = 0$,*
(b) *$\mathcal{H}$ is a subgroup of $\mathrm{Aut}(G)$ acting quasi-transitively on G such that h is $\mathcal{H}$-*difference-invariant *in the sense that*
$$h(\alpha v) - h(\alpha u) = h(v) - h(u), \qquad \alpha \in \mathcal{H},\ u, v \in V,$$
(c) *for $v \in V$, there exist $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$.*

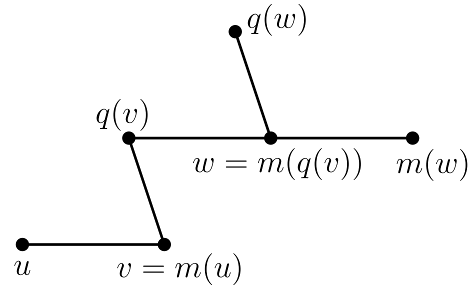

FIGURE 3.1. An illustration of the notation of equations (3.2)–(3.3).

A graph height function $(h, \mathcal{H})$ *of* G *is called* transitive *if* $\mathcal{H}$ *acts transitively on* G.

The properties of normality and unimodularity of the group $\mathcal{H}$ are discussed in [8], but do not appear to be especially relevant to the current work.

Secondly we remind the reader of the definition of a harmonic function on a graph $G = (V, E)$. A function $h : V \to \mathbb{R}$ is called *harmonic* if

$$h(v) = \frac{1}{\deg(v)} \sum_{u \sim v} h(u), \qquad v \in V.$$

There are occasional references to the Cayley graphs of finitely generated groups in this paper, and the reader is referred to [11, 12] for background material.

3. GENERAL RESULTS

Let $G = (V, E)$ be a graph. For $h : V \to \mathbb{R}$, we define two functions $m : V \to V$ and $M : V \to \mathbb{R}$, depending on h, by

$$m(u) \in \operatorname{argmax}\{h(x) - h(u) : x \sim u\}, \quad M_u = h(m(u)) - h(u), \qquad u \in V. \tag{3.1}$$

There may be more than one candidate vertex for $m(u)$, and hence more than one possible value for the term $M_{m(u)}$, which will appear later.

Let $\mathcal{Q}_{\mathrm{h}} \subseteq \mathcal{Q}_3$ be the set of infinite, cubic, quasi-transitive graphs G with the following properties: there exists $h : V \to \mathbb{R}$ such that h is harmonic and, for $u \in V$,

$$M_{m(u)} - M_u < \min\{M_u, M_{q(v)}\}, \tag{3.2}$$

$$2M_{q(v)} > M_v - M_u + M_{m(q(v))}, \tag{3.3}$$

where $q(v)$ is the unique neighbour of $v := m(u)$ other than u and $m(v)$. (The notation is illustrated in Figure 3.1.) Since h is assumed harmonic, we have $M_u \geq 0$ for $u \in V$, and hence $M_u > 0$ by (3.2).

Conditions (3.2)–(3.3) will be used in the proof of part (a) of the following theorem. Less obscure but still sufficient conditions are contained in Remark 3.2, following.

Theorem 3.1. *We have that $\mu(G) \ge \phi$ if any of the following hold.*

(a) $G \in \mathcal{Q}_{\mathrm{h}}$.
(b) $G \in \mathcal{G}_3$ *has a transitive graph height function.*
(c) $G \in \mathcal{Q}_{3,g}$ *where $g \ge 4$, and there exists a harmonic function h on G satisfying* (3.2).

Remark 3.2. *Condition* (3.2) *holds whenever there exists $A > 0$ and a harmonic function $h : V \to \mathbb{R}$ such that, for $u \in V$, $A < M_u \le 2A$. Similarly, both* (3.2) *and* (3.3) *hold whenever there exists $A > 0$ such that, for $u \in V$, $2A < M_u \le 3A$.*

Example 3.3. *Here are three examples of Theorem 3.1 in action.*

(a) *The hexagonal lattice supports a harmonic function h with $M_u \equiv 1$.*
(b) *The Cayley graph of a finitely presented group $\Gamma = \langle S \mid R \rangle$ with $|S| = 3$ has a transitive graph height function whenever it has a group height function (in the language of* [11]*, where infinitely many such examples are given). See Theorem 6.3 for a sufficient condition on a transitive cubic graph to possess a transitive graph height function.*
(c) *The Archimedean lattice* $[4, 6, 12]$ *lies in $\mathcal{Q}_{3,4}$ and possesses a harmonic function satisfying* (3.2)*. This is illustrated in Figure 6.1. See also Remark 8.8.*

The proof of Theorem 3.1 is found in Section 5.

4. Examples of infinite, transitive, cubic graphs

4.1. **Cubic graphs with $\mu \ge \phi$.** Here are some examples of infinite, cubic graphs, to many of which Theorem 3.1 may be applied. Each item is prefixed by the part of the theorem that applies. Most of the examples are transitive, and all are quasi-transitive.

A. (b) The 3-regular tree has connective constant 2.
B. (a) The ladder graph $\mathbb{L}$ (see Figure 5.1) has $\mu = \phi$. This exact value is elementary and well known; see, for example, [10, p. 184].
C. (a) The hexagonal lattice $\mathbb{H}$ has $\mu = \sqrt{2 + \sqrt{2}} > \phi$. See [4].
D. (a) It is explained in [9, Ex. 4.2] that the square/octagon lattice $[4, 8, 8]$ satisfies $\mu > \phi$.
E. (c) The Archimedean $[4, 6, 12]$ lattice has connective constant at least ϕ. See Example 3.3(c) and Remark 8.8.
F. (b) The Cayley graph of the lamplighter group has a so-called group height function, and hence a transitive graph height function. See Example 3.3(b) and [11, Ex. 5.3].
G. The following examples concern so-called Fisher graphs (see [7] and Section 7). For $G \in \mathcal{G}_3$, the *Fisher graph* G_{F} $(\in \mathcal{Q}_3)$ is obtained by replacing each

vertex by a triangle. It is shown at [7, Thm 1] that the value of $\mu(G_{\mathrm{F}})$ may be deduced from that of $\mu(G)$, and furthermore that $\mu(G_{\mathrm{F}}) > \phi$ whenever $\mu(G) > \phi$.

H. In particular, the Fisher graph $\mathbb{H}_{\mathrm{F}}$ of $\mathbb{H}$ satisfies $\mu(\mathbb{H}_{\mathrm{F}}) > \phi$.

I. The Archimedean lattices mentioned above are the hexagonal lattice $[6, 6, 6]$, the square/octagon lattice $[4, 8, 8]$, together with $[4, 6, 12]$, and $\mathbb{H}_{\mathrm{F}} = [3, 12, 12]$. To this list we may add the ladder graph $\mathbb{L} = [4, 4, \infty]$.

These are examples of so-called transitive, TLF-planar graphs [18], and all such graphs are shown in Section 8 to satisfy $\mu \geq \phi$.

J. More generally, if $G \in \mathcal{G}_d$ where $d \geq 3$, and

$$\frac{1}{\mu(G)} \leq \begin{cases} \dfrac{1}{\phi^r} & \text{if } d = 2r + 1, \\ \dfrac{2}{\phi^{r+1}} & \text{if } d = 2r, \end{cases}$$

then its (generalized) Fisher graph satisfies $\mu(G_{\mathrm{F}}) \geq \phi$. See Proposition 7.5.

Since $\mu \leq d - 1$, the above display can be satisfied only if $d \leq 10$.

K. The Cayley graph G of the group $\Gamma = \langle S \mid R\rangle$, where $S = \{a, b, c\}$ and $R = \{c^2, ab, a^3\}$, is the Fisher graph of the 3-regular tree, and hence $\mu(G) > \phi$. The exact value of $\mu(G)$ may be calculated by [7, Thm 1] (see also Proposition 7.5(a) and [6, Ex. 5.1]).

We note that the $[3, 12, 12]$ lattice is a quotient graph of G by adding the further relator $(ac)^6$. Since the last lattice has connective constant at least ϕ, so does G (see [9, Cor. 4.1]).

L. The Cayley graph G of the group $\Gamma = \langle S \mid R\rangle$, where $S = \{a, b, c\}$ and $R = \{a^2, b^2, c^2, (ac)^2\}$, is the generalized Fisher graph of the 4-regular tree. The connective constant $\mu(G)$ may be calculated exactly, as in Theorem 7.4, and satisfies $\mu > \phi$.

Since the ladder graph $\mathbb{L}$ is the quotient graph of G obtained by adding the further relator $(bc)^2$, we have by [9, Cor. 4.1] that $\mu(G) > \phi$. (see [9]).

4.2. **Open questions.** Here are two cases, one specific and the other more general, in which we are unable to show that $\mu \geq \phi$.

A. We are unable to show $\mu(G) \geq \phi$ for the Cayley graph G of the Grigorchuk group. Our best inequality is $\mu(G) \geq 12^{1/6} \approx 1.513$. See Example 7.3, and also [12, Sect. 5].

B. Let G be the Cayley graph of an infinite, finitely generated, virtually abelian group $\Gamma = \langle S \mid R\rangle$ with $|S| = 3$. Is it generally true that $\mu(G) \geq \phi$? Whereas such groups are abelian-by-finite, the finite-by-abelian case is fairly immediate (see Theorem 6.6).

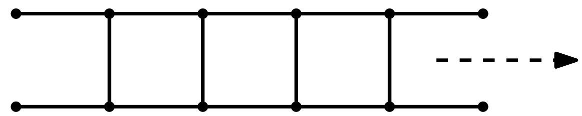

FIGURE 5.1. The singly infinite ladder graph $\mathbb{L}_+$. The doubly infinite ladder $\mathbb{L}$ extends to infinity both leftwards and rightwards.

A method for constructing such graphs was described by Biggs [2, Sect. 19] and developed by Seifter [19, Thm 2.2].

