The Random-Cluster Model

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With ? Illustrations

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Cees Fortuin (1971)

Piet Kasteleyn (19??)

Preface (draft)

The random-cluster model was invented by Cees Fortuin and Piet Kasteleyn around ?1969 as a unification of percolation, Ising, and Potts models, and as an extrapolation of electrical networks. Their original motivation was to generalise the series and parallel laws satisfied by such systems. In so doing, they initiated a study in stochastic geometry which has exhibited beautiful structure in its own right, and which has proved a central tool in the pursuit of one of the oldest theories of classical statistical mechanics, namely, to model and analyse the ferromagnet and especially its phase transition.

We start with a finite graph $G = (V, E)$. The configuration space is the set of all subsets of E, which we represent as the set $\Omega = \{0, 1\}^E$. An edge e is termed *open* in the configuration ω $(\in \Omega)$ if $\omega(e) = 1$, and it is termed *closed* if $\omega(e) = 0$. Thus the random-cluster model is an *edge* model, in contrast to the Ising and Potts models which assign spins to the *vertices* of G. The subject of study is the subgraph of G induced by the set of open edges of a configuration chosen at random from Ω according to a certain probability measure. Of particular importance is the existence of paths of open edges joining given vertices x and y, and thus the random-cluster model is a model in stochastic geometry.

The random-cluster model is a parametric family of probabiltiy measures $\phi_{p,q}$ on Ω, the two parameters being denoted as $p \in [0, 1]$ and $q \in (0, \infty)$. The parameter p amounts to a measure of the density of open edges, and the parameter q is a 'cluster-weighting' factor. Thus an increase in p leads to a greater number of open edges, and an increase in q leads to a greater number of open clusters (that is, components of open edges). When $q = 1$, $\phi_{p,q}$ is a product measure, and the ensuing probability space is usually termed a percolation model or a random graph depending on the context. The integer values $q = 2, 3, \dots$ correspond in a certain way to the Potts model on G with q local states, and thus $q = 2$ corresponds to the Ising model. The nature of these 'correspondences', as desribed in Section 1.4, are that 'correlation functions' of the Potts model may be expressed as 'connectivity functions' of the random-cluster model.

Herein lies a major strength of the random-cluster model. Geometrical aspects of some complexity have been derived in the study of percolation, and some of these may be adapted and extended to more general random-cluster models, thereby obtaining results of importance for Ising and Potts models (+ more). Such has been the value of the random-cluster model in studying Ising and Potts that it is sometimes called merely the 'FK representation' of the latter systems.

The celebrated FKG inequality was born of the random-cluster model, and is a principal technique when $q \geq 1$; it fails to hold when $q < 1$. For percolation (that is, when $q = 1$), the FKG inequality is complemented by the BK (or 'disjoint occurrence') inequality, and it is largely the absence of such a complement for general random-cluster models that is a principal cause of difficulty. Nevertheless, methods of stochastic domination play a large part in understanding random-cluster models for all p, q. (?) There is reason to believe that $\phi_{p,q}$ satisfies some form of disjoint-occurrence inequality when $q < 1$.

A principal feature of the model is the presence of a phase transition. Since singularities may occur only on infinite graphs, one requires a definition of the random-cluster model on an infinite graph. This may be achieved as for other systems either by passing to an infinite-volume weak limit, or by studying measures which satisfy consistency conditions of Dobrushin–Lanford–Ruelle (DLR) type. The former construction depends on a notion of boundary condition, of which the two extreme examples are called 'free'; and 'wired'. Such infinite-volume limits are justified when $q \geq 1$ by use of the FKG inequality. Thermodynamic limits are not understood when $q < 1$; the picture would be much clearer is one could prove a suitable form of the BK inequality [check, and maybe give the theory somewhere]. Mention lattices $\mathbb{Z}^d$.

Following the introductory material of Chapter 1, the fundamental properties of random-cluster measures on finite graphs are summarised in Chapter 2, including accounts of stochastic ordering, positive association, and exponential steepness. Infinite-volume measures in their two forms are studied in Chapter 3.

The critical point $p_c(q)$ is introduced in Chapter 4 following a definition of the percolation probabiltiy. When $p < p_c(q)$, one expects that the size of the open cluster containing a given vertex x of $\mathbb{Z}^d$ is controlled by exponentially-decaying probabilities. This is unproven in general, although partial results are available. One expects similarly an exponential decay of the two-point connectivity probability $\phi_{p,q}(x \leftrightarrow y,\ x \nleftrightarrow \infty)$ when $p > p_c(q)$. The picture in the supercritical regime $p > p_c(q)$ is moderately well understood under the extra hypothesis that p exceeds the limit of certain slab critical points (check) but, unlike percolation, we have no proof that the latter limit equals $p_c(q)$.

The self-duality of the two-dimensional square lattice $\mathbb{Z}^2$ is complemented by a duality relation for random-cluster measures on planar graphs, and this leads to a fuller understanding of the two-dimensional case, as described in Chapter 5. There remain important open problems, of which the principal one is to obtain a clear proof of the the 'exact calculation' $p_c(q) = \sqrt{q}/(1 + \sqrt{q})$. This calculation is accepted by probabilists when $q = 1$ (percolation), $q = 2$ (Ising), and when q is large, but the "exact solutions" of mathematical physics seem to have no complete counterpart in rigorous mathematics for general values of $q \geq 1$.

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In Chapter 6, we consider duality in three and more dimensions. The dual model is now a probabiltiy measure on surfaces, and therefore certain topological complications arise. The conclusions are valid at present only when either p or q is sufficiently large. One obtains thus the existence of non-translation-invariant random-cluster measures when p is large and $q \geq 1$, and the discontinuity of the phase transition when q is large and $p \simeq p_{\mathrm{c}}(q)$. Expand.

Open problems. Later chapters include Chapter 7 on dynamics, Chapter 8 on flows and the Ising model, and Chapter 9 on graphs other than lattices. More . . .

Appendix on history, reproduced with permission of PK.

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Random-Cluster Measures

1.1 Introduction

(This section to be re-written)

During a classical period, probabilists studied the behaviour of *independent* random variables. The ensuing theory is rich, and is linked through theory and application to areas of pure/applied mathematics and to other sciences. It is however unable to answer important questions from a variety of sources concerning large families of *dependent* random variables. Dependence comes in many forms, and one of the targets of modern probability theory has been to derive robust techniques for studying it. The voice of statistical physics has been especially loud in the call for rigour in this general area. In a typical scenario, we are provided with an infinity of random variables, indexed by the vertices of some graph such as the cubic lattice, and which have some dependence structure governed by the geometry of the graph. Thus mathematicians and physicists have had further cause to relate probability and geometry. One major outcome of the synthesis of ideas from physics and probability is the theory of Gibbs states, [**79**], which is now established as a significant branch of probability theory.

A classic example of a Gibbs state is the (Lenz–)Ising model [**114**] for a ferromagnet. When formulated on the bounded region B of the square lattice $\mathbb{L}^2$, a random variable σ_x taking values -1 and $+1$ is assigned to each vertex x of B, and the probability of the configuration σ is proportional to $\exp(-\beta H(\sigma))$, where $\beta > 0$ and the 'energy' $H(\sigma)$ is the negative of the sum of $\sigma_x\sigma_y$ over all neighbouring pairs x, y of B. This 'starter model' has proved extraordinarily successful in generating beautiful and relevant mathematics, and has been useful and provocative in the mathematical theory of phase transitions and cooperative phenomena (see, for example, [**68**]).

There are many possible generalisations of the Ising model in which the σ_x may take a general number q of values, rather than $q = 2$ only. One such generalisation, the so-called Potts model [**165**], has attracted especial interest amongst physicists, and has displayed a complex and varied structure; for example, when q is large, it enjoys a discontinuous phase transition, in contrast to the continuous transition believed to take place for small q. Ising/Potts models are the first of three principal ingredients in the story of random-cluster models. Note that they are 'vertex models' in the sense that they involve random variables σ_x indexed by the vertices x of the underlying graph.

The '(bond) percolation model' was inspired by problems of physical type, and emerged from the mathematics literature of the 1950s [**41**, **194**]. In this model for a porous medium, each edge of a graph is declared 'open' (to the passage of fluid) with probability p, and 'closed' otherwise, different edges having independent states. The problem is to determine the typical large-scale properties of connected components of open edges, as the parameter p varies. Percolation theory is now a mature part of probability, at the core of the study of random media and interacting systems, and it is the second ingredient in the story of random-cluster models. Note that bond percolation is an 'edge model', in that the random variables are indexed by the set of edges of the underlying graph. [There is a variant termed 'site percolation' in which the vertices are open/closed at random rather than the edges.]

The third and final ingredient preceded the first two, and is the theory of electrical networks. Dating back at least to the 1847 paper [**132**] of Kirchhoff, this sets down a method for calculating macroscopic properties of an electrical network in terms of its local structure. In particular, it explains the relevance of counts of certain types of spanning trees of the graph. In the modern vernacular, an electrical network on a graph G may be studied via the properties of a 'uniformly random spanning tree' on G (see [**21**]).

These three ingredients seemed fairly distinct until Fortuin and Kasteleyn discovered, around 1970, [**72**, **73**, **74**, **120**], that each features in a certain way within a family of probability measures of 'edge models', parameterised by two quantities, $p \in [0, 1]$ and $q \in (0, \infty)$. [In actuality, electrical networks arise as a weak limit of such measures.] These models they termed 'random-cluster models', and they developed the basic theory — correlation inequalities and the like — in a series of papers published thereafter. The true power of random-cluster models as a mechanism for studying Ising/Potts models has emerged only gradually over the intervening thirty years.

We note in passing that the genesis of the random-cluster model lay in Kasteleyn's observation that each of the three ingredients above satisfies certain series/parallel laws: any two edges in series (or parallel) may be replaced by a single edge in such a way that, if the interaction function is adapted accordingly, then the distributions of large-scale properties remain unchanged.

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The family of random-cluster measures (that is, probability measures which govern random-cluster models) is not an extension of the Potts measures. The relationship is more sophisticated, and is such that *correlations* for Potts models correspond to *connections* in random-cluster models. Thus the *correlation structure* of a Potts model may be studied via the *stochastic geometry* of a corresponding random-cluster model. The intuition behind this geometrical study comes often from percolation, of which the random-cluster model is indeed an extension.

It turns out that, in many situations involving ferromagnetic Ising/Potts models, the best way forward is via the random-cluster model. As examples of this we mention the existence of discontinuous phase transitions [**135**], exact computations in two dimensions [**18**], the verification of the Wulff construction for Ising droplets [**48**], and the Dobrushin theory of interfaces and non-translation-invariant measures [**82**]. As a major exception to the mantra 'everything worth doing for Ising/Potts is done best via random-cluster', we remind the reader of the so-called random-current expansion for the Ising model, wielded with effect in [**2**, **3**, **7**] and described in Chapter 7. The random-current method appears to be Ising-specific, and has enabled a deep analysis of the Ising model unparalleled in more general Potts systems.

The primary target of this review is to summarise and promote the theory of random-cluster models for its own sake. In doing so, we encounter many results having direct impact on Ising/Potts systems, but we shall not stress such connections. Some of the theory has been discovered several times by apparently independent teams; whilst making a serious attempt to list key references, we apologise for unwitting omissions of which there will certainly be a few. The large number of references to work of the author is attributable in part to the fact that he is acquainted with these contributions.

It is a lesson in humility to return to the original Fortuin–Kasteleyn papers [**72**, **74**], and especially [**73**], where so much of the basic theory was first presented. These authors may not have followed the slickest of routes, but they understood rather well the object of their study. Amongst the many papers of general significance since, we highlight: [**6**], which brought the topic back to the fore; [**64**], where the coupling between Potts and random-cluster models was so beautifully managed; [**90**], where the random-cluster model was studied systematically on infinite grids; [**102**], which links the theory to several other problems of interest in statistical mechanics; and [**81**], where random-cluster models are placed in the perspective of stochastic geometry as a tool for studying phase transitions.

This review is restricted mostly to core material for random-cluster models on the nearest-neighbour cubic lattice in a general number d of dimensions. Only in passing do we mention such subjects as extensions to long-range systems [**6**], mean-field behaviour in high dimensions [**129**], and mixing properties [**10**]. Neither do we stress the impact that graphical methods of the random-cluster type have had on a variety of other disordered systems, such as the Ashkin–Teller model [**16**, **102**, **161**, **172**, **192**], the Widom–Rowlinson model [**51**, **54**, **80**, **102**, **190**], or on methods for simulating disordered physical systems [**55**, **56**, **185**, **193**].

Random-cluster methods may be adapted to systems with random interactions [**5**, **93**, **102**], and even to non-ferromagnetic systems of Edwards–Anderson spin-glass type [**63**, **155**, **156**] where, for example, they have been used to prove that, for a given set $\{J_e\}$ of positive or negative interactions, uniqueness of the infinite-volume Gibbs measure for the ferromagnetic system having interactions $\{|J_e|\}$

implies uniqueness for the original system.

Amongst earlier papers on random-cluster models, the following include a degree of review material: [**6**, **29**, **81**, **87**, **102**, **146**].

1.2 Random-cluster model

Let $G = (V, E)$ be a finite graph. The graphs considered here will usually have neither loops nor multiple edges, but we make no such general assumption. An edge e having endvertices x and y is written as $e = \langle x, y\rangle$. A random-cluster measure on G is a member of a certain class of probability measures on the set of subsets of the edge set E. We take as state space the set $\Omega = \{0, 1\}^E$, members of which are vectors $\omega = (\omega(e) : e \in E)$. We speak of the edge e as being *open* (in ω) if $\omega(e) = 1$, and as being *closed* if $\omega(e) = 0$. For $\omega \in \Omega$, let $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ denote the set of open edges. There is a one–one correspondence between vectors ω ($\in \Omega$) and subsets F ($\subseteq E$), given by $F = \eta(\omega)$. Let $k(\omega)$ be the number of connected components (or 'open clusters') of the graph $(V, \eta(\omega))$, and note that $k(\omega)$ includes a count of isolated vertices, that is, of vertices incident to no open edge. We assign to Ω the σ-field $\mathcal{F}$ of all its subsets.

A *random-cluster measure* on G has two parameters satisfying $0 \le p \le 1$ and $q > 0$, and is the measure $\phi_{p,q}$ on the measurable pair $(\Omega, \mathcal{F})$ given by

$$\phi_{p,q}(\omega) = \frac{1}{Z_{\mathrm{RC}}}\left\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\}q^{k(\omega)}, \qquad \omega \in \Omega, \tag{1.1}$$

1.first where the 'partition function', or 'normalising constant', Z_{RC} is given by

$$Z_{\mathrm{RC}} = Z_{\mathrm{RC}}(p, q) = \sum_{\omega\in\Omega}\left\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\}q^{k(\omega)}. \tag{1.2}$$

1.second This measure differs from product measure through the inclusion of the term $q^{k(\omega)}$. Note the difference between the cases $q \le 1$ and $q \ge 1$: the former favours fewer clusters, whereas the latter favours many clusters. When $q = 1$, edges are open/closed independently of one another. This very special case has been studied in detail under the titles *percolation* and *random graphs*; see [**37**, **92**, **115**]. Perhaps the most important values of q are the integers, since the random-cluster model with $q \in \{2, 3, \dots\}$ corresponds, in a way described in the next two sections, to the Potts model with q local states. The bulk of the work presented in this book is devoted to the theory of random-cluster measures when $q \ge 1$. The case $q < 1$ seems to be harder mathematically and less important physically. There is some interest in the limit as $q \downarrow 0$; see Section 1.5.

We shall sometimes write $\phi_{G,p,q}$ for $\phi_{p,q}$, when the choice of graph G is to be stressed. Samples from random-cluster measures on $\mathbb{Z}^2$ are presented in Figure 1.1.

When $q = 1$, the measure $\phi_{p,q}$ is a product measure with density p. We write $\phi_p = \phi_{p,1}$ for this special case.

1.3 Ising and Potts models

In a famous experiment, a piece of iron is exposed to a magnetic field. The field is increased from zero to a maximum, and then diminishes to zero. If the temperature is sufficiently low, the iron retains some 'residual magnetisation', otherwise it does not. There is a critical temperature for this phenomenon, often called the *Curie point* after Pierre Curie, who reported this discovery in his 1895 thesis, [**58**][1]. The famous (Lenz–)Ising model for such ferromagnetism, [**114**], may be summarised as follows. One supposes that particles are positioned at the points of some lattice in Euclidean space. Each particle may be in either of two states, representing the physical states of 'spin up' and 'spin down'. Spin-values are chosen at random according to a certain probability measure, known as a *Gibbs state*, which is governed by interactions between neighbouring particles. The relevant probability measure is given as follows.

Let $G = (V, E)$ be a finite graph representing part of the lattice. We think of each vertex $v \in V$ as being occupied by a particle having a random spin. Since spins are assumed to come in two basic types, we take as sample space the set $\Sigma = \{-1, +1\}^V$. The appropriate probability mass function $\lambda_{\beta,J,h}$ on Σ has three parameters satisfying $0 \le \beta, J < \infty$ and $h \in \mathbb{R}$, and is given by

$$\lambda_{\beta,J,h}(\sigma) = \frac{1}{Z_{\mathrm{I}}} e^{-\beta H(\sigma)}, \qquad \sigma \in \Sigma, \tag{1.3}$$

1.three where the partition function Z_{I} and the 'Hamiltonian' $H : \Sigma \to \mathbb{R}$ are given by

$$Z_{\mathrm{I}} = \sum_{\sigma\in\Sigma} e^{-\beta H(\sigma)}, \qquad H(\sigma) = -J \sum_{e=\langle x,y\rangle\in E} \sigma_x\sigma_y - h\sum_{x\in V}\sigma_x. \tag{1.4}$$

1.four The physical interpretation of β is as the reciprocal $1/T$ of temperature T, of J as the strength of interaction between neighbours, and of h as the external field. For reasons of simplicity, we shall consider only the case of zero external field, and we assume henceforth that $h = 0$. Each edge has equal interaction strength J in the above formulation. Since β and J occur only as a product βJ, the measure $\lambda_{\beta,J,0}$ has effectively only a single parameter βJ. In a more complicated measure not

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[1] The existence of such a temperature was discovered before 1832 by Pouillet, see [**118**]. This is another example of Stigler's Law, [**184**].

