CONNECTIVE CONSTANTS AND HEIGHT FUNCTIONS FOR CAYLEY GRAPHS

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Abstract. The connective constant $\mu(G)$ of an infinite transitive graph G is the exponential growth rate of the number of self-avoiding walks from a given origin. In earlier work of Grimmett and Li, a locality theorem was proved for connective constants, namely, that the connective constants of two graphs are close in value whenever the graphs agree on a large ball around the origin. A condition of the theorem was that the graphs support so-called 'unimodular graph height functions'. When the graphs are Cayley graphs of infinite, finitely generated groups, there is a special type of unimodular graph height function termed here a '*group* height function'. A necessary and sufficient condition for the existence of a group height function is presented, and may be applied in the context of the bridge constant, and of the locality of connective constants for Cayley graphs. Locality may thereby be established for a variety of infinite groups including those with strictly positive deficiency.

It is proved that a large class of Cayley graphs support unimodular graph height functions, that are in addition *harmonic* on the graph. This implies, for example, the existence of unimodular graph height functions for the Cayley graphs of finitely generated solvable groups. It turns out that graphs with non-unimodular automorphism subgroups also possess graph height functions, but the resulting graph height functions need not be harmonic.

Group height functions, as well as the graph height functions of the previous paragraph, are non-constant harmonic functions with linear growth and an additional property of having periodic differences. The existence of such functions on Cayley graphs is a topic of interest beyond their applications in the theory of self-avoiding walks.

1. Introduction, and summary of results

The main purpose of this article is to study aspects of 'locality' for the connective constants of Cayley graphs of finitely presented groups. The locality question may be posed as follows: if two Cayley graphs are locally isomorphic in the sense that they

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agree on a large ball centred at the identity, then are their connective constants close in value? The current work may be viewed as a continuation of the study of locality for connective constants of quasi-transitive graphs reported in [10]. The locality of critical points is a well developed topic in the theory of disordered systems, and the reader is referred, for example, to [5, 29, 31] for related work about percolation on Cayley graphs.

The self-avoiding walk (SAW) problem was introduced to mathematicians in 1954 by Hammersley and Morton [17]. Let G be an infinite, connected, transitive graph. The number of n-step SAWs on G from a given origin grows in the manner of $\mu^{n(1+o(1))}$ for some growth rate $\mu = \mu(G)$ called the *connective constant* of the graph G. The value of $\mu(G)$ is not generally known, and a substantial part of the literature on SAWs is targeted at properties of connective constants. The current paper may be viewed in this light, as a continuation of the series of papers [8, 9, 12, 10, 11].

The principal result of [10] is as follows. Let G, G' be infinite, transitive graphs, and write $S_K(v, G)$ for the K-ball around the vertex v in G. If $S_K(v, G)$ and $S_K(v', G')$ are isomorphic as rooted graphs, then

$$|\mu(G) - \mu(G')| \le \epsilon_K(G), \tag{1.1}$$

where $\epsilon_K(G) \to 0$ as $K \to \infty$. This is proved subject to a condition on G and G', namely that they support so-called 'unimodular graph height functions'.

Cayley graphs of finitely generated groups provide a category of transitive graphs of special interest. They possess an algebraic structure in addition to their graphical structure, and this algebraic structure provides a mechanism for the study of their graph height functions. A necessary and sufficient condition is given in Theorem 4.1 for the existence of a so-called 'group height function', and it is pointed out that a group height function is a unimodular graph height function, but not *vice versa*. The class of Cayley groups that possess group height functions includes all infinite, finitely generated, free solvable groups and free nilpotent groups, and groups with fewer relators than generators.

We turn briefly to the topic of harmonic functions. The study of the existence and structure of non-constant harmonic functions on Cayley graphs has acquired prominence in geometric group theory through the work of Kleiner and others, see [23, 33]. The group height functions of Section 4 are harmonic with linear growth. Thus, one aspect of the work reported in this paper is the construction, on certain classes of Cayley graphs, of linear-growth harmonic functions with the additional property of having differences that are invariant under the action of a subgroup of automorphisms. Such harmonic functions do not appear to contribute to the discussion of the Liouville property (see, for example, [24, Defn 2.1.10]), since both their positive and negative parts are unbounded. For recent articles on the related aspect of geometric group theory, the reader is referred to [30, 35].

This paper is organized as follows. Graphs, self-avoiding walks, and Cayley graphs are introduced in Section 2. Graph height functions and the locality theorem of [10] are reviewed in Section 3, and a principal tool is presented at Theorem 3.4. Graphs with non-unimodular automorphism subgroups may be handled by similar means (see Theorem 3.5), but the resulting graph height functions need not be harmonic.

Group height functions are the subject of Section 4, and a necessary and sufficient condition is presented in Theorem 4.1 for the existence of a group height function. Section 5 is devoted to existence conditions for height functions, leading to existence theorems for virtually solvable groups. In Section 6 is presented a theorem for the convergence of connective constants subject to the addition of further relators. This parallels the Grimmett–Marstrand theorem [14] for the critical percolation probabilities of slabs of $\mathbb{Z}^d$ (see also [11, Thm 5.2]). Sections 7–8 contain the proofs of Theorems 3.4 and 3.5, respectively.

2. Graphs, self-avoiding walks, and groups

The graphs $G = (V, E)$ considered here are infinite, connected, and usually simple. An undirected edge e with endpoints u, v is written as $e = \langle u, v\rangle$, and if directed from u to v as $[u, v\rangle$. If $\langle u, v\rangle \in E$, we call u and v *adjacent* and write $u \sim v$. The set of neighbours of $v \in V$ is denoted $\partial v := \{u : u \sim v\}$.

The *degree* $\deg(v)$ of vertex v is the number of edges incident to v, and G is called *locally finite* if every vertex-degree is finite. The *graph-distance* between two vertices u, v is the number of edges in the shortest path from u to v, denoted $d_G(u, v)$.

The automorphism group of the graph $G = (V, E)$ is denoted $\mathrm{Aut}(G)$. A subgroup $\Gamma \le \mathrm{Aut}(G)$ is said to *act transitively* on G if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma v = w$. It is said to *act quasi-transitively* if there is a finite set W of vertices such that, for $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $\gamma v = w$. The graph is called (*vertex-*)*transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively). For $\Gamma \le \mathrm{Aut}(G)$ and a vertex $v \in V$, the orbit of v under Γ is written Γv.

An (*n-step*) *walk* w on G is an alternating sequence $(w_0, e_0, w_1, e_1, \dots, e_{n-1}, w_n)$, where $n \ge 0$, of vertices w_i and edges $e_i = \langle w_i, w_{i+1}\rangle$, and its *length* $|w|$ is the number of its edges. The walk w is called *closed* if $w_0 = w_n$. A *cycle* is a closed walk w satisfying $n \ge 3$ and $w_i \ne w_j$ for $1 \le i < j \le n$.

An (*n-step*) *self-avoiding walk* (SAW) on G is a walk containing n edges no vertex of which appears more than once. Let $\Sigma_n(v)$ be the set of n-step SAWs starting at v, with cardinality $\sigma_n(v) := |\Sigma_n(v)|$. *Assume that G is transitive*, and select a vertex of G which we call the *identity* or *origin*, denoted $\mathbf{1} = \mathbf{1}_G$, and let $\sigma_n = \sigma_n(\mathbf{1})$. It is standard (see [17, 28]) that

$$\sigma_{m+n} \le \sigma_m \sigma_n, \tag{2.1}$$

whence, by the subadditive limit theorem, the *connective constant*

$$\mu = \mu(G) := \lim_{n\to\infty} \sigma_n^{1/n}$$

exists. See [2, 28] for recent accounts of the theory of SAWs.

We turn now to finitely generated groups and their Cayley graphs. Let Γ be a group with generator set S satisfying $|S| < \infty$ and $\mathbf{1} \notin S$, where $\mathbf{1} = \mathbf{1}_\Gamma$ is the identity element. We write $\Gamma = \langle S \mid R\rangle$ with R a set of relators, and we adopt the following convention for the inverses of generators. For the sake of concreteness, we consider S as a set of symbols, and any information concerning inverses is encoded in the relator set; it will always be the case that, using this information, we may identify the inverse of $s \in S$ as another generator $s' \in S$. For example, the free abelian group of rank 2 has presentation $\langle x, y, X, Y \mid xX, yY, xyXY\rangle$, and the infinite dihedral group $\langle s_1, s_2 \mid s_1^2, s_2^2\rangle$. Such a group is called *finitely generated* (in that $|S| < \infty$), and *finitely presented* if, in addition, $|R| < \infty$.