5. Proof of Theorem 3.1

Proof of part (a). Let $\mathbb{L}_+$ be the singly-infinite ladder graph of Figure 5.1. An *extendable* SAW is a SAW starting at 0 that, at each stage, steps either to the right (that is, *horizontally*) or between layers (that is, *vertically*). Note that the first step of an extendable walk is necessarily horizontal, and every vertical step is followed by a horizontal step. Let $\mathbb{E}_n$ be the set of n-step extendable SAWs on $\mathbb{L}_+$. It is elementary, by considering the first two steps, that $\eta_n = |\mathbb{E}_n|$ satisfies the recursion

$$\eta_n = \eta_{n-1} + \eta_{n-2}, \qquad n \geq 3,$$

whence

$$\lim_{n\to\infty} \eta_n^{1/n} = \phi. \tag{5.1}$$

Let 0 be a root of $G = (V, E) \in \mathcal{Q}_{\mathrm{h}}$, and let $h : V \to \mathbb{R}$ be harmonic such that (3.2)–(3.3) hold. We shall construct an injection $f : \mathbb{E}_n \to \Sigma_n(0)$, and the claim will follow.

Definition 5.1. *For $\pi = (\pi_0, \pi_1, \pi_2, \dots) \in \mathbb{E}_n$, we let $f(\pi) = (f_0, f_1, f_2, \dots)$ be the n-step walk on G given as follows.*

1. *We set $f_0 = 0$ and $f_1 = m(0)$.*
2. *If the second edge of π is horizontal (respectively, vertical), we set $f_2 = m(m(0))$ (respectively, $f_2 = q(m(0))$).*
3. *Assume $k \geq 1$ and $(f_0, f_1, \dots, f_k)$ have been defined.*
   (a) *If $\langle \pi_{k-1}, \pi_k \rangle$ is vertical, then $\langle \pi_k, \pi_{k+1} \rangle$ is horizontal, and we set $f_{k+1} = m(f_k)$.*
   (b) *Assume $\langle \pi_{k-1}, \pi_k \rangle$ is horizontal. If $\langle \pi_k, \pi_{k+1} \rangle$ is horizontal (respectively, vertical), we set $f_{k+1} = m(f_k)$ (respectively, $f_{k+1} = q(f_k)$).*

Lemma 5.2. *The function f is an injection from $\mathbb{E}_n$ to $\Sigma_n(0)$.*

Proof of Lemma 5.2. Since h is harmonic,

$$\big(h(u) - h(a)\big) + \big(h(u) - h(b)\big) = M_u, \qquad u \in V, \tag{5.2}$$

where a, b, $c = m(u)$ are the three neighbours of u. By (3.2) and (5.2),

$$h(q(v)) - h(u) = 2M_u - M_v > 0, \tag{5.3}$$

$$h(w) - h(v) = M_{q(v)} + M_u - M_v > 0, \tag{5.4}$$

and, by (3.3),

$$h(q(w)) - h(v) = (M_u - M_v) + M_{q(v)} + (M_{q(v)} - M_w) > 0, \tag{5.5}$$

for $u \in V$, where $v = m(u)$ and $w = m(q(v))$. See Figure 3.1.

Let S_k be the statement that

(a) $f_0, f_1, \dots, f_k$ are distinct, and
(b) if $\langle \pi_{k-1}, \pi_k \rangle$ is horizontal, then $h(f_k) > h(f_i)$ for $0 \le i \le k-1$, and
(c) if $\langle \pi_{k-1}, \pi_k \rangle$ is vertical, then $h(f_k) > h(f_i)$ for $0 \le i \le k-2$.

If S_k holds for every k, then the f_k are distinct, whence $f(\pi)$ is a SAW. Furthermore, $f(\pi) \ne f(\pi')$ if $\pi \ne \pi'$, and the claim of the lemma follows. We shall prove the S_k by induction.

Evidently, S_0 and S_1 hold. Let $K \ge 3$ be such that S_k holds for $k < K$, and consider S_K. Let $e_i = \langle \pi_{K-i-1}, \pi_{K-i} \rangle$ for $0 \le i \le K-1$.

1. Suppose first that e_0 is vertical, so that e_1 is horizontal. By (5.3) with $u = f_{K-2}$ and $v = m(f_{K-2}) = f_{K-1}$, we have that $h(f_K) > h(f_{K-2})$.
   (a) If e_2 is horizontal, the claim follows by S_{K-2}.
   (b) Assume e_2 is vertical (so that, in particular, $K \ge 4$). We need also to show that $h(f_K) > h(f_{K-3})$. In this case, we take $u = f_{K-4}$, $v = m(f_{K-4}) = f_{K-3}$, and $w = m(q(v)) = f_{K-1}$ in (5.5), thereby obtaining that $h(f_K) > h(f_{K-3})$ as required.
2. Assume next that e_0 is horizontal.
   (a) If e_1 is horizontal, the relevant claims of S_K follow by S_{K-1} and the fact that $f_K = m(f_{K-1})$.
   (b) If e_1 is vertical, then e_2 is horizontal. By (5.4), $h(f_K) > h(f_{K-2})$, and the claim follows by S_{K-1} and S_{K-2}.

This completes the induction. □

By Lemma 5.2, $|\Sigma_n(0)| \ge \eta_n$, and part (a) follows by (5.1). □

Proof of part (b). Let $G \in \mathcal{G}_3$ and let $(h, \mathcal{H})$ be a transitive graph height function. For $u \in V$, let $M = \max\{h(v) - h(u) : v \sim u\}$ as in (3.1). We have that $M > 0$

and, by transitivity, M does not depend on the choice of u. Since h is $\mathcal{H}$-difference-invariant, the neighbours of any $v \in V$ may be listed as v_1, v_2, v_3 where

$$h(v_i) - h(v) = \begin{cases} M & \text{if } i = 1, \\ -M & \text{if } i = 2, \\ \eta & \text{if } i = 3, \end{cases}$$

where η is a constant satisfying $|\eta| \le M$. By the transitive action of $\mathcal{H}$, we have that $-\eta \in \{-M, \eta, M\}$, whence $\eta \in \{-M, 0, M\}$.

If $\eta = 0$, h is harmonic and satisfies (3.2)–(3.3), and the claim follows by part (a). If $\eta = M$, it is easily seen that the construction of Definition 5.1 results in an injection from $\mathbb{E}_n$ to $\Sigma_n(v)$. If $\eta = -M$, we replace h by $-h$ to obtain the same conclusion. □

Proof of part (c). This is a minor variant of the proof of part (a), in which we eliminate the appeal to (5.5) in paragraph 1(b). Let T_k be the statement that

(a) $f_0, f_1, \dots, f_k$ are distinct, and
(b) if $\langle \pi_{k-1}, \pi_k \rangle$ is horizontal, then $h(f_k) > h(f_i)$ for $0 \le i \le k-1$, and
(c) if $\langle \pi_{k-1}, \pi_k \rangle$ is vertical, then $h(f_k) > h(f_i)$ for $0 \le i \le k-4$ and $i = k-2$.

Thus T_k varies from S_k only in the latter's claim that $h(f_K) > h(f_{K-3})$ in part (c). The above proof is valid with S_k replaced by T_k, except at the appeal to (5.5) in paragraph 1(b). In the present case, we argue as follows at the corresponding stage.

Firstly, $f_K \ne f_{K-3}$ since G has girth at least 4. Secondly, by the equality of (5.5),

$$\begin{aligned} h(q(w)) - h(u) &= (M_u - M_v) + M_{q(v)} + (M_{q(v)} - M_w) + M_u \\ &= (2M_u - M_v) + (2M_{q(v)} - M_w), \end{aligned}$$

where $u = f_{K-4}$, $v = m(f_{K-4}) = f_{K-3}$, $w = m(q(v)) = f_{K-1}$, and $q(w) = f_K$. By (3.2), $h(f_K) > h(f_{K-4})$, as required. □

6. Transitive graph height functions

By Theorem 3.1(b), the possession of a transitive graph height function suffices for the inequality $\mu(G) \ge \phi$. It is not currently known which $G \in \mathcal{G}_3$ possess graph height functions, and it is shown in [11] that the Cayley graph of the Grigorchuk group has no graph height function at all. We pose a weaker question here. Suppose $G \in \mathcal{G}_3$ possesses a graph height function $(h, \mathcal{H})$. Under what further condition does G possess a *transitive* graph height function? A natural candidate function $g : V \to \mathbb{Z}$ is obtained as follows.

Proposition 6.1. *Let Γ act transitively on $G = (V, E) \in \mathcal{G}_d$ where $d \ge 3$. Assume that $(h, \mathcal{H})$ is a graph height function of G, where $\mathcal{H} \trianglelefteq \Gamma$ and $[\Gamma : \mathcal{H}] < \infty$. Let*

$\gamma_i \in \Gamma$ be representatives of the cosets, so that $\Gamma/\mathcal{H} = \{\gamma_i \mathcal{H} : i \in I\}$, and let

$$g(v) = \sum_{i \in I} h(\gamma_i v), \qquad v \in V. \tag{6.1}$$

The function $g : V \to \mathbb{Z}$ is Γ-difference-invariant.

Proof. The function g is given in terms of the representatives γ_i of the cosets, but its differences $g(v) - g(u)$ do not depend on the choice of the γ_i. To see this, suppose γ_1 is replaced in (6.1) by some $\gamma_1' \in \gamma_1 \mathcal{H}$. Since $\mathcal{H}$ is a normal subgroup, $\gamma_1' = \eta\gamma_1$ for some $\eta \in \mathcal{H}$. The new function g' satisfies

$$g'(v) - g(v) = h(\gamma_1' v) - h(\gamma_1 v) = h(\eta\gamma_1 v) - h(\gamma_1 v),$$

so that

$$[g'(v) - g'(u)] - [g(v) - g(u)] = [h(\eta\gamma_1 v) - h(\gamma_1 v)] - [h(\eta\gamma_1 u) - h(\gamma_1 u)] = 0,$$

since $\eta \in \mathcal{H}$ and h is $\mathcal{H}$-difference-invariant.