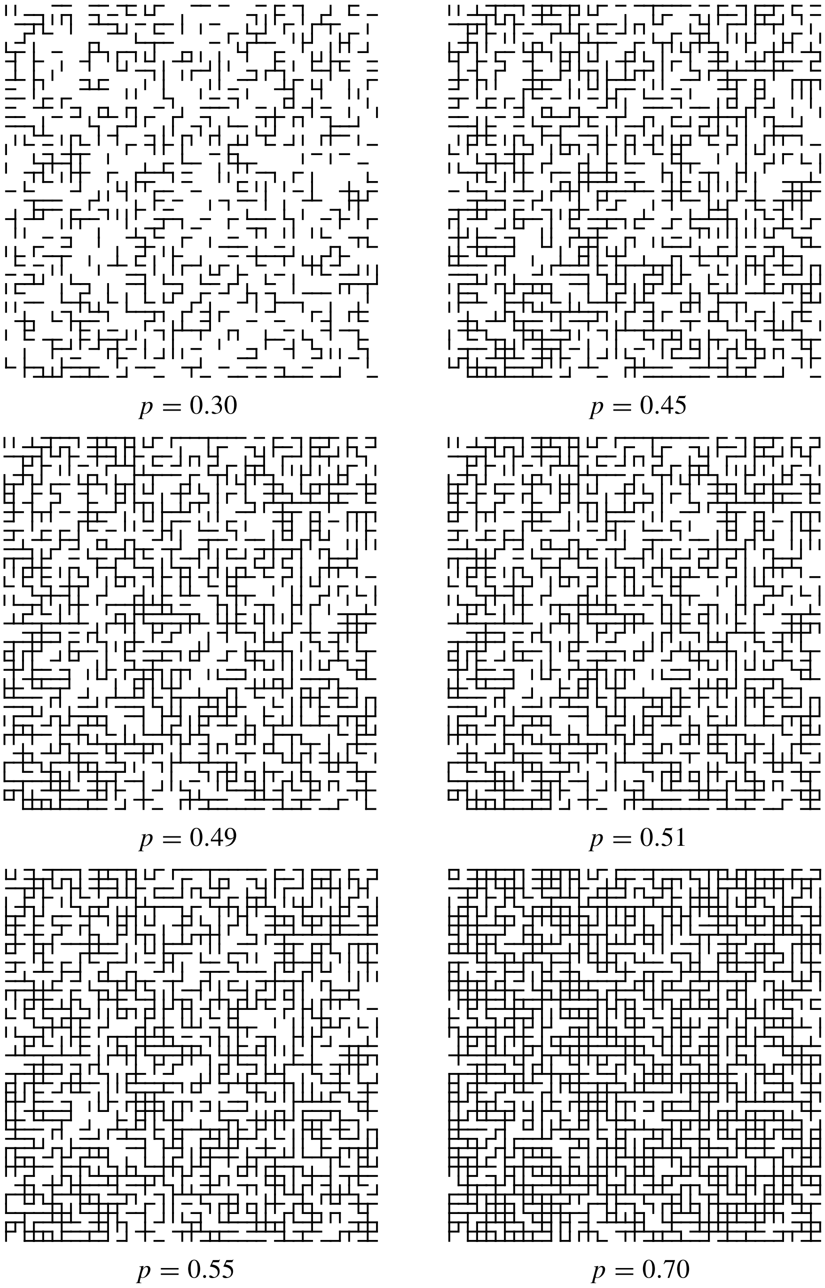


Figure 1.1. Samples from the random-cluster measure with $q = 1$ on the box $[0, 40]^2$ of the square lattice. We have set $q = 1$ for ease of programming, the measure being of product form in this case. The critical value is $p_c(1) = \frac{1}{2}$. Samples with more general values of q may be obtained by the method of 'coupling from the past', as described in Section 7.4.

oldfig1.1

studied here, different edges e are permitted to have different interaction strengths J_e, see Chapter 7.

As pointed out by Baxter, [**18**], the Ising model permits an infinity of generalisations. Of these, the extension to so-called 'Potts models' has proved especially fruitful. Whereas the Ising model permits only two possible spin-values at each vertex, the Potts model [**165**] permits a general number $q \in \{2, 3, \dots\}$, and is governed by a probability measure given as follows.

Let q be an integer satisfying $q \ge 2$, and take as sample space the set of vectors $\Sigma = \{1, 2, \dots, q\}^V$. Thus each vertex of G may be in any of q states. For an edge $e = \langle x, y\rangle$ and a configuration $\sigma = (\sigma_x : x \in V) \in \Sigma$, we write $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ where $\delta_{i,j}$ is the Kronecker delta. The relevant probability measure is now given by

$$\pi_{\beta,J,q}(\sigma) = \frac{1}{Z_{\mathrm{P}}} e^{-\beta H'(\sigma)}, \qquad \sigma \in \Sigma, \tag{1.5}$$

1.potts where Z_{P} is the appropriate normalising constant and the Hamiltonian H' is given by

$$H'(\sigma) = -\sum_{e=\langle x,y\rangle \in E} J\delta_e(\sigma). \tag{1.6}$$

1.pottsham In the special case $q = 2$, the multiplicative formula

$$\delta_{\sigma_x,\sigma_y} = \tfrac{1}{2}(1 + \sigma_x\sigma_y), \quad \sigma_x, \sigma_y \in \{-1, +1\}, \tag{1.7}$$

1.mult is valid. It is now easy to see in this case that the ensuing Potts model is simply the Ising model with an adjusted value of J.

1.4 Random-cluster and Ising/Potts coupled

It was Fortuin and Kasteleyn [**72**, **73**, **74**, **120**] who discovered that Potts models may be recast as random-cluster models, and furthermore that the relationship between the two systems facilitates an extended study of phase transitions in Potts models. Their methods were essentially combinatorial. In the more modern game, we construct the two systems on a common probability space, and then observe their relationship through their realisations. There may in principle be many ways to do this, but the standard coupling reported in [**64**] is of special value.

Let $q \in \{2, 3, \dots\}$, $0 \le p \le 1$, and let $G = (V, E)$ be a finite graph, as before. We consider the product sample space $\Sigma \times \Omega$ where $\Sigma = \{1, 2, \dots, q\}^V$ and $\Omega = \{0, 1\}^E$ as above. We now define a probability mass function μ on $\Sigma \times \Omega$ by

$$\mu(\sigma, \omega) \propto \prod_{e \in E} \bigl\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\bigr\}, \qquad (\sigma, \omega) \in \Sigma \times \Omega, \tag{1.8}$$

1.coupldef where, as before, $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ for $e = \langle x, y\rangle \in E$. The constant of proportionality is exactly that which ensures the normalisation

$$\sum_{\sigma,\omega} \mu(\sigma, \omega) = 1.$$

Elementary calculations reveal the following facts.

old1.1 **Theorem 1.9 (Marginal measures of μ) [64].** *Let $q \in \{2, 3, \dots\}$ and $p = 1 - e^{-\beta J}$.*

(a) Marginal on Σ. *The marginal measure $\mu_1(\sigma) = \sum_{\omega\in\Omega} \mu(\sigma, \omega)$ on Σ is the Potts measure*

$$\mu_1(\sigma) = \frac{1}{Z_\mathrm{P}} \exp\left\{\beta \sum_{e\in E} J\delta_e(\sigma)\right\}, \qquad \sigma \in \Sigma.$$

(b) Marginal on Ω. *The marginal measure $\mu_2(\omega) = \sum_{\sigma\in\Sigma} \mu(\sigma, \omega)$ on Ω is the random-cluster measure*

$$\mu_2(\omega) = \frac{1}{Z_\mathrm{RC}} \left\{\prod_{e\in E} p^{\omega(e)}(1 - p)^{1-\omega(e)}\right\} q^{k(\omega)}, \qquad \omega \in \Omega.$$

(c) Partition functions. *We have that*

$$\sum_{\omega\in\Omega} \left\{\prod_{e\in E} p^{\omega(e)}(1 - p)^{1-\omega(e)}\right\} q^{k(\omega)} = \sum_{\sigma\in\Sigma} \prod_{e\in E} \exp\left\{\beta J(\delta_e(\sigma) - 1)\right\}, \tag{1.10}$$

1.partfns *which is to say that*

$$Z_\mathrm{RC}(p, q) = e^{-\beta J|E|} Z_\mathrm{P}(\beta, J, q). \tag{1.11}$$

1.partfns2

The conditional measures of μ are given as follows.

old1.1a **Theorem 1.12 (Conditional measures of μ) [64].** *Let $q \in \{2, 3, \dots\}$ and $p = 1 - e^{-\beta J}$.*

(a) *Given ω, the conditional measure $\mu(\cdot \mid \omega)$ on Σ is obtained by putting random spins on entire clusters of ω (of which there are $k(\omega)$). These spins are constant on given clusters, are independent between clusters, and each is uniformly distributed on the set $\{1, 2, \dots, q\}$.*

(b) *Given σ, the conditional measure $\mu(\cdot \mid \sigma)$ on Ω is obtained as follows. If $e = \langle x, y\rangle$ is such that $\sigma_x \neq \sigma_y$, we set $\omega(e) = 0$. If $\sigma_x = \sigma_y$, we set*

$$\omega(e) = \begin{cases} 1 & \textit{with probability } p, \\ 0 & \textit{otherwise}, \end{cases}$$

the values of different $\omega(e)$ being (conditionally) independent random variables.

In conclusion, the measure μ is a coupling of a Potts measure $\pi_{\beta,J,q}$ on V, together with the random-cluster measure $\phi_{p,q}$ on Ω. The parameters of these measures are related by the equation $p = 1 - e^{-\beta J}$. Since $0 \le p \le 1$, this is possible only if $\beta J \ge 0$.

This special coupling may be used in a particularly simple way to show that correlations in Potts models correspond to open connections in random-cluster models. When extended to infinite graphs, this will imply that the phase transition of a Potts model corresponds to the creation of an infinite open cluster in the random-cluster model. Thus arguments of stochastic geometry, and particularly those developed for the percolation model, may be harnessed directly in order to understand the correlation structure of the Potts system. The basic step is as follows.

We write $\{x \leftrightarrow y\}$ for the set of all $\omega \in \Omega$ for which there exists an open path joining vertex x to vertex y. The complement of the event $\{x \leftrightarrow y\}$ is denoted $\{x \nleftrightarrow y\}$.

The 'two-point correlation function' of the Potts measure $\pi_{\beta,J,q}$ on the finite graph $G = (V, E)$ is defined to be the function $\tau_{\beta,J,q}$ given by

$$\tau_{\beta,J,q}(x, y) = \pi_{\beta,J,q}(\sigma_x = \sigma_y) - \frac{1}{q}, \qquad x, y \in V. \tag{1.13}$$

1.corrconn The term q^{-1} is the probability that two independent and uniformly distributed spins are equal. Thus[2]

$$\tau_{\beta,J,q}(x, y) = \frac{1}{q}\pi_{\beta,J,q}(q\delta_{\sigma_x,\sigma_y} - 1). \tag{1.14}$$

1.corrconn2 The 'two-point connectivity function' of the random-cluster measure $\phi_{p,q}$ is defined as the function $\phi_{p,q}(x \leftrightarrow y)$ for $x, y \in V$, that is, the probability that x and y are joined by a path of open edges. It turns out that these 'two-point functions' are (except for a constant factor) the same.

old1.2 **Theorem 1.15 (Correlation/connection) [120].** *If $q \in \{2, 3, \dots\}$ and $p = 1 - e^{-\beta J}$ satisfies $0 \le p \le 1$, then*

$$\tau_{\beta,J,q}(x, y) = (1 - q^{-1})\phi_{p,q}(x \leftrightarrow y), \qquad x, y \in V.$$

The theorem may be generalised as follows. Suppose we are studying the Potts model, and are interested in some 'observable' $f : \Sigma \to \mathbb{R}$. The mean value of

[2] If μ is a probability measure and X a random variable, th expectation of X with respect to μ is written $\mu(X)$.

$f(\sigma)$ satisfies

$$\pi_{\beta,J,q}(f) = \sum_{\sigma} f(\sigma)\pi_{\beta,J,q}(\sigma) = \sum_{\sigma,\omega} f(\sigma)\mu(\sigma,\omega)$$
$$= \sum_{\omega} F(\omega)\phi_{p,q}(\omega) = \phi_{p,q}(F)$$

where $F : \Omega \to \mathbb{R}$ is given by

$$F(\omega) = \mu(f \mid \omega) = \sum_{\sigma} f(\sigma)\mu(\sigma \mid \omega).$$

The above Theorem 1.15 is obtained with $f(\sigma) = \delta_{\sigma_x,\sigma_y} - q^{-1}$, where $x, y \in V$.

The Potts models considered above have zero external field. Some complications arise when an external field is added; see the discussions in [**11**, **29**].

Proof of Theorem 1.9. (a) Let $\sigma \in \Sigma$ be given. Then

$$\sum_{\omega\in\Omega} \prod_{e\in E} \bigl\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\bigr\} = \prod_{e\in E} \bigl\{1 - p + p\delta_e(\sigma)\bigr\}.$$

Now $p = 1 - e^{-\beta J}$ and

$$1 - p + p\delta = \exp\{\beta J(\delta - 1)\}, \qquad \delta \in \{0, 1\},$$

whence

$$\sum_{\omega\in\Omega} \prod_{e\in E} \bigl\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\bigr\} = \prod_{e\in E} \exp\{\beta J(\delta_e(\sigma) - 1)\}. \tag{1.16}$$

1.pottssum Viewed as a set of weights on Σ, the latter expression generates the Potts measure.

(b) Let $\omega \in \Omega$ be given. We have that

$$\prod_{e\in E} \bigl\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\bigr\} = p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|} f(\omega,\sigma), \tag{1.17}$$

1.rcelig where $f(\omega, \sigma)$ is the indicator function that $\delta_e(\sigma) = 1$ whenever $\omega(e) = 1$. We have that $f(\omega, \sigma) = 1$ if and only if σ is constant on every open cluster of ω. There are $k(\omega)$ such clusters, and therefore $q^{k(\omega)}$ qualifying spin-vectors σ. Thus

$$\sum_{\sigma\in\Sigma} \prod_{e\in E} \bigl\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\bigr\} = p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|} q^{k(\omega)}. \tag{1.18}$$

1.rcsum This set of weights on Ω generates the random-cluster measure.

(c) We obtain the same answer if we sum (1.16) over all ω, or we sum (1.18) over all σ. $\square$

Proof of Theorem 1.12. (a) Let $\omega \in \Omega$ be given. From (1.17)–(1.18),

$$\mu(\sigma \mid \omega) = \frac{f(\omega, \sigma)}{q^{k(\omega)}}, \qquad \sigma \in \Sigma,$$

whence the conditional measure is uniform on those σ with $f(\omega, \sigma) = 1$.
(b) Let $\sigma \in \Sigma$ be given. By (1.8),

$$\mu(\omega \mid \sigma) = K_\sigma \prod_{e \in E : \delta_e(\sigma)=0} \delta_{\omega(e),0} \prod_{e \in E : \delta_e(\sigma)=1} \big\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\big\},$$

where K_σ depends on σ only. Thus $\mu(\omega \mid \sigma)$ is a product measure on Ω with

$$\omega(e) = 1 \quad \text{with probability} \begin{cases} 0 & \text{if } \delta_e(\sigma) = 0, \\ p & \text{if } \delta_e(\sigma) = 1, \end{cases}$$

as required. □

Proof of Theorem 1.15. The indicator function of an event A is denoted 1_A. We have by using conditional probabilities that

$$\begin{aligned} \tau_{\beta,J,q}(x, y) &= \sum_{\sigma,\omega} \big\{1_{\{\sigma_x=\sigma_y\}}(\sigma) - q^{-1}\big\}\mu(\sigma, \omega) \\ &= \sum_\omega \phi_{p,q}(\omega) \sum_\sigma \mu(\sigma \mid \omega)\big\{1_{\{\sigma_x=\sigma_y\}}(\sigma) - q^{-1}\big\} \\ &= \sum_\omega \phi_{p,q}(\omega)\big\{(1-q^{-1})1_{\{x\leftrightarrow y\}}(\omega) + 0 \cdot 1_{\{x \nleftrightarrow y\}}(\omega)\big\} \\ &= (1-q^{-1})\phi_{p,q}(x \leftrightarrow y), \end{aligned}$$

where μ is the above coupling of the Potts and random-cluster measures. □

1.5 The limit as $q \downarrow 0$

Let $G = (V, E)$ be a finite connected graph, and let $\phi_{p,q}$ be the random-cluster measure on the associated sample space $\Omega = \{0, 1\}^E$. We consider in this section the set of weak limits which may arise as $q \to 0$. In preparation, we introduce three graph-theoretic terms.

A subset F of the edge-set E is called:

- a *forest* of G if the graph (V, F) contains no circuit,
- a *spanning tree* of G if (V, F) is connected and contains no circuit,
- a *connected subgraph* of G if (V, F) is connected.

In each case we consider the graph (V, F) containing every vertex of V; thus, sets F of edges satisfying one of the above conditions are sometimes termed *spanning*. Note that F is a spanning tree if and only if it is both a forest and a connected subgraph. For $\Omega = \{0, 1\}^E$ and $\omega \in \Omega$, we call ω a forest (respectively, spanning tree, connected subgraph) if $\eta(\omega)$ is a forest (respectively, spanning tree, connected subgraph). Write Ω_{for}, Ω_{st}, Ω_{cs} for the subsets of Ω containing all forests, spanning trees, and connected subgraphs, respectively. We write USF, UST, UCS for the uniform probability measures on the respective sets Ω_{for}, Ω_{st}, Ω_{cs}.

We consider first the weak limit of $\phi_{p,q}$ as $q \downarrow 0$ for fixed $p \in (0, 1)$. This limit may be ascertained by observing that the dominant terms in the partition function

$$Z_{\text{RC}}(p, q) = \sum_{\omega \in \Omega} p^{|\eta(\omega)|}(1-p)^{|E \setminus \eta(\omega)|} q^{k(\omega)}$$

are those for which $k(\omega)$ is a minimum, that is, those with $k(\omega) = 1$. It follows that $\lim_{q \downarrow 0} \phi_{p,q}$ is precisely the product measure $\phi_p = \phi_{p,1}$ (that is, percolation with intensity p) *conditioned on the resulting graph* $(V, \eta(\omega))$ *being connected*. That is, $\phi_{p,q} \Rightarrow \phi_\beta^{\text{cs}}$ as $q \downarrow 0$, where

$$\phi_\beta^{\text{cs}}(\omega) = \begin{cases} \dfrac{1}{Z^{\text{cs}}} \beta^{|\eta(\omega)|} & \text{if } \omega \in \Omega_{\text{cs}}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.19}$$

1.connsub $\beta = p/(1-p)$, and Z^{cs} is the appropriate normalising constant. In the special case $p = \frac{1}{2}$, we have that $\phi_{p,q} \Rightarrow$ UCS.