The *Cayley graph* of $\Gamma = \langle S \mid R\rangle$ is the simple graph $G = G(S, R)$ with vertex-set Γ, and an (undirected) edge $\langle \gamma_1, \gamma_2\rangle$ if and only if $\gamma_2 = \gamma_1 s$ for some $s \in S$. Further properties of Cayley graphs are presented as needed in Section 4. See [1] for an account of Cayley graphs, and [27] for a short account. The books [19] and [25, 32] are devoted to geometric group theory, and general group theory, respectively.

The set of integers is written $\mathbb{Z}$, the natural numbers as $\mathbb{N}$, and the rationals as $\mathbb{Q}$.

3. Graph height functions

We recall from [10] the definition of a graph height function for a transitive graph, and then we review the locality theorem (the proof of which may be found in [10]). This is followed by Theorem 3.4, which is one of the main tools of this work.

Let $\mathcal{G}$ be the set of all infinite, connected, transitive, locally finite, simple graphs, and let $G = (V, E) \in \mathcal{G}$. Let $\mathcal{H}$ be a subgroup of $\mathrm{Aut}(G)$. A function $F : V \to \mathbb{R}$ is said to be $\mathcal{H}$-*difference-invariant* if

$$F(v) - F(w) = F(\gamma v) - F(\gamma w), \qquad v, w \in V,\ \gamma \in \mathcal{H}. \tag{3.1}$$

Definition 3.1. *A* graph height function *on G is a pair $(h, \mathcal{H})$, where $\mathcal{H} \le \mathrm{Aut}(G)$ acts quasi-transitively on G and $h : V \to \mathbb{Z}$, such that*

(a) *$h(\mathbf{1}) = 0$,*
(b) *h is $\mathcal{H}$-difference-invariant,*
(c) *for $v \in V$, there exist $u, w \in \partial v$ such that $h(u) < h(v) < h(w)$.*

The graph height function $(h, \mathcal{H})$ is called unimodular *if $\mathcal{H}$ is unimodular.*

We remind the reader of the definition of the unimodularity of a subgroup $\mathcal{H} \le \mathrm{Aut}(G)$. The ($\mathcal{H}$-)*stabilizer* Stab_v ($= \mathrm{Stab}_v^{\mathcal{H}}$) of a vertex v is the set of all $\gamma \in \mathcal{H}$ for which $\gamma v = v$. As shown in [36] (see also [4, 27, 34]), when viewed as a topological group with the topology of pointwise convergence, $\mathcal{H}$ is unimodular if and only if

$$|\mathrm{Stab}_u v| = |\mathrm{Stab}_v u|, \qquad v \in V,\ u \in \mathcal{H}v. \tag{3.2}$$

We follow [27, Chap. 8] by *defining* $\mathcal{H}$ to be *unimodular* (on G) if (3.2) holds.

We sometimes omit the reference to $\mathcal{H}$ and refer to such h as a graph height function. In Section 4 is defined the related concept of a *group height function* for the Cayley graph of a finitely presented group. We shall see that every group height function is a graph height function, but not *vice versa*.

Remark 3.2 (Linear growth)**.** *A graph height function $(h, \mathcal{H})$ on G has linear growth in that $h(\gamma^n \mathbf{1}) = nh(\gamma\mathbf{1})$ for $\gamma \in \mathcal{H}$.*

Associated with the graph height function $(h, \mathcal{H})$ are two integers d, r given as follows. We set

$$d = d(h) = \max\big\{|h(u) - h(v)| : u, v \in V,\ u \sim v\big\}. \tag{3.3}$$

If $\mathcal{H}$ acts transitively, we set $r = 0$. Assume $\mathcal{H}$ does not act transitively, and let $r = r(h, \mathcal{H})$ be the least integer r such that the following holds. For $u, v \in V$ in different orbits of $\mathcal{H}$, there exists $v' \in \mathcal{H}v$ such that $h(u) < h(v')$, and a SAW $\nu(u, v')$ from u to v', with length r or less, all of whose vertices x, other than its endvertices, satisfy $h(u) < h(x) < h(v')$. We recall [10, Prop. 3.2] where it is proved, *inter alia*, that $r < \infty$.

We state next the locality theorem for transitive graphs. The *ball* $S_k = S_k(G)$, with centre $\mathbf{1} = \mathbf{1}_G$ and radius k, is the subgraph of G induced by the set of its vertices within graph-distance k of $\mathbf{1}$. For $G, G' \in \mathcal{G}$, we write $S_k(G) \simeq S_k(G')$ if there exists a graph-isomorphism from $S_k(G)$ to $S_k(G')$ that maps $\mathbf{1}_G$ to $\mathbf{1}_{G'}$, and we let

$$K(G, G') = \max\big\{k : S_k(G) \simeq S_k(G')\big\}, \qquad G, G' \in \mathcal{G}.$$

For $D \ge 1$ and $R \ge 0$, let $\mathcal{G}_{D,R}$ be the set of all $G \in \mathcal{G}$ which possess a unimodular graph height function h satisfying $d(h) \le D$ and $r(h, \mathcal{H}) \le R$.

For $G \in \mathcal{G}$ with a given unimodular graph height function $(h, \mathcal{H})$, there is a subset of SAWs called *bridges* which are useful in the study of the geometry of SAWs on G. The SAW $\pi = (\pi_0, \pi_1, \ldots, \pi_n) \in \Sigma_n(v)$ is called a *bridge* if

$$h(\pi_0) < h(\pi_i) \le h(\pi_n), \qquad 1 \le i \le n, \tag{3.4}$$

and the total number of such bridges is denoted $b_n(v)$. It is easily seen (as in [18]) that $b_n := b_n(\mathbf{1})$ satisfies

$$b_{m+n} \ge b_m b_n, \tag{3.5}$$

from which we deduce the existence of the *bridge constant*

$$\beta = \beta(G) = \lim_{n\to\infty} b_n^{1/n}. \tag{3.6}$$

The definition of β depends on the choice of height function, but it turns out that, under reasonable conditions, its value does not.

Theorem 3.3 (Bridges and locality for transitive graphs, [10])**.**

(a) *If $G \in \mathcal{G}$ supports a unimodular graph height function $(h, \mathcal{H})$, then $\beta(G) = \mu(G)$.*

(b) *Let $D \geq 1$, $R \geq 0$, and let $G \in \mathcal{G}$ and $G_m \in \mathcal{G}_{D,R}$ for $m \geq 1$ be such that $K(G, G_m) \to \infty$ as $m \to \infty$. Then $\mu(G_m) \to \mu(G)$.*

The main thrust of the current paper is to identify classes of finitely generated groups whose Cayley graphs support graph height functions, and one of our main tools is the following theorem, of which the proof is given in Section 7.

Theorem 3.4. *Let $G = (V, E) \in \mathcal{G}$. Suppose there exist*

(a) *a subgroup $\Gamma \leq \mathrm{Aut}(G)$ acting transitively on V,*

(b) *a normal subgroup $\mathcal{H} \trianglelefteq \Gamma$ satisfying $[\Gamma : \mathcal{H}] < \infty$, which is unimodular,*

(c) *a function $F : \mathcal{H}\mathbf{1} \to \mathbb{Z}$ that is $\mathcal{H}$-difference-invariant and non-constant.*

Then,

(i) *there exists a unique harmonic, $\mathcal{H}$-difference-invariant function ψ on G that agrees with F on $\mathcal{H}\mathbf{1}$.*

(ii) *there exists a harmonic, $\mathcal{H}$-difference-invariant function ψ' that increases everywhere, in that every $v \in V$ has neighbours u, w such that $\psi'(u) < \psi'(v) < \psi'(w)$,*

(iii) *the function ψ of part (i) takes rational values, and the ψ' of part (ii) may be taken to be rational also; therefore, there exists a harmonic, unimodular graph height function of the form $(h, \mathcal{H})$.*

The first part of condition (c) is to be interpreted as saying that (3.1) holds for $v, w \in \mathcal{H}\mathbf{1}$ and $\gamma \in \mathcal{H}$. Since G is transitive, the choice of origin $\mathbf{1}$ is arbitrary, and hence the orbit $\mathcal{H}\mathbf{1}$ may be replaced by any orbit of $\mathcal{H}$.