We show as follows that g is Γ-difference-invariant. Let $\alpha \in \Gamma$, and write $\alpha = \gamma_j \eta$ for some $j \in I$ and $\eta \in \mathcal{H}$. Since $\Gamma/\mathcal{H}$ can be written in the form $\{\gamma_i\gamma_j \mathcal{H} : i \in I\}$,

$$\begin{aligned} g(\alpha v) - g(\alpha u) &= \sum_i \Big[h(\gamma_i\gamma_j\eta v) - h(\gamma_i\gamma_j\eta u) \Big] \\ &= g(\eta v) - g(\eta u) \\ &= g(v) - g(u), \end{aligned}$$

since g is $\mathcal{H}$-difference-invariant. □

If the function g of (6.1) is non-constant, it follows that $(g - g(\mathbf{1}), \Gamma)$ is a transitive graph height function, implying by Theorem 3.1(b) that $\mu(G) \geq \phi$. This is not invariably the case, as the following example indicates.

Example 6.2. *Consider the Archimedean lattice $\mathbb{A} = [4, 6, 12]$ of Figure 6.1. Then $\mathbb{A}$ is transitive and cubic, but it has no transitive graph height function. This is seen by examining the structure of $\mathbb{A}$. There is a variety of ways of showing $\mu(\mathbb{A}) \geq \phi$, and we refer the reader to the stronger inequality of Remark 8.8.*

Theorem 6.3. *Let Γ act transitively on $G = (V, E) \in \mathcal{G}_3$. Let $(h, \mathcal{H})$ be a graph height function of G, where $\mathcal{H} \trianglelefteq \Gamma$ and $[\Gamma : \mathcal{H}] < \infty$. Pick $\gamma_i \in \Gamma$ such that $\Gamma/\mathcal{H} = \{\gamma_i \mathcal{H} : i \in I\}$, and let $g : V \to \mathbb{Z}$ be given by (6.1). If there exists a constant $C < \infty$ such that*

$$d_G(v, \gamma_i v) \leq C, \qquad v \in V,\ i \in I, \tag{6.2}$$

then $(g - g(\mathbf{1}), \Gamma)$ is a transitive graph height function.

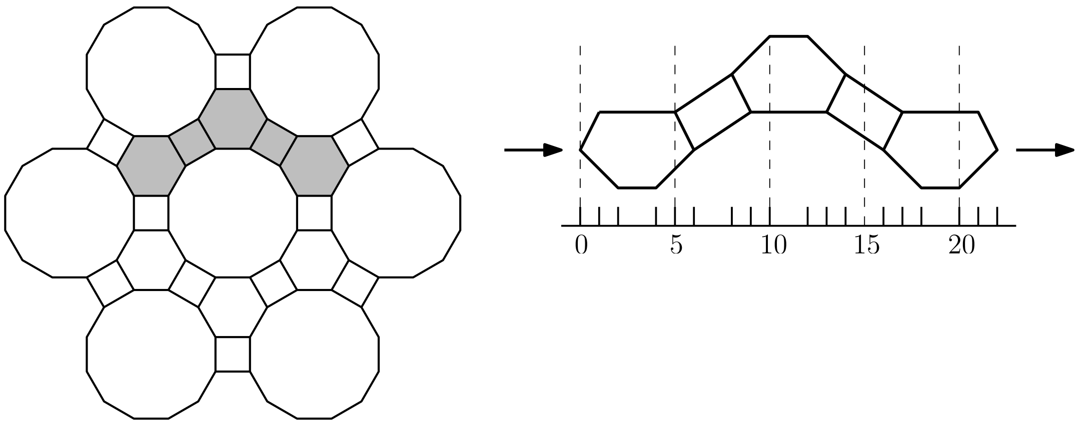

FIGURE 6.1. The left figure depicts part of the Archimedean lattice $\mathbb{A} = [4, 6, 12]$. Potentials may be assigned to the vertices as illustrated in the right figure, and the potential differences are duplicated by translation and reflection (in a horizontal axis). The resulting harmonic function satisfies (3.2).

Proof. Since $(h, \mathcal{H})$ is a graph height function, there exists $v \in V$ such that $h(v) > 2C|I|\delta$, where

$$\delta := \max\{|h(v) - h(u)| : u \sim v\}.$$

By (6.2), $g(v) > g(\mathbf{1})$ a.s required.

Condition (6.2) may be weakened as follows. Let $v_0 \in V$, and

$$D(v) = \max\{d_G(v, \gamma_i v) : i \in I\}, \qquad D_m = \max\{D(v) : d_G(v_0, v) = m\}.$$

It suffices that there exist $v_0 \in V$ and $m \geq 1$ such that

$$(D_m + D_0)\delta < m. \tag{6.3}$$

The proof is elementary and is omitted. $\square$

Corollary 6.4. *Let $\Gamma = \langle S \mid R \rangle$ be an infinite, finitely-generated group. Let $\mathcal{H} \trianglelefteq \Gamma$ be a finite-index normal subgroup, and let $(h, \mathcal{H})$ be a graph height function of the Cayley graph G (so that it is a 'strong' graph height function, see [11]). Pick $\gamma_i \in \Gamma$ such that $\Gamma/\mathcal{H} = \{\gamma_i \mathcal{H} : i \in I\}$, and let $g : V \to \mathbb{Z}$ be given by (6.1). If*

$$\max_{1 \leq i \leq k} \big|[\gamma_i]\big| < \infty, \tag{6.4}$$

where $[\gamma_i] = \{g^{-1}\gamma_i g : g \in \Gamma\}$ is the conjugacy class of γ_i, then $(g - g(\mathbf{1}), \Gamma)$ is a transitive graph height function.

Proof. Since $d_G(g, \gamma_i g) = d_G(\mathbf{1}, g^{-1}\gamma_i g)$, condition (6.2) holds by (6.4). $\square$

Example 6.5. *An* FC-group *is a group all of whose conjugacy classes are finite (see, for example, [20]). Clearly, (6.4) holds for FC-groups.*

We note a further situation in which there exists a transitive graph height function.

Theorem 6.6. *Let Γ act transitively on $G = (V, E) \in \mathcal{G}_d$ where $d \geq 3$, and let $(h, \mathcal{H})$ be a graph height function on G. If there exists a short exact sequence $\mathbf{1} \to K \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} \mathcal{H} \to \mathbf{1}$ with $|K| < \infty$, then G has a transitive graph height function.*

Proof. Suppose such an exact sequence exists. Fix a root $v_0 \in V$, find $\gamma \in \Gamma$ such that $v = \gamma v_0$, and define $g(v) := h(\beta_\gamma v_0)$.

Certainly $g(v_0) = 0$ and g is non-constant. It therefore suffices to show that g is Γ-difference-invariant. Let $u, v \in V$ and find $\gamma, \gamma' \in \Gamma$ such that $\gamma v = \gamma' u = v_0$. For $\rho \in \Gamma$,

$$\begin{aligned} g(\rho v) - g(\rho u) &= h(\beta_{\rho\gamma} v_0) - h(\beta_{\rho\gamma'} v_0) \\ &= h(\beta_\rho \beta_\gamma v_0) - h(\beta_\rho \beta_{\gamma'} v_0) \\ &= h(\beta_\gamma v_0) - h(\beta_{\gamma'} v_0) \qquad \text{since } \beta_\rho \in \mathcal{H} \\ &= g(v) - g(u), \end{aligned}$$

and the proof is complete. □

7. Graphs with girth 3 or 4

We recall the subset $\mathcal{G}_{d,g}$ of $\mathcal{G}$ containing graphs with degree d and girth g. Our next theorem is concerned with $\mathcal{G}_{3,3}$, and the following (Theorem 7.2) with $\mathcal{G}_{3,4}$.

Theorem 7.1. *For $G \in \mathcal{G}_{3,3}$, we have that*

$$x_1 \leq \mu(G) \leq x_2, \tag{7.1}$$

where $x_1, x_2 \in (1, 2)$ satisfy

$$\frac{1}{x_1^2} + \frac{1}{x_1^3} = \frac{1}{\sqrt{2}}, \tag{7.2}$$

$$\frac{1}{x_2^2} + \frac{1}{x_2^3} = \frac{1}{2}. \tag{7.3}$$

Moreover, the upper bound x_2 is sharp.

The bounds of (7.2)–(7.3) satisfy $x_1 \approx 1.529$ and $x_2 \approx 1.769$, so that $\phi \in (x_1, x_2)$. The upper bound x_2 is achieved by the Fisher graph of the 3-regular tree (see Proposition 7.5 and [6, 7]).

Theorem 7.2. *For $G \in \mathcal{G}_{3,4}$, we have that*

$$y_1 \le \mu(G) \le y_2, \tag{7.4}$$

where

$$y_1 = 12^{1/6}, \tag{7.5}$$

and $y_2 = 1/x$ where x is the largest real root of the equation

$$2x(x + x^2 + x^3) = 1. \tag{7.6}$$

Moreover, the upper bound y_2 of (7.4) is sharp.

The lower bound of (7.5) satisfies $12^{1/6} \approx 1.513 < 1.618 \approx \phi$. The upper bound is approximately $y_2 \approx 1.899$, and is achieved by the Fisher graph of the 4-regular tree (see Proposition 7.5). The proofs of Theorems 7.1 and 7.2 are given later in this section.

Example 7.3 (Grigorchuk graph)**.** *The Cayley graph G of the Grigorchuk group, given in* [12, Sect. 5]*, lies in $\mathcal{G}_{3,4}$, and therefore $y_1 \le \mu(G) \le y_2$. We ask if $\mu(G) \ge \phi$.*

The emphasis of the current paper is upon lower bounds for connective constants of cubic graphs. The upper bounds of Theorems 7.1–7.2 are included as evidence of the accuracy of the lower bounds, and in support of the unproven possibility that $\mu \ge \phi$ in each case. We note a more general result (derived from results of [6, 21]) for upper bounds of connective constants as follows.