Further limits arises if we allow p to converge to 0 with q, as follows. Suppose $p = p_q$ is related to q in such a way that $p \to 0$ and $q/p \to 0$ as $q \to 0$; thus p approaches zero slower than does q. We may write Z_{RC} in the form

$$Z_{\text{RC}}(p, q) = (1-p)^{|E|} \sum_{\omega \in \Omega} \left(\frac{p}{1-p}\right)^{|\eta(\omega)|+k(\omega)} \left(\frac{q(1-p)}{p}\right)^{k(\omega)}.$$

Note that $p/(1-p) \to 0$ and $q(1-p)/p \to 0$ as $q \to 0$. Now $k(\omega) \ge 1$ and $|\eta(\omega)| + k(\omega) \ge |V|$ for all $\omega \in \Omega$; these two inequalities are satisfied simultaneously with equality if and only if $\omega \in \Omega_{\text{st}}$. It follows that, in the limit as $q \to 0$, the 'mass' is concentrated on spanning trees, and it is easily seen that the limit mass is uniformly distributed. That is, $\phi_{p,q} \Rightarrow$ UST.

Another limit emerges if p approached zero at the same rate as does q. Take $p = \alpha q$ where $\alpha \in (0, \infty)$ is constant, and consider the limit as $q \downarrow 0$. This time we write

$$Z_{\text{RC}}(p, q) = (1-\alpha q)^{|E|} \sum_{\omega \in \Omega} \left(\frac{\alpha}{1-\alpha q}\right)^{|\eta(\omega)|} q^{|\eta(\omega)|+k(\omega)}.$$

We have that $|\eta(\omega)| + k(\omega) \geq |V|$ with equality if and only if $\omega \in \Omega_{\text{for}}$, and it follows that $\phi_{p,q} \Rightarrow \phi_\alpha^{\text{for}}$, where

$$\phi_\alpha^{\text{for}}(\omega) = \begin{cases} \dfrac{1}{Z^{\text{for}}}\alpha^{|\eta(\omega)|} & \text{if } \omega \in \Omega_{\text{for}}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.20}$$

1.forsub and Z^{for} is the appropriate normalising constant. In the special case $\alpha = 1$, we have that $\phi_{p,q} \Rightarrow$ USF.

Finally, if p approaches zero faster than does q, in that $p/q \to 0$ as $p, q \to 0$, it is easily seen that the limit measure is concentrated on the empty set of edges. We summarise the three special cases above in a theorem.

1.qto0 **Theorem 1.21.** *We have in the limit as $q \downarrow 0$ that*:

$$\phi_{p,q} \Rightarrow \begin{cases} \text{UCS} & \textit{if } p = \frac{1}{2}, \\ \text{UST} & \textit{if } p \to 0 \textit{ and } q/p \to 0, \\ \text{USF} & \textit{if } p = q. \end{cases}$$

The spanning-tree limit is especially interesting for historical and mathematical reasons. As explained in the appendix, the random-cluster model originated in a systematic study by Fortuin and Kasteleyn of systems of a certain type which satisfy certain parallel and series laws. Electrical networks are the best known such systems — two connections in parallel (respectively, series) of resistances r_1 and r_2 may be replaced by a single connection with resistance $(r_1^{-1} + r_2^{-1})^{-1}$ (respectively, $r_1 + r_2$). Fortuin and Kasteleyn [**74**] realised that the electrical-network theory of a graph G is related to the limit as $q \downarrow 0$ of the random-cluster model on G, where p is chosen by $p = \sqrt{q}$ (CHECK). (This choice of p is convenient, but actually one requires only that $q/p \to 0$, see [**100**].) It has been known since Kirchhoff's theorem, [**132**], that the electrical currents which flow in a network may be expressed in terms of counts of spanning trees. We return to this discussion in Section 2.9.

The theory of the uniform-spanning-tree measure UST is beautiful in its own right (see [**21**]), and is linked in an important way to the emerging field of stochastic growth processes of 'stochastic Löwner evolution' (SLE) type (see [**141**, **170**]), to which we return in Section 5.6.

Further discussions of USF and UCS may be found in [**99**, **158**].

1.6 Basic notation

The set of natural numbers $\{1, 2, \dots\}$ is denoted $\mathbb{N}$, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

We introduce next some of the basic notation necessary for a study of random-cluster measures. Let $G = (V, E)$ be a graph, with finite or countably infinite vertex set V and edge set E. If two vertices x and y are joined by an edge e, we write $x \sim y$, and $e = \langle x, y \rangle$, and we say that x is *adjacent* to y. The (graph-theoretic) distance $\delta(x, y)$ from x to y is defined to be the number of edges in a shortest path of G from x to y.

The configuration space of the random-cluster model on G is the set $\Omega = \{0, 1\}^E$, points of which are represented as a vector $\omega = (\omega(e) : e \in E)$ and called *configurations*. For $\omega \in \Omega$, we call an edge e *open* (or ω-open, when the role of ω is to be emphasized) if $\omega(e) = 1$, and *closed* (or ω-closed) if $\omega(e) = 0$. We speak of a set F of edges as being 'open' (respectively, 'closed') in the configuration ω if $\omega(f) = 1$ (respectively, $\omega(f) = 0$) for all $f \in F$. For $e \in E$, we write $J_e = \{\omega : \omega(e) = 1\}$, the event that the edge e is open; thus, $J_e(\omega) = \omega(e)$.

The indicator function of a subset A of Ω is the function $1_A : \Omega \to \{0, 1\}$ given by

$$1_A(\omega) = \begin{cases} 0 & \text{if } \omega \notin A, \\ 1 & \text{if } \omega \in A. \end{cases}$$

A function $X : \Omega \to \mathbb{R}$ is called a *cylinder* function if there exists a finite subset F of E such that $X(\omega) = X(\omega')$ whenever $\omega(e) = \omega'(e)$ for $e \in F$. A subset A of Ω is called a *cylinder event* if its indicator function is a cylinder function. We take $\mathcal{F}$ to be the σ-field of subsets of Ω generated by the cylinder events, and we shall consider certain probability measures on the measurable pair $(\Omega, \mathcal{F})$. If G is finite, then $\mathcal{F}$ is the set of all subsets of Ω; all events are cylinder events, and all functions are cylinder functions. The complement of an event A is written $\overline{A}$.

For $W \subseteq V$, we write E_W for the set of edges of G having both endvertices in W. We write $\mathcal{F}_W$ (respectively, $\mathcal{T}_W$) for the smallest σ-field of $\mathcal{F}$ with respect to which each of the random variables $\omega(e)$, $e \in E_W$ (respectively, $e \notin E_W$), is measurable.

There is a natural partial order on the set Ω of configurations given by $\omega_1 \le \omega_2$ if and only if $\omega_1(e) \le \omega_2(e)$ for all $e \in E$. Rather than working always with the vector ω ($\in \Omega$), we shall sometimes work with its set of open edges, given by

$$\eta(\omega) = \{e \in E : \omega(e) = 1\}. \tag{1.22}$$

old1.3 Clearly,

$$\omega_1 \le \omega_2 \qquad \text{if and only if} \qquad \eta(\omega_1) \subseteq \eta(\omega_2).$$

The smallest (respectively, largest) configuration is that with $\omega(e) = 0$ (respectively, $\omega(e) = 1$) for all e, which we denote by 0 (respectively, 1). The function $X : \Omega \to \mathbb{R}$ is called *increasing* if $X(\omega_1) \le X(\omega_2)$ whenever $\omega_1 \le \omega_2$. Similarly, X is *decreasing* if $-X$ is increasing. Note that every increasing function $X : \Omega \to \mathbb{R}$ is necessarily bounded since $X(0) \le X(\omega) \le X(1)$ for all $\omega \in \Omega$.

A *path* of G is an alternating sequence $x_0, e_0, x_1, e_1, \dots, e_{n-1}, x_n$ of distinct vertices x_i and edges $e_i = \langle x_i, x_{i+1} \rangle$. Such a path has *length* n and is said to connect x_0 to x_n. A *circuit* of G is an alternating sequence $x_0, e_0, x_1, e_1, \dots, e_{n-1}, x_n, e_n, x_0$ of vertices and edges such that $x_0, e_0, \dots, e_{n-1}, x_n$ is a path and $e_n =$

$\langle x_n, x_0 \rangle$; such a circuit has length $n+1$. For $\omega \in \Omega$, we call a path or circuit *open* if all its edges are open, and *closed* if all its edges are closed. Two subgraphs of G are called *edge-disjoint* if they have no edges in common, and *disjoint* if they have neither edges nor vertices in common.

Let $\omega \in \Omega$. Consider the random subgraph of G containing the vertex set V and the open edges only, that is, the edges in $\eta(\omega)$. The connected components of this graph are called *open clusters*. We write C_x for the open cluster containing the vertex x, and we call C_x the *open cluster at* x. The vertex set of C_x is the set of all vertices of G which are connected to x by open paths, and the edges of C_x are those edges of $\eta(\omega)$ which join pairs of such vertices. Occasionally we shall use the term C_x to represent the *set of vertices* joined to x by open paths, rather than the graph of this open cluster. We shall be interested in the size of C_x, and we denote by $|C_x|$ the number of vertices in C_x. We note that $C_x = \{x\}$ whenever x is an isolated vertex, which is to say that x is incident to no open edge. We denote by $k(\omega)$ the number of open clusters in the configuration ω, that is, $k(\omega)$ is the number of components of the graph $(V, \eta(\omega))$. The random variable k plays an important role in the definition of a random-cluster measure, and the reader is warned of the importance of including in k a count of the number of isolated vertices of the graph.

Let $\omega \in \Omega$. If A and B are sets of vertices of G, we shall write '$A \leftrightarrow B$' if there exists an open path joining some vertex in A to some vertex in B; if $A \cap B \neq \varnothing$ then $A \leftrightarrow B$ trivially. Thus, for example, $C_x = \{y \in V : x \leftrightarrow y\}$. We shall write '$A \not\leftrightarrow B$' if there exists no open path from any vertex of A to any vertex of B, and '$A \leftrightarrow B$ off D' if there exists an open path joining some vertex in A to some vertex in B which uses no vertex in the set D.

If W is a set of vertices of the graph, we write ∂W for the *boundary* of A, being the set of vertices in A which are adjacent to some vertex not in A,

$$\partial W = \big\{x \in W : \text{there exists } y \notin W \text{ such that } x \sim y\big\}.$$

We write $\Delta_e W$ for the set of edges of G having exactly one endvertex in W, and we call $\Delta_e W$ the *edge-boundary* of W.

We shall be mostly interested in the case when G is a subgraph of a d-dimensional lattice with $d \geq 2$. Rather than embarking on a debate of just what constitutes a 'lattice-graph', we shall, almost without exception, consider only the case of the (hyper)cubic lattice. This restriction enables a clear exposition of the theory and open problems without suffering the complications which arise through allowing greater generality.

Let d be a positive integer. We write $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ for the set of all integers, and $\mathbb{Z}^d$ for the set of all d-vectors $x = (x_1, x_2, \dots, x_d)$ with integral coordinates. For $x \in \mathbb{Z}^d$ we generally write x_i for the ith coordinate of x, and we define

$$\delta(x, y) = \sum_{i=1}^{d} |x_i - y_i|.$$

The *origin* of $\mathbb{Z}^d$ is denoted 0.

We turn $\mathbb{Z}^d$ into a graph, called the *d-dimensional cubic lattice*, by adding edges between all pairs x, y of points of $\mathbb{Z}^d$ with $\delta(x, y) = 1$. We denote this lattice by $\mathbb{L}^d$, and we write $\mathbb{Z}^d$ for the set of vertices of $\mathbb{L}^d$, and $\mathbb{E}^d$ for the set of its edges. Thus $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$. We shall often think of $\mathbb{L}^d$ as a graph embedded in $\mathbb{R}^d$, the edges being straight line segments between their endvertices. The *edge-set* $\mathbb{E}_V$ of V $(\subseteq \mathbb{Z}^d)$ is the set of all edges of $\mathbb{L}^d$ both of whose endvertices lie in V.

Let x, y be vertices of $\mathbb{L}^d$. The (graph-theoretic) distance from x to y is simply $\delta(x, y)$, and we write $|x|$ for the distance $\delta(0, x)$ from the origin to x. We shall make occasional use of another distance function on $\mathbb{Z}^d$, namely

$$\|x\| = \max\big\{|x_i| : 1 \le i \le d\big\}, \qquad x \in \mathbb{Z}^d,$$

and we note that

$$\|x\| \le |x| \le d\|x\|, \qquad x \in \mathbb{Z}^d.$$

A *box* of $\mathbb{L}^d$ is a subset of $\mathbb{Z}^d$ of the form

$$\Lambda(a, b) = \{x \in \mathbb{Z}^d : a_i \le x_i \le b_i \text{ for } i = 1, 2, \dots, d\},$$

where $a, b \in \mathbb{Z}^d$; we sometimes write

$$\Lambda(a, b) = \prod_{i=1}^{d} [a_i, b_i]$$

as a convenient shorthand. The expression $\Lambda(a, b)$ is used also to denote the graph with vertex set $\Lambda(a, b)$ together with those edges of $\mathbb{L}^d$ which join two points in $\Lambda(a, b)$. For $x \in \mathbb{Z}^d$, we write $x + \Lambda(a, b)$ for the translate by x of the box $\Lambda(a, b)$. We denote by $\Lambda(n)$ the box with side-length $2n$ and centre at the origin:

$$\Lambda(n) = [-n, n]^d = \{x \in \mathbb{Z}^d : \|x\| \le n\}. \tag{1.23}$$

oldo1.3 Note that $\partial\Lambda(n) = \Lambda(n) \setminus \Lambda(n-1)$.

In taking what is called a 'thermodynamic limit', one works often on a finite box Λ of $\mathbb{Z}^d$, and then takes the limit as $\Lambda \uparrow \mathbb{Z}^d$. Such a limit is to be interpreted along a sequence $\boldsymbol{\Lambda} = (\Lambda_n : n \ge 1)$ of boxes such that Λ_n is non-decreasing in n and, for all m, $\Lambda_n \supseteq [-m, m]^d$ for all large n.

When working with an infinite graph $G = (V, E)$, the configuration space is the infinite set $\Omega = \{0, 1\}^{\mathbb{E}^d}$, which we endow with the σ-field $\mathcal{F}$ generated by the cylinder events.

For any random variable X and appropriate probability measure μ, we write $\mu(X)$ for the expectation of X,

$$\mu(X) = \int X \, d\mu.$$

We write $\lfloor a \rfloor$ and $\lceil a \rceil$ for the integer part of the real number a, and the least integer not less than a, respectively. Finally, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

2

Fundamental Properties

2.1 Conditional probabilities

Let $\phi_{G,p,q}$ be the random-cluster measure on the finite graph $G = (V, E)$. The state of a given edge e depends on the configuration on the remainder $E \setminus \{e\}$ of the graph. The relevant conditional probabilities may be described in the following useful manner.

For $e \in E$, we denote by $G \setminus e$ (respectively, $G.e$) the graph obtained from G by deleting (respectively, contracting) the edge e. We write $\Omega_{\langle e \rangle} = \{0, 1\}^{E \setminus \{e\}}$; for $\omega \in \Omega$ we define $\omega_{\langle e \rangle} \in \Omega_{\langle e \rangle}$ by $\omega_{\langle e \rangle}(f) = \omega(f)$ for $f \neq e$. For $e = \langle x, y \rangle$, we write K_e for the event that x and y are joined by an open path not using e.

old2.1 **Theorem 2.1 (Conditional probabilities) [73].** *Let $0 \leq p \leq 1$, $q > 0$, and let $e \in E$. We have that*

$$\phi_{G,p,q}\big(\omega \,\big|\, \omega(e) = j\big) = \begin{cases} \phi_{G \setminus e,p,q}(\omega_{\langle e \rangle}) & \text{if } j = 0, \\ \phi_{G.e,p,q}(\omega_{\langle e \rangle}) & \text{if } j = 1, \end{cases} \tag{2.2}$$

old2.2 *and*

$$\phi_{G,p,q}\big(\omega(e) = 1 \,\big|\, \omega_{\langle e \rangle}\big) = \begin{cases} p & \text{if } \omega_e \in K_e, \\ \dfrac{p}{p + (1-p)q} & \text{if } \omega_e \notin K_e. \end{cases} \tag{2.3}$$

old2.3

That is to say, the effect of conditioning on the absence or presence of an edge e is to replace the measure $\phi_{G,p,q}$ by the random-cluster measure on the respective graph $G \setminus e$ or $G.e$. Secondly, the conditional probability that e is open, given the configuration elsewhere, depends only on whether or not K_e occurs, and is then given by the stated formula. It follows from (2.3) that

$$0 < \phi_{G,p,q}(\omega(e) = 1 \mid \omega_{\langle e \rangle}) < 1 \qquad \text{if } 0 < p < 1 \text{ and } q > 0.$$

Thus, given $\omega_{\langle e \rangle}$, either of the two possible states of e occurs with a strictly positive probability. This is known as the 'finite-energy property', and is related to the property of 'insertion tolerance' referred to in some of the literature.

We shall sometimes need to condition on the states of more than one edge. Towards this end, we state next a more general property than (2.2), beginning with a brief discussion of boundary conditions; more on the latter topic may be found in Section 3.2. For $F \subseteq E$, we write $\mathcal{F}_F$ for the σ-field generated by the set $\{\omega(f) : f \in F\}$ of random variables. Let $\xi \in \Omega$, and let Ω_F^ξ be the subset of Ω containing all configurations ψ satisfying $\psi(e) = \xi(e)$ for all $e \notin F$. We define the random-cluster measure $\phi_{F,p,q}^\xi$ on Ω by

(2.4)
$$\phi_{F,p,q}^\xi(\omega) = \begin{cases} \dfrac{1}{Z_F^\xi(p,q)} \left\{ \prod_{e \in F} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,F)} & \text{if } \omega \in \Omega_F^\xi, \\ 0 & \text{otherwise,} \end{cases}$$

old2.4 where $k(\omega, F)$ is the number of components of the graph $(G, \eta(\omega))$ which intersect the set of endpoints of F, and where

$$Z_F^\xi(p,q) = \sum_{\omega \in \Omega_F^\xi} \left\{ \prod_{e \in F} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,F)}. \tag{2.5}$$

old2.5 Note that $\phi_{F,p,q}^\xi(\Omega_F^\xi) = 1$.

old2.6 **Theorem 2.6.** *Let $0 \le p \le 1$, $q > 0$, and $F \subseteq E$. Let X be a random variable which is $\mathcal{F}_F$-measurable. Then*

$$\phi_{G,p,q}(X \mid \mathcal{F}_{\overline{F}})(\xi) = \phi_{F,p,q}^\xi(X) \qquad \textit{for all } \xi \in \Omega.$$

In other words, given the states of edges not belonging to F, the conditional measure on F is a random-cluster measure with the difference that we retain open connections of ξ using edges not belonging to F.

Proof of Theorem 2.1. First we fix the edge e. By expanding the conditional probability in (2.2), we obtain

$$\phi_{G,p,q}\big(\omega \,\big|\, \omega(e) = j\big) = \begin{cases} \phi_{G,p,q}(\omega_e)/\phi_{G,p,q}(\overline{J_e}) & \text{if } j = 0, \\ \phi_{G,p,q}(\omega^e)/\phi_{G,p,q}(J_e) & \text{if } j = 1, \end{cases}$$

for all $\omega \in \Omega$, where $J_e = \{\omega(e) = 1\}$. We expand the ratios to obtain the required formulae.