One application of Theorem 3.4, or more precisely of its method of proof, is the proof of the existence of graph height functions for graphs with quasi-transitive, non-unimodular automorphism subgroups. See Section 8 for the proof of the following.

Theorem 3.5. *Let $G = (V, E) \in \mathcal{G}$. Suppose there exist a subgroup $\Gamma \leq \mathrm{Aut}(G)$ acting transitively on V, and a normal subgroup $\mathcal{H} \trianglelefteq \Gamma$ satisfying $[\Gamma : \mathcal{H}] < \infty$, such that $\mathcal{H}$ is non-unimodular. Then G has a graph height function $(h, \mathcal{H})$, which is not generally harmonic.*

The proofs of Theorems 3.4 and 3.5 are inspired in part by the proofs of [26, Sect. 3] where, *inter alia*, it is explained that some graphs support harmonic maps, taking values in a function space, with a property of equivariance in norm. In this paper, we study $\mathcal{H}$-difference-invariant, integer-valued harmonic functions.

4. Group height functions

We consider Cayley graphs of finitely generated groups next, and a type of graph height function called a 'group height function'. Let Γ be a finitely generated group with presentation $\langle S \mid R\rangle$, as in Section 2. A group height function on a Cayley graph G of Γ may in fact be defined as a function on the group Γ itself, but it will be convenient that it acts on the same domain as a graph height function (that is, on G rather than on Γ). When viewed as a function on the group Γ, a group height function is essentially a surjective homomorphism to $\mathbb{Z}$, and such functions are of importance in group theory (see Remark 4.2).

Each relator $\rho \in R$ is a word of the form $\rho = t_1 t_2 \cdots t_r$ with $t_i \in S$ and $r \geq 1$, and we define the vector $u(\rho) = (u_s(\rho) : s \in S)$ by

$$u_s(\rho) = |\{i : t_i = s\}|, \qquad s \in S.$$

Let C be the $|R| \times |S|$ matrix with row vectors $u(\rho)$, $\rho \in R$, called the *coefficient matrix* of the presentation $\langle S \mid R\rangle$. Its null space $\mathcal{N}(C)$ is the set of column vectors $\gamma = (\gamma_s : s \in S)$ such that $C\gamma = \mathbf{0}$. Since C has integer entries, $\mathcal{N}(C)$ is non-trivial if and only if it contains a non-zero vector of integers (that is, an integer vector other than the zero vector $\mathbf{0}$). If $\gamma \in \mathbb{Z}^S$ is a non-zero element of $\mathcal{N}(C)$, then γ gives rise to a function $h : V \to \mathbb{Z}$ defined as follows. Any $v \in V$ may be expressed as a word in the alphabet S, which is to say that $v = s_1 s_2 \cdots s_m$ for some $s_i \in S$ and $m \geq 0$. We set

$$h(v) = \sum_{i=1}^{m} \gamma_{s_i}. \tag{4.1}$$

Any function h arising in this way is called a *group height function* of the presentation (or of the Cayley graph). We see next that a group height function is well defined by (4.1), and is indeed a graph height function in the sense of Definition 3.1. A graph height function, even if unimodular, need not be a group height function (see, for instance, Example (d) following Remark 4.2).

Theorem 4.1. *Let G be the Cayley graph of the finitely generated group $\Gamma = \langle S \mid R\rangle$, with coefficient matrix C.*

(a) *Let $\gamma = (\gamma_s : s \in S) \in \mathcal{N}(C)$ satisfy $\gamma \in \mathbb{Z}^S$, $\gamma \neq \mathbf{0}$. The group height function h given by (4.1) is well defined, and gives rise to a unimodular graph height function (h, Γ) on G.*

(b) *The Cayley graph $G(S,R)$ of the presentation $\langle S \mid R\rangle$ has a group height function if and only if* $\operatorname{rank}(C) < |S|$.
(c) *A group height function is a group invariant in the sense that, if h is a group height function of G, then it is also a group height function for the Cayley graph of any other presentation of Γ.*
(d) *A group height function h of G is harmonic, in that*

$$h(v) = \frac{1}{\deg(v)} \sum_{u \sim v} h(u), \qquad v \in V.$$

Since the group height function h of (4.1) is a graph height function, and Γ acts transitively,

$$d(h) = \max\{\gamma_s : s \in S\}, \tag{4.2}$$

in agreement with (3.3). In the light of Theorem 4.1(c), we may speak of a group possessing a group height function.

Remark 4.2. *The quantity $b(\Gamma) := |S| - \operatorname{rank}(C)$ is in fact an invariant of Γ, and may be called the* first Betti number *since it equals the power of $\mathbb{Z}$ in the abelianization $\Gamma/[\Gamma,\Gamma]$ (see, for example,* [25, Chap. 8]*). Group height functions are a standard tool of group theorists, since they are (when the non-zero γ_s are coprime) surjective homomorphisms from Γ to $\mathbb{Z}$. This fact is used, for example, in the proof of* [13, Thm 4.1]*, which asserts that the Cayley graph of an infinite, finitely generated, elementary amenable group possesses a harmonic (unimodular) graph height function. Further details may be found in* [21, 22].

Although some of the arguments of the current paper are standard within group theory, we prefer to include sufficient details to aid readers from other backgrounds.

It follows in particular from Theorem 4.1 that G has a group height function if $|R| < |S|$, which is to say that the presentation $\Gamma = \langle S \mid R\rangle$ has strictly positive deficiency (see [32, p. 419]). Free groups provide examples of such groups.

Consider for illustration the examples of [10, Sect. 3].

(a) The *hypercubic lattice* $\mathbb{Z}^n$ is the Cayley group of an abelian group with $|S| = 2n$, $|R| = n + \binom{n}{2}$, and $\operatorname{rank}(C) = n$. It has a group height function (in fact, it has many, indexed by the non-zero, integer-valued elements of $\mathcal{N}(C)$).
(b) The 3-*regular tree* is the Cayley graph of the group with $S = \{s_1, s_2, t\}$, $R = \{s_1 t, s_2^2\}$, and $\operatorname{rank}(C) = 2$. It has a group height function.
(c) The *discrete Heisenberg group* has $|S| = |R| = 6$ and $\operatorname{rank}(C) = 4$. It has a group height function.
(d) The *square/octagon lattice* is the Cayley graph of a finitely presented group with $|S| = 3$ and $|R| = 5$, and this does not satisfy the hypothesis of Theorem 4.1(b) (since $\operatorname{rank}(C) = 3$). This presentation has no group height function.

Neither does the lattice have a graph height function with automorphism subgroup acting transitively, but nevertheless it possesses a unimodular graph height function in the sense of Definition 3.1, as explained in [10, Sect. 3].

(e) The *hexagonal lattice* is the Cayley graph of the finitely presented group with $S = \{s_1, s_2, s_3\}$ and $R = \{s_1^2, s_2s_3, s_1s_2^2s_1s_3^2\}$. Thus, $|R| = |S| = 3$, $\mathrm{rank}(C) = 2$, and the graph has a group height function.

A discussion is presented in Section 5 of certain types of infinite groups whose Cayley graphs have group or graph height functions. We describe next some illustrative examples and a question. The next proposition is extended in Theorem 5.2.

Proposition 4.3. *Any finitely generated group which is infinite and abelian has a group height function h with $d(h) = 1$.*

Example 4.4. *The* infinite dihedral group $\mathrm{Dih}_\infty = \langle s_1, s_2 \mid s_1^2, s_2^2\rangle$ *is an example of an infinite, finitely generated group Γ which has no group height function. The Cayley graph of Γ is the line $\mathbb{Z}$ with (harmonic) unimodular graph height function $(h, \mathcal{H})$, where h is the identity and $\mathcal{H}$ is the group of shifts. This example of a solvable group is extended in Theorem 5.1.*

Example 4.5. *The* Higman group Γ *of* [20] *is an infinite, finitely presented group with presentation $\Gamma = \langle S \mid R\rangle$ where*

$$S = \{a, b, c, d, a', b', c', d'\},$$
$$R = \{aa', bb', cc', dd'\} \cup \{a'ba(b')^2, b'cb(c')^2, c'dc(d')^2, d'ad(a')^2\}.$$

The quotient of Γ by its maximal proper normal subgroup is an infinite, finitely generated, simple group. By Theorem 4.1(b), Γ has no group height function.