Theorem 7.4. *For $G \in \mathcal{G}_{d,g}$ where $d, g \ge 3$, we have that $\mu(G) \le y$ where $\zeta := 1/y$ is the smallest positive real root of the equation*

$$(d-2)\frac{M_1(\zeta)}{1 + M_1(\zeta)} + \frac{M_2(\zeta)}{1 + M_2(\zeta)} = 1, \tag{7.7}$$

where

$$M_1(\zeta) = \zeta, \qquad M_2(\zeta) = 2(\zeta + \zeta^2 + \cdots + \zeta^{g-1}). \tag{7.8}$$

The upper bound y is sharp, and is achieved by the free product graph $F := K_2 * K_2 * \cdots * K_2 * \mathbb{Z}_g$, with $d-2$ copies of the complete graph K_2 on two vertices and one copy of the cycle $\mathbb{Z}_g$ of length g.

The extremal graph of this theorem is the (simple) Cayley graph F of the free product group $\langle S \mid R \rangle$ with $S = \{a_1, a_2, \dots, a_{d-2}, b\}$ and $R = \{a_1^2, a_2^2, \dots, a_{d-2}^2, b^g\}$.

The proofs follow. Let $G \in \mathcal{G}_d$ where $d \ge 3$. The (generalized) *Fisher graph* G_{F} is obtained from G by replacing each vertex by a d-cycle, as illustrated in Figure 7.1. The Fisher transformation originated in the work of Fisher [5] on the Ising model. The connective constants of G and G_{F} are related as follows.

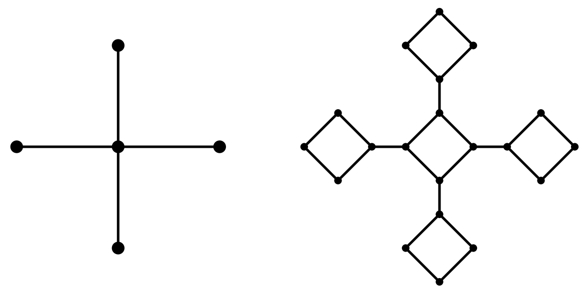

FIGURE 7.1. Each vertex of G is replaced in the Fisher graph G_{F} by a cycle.

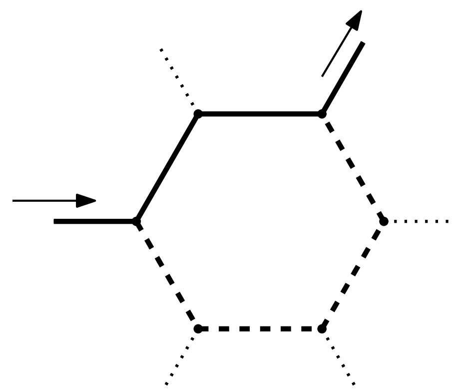

FIGURE 7.2. The entry and exit of a SAW at a Fisher 6-cycle. It follows either two edges clockwise, or 4 edges anticlockwise.

Proposition 7.5. *Let $G \in \mathcal{G}_d$ where $d \geq 3$.*

(a) [7, Thm 1] *If $d = 3$,*

$$\frac{1}{\mu(G_{\mathrm{F}})^2} + \frac{1}{\mu(G_{\mathrm{F}})^3} = \frac{1}{\mu(G)}. \tag{7.9}$$

(b) *If $d = 2r \geq 4$ is even,*

$$\frac{2}{\mu(G_{\mathrm{F}})^{r+1}} \leq \frac{1}{\mu(G)}. \tag{7.10}$$

(c) *If $d = 2r+1 \geq 5$ is odd,*

$$\frac{1}{\mu(G_{\mathrm{F}})^{r+1}} + \frac{1}{\mu(G_{\mathrm{F}})^{r+2}} \leq \frac{1}{\mu(G)}. \tag{7.11}$$

Proof of Proposition 7.5. We use the methods of [7], where a proof of part (a) appears at Theorem 1. Consider SAWs on G and G_{F} that start and end at midpoints of edges.

Let π be such a SAW on G. When π reaches a vertex v of G, it can be directed around the corresponding d-cycle C of G_{F}. There are $d-1$ possible exit points for C relative to the entry point. For each, the SAW may be redirected around C either clockwise or anticlockwise (as illustrated in Figure 7.2). If the exit lies s ($\le d/2$) edges along C from the entry, a single step of π becomes a walk of length either $s+1$ or $d-s+1$. Such a substitution is made at each vertex of π. It is easily checked that (i) the outcome is a SAW π' on G_{F}, and (ii) by observation of π', one may recover the choices made at each v.

Let Z (respectively, Z_{F}) be the generating function of SAWs from a given midpoint of G (respectively, G_{F}). Let $d = 2r \ge 4$ (the case of odd d is similar). By adapting the arguments of [7], we obtain that

$$Z\big(\min\{\zeta^2+\zeta^d, \zeta^3+\zeta^{d-1}, \dots, 2\zeta^{r+1}\}\big) \le Z_{\mathrm{F}}(\zeta), \qquad \zeta \ge 0. \tag{7.12}$$

The radius of convergence of Z_{F} is $1/\mu(G_{\mathrm{F}})$, and (7.10) follows from (7.12) on letting $\zeta \uparrow 1/\mu(G_{\mathrm{F}})$ and noting that the minimum in (7.12) is achieved by $2\zeta^{r+1}$. □

Lemma 7.6. *Let $G = (V, E) \in \mathcal{G}_{3,3}$.*

(a) *For $v \in V$, there exists exactly one triangle passing through v.*
(b) *If each such triangle of G is contracted to a single vertex, the ensuing graph G' satisfies $G' \in \mathcal{G}_3$.*

Proof. (a) Assume the contrary: each $u \in V$ lies in two or more triangles. Since $\deg(u) = 3$, there exists $v \in V$ such that $\langle u, v\rangle$ lies in two distinct triangles, and we write w_1, w_2 for the vertices of these triangles other than u, v. Since each w_i has degree 3, we have than $w_1 \sim w_2$. This implies that G is finite, which is a contradiction.

(b) Let $\mathcal{T}$ be the set of triangles in G, so that the elements of $\mathcal{T}$ are vertex-disjoint. We contract each $T \in \mathcal{T}$ to a vertex, thus obtaining the graph $G' = (V', E')$. Since each vertex of G' arises from a triangle of G, the graph G' is cubic, and G is the Fisher graph of G'. Since G is infinite, so is G'.

We show next that G' is transitive. Let $v_1', v_2' \in V'$, and write $T_i = \{a_i, b_i, c_i\}$, $i = 1, 2$, for the corresponding triangles of G. Since G is transitive, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a_1) = a_2$. By part (a), $\alpha(T_1) = T_2$. Since $\alpha \in \mathrm{Aut}(G)$, it induces an automorphism $\alpha' \in \mathrm{Aut}(G')$ such that $\alpha'(v_1') = v_2'$, as required.

Finally, we show that G' is simple. If not, there exist two vertex-disjoint triangles of G, T_1 and T_2 say, with two edges between their vertex-sets. Each vertex in these two edges belongs to two faces of size 3 and 4. By transitivity, every vertex has this property. By consideration of the various possible cases, one arrives at a contradiction. □

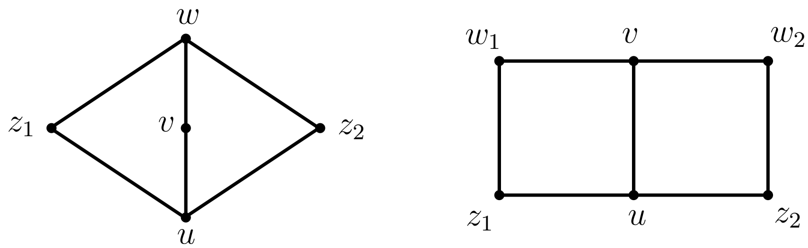

FIGURE 7.3. The two situations in the proof of Lemma 7.7.

Proof of Theorem 7.1. Since G is the Fisher graph of $G' \in \mathcal{G}_3$, by Proposition 7.5(a),

$$\frac{1}{\mu(G)^2} + \frac{1}{\mu(G)^3} = \frac{1}{\mu(G')}.$$

By [9, Thm 4.1],

$$\sqrt{2} \le \mu(G') \le 2,$$

and (7.1) follows. When G' is the 3-regular tree T_3, we have $\mu(G') = 2$, and the upper bound is achieved. □

The following lemma is preliminary to the proof of Theorem 7.2.

Lemma 7.7. *Let $G = (V, E) \in \mathcal{G}_{3,4}$. If G is not the doubly infinite ladder graph $\mathbb{L}$, each $v \in V$ belongs to exactly one quadrilateral.*

Proof. Let $G = (V, E) \in \mathcal{G}_{3,4}$ and $v \in V$. Assume v belongs to two or more quadrilaterals. We will deduce that $G = \mathbb{L}$.

By transitivity, there exist two (or more) quadrilaterals passing through every vertex v, and we pick two of these, denoted $C_{v,1}$, $C_{v,2}$. Since v has degree 3, exactly one of the following occurs (as illustrated in Figure 7.3).

(a) $C_{v,1}$ and $C_{v,2}$ share two edges incident to v.
(b) $C_{v,1}$ and $C_{v,2}$ share exactly one edge incident to v.

Assume first that Case (a) occurs, and let Π_x be the property that $x \in V$ belongs to three quadrilaterals, any two of which share exactly one incident edge of x, these $\binom{3}{2} = 3$ edges being distinct.

Let $\langle u, v\rangle$ and $\langle w, v\rangle$ be the two edges shared by $C_{v,1}$ and $C_{v,2}$, and write $C_{v,i} = (u, v, w, z_i)$, $i = 1, 2$. Note that Π_u occurs, so that Π_x occurs for every x by transitivity.

Let x be the adjacent vertex of v other than u and w. Note that $x \notin \{z_1, z_2\}$ and $x \not\sim u, w$, since otherwise G would have girth 3. By Π_v, either $x \sim z_1$ or $x \sim z_2$. Assume without loss of generality that $x \sim z_1$. If $x \sim z_2$ in addition, then G is finite, which is a contradiction. Therefore, $x \not\sim z_2$.