Similarly,

$$\begin{aligned}
\phi_{G,p,q}\big(\omega(e) = 1 \,\big|\, \omega_{\langle e\rangle}\big) &= \frac{\phi_{G,p,q}(\omega^e)}{\phi_{G,p,q}(\omega^e) + \phi_{G,p,q}(\omega_e)} \\
&= \frac{\{p/(1-p)\}^{|\eta(\omega^e)|} q^{k(\omega^e)}}{\{p/(1-p)\}^{|\eta(\omega^e)|} q^{k(\omega^e)} + \{p/(1-p)\}^{|\eta(\omega_e)|} q^{k(\omega_e)}} \\
&= \begin{cases} \dfrac{p/(1-p)}{\{p/(1-p)\} + 1} & \text{if } \omega_e \in K_e, \\[2ex] \dfrac{p/(1-p)}{\{p/(1-p)\} + q} & \text{if } \omega_e \notin K_e, \end{cases}
\end{aligned}$$

as required. □

Proof of Theorem 2.6. This holds by repeated application of (2.2), one application for every edge not belonging in F. □

2.2 Stochastic ordering of measures

The stochastic ordering of probability measures provides a technique which is fundamental to the study of random-cluster measures. Let E be a finite or countably infinite set; let $\Omega = \{0, 1\}^E$, and let $\mathcal{F}$ be the σ-field generated by the cylinder events of Ω. If $|E| < \infty$, then $\mathcal{F}$ is the power set of E. In applications of the arguments of this section, E will be the edge-set of a graph, but the graphical structure is not relevant at this stage.

Note first that Ω is a partially ordered set with partial order $\omega_1 \le \omega_2$ if $\omega_1(e) \le \omega_2(e)$ for all e. A random variable $X : \Omega \to \mathbb{R}$ is called *increasing* if $X(\omega_1) \le X(\omega_2)$ whenever $\omega_1 \le \omega_2$. A non-empty event $A \in \mathcal{F}$ is called *increasing* if its indicator function 1_A is increasing. The word 'decreasing' should be interpreted in the natural way. The set Ω, equipped with the topology of open sets generated by the cylinder events, is a metric space. We speak of a random variable $X : \Omega \to \mathbb{R}$ as being 'continuous' if it is a continuous function on this metric space. Since Ω is compact, any continuous function on Ω is necessarily bounded. In addition, any increasing function $X : \Omega \to \mathbb{R}$ is necessarily bounded since $X(0) \le X(\omega) \le X(1)$ for all $\omega \in \Omega$.

Given two probability measures μ_1, μ_2 on Ω, we write $\mu_1 \le_{\text{st}} \mu_2$ (or $\mu_2 \ge_{\text{st}} \mu_1$), and say that μ_1 is stochastically smaller than μ_2, if $\mu_1(X) \le \mu_2(X)$ for all increasing continuous random variables X on Ω. (Recall that $\mu(X)$ denotes the expectation of X under μ, that is, $\mu(X) = \int X \, d\mu$.)

Discuss Strassen, Lindvall, if not elsewhere.

We call a probability measure μ on $(\Omega, \mathcal{F})$ *strictly positive* if $\mu(\omega) > 0$ for all $\omega \in \Omega$. For $\omega_1, \omega_2 \in \Omega$, we denote by $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ the 'maximum' and 'minimum' configurations given by

$$\left.\begin{array}{l}\omega_1 \vee \omega_2(e) = \max\{\omega_1(e), \omega_2(e)\} \\ \omega_1 \wedge \omega_2(e) = \min\{\omega_1(e), \omega_2(e)\}\end{array}\right. \quad \text{for } e \in E.$$

We suppose for the remainder of this section that E is finite. There is a useful sufficient condition for the stochastic inequality $\mu_1 \leq_{\text{st}} \mu_2$, as follows.

old2.7 **Theorem 2.7 (Holley inequality) [112].** *Let μ_1 and μ_2 be strictly positive probability measures on $(\Omega, \mathcal{F})$ such that*

$$(2.8) \qquad \mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2) \qquad \textit{for all } \omega_1, \omega_2 \in \Omega.$$

old2.8 *Then*

$$\mu_1(X) \leq \mu_2(X) \qquad \textit{for increasing functions } X : \Omega \to \mathbb{R},$$

which is to say that $\mu_1 \leq_{\text{st}} \mu_2$.

Let $S \subseteq \Omega^2$ be the set of all ordered pairs (π, ω) of configurations satisfying $\pi \leq \omega$. In the proof of Theorem 2.7, we shall construct a probability measure κ on Ω^2 such that:

(a) the first and second marginal probability measures of κ are μ_1 and μ_2, respectively,

(b) $\kappa(S) = 1$.

It is an immediate consequence that $\mu_1 \leq_{\text{st}} \mu_2$, and an explanation of this step is provided at the end of the proof. We refer to κ as a *coupling* of μ_1 and μ_2. There is a variety of couplings of measures which play roles in the theory of random-cluster measures. Another may be found in the proof of Theorem (2.72).

One needs to check condition (2.8) in order to apply Theorem 2.7. It is not necessary to check (2.8) for *all* pairs ω_1, ω_2 but only for those pairs which disagree at two or fewer edges. We explain this remark next. The *Hamming distance* between two configurations is given by

$$(2.9) \qquad H(\omega_1, \omega_2) = \sum_{e \in E} |\omega_1(e) - \omega_2(e)|, \qquad \omega_1, \omega_2 \in \Omega.$$

2.hamm

2.onetwo **Theorem 2.10.** *A pair μ_1, μ_2 of strictly positive probability measures on $(\Omega, \mathcal{F})$ satisfies (2.8) if and only if it satisfies*:

(2.11)

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2) \qquad \textit{whenever } 1 \leq H(\omega_1, \omega_2) \leq 2.$$

2.onetwocond

Proof of Theorem 2.7. The theorem amounts to a 'mere' numerical inequality involving a finite number of positive reals. It may in principle be proved in a totally elementary manner, using essentially no general mechanism. The proof given here proceeds by constructing certain reversible Markov chains. There is some extra mechanism required, but the method is beautiful, and in addition yields a structure which finds applications elsewhere.

The main step of the proof is designed to show that, under condition (2.8), μ_1 and μ_2 may be 'coupled' in such a way that the component with marginal measure μ_1 lies beneath (in the sense of sample realisations) that with marginal measure μ_2. This is achieved by constructing a certain Markov chain with the coupled measure as invariant measure.

Here is a preliminary calculation. Let μ be a strictly positive probability measure on $(\Omega, \mathcal{F})$. We may construct a reversible Markov chain with state space Ω and unique invariant measure μ, in the following way. We do this by choosing a suitable generator (or 'Q-matrix') satisfying the detailed balance equations. The dynamics of the chain involve the 'switching on or off' of components of the current state. For $\omega \in \Omega$, let ω^e and ω_e be the configurations obtained from ω by 'switching on' and 'switching off' the state of the edge e, respectively. That is,

$$\omega^e(f) = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 1 & \text{if } f = e, \end{cases} \qquad \omega_e(f) = \begin{cases} \omega(f) & \text{if } f \neq e, \\ 0 & \text{if } f = e, \end{cases} \qquad \text{for } f \in E.$$

We define the function $G : \Omega^2 \to \mathbb{R}$ by

$$G(\omega_e, \omega^e) = 1, \quad G(\omega^e, \omega_e) = \frac{\mu(\omega_e)}{\mu(\omega^e)}, \tag{2.12}$$

old2.9 for all $\omega \in \Omega$ and $e \in E$; define $G(\omega, \omega') = 0$ for all other pairs ω, ω' with $\omega \neq \omega'$. The diagonal elements $G(\omega, \omega)$ are chosen in such a way that

$$\sum_{\omega'} G(\omega, \omega') = 0 \qquad \text{for all } \omega \in \Omega.$$

It is elementary that

$$\mu(\omega) G(\omega, \omega') = \mu(\omega') G(\omega', \omega) \qquad \text{for all } \omega, \omega' \in \Omega,$$

and therefore G generates a Markov chain on the state space Ω which is reversible with respect to μ. The chain is irreducible, for the following reason. For $\omega, \omega' \in \Omega$, one may add edges one by one to $\eta(\omega)$ thus arriving at the vector of all 1's, and then one may remove edges one by one until one arrives at ω'. By (2.12), each such transition has a strictly positive intensity, whence the chain is irreducible. It follows that the chain has unique invariant measure μ. An account of the general theory of reversible Markov chains may be found in [**98**, Section 6.5].

We follow next a similar route for *pairs* of configurations. Let μ_1 and μ_2 satisfy the hypotheses of the theorem, and let S be the set of all pairs (π, ω) of configurations in Ω satisfying $\pi \le \omega$. We define $H : S \times S \to \mathbb{R}$ by

$$H(\pi_e, \omega; \pi^e, \omega^e) = 1, \tag{2.13}$$

$$H(\pi, \omega^e; \pi_e, \omega_e) = \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)}, \tag{2.14}$$

$$H(\pi^e, \omega^e; \pi_e, \omega^e) = \frac{\mu_1(\pi_e)}{\mu_1(\pi^e)} - \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)}, \tag{2.15}$$

old2.12 for all $(\pi, \omega) \in S$ and $e \in E$; all other off-diagonal values of H are set to 0. The diagonal terms $H(\pi, \omega; \pi, \omega)$ are chosen in such a way that

$$\sum_{\pi', \omega'} H(\pi, \omega; \pi', \omega') = 0 \qquad \text{for all } (\pi, \omega) \in S.$$

Equation (2.13) specifies that, for $\pi \in \Omega$ and $e \in E$, the edge e is acquired by π (if it does not already contain it) at rate 1; any edge so acquired is added also to ω if it does not already contain it. (Here, we speak of a configuration ψ 'containing the edge e' if $\psi(e) = 1$.) Equation (2.14) specifies that, for $\omega \in \Omega$ and $e \in E$ with $\omega(e) = 1$, the edge e is removed from ω (and also from π if $\pi(e) = 1$) at the rate given in (2.14). For e with $\pi(e) = 1$, there is an additional rate given in (2.15) at which e is removed from π but not from ω. This additional rate is indeed non-negative, since the required inequality

$$\mu_1(\pi_e)\mu_2(\omega^e) \ge \mu_1(\pi^e)\mu_2(\omega_e) \quad \text{for } \pi \le \omega \tag{2.16}$$

2.cond follows from assumption (2.8) with $\omega_1 = \pi^e$ and $\omega_2 = \omega_e$.

Let $(Y_t, Z_t)_{t \ge 0}$ be a Markov chain on S having generator H, and set $(Y_0, Z_0) = (0, 1)$, where 0 (respectively, 1) is the state of all 0's (respectively, 1's). We write $\mathbb{P}$ for the appropriate probability measure. Since all transitions retain the ordering of the two components of the state, we have that $\mathbb{P}(Y_t \le Z_t \text{ for all } t) = 1$. By examination of (2.13)–(2.15) we see that $Y = (Y_t)_{t \ge 0}$ is a Markov chain with generator given by (2.12) with $\mu = \mu_1$, and that $Z = (Z_t)_{t \ge 0}$ arises similarly with $\mu = \mu_2$. Here is a brief explanation of this elementary step in the case of Y, a similar argument holding for Z. For $\pi \in \Omega$ and $e \in E$, we have that

$$\begin{aligned}
&\mathbb{P}(Y_{t+h} = \pi^e \mid Y_t = \pi_e) \\
&\qquad = \sum_{\omega \in \Omega} \mathbb{P}\Big(Y_{t+h} = \pi^e \,\Big|\, (Y_t, Z_t) = (\pi_e, \omega)\Big)\mathbb{P}(Z_t = \omega \mid Y_t = \pi_e) \\
&\qquad = \sum_{\omega \in \Omega} (h + \mathrm{o}(h))\mathbb{P}(Z_t = \omega \mid Y_t = \pi_e) \qquad \text{by (2.13)} \\
&\qquad = h + \mathrm{o}(h).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\mathbb{P}(Y_{t+h} = \pi_e \mid Y_t = \pi^e) \\
&\quad = \sum_{\omega,\omega' \in \Omega} \mathbb{P}\Big((Y_{t+h}, Z_{t+h}) = (\pi_e, \omega') \,\Big|\, (Y_t, Z_t) = (\pi^e, \omega^e)\Big) \mathbb{P}(Z_t = \omega^e \mid Y_t = \pi^e) \\
&\quad = \sum_{\omega \in \Omega} \Big\{ H(\pi^e, \omega^e; \pi_e, \omega_e) + H(\pi^e, \omega^e; \pi_e, \omega^e) \Big\} (h + \mathrm{o}(h)) \mathbb{P}(Z_t = \omega^e \mid Y_t = \pi^e) \\
&\quad = \sum_{\omega \in \Omega} \frac{\mu_1(\pi_e)}{\mu_1(\pi^e)} (h + \mathrm{o}(h)) \mathbb{P}(Z_t = \omega^e \mid Y_t = \pi^e) \qquad \text{by (2.14) and (2.15)} \\
&\quad = \frac{\mu_1(\pi_e)}{\mu_1(\pi^e)} (h + \mathrm{o}(h)).
\end{aligned}
$$

Let κ be an invariant measure for the paired chain $(Y_t, Z_t)_{t \ge 0}$. Since Y and Z have (respective) unique invariant measures μ_1 and μ_2, it follows that the marginals of κ are μ_1 and μ_2. Since $\mathbb{P}(Y_t \le Z_t \text{ for all } t) = 1$, we have that

$$\kappa(S) = \kappa\big(\{(\pi, \omega) : \pi \le \omega\}\big) = 1,$$

and κ is the required 'coupling' of μ_1 and μ_2.

Let $(\pi, \omega) \in S$ be chosen according to the measure κ. Then

$$\mu_1(X) = \kappa(X(\pi)) \le \kappa(X(\omega)) = \mu_2(X),$$

for any increasing function X. Therefore $\mu_1 \le_{\mathrm{st}} \mu_2$. □

Proof of Theorem 2.10. The implication in one direction is obvious. Suppose then that (2.11) holds. Since (2.8) is a triviality when $H(\omega_1, \omega_2) = 0$, it suffices to prove it for $H(\omega_1, \omega_2) \ge 3$, and the proof of this is by induction on $H(\omega_1, \omega_2)$. Let $h \ge 3$ and suppose that the inequality of (2.8) holds for all pairs ω_1, ω_2 satisfying $H(\omega_1, \omega_2) < h$. Let $\omega_1, \omega_2 \in \Omega$ be such that $H(\omega_1, \omega_2) = h$, and furthermore such that neither $\omega_1 \le \omega_2$ nor $\omega_1 \ge \omega_2$ (the claim is trivial otherwise). There exist integers a, b such that $a, b \ge 1$ and $a + b = h$, and disjoint subsets $A, B \subseteq E$ with cardinalities a and b respectively, such that

$$
\begin{aligned}
\text{if } e \in A, &\quad (\omega_1(e), \omega_2(e)) = (1, 0), \\
\text{if } e \in B, &\quad (\omega_1(e), \omega_2(e)) = (0, 1), \\
\text{if } e \in E \setminus (A \cup B), &\quad \omega_1(e) = \omega_2(e).
\end{aligned}
$$

We fix an ordering $(e_i : 1 \le i \le |E|)$ of the set E in which edges in A are indexed $1, 2, \dots, a$, and edges in B are indexed $a+1, a+2, \dots, a+b$. A configuration ω may be written as a 'word' $\omega(e_1) \cdot \omega(e_2) \cdot \dots \cdot \omega(e_{|E|})$; we write 0^x for a subword of length x every entry of which is 0, with a similar meaning for 1^y. Since the entries of the configurations $\omega_1, \omega_2, \omega_1 \vee \omega_2, \omega_1 \wedge \omega_2$ are constant off $A \cup B$, we shall omit explicit reference to these values. Thus, for example, $\omega_1 = 1^a \cdot 0^b$ and $\omega_2 = 0^a \cdot 1^b$.

We may assume that $a \le b$, whence $b \ge 2$, since an exactly parallel argument holds when $a > b$. By the induction hypothesis,

$$\begin{aligned} \mu_2(1^a \cdot 1^b)\mu_1(0^a \cdot 1 \cdot 0^{b-1}) &\ge \mu_1(0^a \cdot 1^b)\mu_2(1^a \cdot 1 \cdot 0^{b-1}) \\ &\qquad \text{since } H(0^a \cdot 1^b, 1^a \cdot 1 \cdot 0^{b-1}) = h - 1, \\ \mu_2(1^a \cdot 1 \cdot 0^{b-1})\mu_1(0^{a+b}) &\ge \mu_1(0^a \cdot 1 \cdot 0^{b-1})\mu_2(1^a \cdot 0^b) \\ &\qquad \text{since } H(0^a \cdot 1 \cdot 0^{b-1}, 1^a \cdot 0^b) = a + 1 < h, \end{aligned}$$

whence

$$\begin{aligned} \mu_2(1^a \cdot 1^b)\mu_1(0^a \cdot 1 \cdot 0^{b-1})\mu_1(0^{a+b}) &\ge \mu_1(0^a \cdot 1^b)\mu_2(1^a \cdot 1 \cdot 0^{b-1})\mu_1(0^{a+b}) \\ &\ge \mu_1(0^a \cdot 1^b)\mu_1(0^a \cdot 1 \cdot 0^{b-1})\mu_2(1^a \cdot 0^b). \blacksquare \end{aligned}$$

Check this. The claim follows on dividing by the strictly positive quantity $\mu_1(0^a \cdot 1 \cdot 0^{b-1})$.
$\square$

2.3 Positive association

The set E is assumed finite as in the last section. We might explore a theory of positive association for probability measures on any sample space of the form $\Omega' = S^E$ where S is a finite ordered set (typically, a subset of $\mathbb{N}$). Instead, we consider only the case $\Omega = \{0, 1\}^E$, since this is the situation for random-cluster measures. The more general case is handled similarly. Similarly, although we restrict ourselves in this section to strictly positive measures, some of the results detailed later hold without this assumption.