Remark 4.6. *Since writing this paper, the authors have shown in* [13] *that the Cayley graph of the Higman group does not possess a graph height function.*

Proof of Theorem 4.1. (a) Let γ be as given. To check that h is well defined by (4.1), we must show that $h(v)$ is independent of the chosen representation of v as a word. Suppose that $v = s_1 \cdots s_m = u_1 \cdots u_n$ with $s_i, u_j \in S$, and extend the definition of γ to the directed edge-set of G by

$$\gamma([g, gs\rangle) = \gamma_s, \qquad g \in \Gamma,\ s \in S. \tag{4.3}$$

The walk $(\mathbf{1}, s_1, s_1s_2, \dots, v)$ is denoted as π_1, and $(\mathbf{1}, u_1, u_1u_2, \dots, v)$ as π_2, and the latter's reversed walk as π_2^{-1}. Consider the walk ν obtained by following π_1, followed by π_2^{-1}. Thus ν is a closed walk of G from $\mathbf{1}$.

Any $\rho \in R$ gives rise to a directed cycle in G through $\mathbf{1}$, and we write ΓR for the set of images of such cycles under the action of Γ. Any closed walk lies in the vector space over $\mathbb{Z}$ generated by the directed cycles of ΓR (see, for example, [16, Sect. 4.1]). The sum of the γ_s around any $g\rho \in \Gamma R$ is zero, by (4.3) and the fact that $C\gamma = \mathbf{0}$. Hence

$$\sum_{i=1}^{m} \gamma_{s_i} - \sum_{j=1}^{n} \gamma_{u_j} = 0,$$

as required.

We check next that (h, Γ) is a graph height function. Certainly, $h(\mathbf{1}) = 0$. For $u, v \in V$, write $v = ux$ where $x = u^{-1}v$, so that $h(v) - h(u) = h(x)$ by (4.1). For $g \in \Gamma$, we have that $gv = (gu)x$, whence

$$h(gv) - h(gu) = h(x) = h(v) - h(u). \tag{4.4}$$

Since $\gamma \neq \mathbf{0}$, there exists $s \in S$ with $\gamma_s > 0$. For $v \in V$, we have $h(vs^{-1}) < h(v) < h(vs)$.

(b) The null space $\mathcal{N}(C)$ is non-trivial if and only if $\mathrm{rank}(C) < |S|$. Since C has integer entries and $|S| < \infty$, $\mathcal{N}(C)$ is non-trivial if and only if it contains a non-zero vector of integers.

(c) See Remark 4.2. This may also be proved directly, but we omit the details.

(d) We do not give the details of this, since a more general fact is proved in Proposition 7.1(b). The current proof follows that of the latter proposition with $\mathcal{H} = \Gamma$, $F = h$, and Γ acting on V by left-multiplication. Since this action of Γ has no non-trivial fixed points, Γ is unimodular. □

Proof of Proposition 4.3. See Remark 4.2 for an elementary group-theoretic explanation. Since Γ is infinite and abelian, there exists a generator, σ say, of infinite order. For $s \in S$, let

$$\gamma_s = \begin{cases} 1 & \text{if } s = \sigma, \\ -1 & \text{if } s = \sigma^{-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Since any relator must contain equal numbers of appearances of σ and σ^{-1}, we have that $\gamma \in \mathcal{N}(C)$. Therefore, the function h of (4.1) is a group height function. □

5. Cayley graphs with height functions

The main result of this section is as follows. The associated definitions are presented later, and the proofs of the next two theorems are at the end of this section.

Theorem 5.1. *Let Γ be an infinite, finitely generated group with a normal subgroup Γ^* satisfying $[\Gamma : \Gamma^*] < \infty$. Let $q = \sup\{i : [\Gamma^* : \Gamma^*_{(i)}] < \infty\}$ where $(\Gamma^*_{(i)} : i \geq 1)$ is the derived series of Γ^*. If $q < \infty$ and $[\Gamma^*_{(q)}, \Gamma^*_{(q+1)}] = \infty$, then every Cayley graph of Γ has a unimodular graph height function of the form $(h, \Gamma^*_{(q)})$ which is harmonic.*

The theorem may be applied to any finitely generated, virtually solvable group Γ, and more generally whenever the derived series of Γ^* terminates after finitely many steps at a *finite* perfect group.

In preparation for the proof, we present a general construction of a height function for a group having a normal subgroup. Part (a) extends Proposition 4.3 (see also Remark 4.2).

Theorem 5.2. *Let Γ be an infinite, finitely generated group, and let $\Gamma' \trianglelefteq \Gamma$.*

(a) *If the quotient group Γ/Γ' is infinite and abelian, then Γ has a group height function h with $d(h) = 1$.*
(b) *If the quotient group Γ/Γ' is finite, and Γ' has a group height function, then every Cayley graph of Γ has a harmonic, unimodular graph height function of the form (h, Γ').*

Recall that Γ/Γ' is abelian if and only if Γ' contains the commutator group $[\Gamma, \Gamma]$, of which the definition follows. An example of Theorem 5.2(b) in action is the special linear group $\mathrm{SL}_2(\mathbb{Z})$ of the forthcoming Example 5.4 (see [19, p. 66]).

We turn now towards solvable groups. Let Γ be a group with identity $\mathbf{1}_\Gamma$. The *commutator* of the pair $x, y \in \Gamma$ is the group element $[x, y] := x^{-1}y^{-1}xy$. Let A, B be subgroups of Γ. The *commutator subgroup* $[A, B]$ is defined to be

$$[A, B] = \big\langle [a, b] : a \in A,\ b \in B \big\rangle,$$

that is, the subgroup generated by all commutators $[a, b]$ with $a \in A$, $b \in B$. The *commutator subgroup* of Γ is the subgroup $[\Gamma, \Gamma]$. It is standard that $[\Gamma, \Gamma] \trianglelefteq \Gamma$, and the quotient group $\Gamma/[\Gamma, \Gamma]$ is abelian. The group Γ is called *perfect* if $\Gamma = [\Gamma, \Gamma]$.

Here is an explanation of the terms in Theorem 5.1. Let $\Gamma_{(1)} = \Gamma$. The *derived series* of Γ is given recursively by the formula

$$\Gamma_{(i+1)} = [\Gamma_{(i)}, \Gamma_{(i)}], \qquad i \geq 1. \tag{5.1}$$

The group Γ is called *solvable* if there exists an integer $c \in \mathbb{N}$ such that $\Gamma_{(c+1)} = \{\mathbf{1}_\Gamma\}$. Thus, Γ is solvable if there exists $c \in \mathbb{N}$ such that

$$\Gamma = \Gamma_{(1)} \trianglerighteq \Gamma_{(2)} \trianglerighteq \cdots \trianglerighteq \Gamma_{(c+1)} = \{\mathbf{1}_\Gamma\}.$$

A *virtually solvable group* is a group Γ for which there exists a normal subgroup Γ^* which is solvable and satisfies $[\Gamma : \Gamma^*] < \infty$. The reader is referred to [25, 32] for general accounts of group theory.

Example 5.3. *Here is an example of a finitely generated but not finitely presented group with a group height function. The* lamplighter group *L has presentation $\langle S \mid R\rangle$ where $S = \{a, t, u\}$ and $R = \{a^2, tu\} \cup \{[a, t^n a u^n] : n \in \mathbb{Z}\}$. It has a group height function since the rank of its coefficient matrix is 2. A recent reference to linear-growth harmonic functions on L is [3].*

Example 5.4. *The special linear group* $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ *has a presentation*

$$\Gamma = \langle x, y, u, v \mid xu, yv, x^4, x^2v^3 \rangle, \tag{5.2}$$

where

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

The presentation has no group height function.