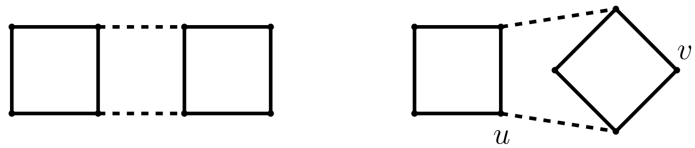

FIGURE 7.4. Two ways in which two quadrilaterals may be joined by two edges.

Let y be the incident vertex of z_2 other than u and w, and note that $y \notin \{u, v, w, x, z_1, z_2\}$. By Π_{z_2}, there exists a quadrilateral containing both $\langle y, z_2\rangle$ and $\langle z_2, u\rangle$. Since u has degree 3, either $y \sim z_1$ or $y \sim v$. However, neither is possible since both z_2 and v have degree 3. Therefore, Case (a) does not occur.

Assume Case (b) occurs, and write $C_{v,i} = (u, v, w_i, z_i)$, $i = 1, 2$, for the above two quadrilaterals passing through v. Let Π_x^2 (respectively, Π_x^3) be the property that $x \in V$ belongs to two quadrilaterals (respectively, three quadrilaterals), and each incident edge of x lies in at least one of these quadrilaterals (respectively, every pair of incident edges of x lie in at least one of these quadrilaterals). Since Π_v^2 occurs, by transitivity Π_x^2 occurs for every $x \in V$.

Since G is infinite, there exists a 'new' edge incident to the union of $C_{v,1}$ and $C_{v,2}$. Without loss of generality, we take this as $\langle z_1, x\rangle$ with $x \notin \{u, v, w_1, w_2, z_1, z_2\}$. By $\Pi_{z_1}^2$, there exists a quadrilateral of the form (z_1, x, y, z). Since G is simple with degree 3 and girth 4, and $d_G(y, z_1) = 2$, $y \notin \{z_1, u, v, w_1, w_2\}$.

We claim that $y \neq z_2$, as follows. If $y = z_2$, then Π_u^3 occurs, whence $\Pi_{z_1}^3$ occurs by transitivity. Therefore, there exists a quadrilateral passing through the two edges $\langle x, z_1\rangle$, $\langle z_1, w_1\rangle$, and we denote this (x, z_1, w_1, y'). It is immediate that $y' \notin \{u, v, w_2, z_2\}$ since G is simple with degree 3 and girth 4, and therefore y' is a 'new' vertex. By $\Pi_{w_1}^3$, $y' \sim w_2$, and G is finite, a contradiction. Therefore, $y \neq z_2$, and hence y is a 'new' vertex, and $z = w_1$.

We iterate the above procedure, adding at each stage a new quadrilateral to the graph already obtained. It could be that the graph thus constructed is a singly infinite ladder graph with two 'terminal' vertices of degree 2. If so, we then turn attention to these terminal vertices, and use the fact that, by transitivity, there exists $D < \infty$ such that $d_{G\setminus e}(a, b) \leq D$ for every edge $e = \langle a, b\rangle \in E$. $\square$

Proof of Theorem 7.2. If $G = \mathbb{L}$, then $\mu = \phi$, which satisfies (7.4).

Assume that $G \neq \mathbb{L}$. Let $\mathcal{T}$ be the set of quadrilaterals of G, and recall Lemma 7.7. We contract each element of $\mathcal{T}$ to a degree-4 vertex, thus obtaining a graph G'. We claim that

$$G' \in \mathcal{G}_4, \text{ and } G \text{ is the Fisher graph of } G'. \tag{7.13}$$

Suppose for the moment that (7.13) is proved. By [9, Thm 4.1], $\mu(G') \ge \sqrt{3}$, and, by Proposition 7.5(b),

$$\frac{2}{\mu(G)^3} \le \frac{1}{\mu(G')} \le \frac{1}{\sqrt{3}},$$

which implies $\mu(G) \ge 12^{1/6}$.

We prove (7.13) next. It suffices that $G' = (V', E') \in \mathcal{G}_4$, and G is then automatically the required Fisher graph. Evidently, G' has degree 4. We show next that G' is transitive. Let $v_1', v_2' \in V'$, and write $Q_i = (a_i, b_i, c_i, d_i)$, $i = 1, 2$, for the corresponding quadrilaterals of G. Since G is transitive, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a_1) = a_2$. By Lemma 7.7, $\alpha(Q_1) = Q_2$. Since $\alpha \in \mathrm{Aut}(G)$, it induces an automorphism $\alpha' \in \mathrm{Aut}(G')$ such that $\alpha'(v_1') = v_2'$, as required.

Suppose that G' is not simple, in that it has parallel edges. Then there exist two quadrilaterals joined by two edges, which can occur in either of the two ways drawn in Figure 7.4. The first is impossible by Lemma 7.7. In the second, u lies in both a 4-cycle and a 5-cycle having exactly one edge in common, whereas v cannot have this property. This contradicts the fact that G is transitive. In summary, G' is infinite, transitive, cubic, and simple, whence $G' \in \mathcal{G}_3$.

For the sharpness of the upper bound, we refer to the proof of the more general Theorem 7.4, following. □

Proof of Theorem 7.4. Let $G \in \mathcal{G}_{d,g}$ where $d, g \ge 3$, and let F ($\in \mathcal{G}_{d,g}$) be the given free product graph. By [21, Thm 11.6], F covers G. Therefore, there is an injection from SAWs on G with a given root to a corresponding set on F, whence $\mu(G) \le \mu(F)$.

By [6, Thm 3.3], $\mu(F) = 1/\zeta$ where ζ is the smallest positive real root of (7.7). □

8. Transitive TLF-planar graphs

8.1. **Background and main theorem.** There are only few infinite, transitive, cubic graphs that are planar, and each has $\mu \ge \phi$. These graphs belong to the larger class of so-called TLF-planar graphs, and we study such graphs in this section. The basic properties of such graphs were presented in [18], to which the reader is referred for further information.

We use the word *plane* to mean a simply connected Riemann surface without boundaries. An *embedding* of a graph $G = (V, E)$ in a plane $\mathcal{P}$ is a function $\eta : V \cup E \to \mathcal{P}$ such that η restricted to V is an injection and, for $e = \langle u, v\rangle \in E$, $\eta(e)$ is a C^1 image of $[0, 1]$. (Later, we consider planar embeddings in which every face is a regular polygon.) An embedding is ($\mathcal{P}$-)*planar* if the images of distinct edges are disjoint except possibly at their endpoints, and a graph is ($\mathcal{P}$-)*planar* if it possesses a ($\mathcal{P}$-)planar embedding. An embedding is *topologically locally finite* (*TLF*) if the images of the vertices have no accumulation point, and a connected graph is called *TLF-planar* if it possesses a planar TLF embedding. Let $\mathcal{T}_d$ denote the class of transitive, TLF-planar graphs with vertex-degree d. We shall sometimes confuse a TLF-planar graph with its TLF embedding. The *boundary* of $S \subseteq \mathcal{P}$ is given as $\partial S := \overline{S} \cap \overline{\mathcal{P} \setminus S}$, where $\overline{T}$ is the closure of T.

The principal theorem of this section is as follows.

Theorem 8.1. *Let $G \in \mathcal{T}_3$ be infinite. Then $\mu(G) \geq \phi$.*

The principal methods of the proof are as follows: (i) the construction of an injection from extendable SAWs on $\mathbb{L}_+$ to SAWs on G, (ii) a method for verifying that certain paths on G are indeed SAWs, and (iii) a generalization of the Fisher transformation of [7].

A *face* of a TLF-planar graph (or, more accurately, of its embedding) is an arc-connected component of the (topological) complement of the graph. The *size* $k(F)$ of a face F is the number of vertices in its topological boundary, if bounded; an unbounded face has size ∞. Let $G = (V, E) \in \mathcal{T}_d$ and $v \in V$. The *type-vector* $[k_1, k_2, \dots, k_d]$ of v is the sequence of sizes of the d faces incident to v, taken in cyclic order around v. Since G is transitive, the type-vector is independent of choice of v modulo permutation of its elements, and furthermore each entry satisfies $k_i \geq 3$. We may therefore refer to the type-vector $[k_1, k_2, \dots, k_d]$ of G, and we define

$$f(G) = \sum_{i=1}^{d} \left(1 - \frac{2}{k_i}\right),$$

with the convention that $1/\infty = 0$. We shall use the following two propositions.

Proposition 8.2 ([18, p. 2827])**.** *Let $G = (V, E) \in \mathcal{T}_3$.*

(a) *If $f(G) < 2$, G is finite and has a planar TLF embedding in the sphere.*
(b) *If $f(G) = 2$, G is infinite and has a planar TLF embedding in the Euclidean plane.*
(c) *If $f(G) > 2$, G is infinite and has a planar TLF embedding in the hyperbolic plane (the Poincaré disk).*

Moreover, all faces of the above embeddings are regular polygons.

There is a moderately extensive literature concerning the function f and the Gauss–Bonnet theorem for graphs. See, for example, [3, 15, 16].

Proposition 8.3. *The type-vector of an infinite graph $G \in \mathcal{T}_3$ is one of the following:*

A. *$[m, m, m]$ with $m \geq 6$,*
B. *$[m, 2n, 2n]$ with $m \geq 3$ odd, and $m^{-1} + n^{-1} \leq \frac{1}{2}$,*
C. *$[2m, 2n, 2p]$ with $m, n, p \geq 2$ and $m^{-1} + n^{-1} + p^{-1} \leq 1$.*

Recall that the elements of a type-vector lie in $\{3, 4, \dots\} \cup \{\infty\}$.