A probability measure μ on Ω is said to have the *FKG lattice property* if it satisfies the *FKG lattice condition*:

$$(2.17) \qquad \mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \ge \mu(\omega_1)\mu(\omega_2) \qquad \text{for all } \omega_1, \omega_2 \in \Omega.$$

old2.13 It is a consequence of the Holley inequality (Theorem 2.1), as follows, that any strictly positive probability measure with the FKG lattice property satisfies the so-called FKG inequality. A stronger result will appear at Theorem 2.26.

old2.14 **Theorem 2.18 (FKG inequality) [75,112].** *Let μ be a strictly positive probability measure on Ω satisfying the FKG lattice condition. Then*

$$(2.19) \qquad \mu(XY) \ge \mu(X)\mu(Y) \qquad \textit{for increasing functions } X, Y : \Omega \to \mathbb{R}.$$

old2.15

Proof. Since inequality (2.19) is 'merely' an inequality involving a finite set of real numbers, it may in principle be proved in a totally elementary manner, [**167**]. Here we follow the more interesting route of Holley's inequality, Theorem 2.7. Assume that μ satisfies the FKG lattice condition (2.17), and let X and Y be increasing functions. Let $a > 0$ and $Y' = Y + a$. Since

$$\mu(XY') - \mu(X)\mu(Y') = \mu(XY) - \mu(X)\mu(Y),$$

it suffices to prove (2.19) with Y replaced by Y'. We may pick a sufficiently large that $Y(\omega) > 0$ for all $\omega \in \Omega$. Thus it suffices to prove (2.19) under the additional hypothesis that Y is strictly positive, and we assume henceforth that this holds. Define strictly positive probability measures μ_1 and μ_2 on $(\Omega, \mathcal{F})$ by $\mu_1 = \mu$ and

$$\mu_2(\omega) = \frac{Y(\omega)\mu(\omega)}{\sum_{\omega'} Y(\omega')\mu(\omega')} \qquad \text{for } \omega \in \Omega.$$

as required. □

Any probability measure μ satisfying (2.19) is said to have the property of 'positive association' or, more concisely, to be 'positively associated'. We consider in Section 3.1 the positive association of measures on $\Omega = \{0, 1\}^E$ when E is countably infinite.

If X is increasing and Y is decreasing, we may apply (2.19) to X and $-Y$ to find, under the conditions of the theorem, that $\mu(XY) \le \mu(X)\mu(Y)$. In the special case when $X = 1_A$, $Y = 1_B$, the indicator functions of events A and B, we obtain under the conditions of the theorem that

$$(2.20) \qquad \mu(A \cap B) \ge \mu(A)\mu(B) \qquad \text{for increasing events } A, B.$$

old2.16

It is a near triviality to check that any product measure on Ω satisfies the FKG lattice condition, and thus product measures are strongly positively associated. This is the $q = 1$ case of Theorem 2.29.

Let $X = (X_1, X_2, \dots, X_r)$ be a vector of random variables which takes values in $\{0, 1\}^r$. We shall speak of X as being positively associated if it is governed by a measure on $\{0, 1\}^r$ which is itself positively associated. Let $Y = h(X)$ where $h : \{0, 1\}^r \to \{0, 1\}^s$ is a non-decreasing function. It is standard that the vector Y is positively associated whenever X is positively associated, [**??**]. The proof is straightforward, as follows. Let A, B be increasing subsets of $\{0, 1\}^s$. Then

$$\begin{aligned}
\mathbb{P}(Y \in A \cap B) &= \mathbb{P}(X \in h^{-1}A \cap h^{-1}B) \\
&\ge \mathbb{P}(X \in h^{-1}A)\mathbb{P}(X \in h^{-1}B) \\
&= \mathbb{P}(Y \in A)\mathbb{P}(Y \in B),
\end{aligned}$$

since $h^{-1}A$ and $h^{-1}B$ are increasing subsets of $\{0, 1\}^r$.

We turn now to a consideration of the FKG lattice condition. Recall the Hamming distance between configurations defined in (2.9).

new2.1 **Theorem 2.21.** *A strictly positive probability measure μ on $(\Omega, \mathcal{F})$ satisfies the FKG lattice condition if and only (2.17) holds for all pairs $\omega_1, \omega_2 \in \Omega$ with $H(\omega_1, \omega_2) = 2$.*

Proof. The inequality of (2.17) is a triviality when $H(\omega_1, \omega_2) = 1$. The conclusion now follows by Theorem 2.10. □

Check this

The FKG lattice condition is sufficient but not necessary for positive association. It is in fact equivalent for strictly positive measures to a stronger property termed 'strong positive association' (or, sometimes, 'strong FKG'). For $F \subseteq E$ and $\xi \in \Omega$, we write $\Omega_F = \{0, 1\}^F$ and

$$\Omega_F^\xi = \{\omega : \omega(e) = \xi(e) \text{ for all } e \in \overline{F}\}, \tag{2.22}$$

new2.2 the set of configurations 'which agree with ω on the complement of F'. Let μ be a probability measure on $(\Omega, \mathcal{F})$, and define the probability measure μ_F^ξ by

$$\mu_F^\xi(\omega) = \frac{1}{\mu(\Omega_F^\xi)}\mu(\omega)1_{\Omega_F^\xi}(\omega), \qquad \omega \in \Omega, \tag{2.23}$$

new2.3 whenever $\mu(\Omega_F^\xi) > 0$, noting that μ is supported on Ω_F^ξ. We say that μ is *strongly positively associated* if: for all $F \subseteq E$ and all $\xi \in \Omega$ such that $\mu(\Omega_F^\xi) > 0$, the measure μ_F^ξ is positively associated.

viewed as a measure on Ω or Ω_F?

For any subset A of $\Omega_F = \{0, 1\}^F$ and any $\xi \in \Omega$, we write $A \times \xi$ for the set of all configurations $\omega \in \Omega_F^\xi$ such that $(\omega(e) : e \in F) \in A$. We call μ *monotonic* if: for all $F \subseteq E$, all increasing subsets A of Ω_F, and all $\xi, \zeta \in \Omega$ such that $\mu(\Omega_F^\xi), \mu(\Omega_F^\zeta) > 0$,

$$\mu_F^\xi(A \times \xi) \le \mu_F^\zeta(A \times \zeta) \qquad \text{whenever } \xi \le \zeta . TAG$$

That is, μ is monotonic if, when viewed as an inequality involving probability measures on Ω_F, we have that

$$\mu_F^\xi \le_{\text{st}} \mu_F^\zeta \quad \text{whenever} \quad \xi \le \zeta. \tag{2.24}$$

new2.4 We call μ 1-*monotonic* if (2.24) holds for all singleton sets F. That is, μ is 1-monotonic if and only if

$$\mu\big(\omega(f) = 1 \,\big|\, \omega(e) = \xi(e) \text{ for all } e \in E \setminus \{f\}\big) \tag{2.25}$$

new2.5 is non-decreasing in ξ.

new2.7 **Theorem 2.26**[3]**.** *Let μ be a strictly positive probability measure on $(\Omega, \mathcal{F})$. The following are equivalent.*

(a) *μ is strongly positively associated.*

(b) *μ satisfies the FKG lattice condition.*

(c) *μ is monotonic.*

(d) *μ is* 1*-monotonic.*

We give two examples of probability measures which are positively associated but which do not satisfy the statements of the above theorem.

new2.8 **Example 2.27**[4]**.** Let $\epsilon, \delta \in (0, 1)$, and let μ_0, μ_1 be the probability measures on $\{0, 1\}^3$ given by

$$\mu_0(010) = \mu_0(001) = \delta, \quad \mu_0(000) = 1 - 2\delta,$$
$$\mu_1(111) = \mu_1(100) = \tfrac{1}{2}.$$

We set $\mu = \epsilon\mu_0 + (1 - \epsilon)\mu_1$. Note that

$$\mu(011) = \mu(101) = \mu(110) = 0.$$

It may be checked that μ does not satisfy the FKG lattice condition whereas, for sufficiently small positive values of the constants ϵ, δ, μ is strongly positively associated. The measure μ is not strictly positive, but such an example may be arranged by working instead with $\mu' = (1 - \eta)\mu + \eta\mu_2$ where

$$\mu_2(011) = \mu_2(101) = \mu_2(110) = \tfrac{1}{3}$$

and η is small and positive.

new2.9 **Example 2.28**[5]**.** Let X and Y be independent Bernoulli random variables with parameter $\frac{1}{2}$, so that

$$\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = \tfrac{1}{2},$$

and similarly for Y. Let Z be the indicator function of the event $\{X = 1\} \cup \{Y = 1\}$. It is clear that

$$\mathbb{P}(X = 1 \mid Z = 1) > \mathbb{P}(X = 1), \quad \mathbb{P}(X = 1 \mid Y = Z = 1) = \mathbb{P}(X = 1).$$

[3] Closely related material is discussed in [**121**]. The equivalence of (a) and (b) is attributed in [**6**] to J. van den Berg and R. M. Burton (1987). See [**81**] for a further discussion of monotonic measures.

[4] Proposed by J. Steif

[5] Proposed by J. van den Berg

It is easy to deduce that the probability measure μ governing the triple (X, Y, Z) is not monotonic. It is however positively associated since the triple (X, Y, Z) is an increasing function of the independent pair X, Y.

As in the previous example, μ is not strictly positive, a weakness which we shall remedy in a different way than before. Let X', Y', Z' (respectively, X'', Y'', Z'') be Bernoulli random variables with parameter $1-\delta$ (respectively, δ), and assume the maximal amount of independence. The triple

$$(A, B, C) = \big((X \vee X') \wedge X'', (Y \vee Y') \wedge Y'', (Z \vee Z') \wedge Z''\big)$$

is an increasing function of positively associated random variables, and is therefore positively associated. However, for small δ, it is only a (stochastically) small perturbation of the original triple (X, Y, Z), and thus does not have the FKG lattice property. It is easy to check that $\mathbb{P}((A, B, C) = \omega) > 0$ for all $\omega \in \Omega$.

We return now to the case when $G = (V, E)$ is a graph, which for the moment we assume to be finite. Let $\phi_{p,q}$ be the random-cluster measure on G with parameters p and q. We shall see that $\phi_{p,q}$ satisfies the FKG lattice condition whenever $q \ge 1$, and we arrive thus at the following conclusion.

old2.18 **Theorem 2.29 [73].**
(a) *Suppose that* $0 < p < 1$ *and* $q \ge 1$. *The random-cluster measure* $\phi_{p,q}$ *is strictly positive and satisfies the FKG lattice condition.*
(b) *Suppose that* $0 \le p \le 1$ *and* $q \ge 1$. *The random-cluster measure* $\phi_{p,q}$ *is strongly positively associated, and in particular*

$$\begin{aligned} \phi_{p,q}(XY) &\ge \phi_{p,q}(X)\phi_{p,q}(Y) && \textit{for increasing functions } X, Y : \Omega \to \mathbb{R}, \\ \phi_{p,q}(A \cap B) &\ge \phi_{p,q}(A)\phi_{p,q}(B) && \textit{for increasing events } A, B. \end{aligned}$$

It is not difficult to see that $\phi_{p,q}$ is generally not positively associated when $0 < q < 1$, and a simple example is the following. Let G be the graph containing just two vertices and having exactly two parallel edges e and f joining these vertices. Let J_g be the indicator function of the event that the edge g is open. It is an easy computation that

$$\phi_{p,q}(J_e J_f) - \phi_{p,q}(J_e)\phi_{p,q}(J_f) = \frac{(q-1)(1-p)^2}{Z(p,q)^2}, \tag{2.30}$$

old2.19 which is negative if and only if $q < 1$.

Theorem 2.29 applies only to *finite* graphs G, whereas many potential applications concern *infinite* graphs. We shall see in Sections 3.3 and 3.4 how to obtain the required extension.

Proof of Theorem 2.26. Throughout, μ is assumed strictly positive.

(a) $\Rightarrow$ (b). It suffices by Theorem 2.21 to prove (2.17) for two configurations ω_1, ω_2 which disagree on exactly two edges. Let e, f be distinct members of E, and take e and f to be first two edges in a given ordering of E. We shall adopt the notation of the proof of Theorem 2.?. Thus we write $\omega_1 = 0 \cdot 1 \cdot w$ and $\omega_2 = 1 \cdot 0 \cdot w$ for some word w of length $|E| - 2$. By strong positive association, $\alpha(xy) = \mu(x \cdot y \cdot w)$ satisfies

$$\alpha(11)\big[\alpha(00) + \alpha(01) + \alpha(10) + \alpha(11)\big] \geq \big[\alpha(01) + \alpha(11)\big]\big[\alpha(10) + \alpha(11)\big],$$

which simplifies to yield as required that

$$\alpha(11)\alpha(00) \geq \alpha(01)\alpha(10).$$

(b) $\Rightarrow$ (c). For $\omega, \xi \in \Omega$, we define the configuration ω_F^ξ by

$$\omega_F^\xi(e) = \begin{cases} \omega(e) & \text{if } e \in F, \\ \xi(e) & \text{if } e \in \overline{F}. \end{cases} \tag{2.31}$$

new2.6 Thus ω_F^ξ agrees with ω on F and with ξ on its complement. By the Holley inequality, Theorem 2.7, it suffices to prove for $\omega, \rho \in \Omega$ that

$$\mu_F^\zeta((\omega \vee \rho)_F^\zeta)\mu_F^\xi((\omega \wedge \rho)_F^\xi) \geq \mu_F^\zeta(\omega_F^\zeta)\mu_F^\xi(\rho_F^\xi) \qquad \text{when } \xi \leq \zeta.$$

This is, by (2.?), an immediate consequence of the FKG lattice property applied to the pair ω_F^ζ, ρ_F^ξ.

(c) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (a). Assume that μ is 1-monotonic, and let $F \subseteq E$. Let $X, Y : \Omega \to \mathbb{R}$ be increasing functions. By adding a constant to Y as in the proof of Theorem 2.18, we may assume that $Y(\omega) > 0$ for all $\omega \in \Omega$. Fix $\xi \in \Omega$, and consider the two probability measures on Ω_F^ξ given by

$$\mu_1(\omega) = \mu_F^\xi(\omega), \qquad \mu_2(\omega) = \frac{Y(\omega)\mu_F^\xi(\omega)}{\sum_{\zeta \in \Omega_F^\xi} Y(\zeta)\mu_F^\xi(\zeta)} = \frac{Y(\omega)\mu_F^\xi(\omega)}{\mu_F^\xi(Y)}.$$

We now follow the proof of the Holley inequality, Theorem 2.7, in order to obtain the required conclusion. In thus doing we construct a Markov chain on the space S_F of all ordered pairs (π, ζ) with $\pi, \zeta \in \Omega_F^\xi$ and satisfying $\pi \leq \zeta$, and we do this in such a way that the first marginal of the chain has unique invariant measure μ_1 and the second has unique invariant measure μ_2. It will follow as in Theorem 2.7 that $\mu_1 \leq_{\text{st}} \mu_2$, whence

$$\mu_F^\xi(X) = \mu_1(X) \leq \mu_2(X) = \frac{\mu_F^\xi(XY)}{\mu_F^\xi(Y)}$$

as required. We do not give all details of this, since a full proof would involve much repetition of the proof of Theorem 2.7. Instead we concentrate on the salient condition which enables that proof, namely that the equivalent of (2.16) holds. Re-cast in the present setting, (2.16) becomes

$$(2.32) \qquad \mu_1(\pi_e)\mu_2(\zeta^e) \geq \mu_1(\pi^e)\mu_2(\zeta_e) \qquad \text{for } \pi, \zeta \in \Omega_F^\xi,\ \pi \leq \zeta,\ e \in F.$$

2.cond2 Since Y is increasing and positive, we have that $Y(\zeta^e) \geq Y(\zeta_e) > 0$, whence (2.32) is implied by

$$(2.33) \quad \mu_F^\xi(\pi_e)\mu_F^\xi(\zeta^e) \geq \mu_F^\xi(\pi^e)\mu_F^\xi(\zeta_e) \qquad \text{for } \pi, \zeta \in \Omega_F^\xi,\ \pi \leq \zeta,\ e \in F.$$

2.cond3 The latter condition is equivalent to

$$\frac{\mu_F^\xi(\zeta^e)}{\mu_F^\xi(\zeta_e) + \mu_F^\xi(\zeta^e)} \geq \frac{\mu_F^\xi(\pi^e)}{\mu_F^\xi(\pi_e) + \mu_F^\xi(\pi^e)}, \qquad \text{for } \pi, \zeta \in \Omega_F^\xi,\ \pi \leq \zeta,\ e \in F.$$

which is easily seen to be a consequence of 1-monotonicity. □

Proof of Theorem 2.29. If $p \neq 0, 1$, the conclusion of (b) follows from (a) and Theorem 2.26. The two special cases $p = 0, 1$ are trivial, since in the first all the probability is concentrated on the zero vector, and in the second on the unit vector; in each case the conclusion of part (b) is easy to check. Thus we may assume that $0 < p < 1$, in which case $\phi_{p,q}$ is strictly positive. We now check as required that $\phi_{p,q}$ satisfies the FKG lattice condition (2.17). Since the set $\eta(\omega)$ of open edges in a configuration ω satisfies

$$(2.34) \quad |\eta(\omega_1 \vee \omega_2)| + |\eta(\omega_1 \wedge \omega_2)| = |\eta(\omega_1)| + |\eta(\omega_2)| \qquad \text{for } \omega_1, \omega_2 \in \Omega,$$

old2.20 it suffices, on taking logarithms, to prove that

$$(2.35) \qquad k(\omega_1 \vee \omega_2) + k(\omega_1 \wedge \omega_2) \geq k(\omega_1) + k(\omega_2) \qquad \text{for } \omega_1, \omega_2 \in \Omega.$$

old2.21 By Theorem 2.21, we may restrict our attention to pairs ω_1, ω_2 which differ on exactly two edges. Suppose then that $\omega \in \Omega$ and $e, f \in E$ are such that

$$\begin{gathered} \omega_1(e) = 1,\ \omega_2(e) = 0, \\ \omega_1(f) = 0,\ \omega_2(f) = 1, \\ \omega_1(g) = \omega_2(g) = \omega(g) \quad \text{for } g \in E \setminus \{e, f\}. \end{gathered}$$

[The other cases are trivial.] As in the proof of Theorem 2.21, we omit reference to the states of edges other than e and f, and we write $\omega_1 = 10$ and $\omega_2 = 01$. Let L_f be the indicator function of the event that the two endpoints of f lie in different open clusters. Since L_f is a decreasing function, we have that $L_f(10) \leq L_f(00)$. Therefore,

$$k(10) - k(11) \leq k(00) - k(01),$$

which is to say that

$$k(\omega_1) - k(\omega_1 \vee \omega_2) \leq k(\omega_1 \wedge \omega_2) - k(\omega_2)$$

as required for (2.35). □

2.4 Differential formulae

One way of estimating the probability of an event A is via an estimate of its derivative $d\phi_{p,q}(A)/dp$. When $q = 1$, there is a formula for this derivative which has proved very useful, and which is commonly attributed to Russo, see [**17**, **92**, **171**]. This formula may be generalised to random-cluster measures as follows. The proof is an exercise in the differentiation of summations.

old2.49 **Theorem 2.36 [25].** *Let $0 < p < 1$, $q > 0$, and let $\phi_{p,q}$ be the corresponding random-cluster measure on a finite graph $G = (V, E)$. We have that*

$$\frac{d}{dp}\phi_{p,q}(X) = \frac{1}{p(1-p)}\mathrm{cov}_{p,q}(X, |\eta|), \tag{2.37}$$

$$\frac{d}{dq}\phi_{p,q}(X) = \frac{1}{q}\mathrm{cov}_{p,q}(X, k), \tag{2.38}$$

old2.51 *for any random variable $X : \Omega \to \mathbb{R}$, where $|\eta| = |\eta(\omega)| = \sum_{e\in E}\omega(e)$ is the number of open edges of the configuration ω, and* $\mathrm{cov}_{p,q}$ *denotes covariance with respect to $\phi_{p,q}$.*

In most applications of this, we set $X = 1_A$, the indicator function of some given event A, and we obtain that

$$\frac{d}{dp}\phi_{p,q}(A) = \frac{\phi_{p,q}(1_A|\eta|) - \phi_{p,q}(A)\phi_{p,q}(|\eta|)}{p(1-p)}. \tag{2.39}$$

old2.52

We choose to prove this in the wider context of a random-cluster measure having a separate parameter $\widetilde{p}_e$ for each edge e. The extra generality thereby obtained will find an application later. Let $\widetilde{\mathbf{p}} = (\widetilde{p}_e : e \in E)$ be a vector of reals each of which satisfies $0 < \widetilde{p}_e < 1$, and define the probability measure $\phi_{\widetilde{\mathbf{p}},q}$ on $(\Omega, \mathcal{F})$ by

$$\phi_{\widetilde{\mathbf{p}},q}(\omega) = \frac{1}{Z}\left\{\prod_{e\in E}\widetilde{p}_e^{\,\omega(e)}(1-\widetilde{p}_e)^{1-\omega(e)}\right\}q^{k(\omega)}, \qquad \omega \in \Omega, \tag{2.40}$$

old2.53 where Z is the appropriate normalizing factor. The measure $\phi_{p,q}$ is retrieved by setting $\widetilde{p}_e = p$ for all $e \in E$. Thus, in (2.40), each edge e has an associated independent parameter $\widetilde{p}_e$.