The commutator subgroup $\Gamma^{(2)} := [\Gamma, \Gamma]$ *is a normal subgroup of* Γ *with index* 12, *and* $\Gamma^{(2)}$ *is free of rank* 2. *(See* [6] *and* [19, p. 66]*.) By Theorem 5.2(b), every Cayley group of* Γ *has a harmonic, unimodular graph height function.*

Proof of Theorem 5.2. (a) This is an immediate consequence of Remark 4.2. A detailed argument may be outlined as follows. Let $\Gamma = \langle S \mid R \rangle$. If $Q := \Gamma/\Gamma'$ is infinite and abelian, it is generated by the cosets $\{\overline{s} := s\Gamma' : s \in S\}$, and its relators are the words $\overline{s}_1\overline{s}_2 \cdots \overline{s}_r$ as $\rho = s_1 s_2 \cdots s_r$ ranges over R. Choose $\sigma \in S$ with infinite order, and let

$$\gamma_s = \begin{cases} 1 & \text{if } s \in \overline{\sigma}, \\ -1 & \text{if } s^{-1} \in \overline{\sigma}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

It may now be checked that $C\gamma = \mathbf{0}$ where C is the coefficient matrix.
(b) Let G be a Cayley graph of Γ, and let $\Gamma' \trianglelefteq \Gamma$ satisfy $[\Gamma : \Gamma'] < \infty$. By assumption, Γ' has a group height function h'. The subgroup Γ' of Γ acts on G by left-multiplication, and it is unimodular since its elements act with no non-trivial fixed points. We apply Theorem 3.4 with $\mathcal{H} = \Gamma'$ and $F = h'$ to obtain a harmonic, unimodular graph height function (h, Γ') on G. □

Proof of Theorem 5.1. Since $q < \infty$, we have that $[\Gamma : \Gamma^*_{(q)}] < \infty$, and in particular $\Gamma^*_{(q)}$ is finitely generated. Now, $\Gamma^*_{(q)}$ is characteristic in Γ^*, and $\Gamma^* \trianglelefteq \Gamma$, so that $\Gamma^*_{(q)} \trianglelefteq \Gamma$.

By applying Theorem 5.2(a) to the pair $\Gamma^*_{(q+1)} \trianglelefteq \Gamma^*_{(q)}$, there exists a group height function h^*_q on $\Gamma^*_{(q)}$. We apply Theorem 5.2(b) to the pair $\Gamma^*_{(q)} \trianglelefteq \Gamma$ to obtain a harmonic, unimodular graph height function $(h, \Gamma^*_{(q)})$ on Γ. □

6. Convergence of connective constants of Cayley graphs

Let $\Gamma = \langle S \mid R \rangle$ be finitely presented with coefficient matrix C and Cayley graph $G = G(S, R)$. Let $t \in \Gamma$ have infinite order. We consider in this section the effect of adding a new relator t^m, in the limit as $m \to \infty$. Let G_m be the Cayley graph of the group $\Gamma_m = \langle S \mid R \cup \{t^m\} \rangle$.

Theorem 6.1. *If* $\mathrm{rank}(C) < |S| - 1$, *then* $\mu(G_m) \to \mu(G)$ *as* $m \to \infty$.

Proof. The coefficient matrix C_m of G_m differs from C_1 only in the multiplicity of the row corresponding to the new relator, and therefore $\mathcal{N}(C_1) = \mathcal{N}(C_m)$. Since Γ_1 has only one relator more than G, $\operatorname{rank}(C_1) \le \operatorname{rank}(C) + 1$. If $\operatorname{rank}(C) < |S| - 1$, then $\operatorname{rank}(C_1) < |S|$. By Theorem 4.1, we may find $\gamma = (\gamma_s : s \in S) \in \mathcal{N}(C_1)$ such that $\gamma \in \mathbb{Z}^S$, $\gamma \ne \mathbf{0}$. By the above, for $m \ge 1$, $\gamma \in \mathcal{N}(C_m)$, so that G_m has a corresponding group height function h_m. By (4.2), $d(h) = d(h_m) =: D$ for all n, so that $G_m \in \mathcal{G}_{D,0}$ for all m.

The group Γ_m is obtained as the quotient group of Γ by the (normal) subgroup generated by t^m. We apply [10, Thm 5.2] with $\mathcal{A}_m$ the cyclic group generated by t^m. The condition of the theorem holds since t has infinite order. □

Remark 6.2 (Approximating $\mu(G)$)**.** *The question is posed in* [11] *of whether one can obtain rigorous sequences of bounds for $\mu(G)$ which are sharp. Such upper bounds are provided by the subadditive argument of* (2.1)*, namely $\mu \le \sigma_m^{1/m}$ for $m \ge 1$. Theorem 6.1, taken together with* [11, Thm 3.8]*, provides lower bounds. It is however preferable to use the improved lower bound $\mu \ge b_m^{1/m}$, where b_m is the number of m-step bridges. The latter inequality is asymptotically sharp whenever G has a unimodular graph height function (see* [10, Remark 4.5]*), and this is a less restrictive condition than that of Theorem 6.1.*

As examples of finitely generated groups satisfying the conditions of Theorem 6.1, we mention free groups, abelian groups, free nilpotent groups, free solvable groups, and, more widely, nilpotent and solvable groups Γ with presentations $\langle S \mid R\rangle$ whose coefficient matrix C satisfies $b(\Gamma) = |S| - \operatorname{rank}(C) > 1$. Here is an example where Theorem 6.1 cannot be applied, though the conclusion is valid.

Example 6.3. *Let G be the Cayley graph of the* infinite dihedral group $\mathrm{Dih}_\infty = \langle s_1, s_2 \mid s_1^2, s_2^2\rangle$ *of Example 4.4. As noted there, G has no group height function, though it has a unimodular graph height function $(h, \mathcal{H})$ with $d(h) = 1$. Let $\Gamma_m = \mathrm{Dih}_\infty \times J_m$ where $m \ge 3$ and $J_m = \langle a, b \mid ab, a^m\rangle$ is the cyclic group $\{\mathbf{1}, a, a^2, \dots, a^{m-1}\}$. Thus, Γ_m is finitely presented but, by Theorem 4.1(b), it has no group height function. In particular, Theorem 6.1 may not be applied.*

The Cayley graph G is isomorphic to $\mathbb{Z}$. Therefore, we may define a unimodular graph height function $(h', \mathcal{H}')$ on the Cayley graph G_m of Γ_m by $h'(\gamma, a^k) = h(\gamma)$, with $\mathcal{H}'$ generated by the shifts $(\gamma, a^k) \mapsto (\gamma+1, a^k)$ and $(\gamma, a^k) \mapsto (\gamma, a^{k+1})$. Furthermore, $d(h') = d(h) = 1$ and $r(h', \mathcal{H}') = r(h, \mathcal{H}) = 0$. By [10, Thms 5.1, 5.2]*, $\mu(G_m) \to \mu(\mathbb{Z}^2)$ as $m \to \infty$.*

7. Proof of Theorem 3.4

Assume that assumptions (a)–(c) of Theorem 3.4 hold. There are two steps in the proof, namely of the following.

A. (Prop. 7.2) There exists $\psi : V \to \mathbb{Q}$ which is $\mathcal{H}$-difference-invariant, harmonic, non-constant, and takes values in the rationals.
B. (Prop. 7.4) There exists a graph height function which is harmonic on G.

The vertex $\mathbf{1}$ may appear to play a distinguished role in this section. This is in fact not so: since G is assumed transitive, the following is valid with any choice of vertex for the label $\mathbf{1}$. The approach of the proof is inspired in part by the proof of [26, Cor. 3.4]. Let $X = (X_n : n = 0, 1, 2, \dots)$ be a simple random walk on G, with transition matrix

$$P(u, v) = \mathbb{P}_u(X_1 = v) = \frac{1}{\deg(u)}, \qquad u, v \in V,\ v \in \partial u,$$

where $\mathbb{P}_u$ denotes the law of the random walk starting at u.