Proof. See [18, p. 2828] for an identification of the type-vectors in $\mathcal{T}_3$. The inequalities on m, n, p arise from the condition $f(G) \ge 2$. □

8.2. **Proof of Theorem 8.1.** Let $G \in \mathcal{T}_3$ be infinite. By Proposition 8.2, $f(G) \ge 2$. If $f(G) = 2$ then, by Proposition 8.3, the possible type-vectors are precisely those with type-vectors $[6,6,6]$, $[3,12,12]$, $[4,8,8]$, $[4,6,12]$, $[4,4,\infty]$, that is, the hexagonal lattice [4] and its Fisher graph [7], the square/octagon lattice [9], the Archimedean lattice $[4,6,12]$ of Examples 3.3(c) and 6.2, and the doubly infinite ladder graph of Figure 5.1. Each of these has $\mu \ge \phi$.

It remains to prove Theorem 8.1 when $G \in \mathcal{T}_3$ is infinite with $f(G) > 2$. By Proposition 8.3, the cases to be considered are:

A. $[m,m,m]$ where $m > 6$,
B. $[m,2n,2n]$ where $m \ge 3$ is odd and $m^{-1} + n^{-1} < \frac{1}{2}$,
C. $[2m,2n,2p]$ where $m,n,p \ge 2$ and $m^{-1} + n^{-1} + p^{-1} < 1$.

These cases are covered in the following order, as indexed by section number.

§8.3. $\min\{k_i\} \ge 5$, $[k_1,k_2,k_3] \ne [5,8,8]$,
§8.4. $\min\{k_i\} = 3$,
§8.5. $[4,2n,2p]$ where $p \ge n \ge 4$ and $n^{-1} + p^{-1} < \frac{1}{2}$,
§8.6. $[4,6,2p]$ where $p \ge 6$,
§8.7. $[5,8,8]$.

We identify G with a specific planar, TLF embedding in the hyperbolic plane every face of which is a regular polygon. The proof is similar in overall approach to that of Theorem 3.1, as follows. Let $\mathbb{E}_n$ be the set of extendable n-step SAWs from 0 on the singly-infinite ladder graph $\mathbb{L}_+$ of Figure 5.1. Fix a root $v \in V$, and let $\Sigma_n(v)$ be the set of n-step SAWs on G starting at v. We shall construct an injection from $\mathbb{E}_n$ to $\Sigma_n(v)$, and the claim will follow by (5.1).

Consider the alphabet $\{\mathrm{H}, \mathrm{V}\}$. Let W_n of the set of n-letter words w in this alphabet, starting with the letter H, and with no pair of consecutive appearances of the letter V. The set $\mathbb{E}_n$ is in one-to-one correspondence with W_n, where H (respectively, V) denotes a horizontal (respectively, vertical) step on $\mathbb{L}_+$. In building an element of $\Sigma_n(v)$ sequentially, at each stage there is a choice between two new edges, which, in the sense of the embedding, we may call 'right' and 'left'. We shall explain how the word w encodes an element of $\Sigma_n(v)$. The key step is to show that the ensuing paths on G are indeed SAWs so long as the cumulative differences between the aggregate numbers of right and left steps remain sufficiently small.

There follow some preliminary lemmas. Let $G \in \mathcal{T}_d$ be infinite, where $d \ge 3$. A cycle C of G is called *clockwise* if its orientation after embedding is clockwise. Let C be traversed clockwise, and consider the changes of direction at each turn. Since the vertex-degree is d, each turn is along one of $d-1$ possible non-backtracking edges, exactly one of which may be designated *rightwards* and another *leftwards* (the other $d-3$ are neither rightwards nor leftwards). Let $r = r(C)$ (respectively, $l = l(C)$) be the number of right (respectively, left) turns encountered when traversing C clockwise, and let

$$\rho(C) = r(C) - l(C). \tag{8.1}$$

Lemma 8.4. *Let $G \in \mathcal{T}_d$ be infinite with $d \geq 3$. Let C be a cycle of G, and let $\mathcal{F} := \{F_1, F_2, \dots, F_s\}$ be the set of faces enclosed by w. There exists $F \in \mathcal{F}$ such that the boundary of $\mathcal{F} \setminus F$ is a cycle of G. The set of edges lying in $\partial F \setminus C$ forms a path.*

Proof. Let C be a cycle of G, and let $\mathcal{F}' \subseteq \mathcal{F}$ be the subset of faces that share an edge with C. Let I be the (connected) subgraph of G comprising the edges and vertices of the faces in $\mathcal{F}'$, and let I_{d} be its dual graph (with the infinite face omitted). Then I_{d} is finite and connected, and thus has some spanning tree T which is non-empty. Pick a vertex t of T with degree 1, and let F be the corresponding face. The first claim follows since the removal of t from T results in a connected sub-tree. The second claim holds since, if not, the interior of C is disconnected, which is a contradiction. □

Lemma 8.5. *Let $G \in \mathcal{T}_d$ be infinite with $d \geq 3$. For any cycle $C = (c_0, c_1, \dots, c_n)$ of G,*

$$\rho(C) \begin{cases} = 6 + \displaystyle\sum_{i=1}^{s} \big[k(F_i) - 6\big] & \text{if } d = 3, \\ \geq 4 + \displaystyle\sum_{i=1}^{s} \big[k(F_i) - 4\big] & \text{if } d \geq 4, \end{cases} \tag{8.2}$$

where $\mathcal{F} = \{F_1, F_2, \dots, F_s\}$ is the set of faces enclosed by C.

Proof. The proof is by induction on the number $s = s(C)$ of faces enclosed by C. It is trivial when $s = 1$ that $r(C) = k(F_1)$ and $l(C) = 0$, and (8.2) follows in that case.

Let $S \geq 2$ and assume that (8.2) holds for all C with $s(C) < S$. Let $C = (c_0, c_1, \dots, c_n)$ be such that $s(C) = S$, and pick $F \in \mathcal{F}$ as in Lemma 8.4. Let π be the path of edges in $\partial F \setminus C$.

Let C_F (respectively, C') be the boundary cycle of F (respectively, $\mathcal{F} \setminus F$), each viewed clockwise. We write π in the form $\pi = (c_a, \psi_1, \psi_2, \dots, \psi_r, c_b)$ where $a \neq b$, $\psi_i \notin C$. We claim that

$$\rho(C) \begin{cases} = \rho(C') + \rho(C_F) - 6 & \text{if } d = 3, \\ \geq \rho(C') + \rho(C_F) - 4 & \text{if } d \geq 4. \end{cases} \tag{8.3}$$

of which the induction step is a consequence.

Equation (8.3) follows by two observations.

1. The cycle C_F (respectively, C') takes a right (respectively, left) turn at each vertex ψ_i.
2. Consider the turns at a vertex $x \in \{c_a, c_b\}$.
   (a) Suppose $d = 3$. At x, C_F takes a right turn, C' takes a right turn, and C takes a left turn.
   (b) Suppose $d \geq 4$. At x, C_F takes a right turn, C' does not take a left turn, and C does not take a right turn. Furthermore, if C' takes a right turn, then C does not take a left turn.

The proof is complete. □

Lemma 8.6. *Let $G \in \mathcal{T}_d$ be infinite with type-vector $[k_1, k_2, \dots, k_d]$, and let C be a cycle of G.*

(a) *If $d = 3$ and $\min\{k_i\} \geq 6$, then $\rho(C) \geq 6$.*
(b) *If $d = 3$ and $[k_1, k_2, k_3] = [5, 2n, 2n]$ with $n \geq 5$, then $\rho(C) \geq 5$.*
(c) *If $d \geq 4$ and $\min\{k_i\} \geq 4$, then $\rho(C) \geq 4$.*

Proof. (a, c) These are immediate consequences of (8.2).
(b) Suppose $[k_1, k_2, k_3] = [5, 2n, 2n]$ with $n \geq 5$, and let $M = M(C)$ be the number of size-$2n$ faces enclosed by a cycle C. We shall prove $\rho(C) \geq 5$ by induction on $M(C)$. If $M = 0$, then C encloses exactly one size-5 face, and $\rho(C) = 5$. Let $S \geq 2$, and assume $\rho(C) \geq 5$ for any cycle C with $M(C) < S$.

Let C be a cycle with $M(C) = S$, and let F be a size-$2n$ face inside C. Let C' be the boundary of the set obtained by removing F from the inside of C; that is, C' may be viewed as the sum of the cycles C and ∂F with addition modulo 2. Then C' may be expressed as the edge-disjoint union of cycles $C_1, C_2, \dots, C_r$ satisfying $M(C_i) < S$ for $i = 1, 2, \dots, r$.

By (8.2) and the induction hypothesis,

$$\begin{aligned}\rho(C) &= 6 + [2n - 6] + \sum_{i=1}^{r} [\rho(C_i) - 6] \\ &\geq 2n - r.\end{aligned}$$

Each C_i shares an edge with ∂F, and no two such edges have a common vertex. Therefore, $r \leq n$, and the induction step is complete since $n \geq 5$. □

8.3. **Proof that $\mu \geq \phi$ when $\min\{k_i\} \geq 5$ and $[k_1, k_2, k_3] \neq [5, 8, 8]$.** This case covers the largest number of instances. It is followed by consideration of certain other special families of type-vectors. By Proposition 8.3, it suffices to assume

$$\text{either } \min\{k_i\} \geq 6, \text{ or } [k_1, k_2, k_3] = [5, 2n, 2n] \text{ with } n \geq 5. \tag{8.4}$$

As described before Lemma 8.4, we shall construct an injection from the set $\mathbb{E}_n$ of n-step extendable SAWs on $\mathbb{L}_+$ to the set $\Sigma_n(v)$ of SAWs on G starting at $v \in V$. Let W_n be the set of n-letter words in the alphabet $\{\mathrm{H}, \mathrm{V}\}$ beginning H and with no pair of consecutive appearances of V. For $w \in W_n$, we shall define an n-step SAW $\pi(w)$ on G, and the map $\pi : W_n \to \Sigma_n(v)$ will be an injection.