We write J_e for the indicator function of the event $\{\omega(e) = 1\}$, that is, $J_e(\omega) = \omega(e)$ for $\omega \in \Omega$.

old2.54 **Proposition 2.41.** *Let $0 < q < \infty$, and let $X : \Omega \to \mathbb{R}$. Then*

$$\frac{\partial}{\partial\widetilde{p}_e}\phi_{\widetilde{\mathbf{p}},q}(X) = \frac{1}{\widetilde{p}_e(1-\widetilde{p}_e)}\mathrm{cov}(X, J_e) \qquad \textit{for all } e \in E. \tag{2.42}$$

old2.55

Here, 'cov' denotes covariance relative to the probability measure $\phi_{\widetilde{\mathbf{p}},q}$. In the case $q = 1$, equation (2.42) is a version of Russo's formula for percolation, but in a form first discovered by Barlow and Proschan [**17**, p. 10]; see [**92**, p. 38]. It is notable that (2.42) is valid for all values of q. Indeed, (2.42) is valid for any measure of the form of (2.40) with the term $q^{k(\omega)}$ replaced by a general function of ω which does not depend on $\widetilde{p}_e$.

Proof of Proposition 2.41. We may write

$$\phi_{\widetilde{\mathbf{p}},q}(X) = \frac{1}{Z}\sum_{\omega\in\Omega} X(\omega)\widetilde{p}_e^{\,\omega(e)}(1-\widetilde{p}_e)^{1-\omega(e)}\alpha(\omega), \tag{2.43}$$

old2.56 where

$$\alpha(\omega) = \left\{\prod_{f\in E\setminus\{e\}} \widetilde{p}_f^{\,\omega(f)}(1-\widetilde{p}_f)^{1-\omega(f)}\right\} q^{k(\omega)}.$$

Note that α is not a function of $\widetilde{p}_e$. We differentiate (2.43) to obtain

(2.44)

$$\begin{aligned}\frac{\partial}{\partial \widetilde{p}_e}\phi_{\widetilde{\mathbf{p}},q}(X) &= \frac{1}{Z}\sum_{\omega} X(\omega)\widetilde{p}_e^{\,\omega(e)}(1-\widetilde{p}_e)^{1-\omega(e)}\alpha(\omega)\left(\frac{\omega(e)}{\widetilde{p}_e} - \frac{1-\omega(e)}{1-\widetilde{p}_e}\right)\\ &\qquad - \frac{1}{Z^2}\frac{\partial Z}{\partial \widetilde{p}_e}\sum_{\omega} X(\omega)\widetilde{p}_e^{\,\omega(e)}(1-\widetilde{p}_e)^{1-\omega(e)}\alpha(\omega)\\ &= \phi_{\widetilde{\mathbf{p}},q}\left(X\left(\frac{J_e-\widetilde{p}_e}{\widetilde{p}_e(1-\widetilde{p}_e)}\right)\right) - \frac{1}{Z}\frac{dZ}{d\widetilde{p}_e}\phi_{\widetilde{\mathbf{p}},q}(X).\end{aligned}$$

old2.57 Set $X = 1$ to obtain that

$$0 = \phi_{\widetilde{\mathbf{p}},q}\left(\frac{J_e-\widetilde{p}_e}{\widetilde{p}_e(1-\widetilde{p}_e)}\right) - \frac{1}{Z}\frac{\partial Z}{\partial \widetilde{p}_e}. \tag{2.45}$$

old2.58 We eliminate $Z^{-1}\partial Z/\partial\widetilde{p}_e$ from (2.44) and (2.45), and we arrive at (2.42). □

Proof of Theorem 2.36. By Proposition 2.41,

$$\begin{aligned}\sum_{e\in E}\frac{\partial}{\partial\widetilde{p}_e}\phi_{\widetilde{\mathbf{p}},q}(X)\bigg|_{\widetilde{\mathbf{p}}=p} &= \frac{1}{p(1-p)}\sum_{e\in E}\mathrm{cov}(X, J_e)\\ &= \frac{1}{p(1-p)}\mathrm{cov}(X, |\eta|),\end{aligned}$$

and the first result follows. The second part is similar, but rather shorter, since we may choose to calculate the derivative directly in preference to than going via Proposition 2.41. □

2.5 Comparison inequalities

The Holley inequality leads easily to the following comparison inequalities, which were first proved by Fortuin. We assume as before that $\phi_{p,q}$ is the random-cluster measure on the *finite* graph $G = (V, E)$.

old2.24 **Theorem 2.46 (Comparison inequalities) [73].** *It is the case that*

(2.47)
$$\phi_{p_1,q_1} \le_{\mathrm{st}} \phi_{p_2,q_2} \quad \textit{if}:\ q_1 \ge q_2,\ q_1 \ge 1,\ \textit{and}\ p_1 \le p_2,$$

(2.48)
$$\phi_{p_1,q_1} \ge_{\mathrm{st}} \phi_{p_2,q_2} \quad \textit{if}:\ q_1 \ge q_2,\ q_1 \ge 1,\ \textit{and}\ \frac{p_1}{q_1(1-p_1)} \ge \frac{p_2}{q_2(1-p_2)}.$$

old2.26

The first of these inequalities may be strengthened as in the next theorem. Here is some notation. A subset W of the vertex set V is called *spanning* if every edge of E is incident to at least one vertex of W. The degree $\deg(W)$ of a spanning set W is defined to be the maximum degree of its members, that is, the maximum number of edges of G incident to any one vertex in W.

old2.27 **Theorem 2.49 [89].** *For each $\delta > 0$, there exists a continuous function $\gamma(p,q) = \gamma(p,q,\delta)$, which is strictly increasing in p on $(0,1)$, and strictly decreasing in q on $[1,\infty)$, such that the following holds. Let G be a finite graph, and suppose there exists a spanning set W such that* $\deg(W) \le \delta$. *Then*

(2.50)
$$\phi_{p_1,q_1} \le_{\mathrm{st}} \phi_{p_2,q_2} \quad \textit{if} \quad 1 \le q_2 \le q_1 \ \textit{and}\ \gamma(p_1,q_1) \ge \gamma(p_2,q_2).$$

old2.28

check the last in- equality

A major application of this is to be found in Section 4.1, where it is proved that the critical point $p_c(q)$ of an infinite-volume random-cluster model is *strictly* decreasing in q.

Proof of Theorem 2.46. We may assume that $0 < p_1, p_2 < 1$, since the other cases are, as in the proof of Theorem 2.29, straightforward. We may either apply the Holley inequality (Theorem 2.7 or use the positive association of random-cluster measures (Theorem 2.29), as follows. Let $X : \Omega \to \mathbb{R}$ be increasing. Then

$$\begin{aligned}
\phi_{p_2,q_2}(X) &= \frac{1}{Z(p_2,q_2)} \sum_{\omega\in\Omega} X(\omega) p_2^{|\eta(\omega)|}(1-p_2)^{|E\setminus\eta(\omega)|} q_2^{k(\omega)} \\
&= \left(\frac{1-p_2}{1-p_1}\right)^{|E|} \frac{1}{Z(p_2,q_2)} \sum_{\omega\in\Omega} X(\omega)Y(\omega) {p_1}^{|\eta(\omega)|}(1-p_1)^{|E\setminus\eta(\omega)|} {q_1}^{k(\omega)} \\
&= \left(\frac{1-p_2}{1-p_1}\right)^{|E|} \frac{Z(p_1,q_1)}{Z(p_2,q_2)} \phi_{p_1,q_1}(XY)
\end{aligned}$$

∎

where
$$Y(\omega) = \left(\frac{q_2}{q_1}\right)^{k(\omega)} \left(\frac{p_2/(1-p_2)}{p_1/(1-p_1)}\right)^{|\eta(\omega)|}.$$
Setting $X = 1$, we obtain
$$\phi_{p_2,q_2}(1) = 1 = \left(\frac{1-p_2}{1-p_1}\right)^{|E|} \frac{Z(p_1,q_1)}{Z(p_2,q_2)} \phi_{p_2,q_2}(Y),$$
whence, on dividing,
$$\phi_{p_2,q_2}(X) = \frac{\phi_{p_1,q_1}(XY)}{\phi_{p_1,q_1}(Y)}. \tag{2.51}$$

old2.29

Assume now that conditions of (2.47) hold. Since $q_2 \le q_1$ and $p_2/(1-p_2) \ge p_1/(1-p_1)$, and since $k(\omega)$ is a decreasing function and $|\eta(\omega)|$ is increasing, we have that Y is an increasing function on Ω. Since $q_1 \ge 1$, ϕ_{p_1,q_1} is positively associated, whence
$$\phi_{p_1,q_1}(XY) \ge \phi_{p_1,q_1}(X)\phi_{p_1,q_1}(Y), \tag{2.52}$$
old2.30 and (2.51) yields $\phi_{p_2,q_2}(X) \ge \phi_{p_1,q_1}(X)$. Claim (2.47) follows.

Assume now that the conditions of (2.48) hold. We write $Y(\omega)$ in the form
$$Y(\omega) = \left(\frac{q_2}{q_1}\right)^{k(\omega)+|\eta(\omega)|} \left(\frac{p_2/\{q_2(1-p_2)\}}{p_1/\{q_1(1-p_1)\}}\right)^{|\eta(\omega)|}.$$
Note that $k(\omega) + |\eta(\omega)|$ is an increasing function of ω, since the addition of an extra open edge to ω causes $|\eta(\omega)|$ to increase by 1 and $k(\omega)$ to decrease by at most 1. In addition, $|\eta(\omega)|$ is increasing. Since $q_2 \le q_1$ and $p_2/\{q_2(1-p_2)\} \le p_1/\{q_1(1-p_1)\}$ by assumption, we have that Y is a decreasing function. Using the positive association of ϕ_{p_1,q_1} as above, we find that
$$\phi_{p_1,q_1}(XY) \le \phi_{p_1,q_1}(X)\phi_{p_1,q_1}(Y),$$
and (2.51) now implies $\phi_{p_2,q_2}(X) \le \phi_{p_1,q_1}(X)$. Claim (2.48) follows. □

The proof of Theorem 2.49 begins with a subsidiary result. This contains two inequalities, only the first of which will be used in that which follows.

old2.31 **Proposition 2.53 [89].** *For each $\delta > 0$, there exists a continuous function $\alpha(p,q) = \alpha(p,q,\delta)$, which is strictly positive when $0 < p < 1$, $q \ge 1$, such that the following holds. Let G be a finite graph, and suppose there exists a spanning set W such that* $\deg(W) \le \delta$. *Then*
$$\alpha(p,q)\frac{\partial}{\partial p}\phi_{p,q}(A) \le -q\frac{\partial}{\partial q}\phi_{p,q}(A) \le p(1-p)\frac{\partial}{\partial p}\phi_{p,q}(A) \tag{2.54}$$

old2.32 *for all increasing events A.*

Proof. Let A be an increasing event, and write $\theta(p,q) = \phi_{p,q}(A)$. We prove (2.32) following the scheme laid down in [**8**] and developed in [**25**, **86**]. The method proceeds as in the proof of Theorem 2.7 by constructing a Markov chain $Z_t = (X_t, Y_t)$ taking values in the state space Ω^2.

Let $\omega \in \Omega$ and $e \in E$, and let ω_e, ω^e be the configurations given before (2.12). Let $D_e(\omega)$ be the indicator function of the event that the endpoints of e are in different components of the graph $(V, \eta(\omega_e))$, that is, the event that the endpoints of e are connected by no open path that does not use e itself. We define the functions $H, H^A : \Omega^2 \to \mathbb{R}$ as follows. First,

$$H(\omega_e, \omega^e) = 1, \tag{2.55}$$

$$H(\omega^e, \omega_e) = \frac{1-p}{p} q^{D_e(\omega)}, \tag{2.56}$$

old2.34 for $\omega \in \Omega$ and $e \in E$. Secondly, $H(\omega, \omega')$ is set to 0 for other pairs ω, ω' with $\omega \neq \omega'$. Next, we define H^A by

$$H^A(\omega, \omega') = H(\omega, \omega') 1_A(\omega \wedge \omega') \quad \text{if } \omega \neq \omega', \tag{2.57}$$

old2.35 where 1_A is the indicator function of A, and $\omega \wedge \omega'(g) = \min\{\omega(g), \omega'(g)\}$ for $g \in E$ as before. The diagonal terms $H(\omega, \omega)$ and $H^A(\omega, \omega)$ are chosen in such a way that

$$\sum_{\omega' \in \Omega} H(\omega, \omega') = \sum_{\omega' \in \Omega} H^A(\omega, \omega') = 0 \quad \text{for all } \omega \in \Omega.$$

We are now ready to define the Markov chain Z_t. Let $S = \{(\pi, \omega) \in \Omega^2 : \pi \leq \omega\}$, the set of all ordered pairs of configurations, and let $J : S \times S \to \mathbb{R}$ be given by

$$J(\pi_e, \omega; \pi^e, \omega^e) = 1, \tag{2.58}$$

$$J(\pi, \omega^e; \pi_e, \omega_e) = H^A(\omega^e, \omega_e), \tag{2.59}$$

$$J(\pi^e, \omega^e; \pi_e, \omega^e) = H(\pi^e, \pi_e) - H^A(\omega^e, \omega_e), \tag{2.60}$$

old2.41 for all $e \in E$. All other off-diagonal values of J are set to 0, and the diagonal elements are chosen so that

$$\sum_{(\pi', \omega')} J(\pi, \omega; \pi', \omega') = 0 \quad \text{for all } \pi, \omega.$$

The function J will be used as the generator of a Markov chain on the state space S $(\subseteq \Omega^2)$. With J viewed in this way, equation (2.58) specifies that, for $\pi \in \Omega$

and $e \in E$, the edge e is acquired by π (if it does not already contain it) at rate 1; any edge so acquired is added also to ω if it does not already contain it. Equation (2.59) specifies that, for $\omega \in \Omega$ and $e \in \eta(\omega)$, the edge e is removed from ω (and also from π if $e \in \eta(\pi)$) at the rate given by (2.57). For $e \in \eta(\pi)$ ($\subseteq \eta(\omega)$), there is an additional rate at which e is removed from π but not from ω. Note that this additional rate is indeed non-negative, since

$$H(\pi^e, \pi_e) - H^A(\omega^e, \omega_e) = \frac{1-p}{p}\left\{q^{D_e(\pi)} - q^{D_e(\omega)} 1_A(\omega_e)\right\} \geq 0,$$

by (2.56) and (2.57); remember that $q \geq 1$ and $D_e(\omega) \leq D_e(\pi)$ when $\pi \leq \omega$. It is an important observation that the ensuing Markov chain has no possible transition which can depart the set S. That is, if the chain starts in S, then it remains in S for all time.

It is easily seen as in Section 2.2 and [**25**] that there exists a Markov chain $Z_t = (X_t, Y_t)$ on the state space S such that:

(i) Z_t has generator J, that is, for $(\pi, \omega) \neq (\pi', \omega')$,

$$\mathbb{P}\big(Z_{t+h} = (\pi', \omega') \,\big|\, Z_t = (\pi, \omega)\big) = J(\pi, \omega; \pi', \omega')h + \mathrm{o}(h) \qquad \text{as } h \downarrow 0.$$

(ii) X_t converges weakly as $t \to \infty$ to $\phi_{p,q}(\cdot)$;

(iii) Y_t converges weakly as $t \to \infty$ to $\phi_{p,q}(\cdot \mid A)$;

(iv) $X_t \leq Y_t$ for all t.

See [**98**, Chapter 6] for an account of the theory of Markov chains.

Evidently,

$$\phi_{p,q}(A) = \frac{1}{Z(p,q)} \sum_{\omega \in \Omega} 1_A(\omega) \left\{ \prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega)}. \tag{2.61}$$

old2.42 Differentiating with respect to p, one obtains as in Theorem 2.36 that

$$\begin{aligned}
\frac{\partial \theta}{\partial p} &= \frac{1}{p(1-p)} \mathrm{cov}_{p,q}(1_A, |\eta|) \\
&= \frac{1}{p(1-p)} \big\{ \phi_{p,q}(|\eta| 1_A) - \phi_{p,q}(|\eta|)\phi_{p,q}(A) \big\} \\
&= \frac{\theta(p,q)}{p(1-p)} \left\{ \lim_{t \to \infty} \mathbb{P}\big(|\eta(Y_t)| - |\eta(X_t)|\big) \right\} \\
&= \frac{\theta(p,q)}{p(1-p)} \sum_{e \in E} \lim_{t \to \infty} \mathbb{P}\big(X_t(e) = 0,\ Y_t(e) = 1\big),
\end{aligned} \tag{2.62}$$

old2.43 where $|\eta| = |\eta(\omega)| = |\{e : \omega(e) = 1\}|$, '$\mathrm{cov}_{p,q}$' denotes covariance with respect to $\phi_{p,q}$, and $\mathbb{P}$ is the appropriate probability measure for the chain Z_t. A similar calculation using the second part of Theorem 2.36 yields that

$$\frac{\partial \theta}{\partial q} = \frac{1}{q} \mathrm{cov}_{p,q}(1_A, k) = -\frac{1}{q}\theta(p,q) \left\{ \lim_{t \to \infty} \mathbb{P}\big(k(X_t) - k(Y_t)\big) \right\}, \tag{2.63}$$

old2.44 where $k = k(\omega)$ is the number of components of the graph $(V, \eta(\omega))$ as usual.