Let $V_1 = \mathcal{H}\mathbf{1}$ be the orbit of the identity under $\mathcal{H}$, and let P_1 be the transition matrix of the induced random walk on V_1, that is

$$P_1(u, v) = \mathbb{P}_u(X_\tau = v), \qquad u, v \in V_1,$$

where $\tau = \min\{n \geq 1 : X_n \in V_1\}$. It is easily seen that $\mathbb{P}_u(\tau < \infty) = 1$ since, by the quasi-transitive action of $\mathcal{H}$, there exist $\alpha > 0$ and $K < \infty$ such that

$$\mathbb{P}_u(X_k \in V_1 \text{ for some } 1 \leq k \leq K) \geq \alpha, \qquad u \in V. \tag{7.1}$$

We note for later use that, by (7.1), there exist $\alpha' = \alpha'(\alpha, K) \in (0, 1)$ and $A = A(\alpha, K)$ such that

$$\mathbb{P}_u(\tau \geq m) \leq A(1 - \alpha')^m, \qquad m \geq 1,\ u \in V. \tag{7.2}$$

Since $\mathcal{H} \leq \mathrm{Aut}(G)$, P_1 is invariant under $\mathcal{H}$ in the sense that

$$P_1(u, v) = P_1(\gamma u, \gamma v), \qquad \gamma \in \mathcal{H},\ u, v \in V_1. \tag{7.3}$$

Proposition 7.1.

(a) *The transition matrix P_1 is symmetric, in that*

$$P_1(u, v) = P_1(v, u), \qquad u, v \in V_1.$$

(b) *Let $F : V_1 \to \mathbb{Z}$ be $\mathcal{H}$-difference-invariant. Then F is P_1-harmonic in that*

$$F(u) = \sum_{v \in V_1} P_1(u, v) F(v), \qquad u \in V_1.$$

Proof. (a) Since P is reversible with respect to the measure $(\deg(v) : v \in V)$, and $\deg(v)$ is constant on V_1, we have that

$$P(u_0, u_1) P(u_1, u_2) \cdots P(u_{n-1}, u_n) = P(u_n, u_{n-1}) P(u_{n-1}, u_{n-2}) \cdots P(u_1, u_0)$$

for $u_0, u_n \in V_1,\ u_1, \dots, u_{n-1} \in V$. The symmetry of P_1 follows by summing over appropriate sequences (u_i).

(b) It is required to prove that

$$\sum_{v\in V_1} P_1(u,v)[F(u)-F(v)] = 0, \qquad u \in V_1, \tag{7.4}$$

and it is here that we shall use assumption (b) of Theorem 3.4, namely, that $\mathcal{H}$ is unimodular. Since F is $\mathcal{H}$-difference-invariant, there exists $D < \infty$ such that

$$|F(u)-F(v)| \le D d_G(u,v), \qquad u,v \in V_1.$$

By (7.1), the random walk on V_1 has finite mean step-size. It follows that the summation in (7.4) converges absolutely.

Equation (7.4) may be proved by a cancellation of summands, but it is shorter to use the mass-transport principle. Let

$$m(u,v) = P_1(u,v)[F(u)-F(v)], \qquad u,v \in V_1.$$

The sum $\sum_{v\in V_1} m(u,v)$ is absolutely convergent as above, and $m(\gamma u, \gamma v) = m(u,v)$ for $\gamma \in \mathcal{H}$. Since $\mathcal{H}$ is unimodular, by the mass-transport principle (see, for example, [27, Thm 8.7, Cor. 8.11]),

$$\sum_{v\in V_1} m(u,v) = \sum_{w\in V_1} m(w,u), \qquad u \in V_1. \tag{7.5}$$

Now,

$$\begin{aligned} \sum_{w\in V_1} m(w,u) &= \sum_{w\in V_1} P_1(w,u)[F(w)-F(u)] \\ &= -\sum_{w\in V_1} P_1(u,w)[F(u)-F(w)] \qquad \text{by part (a),} \end{aligned}$$

and (7.4) follows by (7.5).

It is usual to assume in the mass-transport principle that $m(u,v) \ge 0$, but it suffices that $\sum_v m(u,v)$ is absolutely convergent. □

Let $\beta > 1$, and let $f : V \to \mathbb{R}$. We write $f = \mathrm{O}(\beta^n)$ if there exists B such that

$$|f(v)| \le B\beta^n \qquad \text{if } d_G(\mathbf{1}, v) \le n, \text{ and } n \ge 1. \tag{7.6}$$

Proposition 7.2. *Let $F : V_1 \to \mathbb{Z}$ be $\mathcal{H}$-difference-invariant, and let*

$$\psi(v) = \mathbb{E}_v[F(X_T)], \qquad v \in V, \tag{7.7}$$

where $T = \inf\{n \ge 0 : X_n \in V_1\}$. Then,

(a) *the function ψ is $\mathcal{H}$-difference-invariant, and agrees with F on V_1,*

(b) *ψ is harmonic on G, in that*

$$\psi(u) = \sum_{v \in V} P(u,v)\psi(v), \qquad u \in V, \tag{7.8}$$

and, furthermore, ψ is the unique harmonic function that agrees with F on V_1 and satisfies $\psi = \mathrm{O}(\beta^n)$ with any $1 \le \beta < 1/(1-\alpha')$, where α' satisfies (7.2),

(c) *ψ takes rational values.*

Remark 7.3. *By Proposition 7.2(a, b), any $\mathrm{O}(\beta^n)$ harmonic extension of F (with suitable β) is $\mathcal{H}$-difference-invariant. Conversely, any $\mathcal{H}$-difference-invariant function f satisfies $f = \mathrm{O}(\beta^n)$ for all $\beta > 1$, whence the function ψ of (7.7) is the unique harmonic extension of F that is $\mathcal{H}$-difference-invariant.*

Proof. (a) The function ψ is $\mathcal{H}$-difference-invariant since the law of the random walk is $\mathcal{H}$-invariant, and

$$\psi(v) - \psi(w) = \mathbb{E}_v[F(X_T)] - \mathbb{E}_w[F(X_T)].$$

It is trivial that $\psi \equiv F$ on V_1.

(b) By conditioning on the first step, ψ is harmonic at any $v \notin V_1$. For $v \in V_1$, it suffices to show that

$$\psi(v) = \sum_{w \in V} P(v,w)\psi(w).$$

Since $\psi \equiv F$ on V_1, and F is P_1-harmonic (by Proposition 7.1), this may be written as

$$\sum_{w \in V_1} P_1(v,w)\psi(w) = \sum_{w \in V} P(v,w)\psi(w), \qquad v \in V_1.$$

Each term equals $\mathbb{E}_v[\psi(W(X_1))]$, where X_1 is the position of the random walk after one step, and $W(X_1)$ is the first element of V_1 encountered having started at X_1.

To establish uniqueness, let ϕ be a harmonic function with $\phi = \mathrm{O}(\beta^n)$ where $1 \le \beta < 1/(1-\alpha')$, such that $\phi \equiv F$ on V_1. Then $Y_n := \phi(X_n)$ is a martingale, and furthermore T is a stopping time with tail satisfying (7.2). By the optional stopping theorem (see, for example, [15, Thm 12.5.1]) and (7.7),

$$\phi(u) = \mathbb{E}_u(Y_T) = \mathbb{E}_u(F(X_T)) = \psi(u),$$

so long as $\mathbb{E}_u(|Y_n|I_{\{T \ge n\}}) \to 0$ as $n \to \infty$, where I_E denotes the indicator function of an event E. To check the last condition, note by (7.6) and (7.2) that

$$\begin{aligned} \mathbb{E}_u(|Y_n|I_{\{T \ge n\}}) &\le B\beta^{n+|u|}\mathbb{P}_u(T \ge n) \\ &\le (AB\beta^{|u|})\beta^n(1-\alpha')^n \to 0 \qquad \text{as } n \to \infty, \end{aligned}$$

where $|u| = d_G(\mathbf{1}, u)$.

(c) The quantity $\psi(v)$ has a representation as a sum of values of the unique solution of a finite set of linear equations with integral coefficients and boundary conditions, and thus $\psi(v) \in \mathbb{Q}$. Some further details follow.