Let $n \geq 1$ and $w = (w_1 w_2 \cdots w_n) \in W_n$, so that in particular $w_1 = \mathrm{H}$. The path $\pi = \pi(w)$ is constructed iteratively as follows. Let (v', v, v'') be a 2-step SAW of G starting at some neighbour v' of v. We assume in the following that the turn in the path (v', v, v'') is rightwards (the other case is similar). We set $\pi'(w) = (v', v, v'')$ if $n = 1$. The first letter of w is $w_1 = \mathrm{H}$, and the second is either T or H, and the latter determines whether the next turn is the same as or opposite to the previous turn. We adopt the rule that:

$$\begin{aligned} &\text{if } (w_1 w_2) = (\mathrm{HT}), && \text{the turn is the same as the previous,} \\ &\text{if } (w_1 w_1) = (\mathrm{HH}), && \text{the turn is the opposite.} \end{aligned} \tag{8.5}$$

For $k \geq 3$, the kth turn of π' is either to the right or the left, and is either the same or opposite to the $(k-1)$th turn. Whether it is the same or opposite is determined as follows:

$$\begin{aligned} &\text{when } (w_{k-2} w_{k-1} w_k) = (\mathrm{HHH}), && \text{it is opposite,} \\ &\text{when } (w_{k-2} w_{k-1} w_k) = (\mathrm{HHV}), && \text{it is the same,} \\ &\text{when } (w_{k-2} w_{k-1} w_k) = (\mathrm{HVH}), && \text{it is opposite,} \\ &\text{when } (w_{k-2} w_{k-1} w_k) = (\mathrm{VHH}), && \text{it is the same,} \\ &\text{when } (w_{k-2} w_{k-1} w_k) = (\mathrm{VHV}), && \text{it is opposite.} \end{aligned} \tag{8.6}$$

The ensuing π' is clearly non-backtracking. The following claim will be useful in showing it is also self-avoiding.

Lemma 8.7. *For any sub-path of π', the numbers of right turns and left turns differ by at most* 3.

Proof. A sub-path of π' corresponds to a sub-word of w. We may assume this sub-word begins H, since if it begins V then the preceding letter is necessarily H. We shall prove the statement for the entire path π' (with corresponding word w), and the same proof works for a sub-path. A *block* is a sub-word B of w of the form $\mathrm{VH}^k\mathrm{V}$, where H^k denotes k (≥ 1) consecutive appearances of H. The block B generates $k+1$ turns in π' corresponding to the letters $\mathrm{H}^k\mathrm{V}$, and B is called *even* (respectively, odd) according to the parity of k.

(a) If B is odd, then, in the corresponding $(k+1)$ turns made by π, the numbers of right and left turns are equal. Moreover, if the first turn is to the right (respectively, left), then the last turn is to the left (respectively, right).

(b) If B is even, the numbers of right and left turns differ by 3. Moreover, the first turn is to the right if and only if the last turn is to the right, and in that case there are 3 more right turns than left turns.

Let B be an odd block. By (a), B makes no contribution to the aggregate difference between the number of right and left turns. Furthermore, the first turn of B equals the first turn following B (since the last turn of B is opposite to the first, and the following sub-word HVH results in a turn equal to the first). We may therefore consider w with all odd blocks removed, and we assume henceforth that w has no odd blocks.

Using a similar argument for even blocks based on (b) above, the effects of two even blocks cancel each other, and we may therefore remove any even number of even blocks from w without altering the aggregate difference. In conclusion, we may assume that w has the form either $\mathrm{H}^a\mathrm{V}\mathrm{H}^b$ or $\mathrm{H}^a\mathrm{V}\mathrm{H}^{2r}\mathrm{V}\mathrm{H}^b$ where $a \geq 1$, $r \geq 1$, $b \geq 0$. Each of these cases is considered separately to obtain the lemma. $\square$

Write $\pi'(w) = (v', v = x_0, v'' = x_1, \dots, x_n)$, and remove the first step to obtain a SAW $\pi(w) = (v = x_0, x_1, \dots, x_n)$. By Lemmas 8.6(a, b) and 8.7, subject to (8.4), $\pi(w)$ contains no cycle and is thus a SAW. This is seen as follows. Suppose $\nu = (x_i, x_{i+1}, \dots, x_j = x_i)$ is a cycle. The cycle has one more turn than the path, and hence, by Lemma 8.7, $|\rho(\nu)| \leq 4$, in contradiction of Lemma 8.6(a, b). Therefore, π maps W_n to $\Sigma_n(v)$. It is an injection since, by examination of (8.5)–(8.6), $\pi(w) \neq \pi(w')$ if $w \neq w'$. We deduce by (5.1) that $\mu(G) \geq \phi$.

The above difference between counting turns on *paths* and *cycles* can be overcome by considering SAWs between midpoints of edges (as in Section 8.7).

8.4. **Proof that $\mu \geq \phi$ when $\min\{k_i\} = 3$.** Assume $\min\{k_i\} = 3$. By Proposition 8.3 and the assumption $f(G) > 2$, the type-vector is $[3, 2n, 2n]$ for some $n \geq 7$. This G is the Fisher graph of the graph $G' \in \mathcal{T}_3$ with type-vector $[n, n, n]$. By Proposition 7.5(a),

$$\frac{1}{\mu(G)^2} + \frac{1}{\mu(G)^3} = \frac{1}{\mu(G')}.$$

It is proved in Section 8.3 that $\mu(G') \geq \phi$, and the inequality $\mu(G) \geq \phi$ follows (see also [7]).

8.5. **Proof that $\mu \geq \phi$ for $[4, 2n, 2p]$ with $p \geq n \geq 4$ and $n^{-1} + p^{-1} < \frac{1}{2}$.** Let $G = (V, E) \in \mathcal{T}_3$ be infinite with type-vector $[4, 2n, 2p]$ where $p \geq n \geq 4$ and $n^{-1} + p^{-1} < \frac{1}{2}$. Note that G has girth 4, and every vertex is incident to exactly one size-4 face. Let G' be the simple graph obtained from G by contracting each size-4 face to a vertex. Then $G' \in \mathcal{T}_4$ is infinite with girth $n \geq 4$ and type-vector $[n, p, n, p]$. Recall Lemma 8.6(c).

Let $v \in V$. As following (8.4), we will construct an injection from W_n to $\Sigma_n(v)$. An edge of G is called *square* if it lies in a size-4 face, and *non-square* otherwise. Let $w = (w_1 w_2 \cdots w_n) \in W_n$. We shall construct a *non-backtracking* n-step path $\pi = \pi(w)$ from v, and then show it is a SAW. If $n = 1$, set $\pi(w) = (v, v')$ where $v' \sim v$. We perform the following construction for $k = 2, 3, \dots, n$.

1. Suppose $(w_{k-1} w_k) = (\mathrm{HV})$. The following edge is always square.
   (a) If the edge e_{k-1} of π corresponding to w_{k-1} is square, then the next edge e_k of π is square.
   (b) Suppose e_{k-1} is non-square. Then the next edge e_k is one of the two possible square edges, chosen as follows. In contracting G to G', the path $(\pi_0, \pi_1, \dots, \pi_{k-1})$ contracts to a non-backtracking path π' on G'. Find the most recent turn at which π' turns either right or left. If, at that turn, π' turns left (respectively, right), the non-backtracking path π on G turns left (respectively, right). If no turn of π' is rightwards or leftwards, then π turns left.
2. Suppose $(w_{k-1} w_k) = (\mathrm{HH})$.
   (a) If the edge e_{k-1} of π corresponding to w_{k-1} is square, then the next edge e_k of π is non-square.
   (b) Suppose e_{k-1} is non-square. Then e_k is one of the two possible square edges, chosen as follows. In the notation of 1(b) above, find the most recent turn at which π' is to either the right or the left. If at that turn, π' turns left (respectively, right), the non-backtracking path π on G turns right (respectively, left). If π' has no such turn, then π turns right.
3. Suppose $(w_{k-1} w_k) = (\mathrm{VH})$. The edge e_{k-1} of G corresponding to w_{k-1} must be square. If the edge previous to e_{k-1} on π is square (respectively, non-square), then π continues to a non-square (respectively, square) edge.

We claim the mapping $\pi : W_n \to \Sigma_n(v)$ is an injection. By construction, $\pi(w) = \pi(w')$ if and only if $w = w'$, and, furthermore, $\pi(w)$ is non-backtracking. It remains to show that each $\pi(w)$ is a SAW. Let $w \in W_n$, and note that $\pi(w)$ is non-backtracking with at most three consecutive square edges (this occurs on encountering VHV preceded by a non-square edge). It suffices, therefore, to show that, after contracting each square face to a vertex, the resulting path $\pi'(w)$ on G' is a SAW.

By considering the different possibilities (illustrated in Figure 8.1), we see that any right (respectively, left) turn in π' is followed (possibly after some straight section) by a left (respectively, right) turn. Therefore, in any sub-walk ν of $\pi'(w)$, the numbers of right and left turns differ by at most 1. By Lemma 8.6(c) or directly, ν cannot form a cycle. Hence $\pi'(w)$ is a SAW, and the proof is complete.

8.6. **Proof that $\mu \ge \phi$ for $[4, 6, 2p]$ with $p \ge 6$.** Let $G \in \mathcal{T}_3$ be infinite with type-vector $[4, 6, 2p]$ where $p \ge 6$. (When $p = 6$, this graph G is drawn in Figure 6.1.)

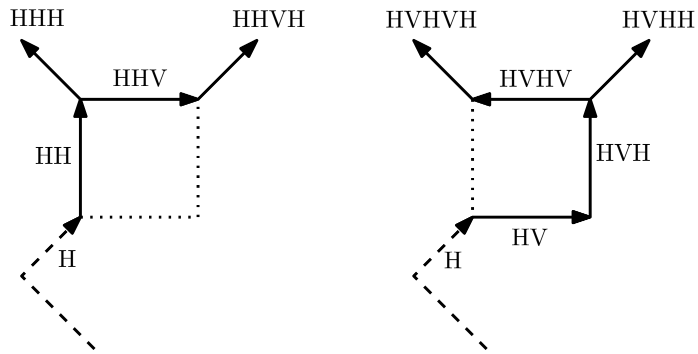

FIGURE 8.1. The dashed line is the projected SAW on G'. After a right (respectively, left) turn, the projection either moves straight or turns left (respectively, right).