By an elementary graph-theoretic argument,

$$k(X_t) - k(Y_t) \le |\eta(Y_t)| - |\eta(X_t)|,$$

whence, by (2.62)–(2.63),

$$-q\frac{\partial\theta}{\partial q} \le p(1-p)\frac{\partial\theta}{\partial p},$$

which is the right-hand inequality of (2.54).

Let δ be a positive integer, and suppose that W is a spanning set of vertices satisfying $\deg(W) \le \delta$. For $x \in V$, let I_x be the indicator function of the event that x is an isolated vertex. Clearly,

$$\mathbb{P}\big(k(X_t) - k(Y_t)\big) \ge \sum_{x\in W} \mathbb{P}\big(I_x(X_t) = 1,\ I_x(Y_t) = 0\big), \tag{2.64}$$

old2.45 since the right-hand side counts the number of vertices of W which are isolated in X_t, but which are not isolated in Y_t. Let $x \in W$, and let e_x be any edge of E which is incident to x. We claim that

$$\nu\mathbb{P}\big(X_t(e_x) = 0,\ Y_t(e_x) = 1\big) \le \mathbb{P}\big(I_x(X_{t+1}) = 1,\ I_x(Y_{t+1}) = 0\big) \tag{2.65}$$

old2.46 for some $\nu = \nu(p, q)$ which is continuous, and is strictly positive on $(0, 1) \times [1, \infty)$; here, ν is allowed to depend on the value of δ but not further upon x, e_x, W, G, or the choice of A. Once (2.65) is proved, the left-hand inequality of (2.54) follows with $\alpha = \nu p(1-p)/\delta$ by summing (2.65) over x and using (2.61)–(2.64) as follows:

$$\begin{aligned}
-q\frac{\partial\theta}{\partial q} &\ge \theta\left\{\lim_{t\to\infty}\sum_{x\in W}\mathbb{P}\big(I_x(X_{t+1}) = 1,\ I_x(Y_{t+1}) = 0\big)\right\}\\
&\ge \theta\nu\left\{\lim_{t\to\infty}\sum_{x\in W}\frac{1}{\delta}\sum_{e\sim x}\mathbb{P}\big(X_t(e) = 0,\ Y_t(e) = 1\big)\right\}\\
&\ge \frac{\theta\nu}{\delta}\left\{\lim_{t\to\infty}\sum_{e\in E}\mathbb{P}\big(X_t(e) = 0,\ Y_t(e) = 1\big)\right\}\\
&= \frac{\nu p(1-p)}{\delta}\frac{\partial\theta}{\partial p},
\end{aligned}$$

where $\sum_{e\sim x}$ sums over all edges e incident to the vertex x.

Finally we prove (2.65). Let E_x be the set of edges of E which are incident to x. Suppose that the event $F_t = \{X_t(e_x) = 0,\ Y_t(e_x) = 1\}$ occurs. Let:

(a) T be the event that, during the time-interval $(t, t+1)$, every edge e of $E_x \setminus \{e_x\}$ with $X_t(e) = 1$ changes its X-state from 1 to 0; the removal of such edges from X may involve also their removal from Y;

(b) U be the event that no edge e of $E_x \setminus \{e_x\}$ with $X_t(e) = 0$ changes its state in the time-interval $(t, t+1)$;

(c) V be the event that the state of the edge e_x remains unchanged during $(t, t+1)$.

By elementary computations using the generator of the chain, there exists a strictly positive and continuous function $\nu_W = \nu_W(p, q)$ on $(0, 1) \times [1, \infty)$, which is allowed to depend on G and W only through the quantity $\deg(W)$, such that

$$\mathbb{P}(T \cap U \cap V \mid F_t) \ge \nu_W \qquad \text{for all } t,$$

uniformly in x, e_x, G, and t. This inequality remains true if we replace ν_W by the positive continuous function ν defined by

$$\nu(p, q, \delta) = \min\bigl\{\nu_W(p, q) : W \text{ a spanning set such that } 0 \le \deg(W) \le \delta\bigr\}.$$

Now, if $F_t \cap T \cap U \cap V$ occurs, then x is isolated in X_{t+1}, but not isolated in Y_{t+1} (since $Y_{t+1}(e_x) = 1$). Therefore (2.65) is valid, and the proof of the proposition is complete. A function ν of the required form may be written down explicitly. $\square$

Proof of Theorem 2.49. Let α be as in Proposition 2.31, and let A be an increasing event. Inequality (2.32) may be stated in the form

$$(\alpha, q).\nabla\phi_{p,q}(A) \le 0 \le (p(1-p), q).\nabla\phi_{p,q}(A), \tag{2.66}$$

old2.47 where

$$\nabla f = \left(\frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}\right) \qquad \text{for functions } f : [0, 1] \times (0, \infty) \to \mathbb{R}.$$

We have in addition by Theorem 2.46 that

$$\frac{\partial \phi_{p,q}(A)}{\partial q} \le 0 \le \frac{\partial \phi_{p,q}(A)}{\partial p}. \tag{2.67}$$

old2.48 The right-hand inequality of (2.66) may be used to obtain (2.48), but we concentrate here on the left-hand inequality. Let $\gamma : (0, 1) \times [1, \infty)$ be a solution of the differential equation $(\alpha, q).\nabla\gamma = 0$ subject to

$$\frac{\partial \gamma}{\partial q} < 0 < \frac{\partial \gamma}{\partial p} \qquad \text{for } 0 < p < 1,\ q > 1.$$

draw figure See Figure 2.1 for a sketch of the contours of γ, that is, the curves on which γ is constant. The contour of γ passing through the point (p, q) has tangent (α, q). The directional derivative of $\phi_{p,q}(A)$ in this direction satisfies

$$\alpha\frac{\partial \phi_{p,q}(A)}{\partial p} + q\frac{\partial \phi_{p,q}(A)}{\partial q} < 0,$$

whence $\phi_{p,q}(A)$ is decreasing as (p, q) moves along the contour of γ in the direction of increasing q. Thus

$$\phi_{p_1,q_1}(A) \le \phi_{p_2,q_2}(A) \qquad \text{if } \gamma(p_1, q_1) = \gamma(p_2, q_2) \text{ and } 1 \le q_2 \le q_1,$$

and (2.50) follows by a glance at Figure 2.1.

2.6 Exponential steepness

Let $G = (V, E)$ be a finite graph and let $\phi_{p,q}$ be the random-cluster measure on $\Omega = \{0, 1\}^E$, where q is assumed to satisfy $q \ge 1$. We derive in this section a lower bound for the derivative with respect to p of the probability $\phi_{p,q}(A)$ of an increasing event A.

Let A be an event, and let $\omega \in \Omega$. We define $H_A(\omega)$ to be the Hamming distance from ω to A, that is,

$$H_A(\omega) = \inf\left\{\sum_e |\omega'(e) - \omega(e)| : \omega' \in A\right\}. \tag{2.68}$$

old2.59 Note that

(2.69)

$$H_A(\omega) = \begin{cases} \inf\left\{\displaystyle\sum_e \{\omega'(e) - \omega(e)\} : \omega' \ge \omega,\ \omega' \in A\right\} & \text{if } A \text{ is increasing,} \\ \inf\left\{\displaystyle\sum_e \{\omega(e) - \omega'(e)\} : \omega' \le \omega,\ \omega' \in A\right\} & \text{if } A \text{ is decreasing.} \end{cases}$$

old2.60

Suppose now that A is increasing. We note two useful facts concerning H_A.

(i) The function $|\eta| + H_A$ is an increasing random variable, since the addition of a single open edge to a configuration ω causes $|\eta(\omega)|$ to increase by 1, and $H_A(\omega)$ to decrease by at most 1.

(ii) We have that $H_A(\omega)1_A(\omega) = 0$ for all ω, since $H_A = 0$ whenever A occurs.

old2.61 **Theorem 2.70 [91, 97].** *Let $q \ge 1$ and $0 < p < 1$. For any increasing event A,*

$$\frac{d}{dp}\{\log \phi_{p,q}(A)\} \ge \frac{\phi_{p,q}(H_A)}{p(1-p)}. \tag{2.71}$$

old2.62

Inequality (2.71) bears a resemblance to a formula valid for percolation which may be written in the present context as

$$\frac{d}{dp}\{\log \phi_{p,1}(A)\} \ge \frac{\phi_{p,1}(N_A \mid A)}{p},$$

where N_A is the number of edges which are pivotal for the increasing event A. See [**92**, p. 44] for further details.

Proof. Applying (i) and (ii) above, we find that

$$\phi_{p,q}(|\eta| 1_A) = \phi_{p,q}\big((|\eta| + H_A)1_A\big) \geq \phi_{p,q}(|\eta| + H_A)\phi_{p,q}(A),$$

by positive association. Hence

$$\phi_{p,q}(|\eta| 1_A) - \phi_{p,q}(|\eta|)\phi_{p,q}(A) \geq \phi_{p,q}(H_A)\phi_{p,q}(A),$$

which we substitute into (2.39) to obtain that

$$\frac{d}{dp}\phi_{p,q}(A) \geq \frac{\phi_{p,q}(A)\phi_{p,q}(H_A)}{p(1-p)},$$

and (2.71) follows. □

Inequality (2.71) is usually used in integrated form. Integrating over the interval $[r, s]$, and using the facts that $p(1-p) \leq \frac{1}{4}$ and that H_A is a decreasing random variable, we find that

(2.72)
$$\begin{aligned}\phi_{r,q}(A) &\leq \phi_{s,q}(A) \exp\left\{-4\int_r^s \phi_{p,q}(H_A)\, dp\right\}\\ &\leq \phi_{s,q}(A)\exp\big\{-4(s-r)\phi_{s,q}(H_A)\big\}, \quad \text{if } r \leq s.\end{aligned}$$

old2.63

A further relation between the probability of A and the mean of H_A facilitates application of (2.72).

old2.64 **Theorem 2.73 [91, 97].** *Let $q \geq 1$ and $0 < r < s < 1$. Then, for any increasing event A,*
(2.74)
$$\phi_{r,q}(H_A \leq k) \leq \left(\frac{q}{s-r}\cdot\frac{(1-r)q}{r+(1-r)q}\right)^k \phi_{s,q}(A) \quad \textit{for } k = 0, 1, 2, \ldots.$$

old2.65

We shall make use of the theorem in the following way. By (2.74) with $C = q^2(1-r)/\{(s-r)(r+(1-r)q)\}$,

$$\phi_{r,q}(H_A) = \sum_{k=0}^{\infty}\phi_{r,q}(H_A > k) \geq \sum_{k=0}^{K}\{1 - C^k\phi_{s,q}(A)\}$$

where $K = \max\{k : C^k \phi_{s,q}(A) \le 1\}$. We sum this as usual, noting that $C > 1$, to find that

$$\phi_{r,q}(H_A) \ge \frac{-\log \phi_{s,q}(A)}{\log C} - \frac{C - \phi_{s,q}(A)}{C-1} \quad \text{if } r < s. \tag{2.75}$$

old2.66 This, in combination with (2.71), provides a mechanism for bounding below the gradient of $\log \phi_{p,q}(A)$, viewed as a function of p.

Here is a sketch of a topic of application of these ideas. Let S and T be disjoint sets of vertices of G, and let $A = \{S \leftrightarrow T\}$ be the event that there exists an open path joining some $s \in S$ to some $t \in T$. Then H_A is the minimum number of closed edges amongst the family of all paths in G from S to T, which is to say that

$$H_A(\omega) = \min\left\{\sum_{e\in\pi}\{1-\omega(e)\} : \pi \in \Pi\right\},$$

where Π is the set of all paths of G having one endvertex in S and the other in T.

Before proceeding to the proofs, we note that Theorem 2.73 is closely related to the 'sprinkling lemma' of [**4**], a version of which is valid for random-cluster models; see also [**92**]. The argument used to prove Theorem 2.73 may also be used to prove the following, which is more in the style of 'sprinkling'.

old2.67 **Theorem 2.76.** *Let $q \ge 1$ and $0 < r < s < 1$. There exists a function $c(r, s, q)$, uniformly bounded when q is bounded away from ∞, such that for any decreasing event A,*

$$\phi_{r,q}(A) \ge \left(\frac{c(r,s,q)}{s-r}\right)^k \phi_{s,q}(H_A \le k) \quad \textit{for } k = 0, 1, 2, \dots. \tag{2.77}$$

old2.68

Check.
Expand c?

Proof of Theorem 2.73. Let $r < s$. We shall employ a suitable coupling of the measures $\phi_{r,q}$ and $\phi_{s,q}$. Let $E = \{e_1, e_2, \dots, e_m\}$ be the edges of the graph G, and let $U_1, U_2, \dots, U_m$ be independent random variables having the uniform distribution on $[0, 1]$. We shall examine the edges in turn, to determine whether they are open or closed for the respective parameters r and s. The outcome will be a pair (π, ω) of configurations each lying in $\Omega = \{0, 1\}^E$ such that $\pi \le \omega$. The configurations π, ω are random in the sense that they are functions of the U_j.

First, we declare

$$\begin{aligned} \pi(e_1) = 1 &\quad \text{if and only if} \quad U_1 < \phi_{r,q}(J_1), \\ \omega(e_1) = 1 &\quad \text{if and only if} \quad U_1 < \phi_{s,q}(J_1), \end{aligned}$$

where J_i is the set of configurations γ ($\in \Omega$) with $\gamma(e_i) = 1$. We have by the comparison inequality (2.17) that $\phi_{r,q}(J_1) \le \phi_{s,q}(J_1)$ since $r < s$, and therefore $\pi(e_1) \le \omega(e_1)$.

Let M be an integer satisfying $1 \le M < m$. Having defined $\pi(e_i)$, $\omega(e_i)$ such that $\pi(e_i) \le \omega(e_i)$ (for $i \le M$), we define $\pi(e_{M+1})$ and $\omega(e_{M+1})$ as follows. We declare

$$\begin{aligned} \pi(e_{M+1}) = 1 \quad &\text{if and only if} \quad U_{M+1} < \phi_{r,q}\big(J_{M+1} \,\big|\, F_M(\pi)\big), \\ \omega(e_{M+1}) = 1 \quad &\text{if and only if} \quad U_{M+1} < \phi_{s,q}\big(J_{M+1} \,\big|\, F_M(\omega)\big), \end{aligned}$$

where $F_M(\gamma)$ is the set of configurations ν satisfying $\nu(e_i) = \gamma(e_i)$ for $1 \le i \le M$. We have by positive association and Theorem (2.48) that

$$\phi_{r,q}(J_{M+1} \mid F_M(\pi)) \le \phi_{s,q}(J_{M+1} \mid F_M(\omega))$$

since $r < s$ and $\pi(e_i) \le \omega(e_i)$ for $1 \le i \le M$; this implies that $\pi(e_{M+1}) \le \omega(e_{M+1})$.

Continuing likewise, we obtain a pair (π, ω) of configurations satisfying:

(a) $\pi \le \omega$,

(b) π is distributed according to the measure $\phi_{r,q}$,

(c) ω is distributed according to the measure $\phi_{s,q}$.

We write $\mathbb{P}$ for the probability measure associated with the U_j.

By Theorem 2.1,

$$\begin{aligned} \phi_{p,q}(J_i \mid \overline{K_i}) &= \frac{p}{p + (1-p)q}, \\ \phi_{p,q}(J_i \mid K_i) &= p, \end{aligned}$$

where K_i is the event that there exists an open path of $E \setminus \{e_i\}$ joining the endpoints of e_i. Using conditional expectations, we deduce that, since $q \ge 1$,

$$\frac{p}{p + (1-p)q} \le \phi_{p,q}(J_i \mid D) \le p \tag{2.78}$$

old2.69 for any event D defined on the states of $E \setminus \{e_i\}$. It follows from the definition of the $\pi(e_i)$ and $\omega(e_i)$ that

$$\begin{aligned} \mathbb{P}\big(\pi(e_{M+1}) = 0 \,\big|\, U_1, U_2, \dots, U_M\big) &= 1 - \phi_{r,q}\big(J_{M+1} \,\big|\, F_m(\pi)\big) \\ &\le \frac{(1-r)q}{r + (1-r)q}. \end{aligned}$$

By a similar argument,

$$\begin{aligned} &\mathbb{P}\Big(\omega(e_{M+1}) = 1, \pi(e_{M+1}) = 0 \,\Big|\, U_1, U_2, \dots, U_M\Big) \\ &\qquad = \phi_{s,q}\big(J_{M+1} \,\big|\, F_M(\omega)\big) - \phi_{r,q}\big(J_{M+1} \,\big|\, F_M(\pi)\big) \ge \frac{s-r}{q}. \end{aligned}$$

A full derivation of the last inequality is obtainable as follows. Using Theorem 2.70 with $A = J_i$ (so that $H_{J_i} = 1 - 1_{J_i}$) together with (2.78),

$$\frac{d}{dp}\phi_{p,q}(J_i) \ge \frac{\phi_{p,q}(J_i)\big(1-\phi_{p,q}(J_i)\big)}{p(1-p)} \ge \frac{1}{p+(1-p)q} \ge \frac{1}{q}.$$

We integrate over the interval $[r, s]$ to obtain that

$$\phi_{s,q}(J_i) - \phi_{r,q}(J_i) \ge \frac{s-r}{q}. \tag{2.79}$$

old2.70 Finally,

$$\begin{aligned}\phi_{s,q}\big(J_{M+1} \,\big|\, F_M(\omega)\big) &- \phi_{r,q}\big(J_{M+1} \,\big|\, F_M(\pi)\big)\\ &\ge \phi_{s,q}\big(J_{M+1} \,\big|\, F_M(\omega)\big) - \phi_{r,q}\big(J_{M+1} \,\big|\, F_M(\omega)\big),\end{aligned}$$

and the claim follows by applying (2.79) with $i = M + 1$ to the graph obtained from G by contracting (respectively, deleting) any edge e_i (for $1 \le i \le M$) with $\omega(e_i) = 1$ (respectively, $\omega(e_i) = 0$). Cf. [**90**, Theorem 2.3].