Let $\vec{G} = (V, \vec{E})$ be the directed graph obtained from $G = (V, E)$ by replacing each $e \in E$ by two edges $\vec{e}$, $-\vec{e}$ with the same endpoints and opposite orientations. Suppose $\delta : \vec{E} \to \mathbb{R}$ satisfies the linear equations

$$\delta(-\vec{e}) + \delta(\vec{e}) = 0, \qquad e \in E, \tag{7.9}$$

$$\sum_{\vec{e} \in W} \delta(\vec{e}) = 0, \qquad W \in \mathcal{W}(G), \tag{7.10}$$

$$\sum_{v \sim u} \delta([u, v\rangle) = 0, \qquad u \in V, \tag{7.11}$$

$$\delta(\alpha\vec{e}) = \delta(\vec{e}), \quad e \in E,\ \alpha \in \mathcal{H}, \tag{7.12}$$

where $\mathcal{W}(G)$ is the set of directed closed walks of G. (Equation (7.10) may be viewed as including (7.9).) By (7.10), the sum

$$\Delta(v) := \sum_{\vec{e} \in l_v} \delta(\vec{e}), \qquad v \in V,$$

is well defined, where l_v is an arbitrary (directed) walk from $\mathbf{1}$ to $v \in V$. Equation (7.11) requires that Δ be harmonic, and (7.12) that Δ be $\mathcal{H}$-difference-invariant.

Since $\mathcal{H}$ acts quasi-transitively, by (7.12), the linear equations (7.9)–(7.11) involve only finitely many variables. Therefore, there exists a finite subset of equations, denoted as Eq, of (7.9)–(7.11) such that δ satisfies (7.9)–(7.12) if only if δ satisfies Eq together with (7.12). In summary, any harmonic, $\mathcal{H}$-difference-invariant function Δ, satisfying $\Delta(\mathbf{1}) = 0$, corresponds to a solution to the finite collection Eq of linear equations.

With F as given, let ψ be given by (7.7). By Remark 7.3, equations (7.9)–(7.12) have a unique solution satisfying

$$\sum_{\vec{e} \in l_v} \delta(\vec{e}) = F(v) - F(\mathbf{1}), \qquad v \in V_1. \tag{7.13}$$

By (7.10), it suffices in (7.13) to consider only the finite set $V_1' \subseteq V_1$ of vertices v within some bounded distance of $\mathbf{1}$ that depends on the pair G, $\mathcal{H}$.

Therefore, Eq possesses a unique solution subject to (7.13) (with V_1 replaced by V_1'). All coefficients and boundary values in Eq and (7.13) are integral, and therefore ψ takes only rational values. □

Proposition 7.4. *Let $F : V_1 \to \mathbb{Z}$ be $\mathcal{H}$-difference-invariant, and non-constant on V_1. There exists a graph height function $(h = h_F, \mathcal{H})$ such that h is harmonic on G.*

Proof. The normality of $\mathcal{H}$ is used in this proof. A vertex $v \in V$ is called a *point of increase* of a function $h : V \to \mathbb{R}$ if v has neighbours u, w such that $h(u) < h(v) < h(w)$. The function h is said to *increase everywhere* if every vertex is a point of increase. For $v \in V$ and a harmonic function h,

$$\begin{aligned} \text{either:}\quad & v \text{ is a point of increase of } h, \\ \text{or:}\quad & h \text{ is constant on } \{v\} \cup \partial v. \end{aligned} \tag{7.14}$$

An $\mathcal{H}$-difference-invariant function h on G is a graph height function if and only if $h(\mathbf{1}) = 0$, h takes integer values, and h increases everywhere.

Let F be as given, and let ψ be given by Proposition 7.2. Thus, $\psi : V \to \mathbb{Q}$ is non-constant on V_1, $\mathcal{H}$-difference-invariant, and harmonic on G. Since ψ is $\mathcal{H}$-difference-invariant, we may replace it by $m\psi$ for a suitable $m \in \mathbb{N}$ to obtain such a function that in addition takes integer values. We shall work with the latter function, and thus we assume henceforth that $\psi : V \to \mathbb{Z}$. Now, ψ may not increase everywhere. By (7.14), ψ has some point of increase $w \in V$.

Let $V_1, V_2, \dots, V_M$ be the orbits of V under $\mathcal{H}$. Find W such that $w \in V_W$. Since Γ acts transitively on G, and $\mathcal{H}$ is a normal subgroup of Γ acting quasi-transitively on G, there exist $\gamma_1, \gamma_2, \dots, \gamma_M \in \Gamma$ such that $\gamma_W = \mathbf{1}_\Gamma$ and

$$V_i = \gamma_i V_W, \qquad i = 1, 2, \dots, M.$$

Let

$$\psi_i(v) = \psi(\gamma_i^{-1} v), \qquad i = 1, 2, \dots, M, \tag{7.15}$$

so that, in particular, $\psi_W = \psi$. Since $w \in V_W$ is a point of increase of ψ, $w_i := \gamma_i w$ is a point of increase of ψ_i, and also $w_i \in V_i$.

Lemma 7.5. *For $i = 1, 2, \dots, M$,*

(a) *$\psi_i : V \to \mathbb{Z}$ is a non-constant, harmonic function, and*
(b) *ψ_i is $\mathcal{H}$-difference-invariant.*

Proof. (a) Since ψ_i is obtained from ψ by shifting the domain according to the automorphism γ_i, ψ_i is non-constant and harmonic.
(b) For $\alpha \in \mathcal{H}$ and $u, v \in V$,

$$\psi_i(\alpha v) - \psi_i(\alpha u) = \psi(\gamma_i^{-1} \alpha v) - \psi(\gamma_i^{-1} \alpha u).$$

Since $\mathcal{H} \trianglelefteq \Gamma$ and $\gamma_i \in \Gamma$, there exists $\alpha_i \in \mathcal{H}$ such that $\gamma_i^{-1} \alpha = \alpha_i \gamma_i^{-1}$. Therefore,

$$\begin{aligned} \psi_i(\alpha v) - \psi_i(\alpha u) &= \psi(\alpha_i \gamma_i^{-1} v) - \psi(\alpha_i \gamma_i^{-1} u) \\ &= \psi(\gamma_i^{-1} v) - \psi(\gamma_i^{-1} u) \qquad \text{since } \psi \text{ is } \mathcal{H}\text{-difference-invariant} \\ &= \psi_i(v) - \psi_i(w) \qquad \text{by (7.15)}, \end{aligned}$$

so that ψ_i is $\mathcal{H}$-difference-invariant. $\square$

Let $\nu : V \to \mathbb{R}$ be $\mathcal{H}$-difference-invariant. For $j = 1, 2, \dots, M$, either every vertex in V_j is a point of increase of ν, or no vertex in V_j is a point of increase of ν. We shall now use an iterative construction in order to find a harmonic, $\mathcal{H}$-difference-invariant function h' for which every w_i is a point of increase. Since the w_i represent the orbits V_i, the ensuing h' increases everywhere.

1. If every w_i is a point of increase of ψ, we set $h' = \psi$.
2. Assume otherwise, and find the smallest j_2 such that w_{j_2} is not a point of increase of ψ. By (7.14), we may choose $c_{j_2} \in \mathbb{Q}$ such that both w and w_{j_2} are points of increase of $h_2 := \psi + c_{j_2}\psi_{j_2}$. If h_2 increases everywhere, we set $h' = h_2$.
3. Assume otherwise, and find the smallest j_3 such that w_{j_3} is not a point of increase of h_2. By (7.14), we may choose $c_{j_3} \in \mathbb{Q}$ such that w, w_{j_2}, and w_{j_3} are points of increase of $h_3 := \psi + c_{j_2}\psi_{j_2} + c_{j_3}\psi_{j_3}$. If h_3 increases everywhere, we set $h' = h_3$.
4. This process is iterated until we find an $\mathcal{H}$-difference-invariant, harmonic function of the form
$$h' = \sum_{l=1}^{M} c_{j_l}\psi_{j_l},$$
with $j_1 = W$, $c_W = 1$, and $c_{j_l} \in \mathbb{Q}$, which increases everywhere.

The function $h' - h'(\mathbf{1})$ may fail to be a graph height function only in that it may take rational rather than integer values. Since the c_{j_l} are rational, there exists $m \in \mathbb{Z}$ such that $h = m(h' - h(\mathbf{1}))$ is a graph height function. □

Proof of Theorem 3.4. By Propositions 7.1 and 7.2, there exists $\psi : V \to \mathbb{Q}$ satisfying (i). The existence of $\psi' : V \to \mathbb{Q}$, in (ii), follows as in Proposition 7.4. Similarly, ψ, ψ' may be taken to be integer-valued, and the unimodularity holds since $\mathcal{H}$ is assumed unimodular. □

8. Proof of Theorem 3.5

Let G, Γ, $\mathcal{H}$ be as given. The idea is to apply Theorem 3.4 to a suitable triple G', Γ', $\mathcal{H}'$, and to extend the resulting graph height function to the original graph G. The required function F of the theorem will be derived from the modular function of G under $\mathcal{H}$.