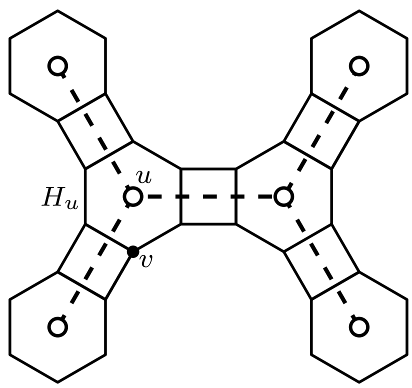

FIGURE 8.2. The graph G with an embedded copy of the graph $[p, p, p]$.

Associated with G is the graph $P := [p, p, p]$ as drawn in Figure 8.2. Each vertex u of P lies within some hexagon of G denoted H_u. Let u be a vertex of P and let v be a vertex of H_u. Let $\pi = (u_0 = u, u_1, \dots, u_n)$ be a SAW on P from u. We shall explain how to associate with π a family of SAWs on G from v. The argument is similar to that of the proof of Proposition 7.5.

A hexagon of G has six vertices, which we denote in consecutive pairs according to approximate compass bearing. For example, $p_{\mathrm{w}}(H)$ is the pair on the west side of H, and similarly p_{nw}, p_{ne}, p_{e}, p_{se}, p_{sw}. For definiteness, we assume that H_u has orientation as in Figure 8.2, and $v \in p_{\mathrm{sw}}(H_u)$, as in Figure 8.3.

Let $\Sigma_n(u)$ be the set of n-step SAWs on P from u, the first edge of which is either north-westwards or eastwards (that is, away from $p_{\mathrm{sw}}(H_u)$). Suppose the first step of the SAW $\pi \in \Sigma_n(u)$ is to the neighbour u_1 that lies eastwards of u (the other case is similar). With the step (u, u_1), we may associate any of four SAWs on G from v to $p_{\mathrm{w}}(H_{u_1})$, namely those illustrated in Figure 8.3. These paths have lengths 2, 3, 5, 6. If u_1 lies to the north-west of u, the corresponding four paths have lengths 3, 4, 4, 5.

We now iterate the above construction. At each step of π, we construct a family of 4 SAWs on G that extend the walk on G to a new hexagon. When this process is complete, the ensuing paths on G are all SAWs, and they are distinct.

Let $Z_P(\zeta)$ (respectively, $Z_G(\zeta)$) be the generating function of SAWs on P from u (respectively, on G from v), subject to above starting assumption. In the above construction, each step of π is replaced by one of four paths, with lengths lying in either $(2,3,5,6)$ or $(3,4,4,5)$, depending on the initial vertex of the segment. Since

$$\zeta^2+\zeta^3+\zeta^5+\zeta^6 \ge \zeta^3+2\zeta^4+\zeta^5 \quad [=\zeta^3(1+\zeta)^2], \qquad \zeta \in \mathbb{R},$$

we have that

$$Z_P(\zeta^3(1+\zeta)^2) \le Z_G(\zeta), \qquad \zeta \ge 0. \tag{8.7}$$

Let $z>0$ satisfy

$$z^3(1+z)^2 = \frac{1}{\mu(P)}. \tag{8.8}$$

Since $1/\mu(P)$ is the radius of convergence of Z_P, (8.7) implies $z \ge 1/\mu(G)$, which is to say that

$$\mu(G) \ge \frac{1}{z}. \tag{8.9}$$

As in Section 8.3, $\mu(P) \ge \phi$. It suffices for $\mu(G) \ge \phi$, therefore, to show that the (unique) root in $(0,\infty)$ of

$$x^3(1+x)^2 = \frac{1}{\phi}$$

satisfies $x \le 1/\phi$, and it is easily checked that, in fact, $x = 1/\phi$.

Remark 8.8 (Archimedean lattice $\mathbb{A} = [4,6,12]$)**.** *The inequality $\mu(\mathbb{A}) \ge \phi$ may be strengthened. In the special case $p=6$, we have that $\mu(P) = \sqrt{2+\sqrt{2}}$; see [4]. By (8.8)–(8.9), $\mu(G) \ge 1.676$.*

8.7. **Proof that $\mu \ge \phi$ for $[5,8,8]$.** Let $G \in \mathcal{T}_3$ be infinite with type-vector $[5,8,8]$. Let G' be the simple graph obtained from G by contracting each size-5 face of G to a vertex. Note that $G' \in \mathcal{T}_5$ is infinite with type-vector $[4,4,4,4,4]$, and recall Lemma 8.6(c). An edge of G is called *pentagonal* if it is belongs to a size-5 face, and *non-pentagonal* otherwise.

We opt to consider SAWs between midpoints of edges. Let m be the midpoint of some non-pentagonal edge of G, and let $\Sigma_n(m)$ be the set of n-step SAWs on G from

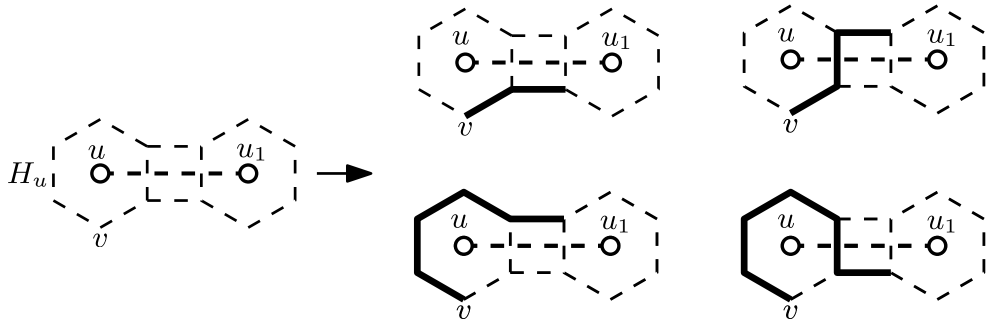

FIGURE 8.3. The step (w, w_1) on P may be mapped to any of the four SAWs on G from v, as drawn on the right.

m. We will find an injection from W_n to $\Sigma_n(m)$. Let $w = (w_1 w_2 \cdots w_n) \in W_n$. We construct as follows a non-backtracking path $\pi(w)$ on G starting from m. The first step of $\pi(w)$ is (v, v') where v' is an arbitrarily chosen midpoint adjacent to m.

For any path π' of G', let $\rho(\pi') = r(\pi') - l(\pi')$, where $r(\pi')$ (respectively, $l(\pi')$ is the number of right (respectively, left) turns of π'. Since paths move between midpoints, this agrees with the previous use of ρ.

We iterate the following for $k = 2, 3, \dots, n$ (cf. the construction of Section 8.5).

1. Suppose $(w_{k-1} w_k) = $ (HV). The following edge is always pentagonal.
   (a) If the position at time $k-1$ is on a pentagonal edge, the next step of π is to the midpoint of the incident pentagonal edge.
   (b) Suppose the position is non-pentagonal. On contracting G to G', the path $\pi(w_1 w_2 \cdots w_{k-1})$ on G, so far, gives rise to a non-backtracking path π' on G'. If $\rho(\pi') < 0$ (respectively, $\rho(\pi') \geq 0$), then the next turn of π is to the left (respectively, right).
2. Suppose $(w_{k-1} w_k) = $ (HH).
   (a) If the position at time $k-1$ is on a pentagonal edge, the next step of π is to the midpoint of the incident non-pentagonal edge.
   (b) Suppose the position is non-pentagonal. In the notation of 1(b) above, if $\rho(\pi') < 0$ (respectively, $\rho(\pi') \geq 0$), then the next turn of π is to the right (respectively, left).
3. Suppose $(w_{k-1} w_k) = $ (VH), and note that the current position is necessarily at the midpoint of some pentagonal edge e_{k-1}. If the precursor of e_{k-1} is pentagonal (respectively,

We claim the mapping $\pi : W_n \to \Sigma_n(m)$ is an injection. It is straightforward that π is an injection from W_n to the set of n-step non-backtracking paths in G from m, and it suffices to show that any $\pi(w)$ is a SAW. For $w \in W_n$, at most three consecutive edges of $\pi(w)$ are pentagonal. It suffices to show that, after contracting

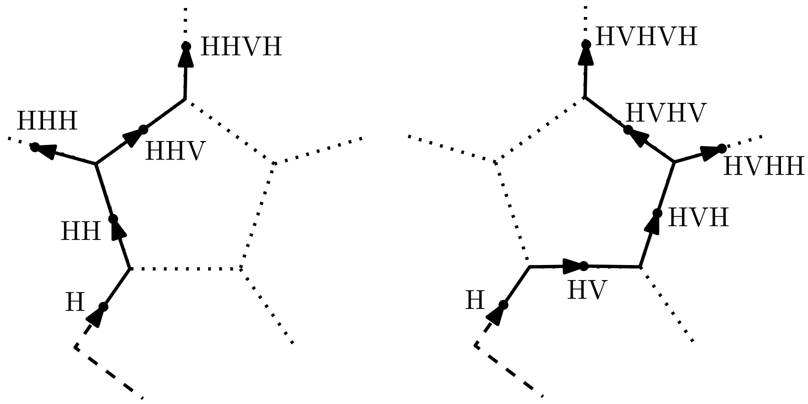

FIGURE 8.4. The dashed line is the projected SAW π' on G', assumed in the figure to satisfy $\rho(\pi') \geq 0$. When $\rho(\pi') \geq 0$ (respectively, $\rho(\pi') < 0$), the projection may move leftwards but not rightwards (respectively, rightwards but not leftwards) at the next pentagon.

each pentagon to a vertex, the ensuing $\pi'(w)$ is a SAW on G'. For any subwalk ν of $\pi'(w)$, it may be checked (as in the proof of Section 8.5) that the numbers of right and left turns differ by at most 1. By Lemma 8.6(c) or directly, ν cannot form a cycle. Hence $\pi'(w)$ is a SAW, and the proof is complete.

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