By [27, Thm 8.10], we may define a positive *weight function* $M : V \to (0, \infty)$ satisfying

$$\frac{M(u)}{M(v)} = \frac{|\mathrm{Stab}_u v|}{|\mathrm{Stab}_v u|}, \qquad u, v \in V, \tag{8.1}$$

where $|\cdot|$ denotes cardinality. The weight function is uniquely defined up to a multiplicative constant, and is automorphism-invariant up to a multiplicative constant. Since G is assumed non-unimodular, M is non-constant on some orbit of $\mathcal{H}$. Without loss of generality, we assume $\mathbf{1}$ lies in such an orbit and that $M(\mathbf{1}) = 1$. See [27, Sect. 8.2] for an account of (non-)unimodularity.

Let $\mathcal{S}$ be the normal subgroup of Γ generated by $\bigcup_{v\in V} \mathrm{Stab}_v$, where $\mathrm{Stab}_v = \mathrm{Stab}_v^{\mathcal{H}}$. Let G' denote the quotient graph $G/\mathcal{S}$ (as in [10, Sect. 2]), which we take to be simple in that every pair of neighbours is connected by just one edge, and any loop is removed.

Lemma 8.1.

(a) $\mathcal{S} \trianglelefteq \mathcal{H}$.
(b) *The function* $F' : V/\mathcal{S} \to (0,\infty)$ *given by* $F'(\mathcal{S}v) = \log M(v)$, $v \in V$, *is well defined, in the sense that* F' *is constant on each orbit in* $\mathcal{S}$.
(c) *The quotient group* $\Gamma' := \Gamma/\mathcal{S}$ *acts transitively on* G', *and* $\mathcal{H}' := \mathcal{H}/\mathcal{S}$ *acts quasi-transitively on* G'.
(d) *The quotient graph* $G' = G/\mathcal{S}$ *satisfies* $G' \in \mathcal{G}$.
(e) $\mathcal{H}'$ *is unimodular on* G'.

Proof. (a) Since $\mathcal{S} \trianglelefteq \Gamma$ and $\mathcal{H} \le \Gamma$, it suffices to show that $\mathcal{S} \le \mathcal{H}$. Now, $\mathcal{S}$ is the set of all products of the form $(\gamma_1\sigma_1\gamma_1^{-1})(\gamma_2\sigma_2\gamma_2^{-1})\cdots(\gamma_k\sigma_k\gamma_k^{-1})$ with $k \ge 0$, $\gamma_i \in \Gamma$, $\sigma_i \in \mathrm{Stab}_{w_i}$, $w_i \in V$. Since $\gamma_i\sigma_i\gamma_i^{-1} \in \mathrm{Stab}_{\gamma_i w_i}$, we have that $\mathcal{S} \le \mathcal{H}$ as required.
(b) If $u = \sigma v$ with $\sigma \in \mathrm{Stab}_w$, then

$$\frac{M(u)}{M(w)} = \frac{|\mathrm{Stab}_u w|}{|\mathrm{Stab}_w u|} = \frac{|\mathrm{Stab}_{\sigma v}(\sigma w)|}{|\mathrm{Stab}_{\sigma w}(\sigma v)|} = \frac{|\mathrm{Stab}_v w|}{|\mathrm{Stab}_w v|} = \frac{M(v)}{M(w)},$$

so that $M(u) = M(v)$. As in part (a), every element of $\mathcal{S}$ is the product of members of the stabilizer groups Stab_w, and the claim follows.
(c) Let $u, v \in V$, and find $\gamma \in \Gamma$ such that $v = \gamma u$. Since $\mathcal{S} \trianglelefteq \Gamma$, $\mathcal{S}\gamma(\mathcal{S}u) = \mathcal{S}\gamma u = \mathcal{S}v$, so that $\mathcal{S}\gamma : \mathcal{S}u \mapsto \mathcal{S}v$. The first claim follows, and the second is similar since $\mathcal{H}$ acts quasi-transitively on G.
(d) Since M is non-constant on the orbit $\mathcal{H}\mathbf{1}$, there exist $v, w \in \mathcal{H}\mathbf{1}$ such that $\xi := M(w)/M(v)$ satisfies $\mu > 1$. Let $\alpha \in \mathcal{H}$ be such that $w = \alpha v$. By (8.1), $M(\alpha^k v)/M(v) = \xi^k$, whence the range of M is unbounded. By part (b), G' is infinite. (The non-constantness of the modular function has been used also in [7].) The graph G' is connected since G is connected, and is transitive by part (c). It is locally finite since its vertex-degree is no greater than that of G.
(e) It suffices for the unimodularity that, for $u \in V$ and $\overline{u} := \mathcal{S}u$, we have that $\mathrm{Stab}_{\overline{u}} := \mathrm{Stab}_{\overline{u}}^{\mathcal{H}'}$ is a single element, namely the identity element $\mathcal{S}$ of $\mathcal{H}'$. Let $\alpha \in \mathcal{H}$ be such that $\mathcal{S}\alpha \in \mathrm{Stab}_{\overline{u}}$. Then $\mathcal{S}\alpha(\mathcal{S}u) = \alpha\mathcal{S}u = \mathcal{S}u$. Therefore, there exists $s \in \mathcal{S}$

such that $\alpha s(u) = u$, so that $\alpha s \in \mathcal{S}$. It follows that $\alpha \in \mathcal{S}$, and hence $\mathcal{S}\alpha = \mathcal{S}$ as required. $\square$

Since M is non-constant on $\mathcal{H}\mathbf{1}$, F' is non-constant on the orbit of $\mathcal{H}'$ containing $\mathcal{S}\mathbf{1}$. By Theorem 3.4 applied to $(G', \Gamma', \mathcal{H}', F')$, G' has a harmonic, unimodular graph height function $(\psi', \mathcal{H}')$ satisfying $\psi'(\mathcal{S}\mathbf{1}) = 0$. Let $\psi : V \to \mathbb{Z}$ be given by $\psi(v) = \psi'(\mathcal{S}v)$. We claim that $(\psi, \mathcal{H})$ is a graph height function on G.

Firstly, for $\alpha \in \mathcal{H}$,

$$\begin{aligned}
\psi(\alpha v) - \psi(\alpha u) &= \psi'(\mathcal{S}\alpha v) - \psi'(\mathcal{S}\alpha u) \\
&= \psi'(\alpha\mathcal{S}v) - \psi'(\alpha\mathcal{S}u) \quad \text{since } \mathcal{S} \trianglelefteq \mathcal{H} \\
&= \psi'(\mathcal{S}v) - \psi'(\mathcal{S}u) \qquad \text{since } (\psi', \mathcal{H}') \text{ is a graph height function} \\
&= \psi(v) - \psi(u),
\end{aligned}$$

whence ψ is $\mathcal{H}$-difference-invariant. Secondly, let $v \in V$, and find $u, w \in \partial v$ such that $\psi'(\mathcal{S}u) < \psi'(\mathcal{S}v) < \psi'(\mathcal{S}w)$. Then $\psi(u) < \psi(v) < \psi(w)$, so that v is a point of increase of ψ. Therefore, $(\psi, \mathcal{H})$ is a graph height function on G.

Finally, we give an example in which the above recipe leads to a graph height function which is not harmonic. Consider the 'grandparent graph' introduced in [36] (see also [27, Example 7.1]) and defined as follows. Let T be an infinite degree-3 tree, and select an 'end' ω. For each vertex v, we add an edge to the unique grandparent of v in the direction of ω. Let $\mathcal{H}$ be the set of automorphisms of the resulting graph G. Note that $\mathcal{H}$ acts transitively on G, and is non-unimodular. The above recipe yields (up to a multiplicative constant which we take to be 1) the graph height function on T which measures the (integer) height of a vertex in the direction of ω. The neighbours of a vertex with height h have average height $h - \frac{7}{8}$, whence h is not harmonic.

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