It follows from the above that

$$\mathbb{P}\Big(\omega(e_{M+1}) = 1 \,\Big|\, \pi(e_{M+1}) = 0,\ U_1, U_2, \dots, U_M\Big) \ge \frac{s-r}{q} \cdot \frac{r+(1-r)q}{(1-r)q}. \tag{2.80}$$

old2.71

Now fix a configuration ξ ($\in \Omega$) and a set B of edges such that $\xi(e) = 0$ for $e \in B$. We claim that

$$\mathbb{P}\big(\pi = \xi,\ \omega(e) = 1 \text{ for } e \in B\big) \ge \left(\frac{s-r}{q} \cdot \frac{r+(1-r)q}{(1-r)q}\right)^{|B|} \mathbb{P}(\pi = \xi). \tag{2.81}$$

old2.72 This follows from the recursive construction of π and ω in terms of the family $U_1, U_2, \dots, U_m$, in the light of the bound (2.80).

Inequality (2.81) implies the claim of the theorem, as follows. Let ξ be a configuration satisfying $H_A(\xi) \le k$. There exists a set $B = B_\xi$ of edges such that

(a) $|B| \le k$,
(b) $\xi(e) = 0$ for $e \in B$,
(c) the configuration obtained from ξ by allocating state 1 to all edges in B lies in the event A.

If more than one such set B exists, we pick the earliest in some deterministic ordering of all subsets of E. Then, by (2.81),

$$\begin{aligned}\phi_{s,q}(A) &\ge \mathbb{P}\big(H_A(\pi) \le k,\ \omega(e) = 1 \text{ for } e \in B_\pi\big)\\ &= \sum_{\xi : H_A(\xi) \le k} \mathbb{P}\big(\pi = \xi,\ \omega(e) = 1 \text{ for } e \in B_\xi\big)\\ &\ge \left(\frac{s-r}{q} \cdot \frac{r+(1-r)q}{(1-r)q}\right)^k \phi_{r,q}(H_A \le k),\end{aligned}$$

and the proof is complete. $\square$

2.7 Partition functions

The partition function of the finite graph $G = (V, E)$ is given by

$$Z_G(p, q) = \sum_{\omega \in \Omega} p^{|\eta(\omega)|}(1-p)^{|E \setminus \eta(\omega)|} q^{k(\omega)}, \qquad 0 \le p \le 1,\ q > 0. \tag{2.82}$$

old2.73 In an approach borrowed from classical statistical mechanics, one studies the random-cluster model and its phase transition via the function Z_G and its derivatives. We shall not explore this route in this work, preferring as we do the probabilistic approach. We shall however have recourse to various arguments based on the behaviour of the partition function, and we therefore note some of its fundamental properties.

The (Whitney) *rank-generating function* of $G = (V, E)$ is the function

$$W_G(u, v) = \sum_{E' \subseteq E} u^{r(G')} v^{c(G')}, \qquad u, v \in \mathbb{R}, \tag{2.83}$$

old2.74 where $r(G') = |V| - k(G')$ is the *rank* of the graph $G' = (V, E')$, and $c(G') = |E'| - |V| + k(G')$ is its *co-rank*; here, $k(G')$ denotes the number of components of the graph G'. The rank-generating function has various useful properties, and occurs in several contexts in graph theory; see [**26**, **186**]. It crops up in other forms also. For example, the function

$$T_G(u, v) = (u-1)^{|V|-1} W_G\big((u-1)^{-1}, v-1\big) \tag{2.84}$$

old2.75 is known as the *dichromatic* (or *Tutte*) *polynomial*, [**186**]. The partition function Z_G of the graph G is easily seen to satisfy

$$Z_G = q^{|V|}(1-p)^{|E|} W_G\left(\frac{p}{q(1-p)}, \frac{p}{1-p}\right), \tag{2.85}$$

old2.76 a relationship which provides a link with other classical quantities associated with a graph. See [**26**, **27**, **72**, **188**] also.

Another way of viewing Z_G is as the moment generating function of the number of clusters in a random graph, that is,

$$Z_G(p, q) = \phi_{p,1}(q^{k(\omega)}). \tag{2.86}$$

old2.77 This indicates a link to percolation on G (that is, product measure on Ω), and to the large-deviation theory of the number of clusters in the percolation model. See [**38**, **177**].

We show next that Z_G does not change a great deal if an edge is removed from G. Let $F \subseteq E$, and write $G_{E \setminus F}$ for the graph G with the edges in F removed. If F is the singleton $\{e\}$, we write $G \setminus e$ for $G \setminus \{e\}$.

old2.78 **Theorem 2.87.** *Let* $0 \le p \le 1$ *and* $q > 0$*. For* $e \in E$*,*

$$1 \wedge q \le \frac{Z_{G\setminus e}(p,q)}{Z_G(p,q)} \le 1 \vee q, \tag{2.88}$$

old2.79 *and, more generally, if* $F \subseteq E$*,*

$$(1 \wedge q)^{|F|} \le \frac{Z_{G\setminus F}(p,q)}{Z_G(p,q)} \le (1 \vee q)^{|F|}. \tag{2.89}$$

old2.80

We give next an application of these inequalities which will find application later. Let $G_i = (V_i, E_i)$, $i = 1, 2$, be finite graphs on disjoint vertex sets V_1, V_2, and write $G_1 \cup G_2$ for the graph $(V_1 \cup V_2, E_1 \cup E_2)$. It is immediate from (2.82) that

$$Z_{G_1 \cup G_2} = Z_{G_1} Z_{G_2}, \tag{2.90}$$

old2.81 where for clarity we have removed explicit mention of p, q. Taken in conjunction with (2.89), this leads easily to a pair of inequalities which we state as a theorem.

old2.82 **Theorem 2.91.** *Let* $G = (V, E)$ *be a finite graph, and let* F *be a set of edges whose removal breaks G into two disjoint graphs* $G_1 = (V_1, E_1)$*,* $G_2 = (V_2, E_2)$*. Thus,* $V = V_1 \cup V_2$ *and* $E = E_1 \cup E_2 \cup F$*. For* $0 \le p \le 1$ *and* $q > 0$*,*

$$Z_{G_1} Z_{G_2} (1 \vee q)^{-|F|} \le Z_G \le Z_{G_1} Z_{G_2} (1 \wedge q)^{-|F|}.$$

Inequalities changed. Check, also consequences.

Proof of Theorem 2.87. Since (2.89) follows by repeated application of (2.88), it suffices to prove the latter. For $\omega \in \Omega$, we write $\omega_{\langle e \rangle}$ for the configuration in $\Omega_{\langle e \rangle} = \{0, 1\}^{E \setminus \{e\}}$ which agrees with ω off e. Clearly,

$$k(\omega) \le k(\omega_{\langle e \rangle}) \le k(\omega) + 1,$$

whence

$$(1 \wedge q) q^{k(\omega)} \le q^{k(\omega_{\langle e \rangle})} \le (1 \vee q) q^{k(\omega)}. \tag{2.92}$$

old2.83 Now,

$$\begin{aligned} Z_{G\setminus e}(p,q) &= \sum_{\omega_{\langle e \rangle} \in \Omega_{\langle e \rangle}} p^{|\eta(\omega_{\langle e \rangle})|} (1-p)^{|E \setminus \eta(\omega_{\langle e \rangle})| - 1} q^{k(\omega_{\langle e \rangle})} \\ &= \sum_{\omega \in \Omega} p^{|\eta(\omega)|} (1-p)^{|E \setminus \eta(\omega)|} q^{k(\omega_{\langle e \rangle})} \end{aligned} \tag{2.93}$$

old2.84 since $p + (1 - p) = 1$. Equations (2.92) and (2.93) may be combined to obtain (2.88). □

We develop next an inequality related to (2.89) which concerns the addition of a vertex, and which will be useful later. Let $G = (V, E)$ be a finite graph as usual, and let $v \notin V$ and $W \subseteq V$. We augment G by adding the vertex v together with edges $\langle v, w \rangle$ for $w \in W$. Let us write $G + v$ for the resulting graph.

2.onevertex **Theorem 2.94.** *Let* $0 \le p \le 1$ *and* $q \ge 1$. *We have in the above situation that*

$$\frac{Z_{G+v}(p,q)}{Z_G(p,q)} \ge q(1-p+pq^{-1})^{|W|}.$$

Proof. Let $\Omega_E = \{0,1\}^E$ and $\Omega_v = \{0,1\}^W$. We identify $\nu \in \Omega_v$ with the edge-configuration on the neighbourhood $\{\langle v, w\rangle : w \in W\}$ of v by $\nu(\langle v,w\rangle) = \nu(w)$. Now,

(2.95)
$$\begin{aligned} Z_{G+v}(p,q) &= \sum_{\omega\in\Omega_E,\, \nu\in\Omega_v} p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}q^{k(\omega)} \\ &\qquad\qquad \times \left[\left\{\prod_{w\in W} p^{\nu(w)}(1-p)^{1-\nu(w)}\right\} q^{1-k(\omega,\nu)}\right] \\ &= Z_G(p,q)\phi_{G,p,q}\left(q\phi_p(q^{-k(\omega,\nu)})\right), \end{aligned}$$

2.newineq where ϕ_p is product measure on Ω_v with density p, and $k(\omega,\nu)$ is the number of open clusters of ω containing members of W. Let $n_1, n_2, \dots, n_r$ be the sizes of the equivalence classes of W under the equivalence relation $w_1 \sim w_2$ if $w_1 \leftrightarrow w_2$ in ω. Then, for given $\omega \in \Omega_E$,

$$\begin{aligned} \phi_p(q^{-k(\omega,\nu)}) &= \prod_{i=1}^r \left[(1-p)^{n_i} + \frac{1}{q}\big(1-(1-p)^{n_i}\big)\right] \\ &\ge \prod_{i=1}^r \left[\frac{1}{q} + \left(1-\frac{1}{q}\right)(1-p)\right]^{n_i} \\ &= \left[\frac{1}{q} + \left(1-\frac{1}{q}\right)(1-p)\right]^{|W|}, \end{aligned} \tag{2.96}$$

2.newineq2 where we have used the elementary inequality

$$\alpha + (1-\alpha)y^n \ge [\alpha + (1-\alpha)y]^n \qquad \text{for } 0 \le \alpha, y \le 1,\ n \in \{1,2,\dots\}.$$

We substitute (2.96) into (2.95) to obtain the claim. □

So far in this section we have considered the effect on the partition function of removing edges from the graph. There is a related result in which, instead, we identify certain vertices. Let $G = (V,E)$ be a finite graph, and let C be a subset of V which separates vertex-sets A and B; that is, $A, B \subseteq V \setminus C$, and every path from some $a \in A$ to some $b \in B$ passes through at least one vertex in C. We write c for a composite vertex formed by identifying all vertices in C, and $\mathcal{A} = (A \cup \{c\}, \mathbb{E}_A)$ (respectively, $\mathcal{B} = (B \cup \{c\}, \mathbb{E}_B)$ for the graph on the vertex set $A \cup \{c\}$ (respectively, $B \cup \{c\}$) and with the edges derived from G. For example, if $a_1, a_2 \in A$, then $\langle a_1, a_2\rangle$ is an edge of $\mathcal{A}$ if and only if it is an edge of G; for $a \in A$, the number of edges of $\mathcal{A}$ between a and c is exactly the number of edges in G between a and members of C.

2.monowired **Lemma 2.97.** *For* $0 \le p \le 1$ *and* $q \ge 1$*,*

$$Z_G \ge \frac{1}{q} Z_{\mathcal{A}} Z_{\mathcal{B}}.$$

Since $Z_G \ge q$ for all G when $q \ge 1$, we deduce in the notation of this lemma that

$$Z_G \ge Z_{\mathcal{A}}. \tag{2.98}$$

2.monowired2

Proof. Let $\omega \in \{0,1\}^E$, and let ω_A and ω_B be the induced configurations in $\Omega_{\mathcal{A}} = \{0,1\}^{\mathbb{E}_{\mathcal{A}}}$ and $\Omega_B = \{0,1\}^{\mathbb{E}_{\mathcal{B}}}$, respectively. It is easily seen that

$$k(\omega) \ge k(\omega_A) + k(\omega_B) - 1,$$

and the claim follows from the definition (2.82) □

The partition function has a convexity property which will prove useful in studying infinite-volume limit measures. Rather than working with Z_G, we work instead with the function $Y_G : \mathbb{R}^2 \to \mathbb{R}$ given by

$$Y_G(\pi, \kappa) = \sum_{\omega \in \Omega} \exp\bigl\{\pi |\eta(\omega)| + \kappa k(\omega)\bigr\}, \tag{2.99}$$

old2.85 a function which is related to Z_G as follows. We set $\pi = \pi(p)$ and $\kappa = \kappa(q)$ where

$$\pi(p) = \log\{p/(1-p)\}, \quad \kappa(q) = \log q, \tag{2.100}$$

old2.86 and then

$$Z_G(p,q) = (1-p)^{|E|} Y_G(\pi(p), \kappa(q)).$$

We write ∇X for the gradient vector of a function $X : \mathbb{R}^2 \to \mathbb{R}$.

old2.87 **Theorem 2.101.** *Let* (π, κ) *be related to* (p,q) *by* (2.100).
(a) *The gradient vector of the function* $\log Y_G(\pi, \kappa)$ *is given by*

$$\nabla\{\log Y_G(\pi, \kappa)\} = \bigl(\phi_{p,q}(|\eta|), \phi_{p,q}(k)\bigr). \tag{2.102}$$

old2.88

(b) *Let* $\mathbf{i} = (i_1, i_2)$ *be a unit vector in* $\mathbb{R}^2$*. Then*

$$\frac{d^2}{d\alpha^2}\Bigl\{\log Y_G\bigl((\pi,\kappa) + \alpha\mathbf{i}\bigr)\Bigr\}\Bigr|_{\alpha=0} = \mathrm{var}_{p,q}(i_1|\eta| + i_2 k) \tag{2.103}$$

old2.89 *where* $\operatorname{var}_{p,q}$ *denotes variance with respect to* $\phi_{p,q}$. *In particular* $\log Y_G$ *is a convex function on* $\mathbb{R}^2$.

More generally, the jth derivative as in (2.103) equals the jth cumulant (or 'semi-invariant') of the random variable $i_1|\eta| + i_2 k$, as detailed in [**98**, p. 185] and [**154**, p. 266].

Proof. (a) It is elementary that

$$\begin{aligned}\frac{\partial}{\partial \pi} \log Y_G(\pi, \kappa) &= \frac{1}{Y_G(\pi, \kappa)} \sum_{\omega \in \Omega} |\eta(\omega)| \exp\bigl\{\pi|\eta(\omega)| + \kappa k(\omega)\bigr\} \\ &= \phi_{p,q}(|\eta|),\end{aligned}$$

with a similar relation for the other derivative.

(b) We have that

$$Y_G\bigl((\pi, \kappa) + \alpha \mathbf{i}\bigr) = \sum_{\omega \in \Omega} \exp\bigl\{\alpha(i_1|\eta(\omega)| + i_2 k(\omega))\bigr\} \exp\bigl\{\pi|\eta(\omega)| + \kappa k(\omega)\bigr\},$$

and (2.103) follows as in part (a). The convexity is a consequence of the fact that variances are negative. □

2.8 Domination by the Ising model

Stochastic domination is an invaluable tool in the study of random-cluster measures; see Theorem 2.46. Since the random-cluster model is an 'edge model', it is usual to make comparisons with other edge models. The relationship when $q \in \{2, 3, \dots\}$ to Potts models suggests a comparison with a 'vertex model', and a hint of how to achieve this is provided by the case of integral q.

Consider the random-cluster model with parameters p and q on the finite graph $G = (V, E)$. If $q \in \{2, 3, \dots\}$, we may generate a Potts model by assigning a uniformly chosen spin-value to each open cluster. The spin configuration thus obtained is governed by the Potts measure with $p = 1 - e^{-\beta J}$. Evidently, this can work only if q is an integer. A weaker conclusion may be obtained if q is not an integer, namely the following. Suppose $0 \le p \le 1$ and $q \ge 1$. We examine each open cluster of the random-cluster model in turn, and we declare it to be *red* with probability $1/q$ and *white* otherwise, different clusters receiving independent colours. Let R be the set of vertices lying in red clusters. If $q \in \{2, 3, \dots\}$, then R has the same distribution as the set of vertices of the corresponding Potts model having a given spin-value. One has for general $q \in (1,\infty)$ that, for $A \subseteq V$,

(2.104)
$$\begin{aligned}\phi_{p,q}(R=A) &= \frac{1}{Z_G(p,q)}(1-p)^{|\Delta_{\mathrm{e}}A|}\\ &\quad\times\left\{\sum_{\omega\in\Omega_A} p^{|\eta(\omega)|}(1-p)^{|\mathbb{E}_A\setminus\eta(\omega)|}q^{k(\omega)}\left(\frac{1}{q}\right)^{k(\omega)}\right\}\\ &\quad\times\left\{\sum_{\omega'\in\Omega_{\overline{A}}} p^{|\eta(\omega')|}(1-p)^{|\mathbb{E}_{\overline{A}}\setminus\eta(\omega')|}q^{k(\omega')}\left(1-\frac{1}{q}\right)^{k(\omega')}\right\}\\ &= \frac{1}{Z_G(p,q)}(1-p)^{|\Delta_{\mathrm{e}}A|}Z_{\overline{A}}(p,q-1)\end{aligned}$$

2.rcpots where $\overline{A} = V \setminus A$, $\Omega_A = \{0,1\}^{\mathbb{E}_A}$, the Z_G, $Z_{\overline{A}}$ are the appropriate partition functions, and $\Delta_{\mathrm{e}}A$ is the set of edges of G having exactly one endvertex in A. When q is an integer, (2.104) reduces to the usual Potts distribution on the set of vertices having a given spin.

The random set $R = R(\omega)$, with distribution given in (2.104), is the first element in the proposed stochastic comparison. The second is the set of $+$ spins in an Ising model with external field, and we recall from Section 1.3 the definition of an Ising model on the graph G. Let $\Sigma = \{-1,+1\}^V$, and let $0 < \beta, J < \infty$ and $-\infty < h < \infty$. The Hamiltonian is the function $H : \Sigma \to \mathbb{R}$ given by

$$H(\sigma) = -J\sum_{e=\langle u,v\rangle}\sigma_u\sigma_v - h\sum_{v\in V}\sigma_v, \qquad \sigma = (\sigma_u : u \in V) \in \Sigma, \tag{2.105}$$

2.alexham and the (Ising) probability measure is as follows:

$$\pi_{\beta,J,h}(\sigma) = \frac{1}{Z_{\mathrm{I}}}e^{-\beta H(\sigma)}, \qquad \sigma \in \Sigma, \tag{2.106}$$

2.alexismeas where $Z_{\mathrm{I}} = Z_{\mathrm{I}}(\beta,J,h)$ is the required normalising constant. We shall be concerned here with the random set $S = S(\sigma) = \{u \in V : \sigma_u = 1\}$, containing all vertices with spin $+1$. Since the quantity J plays no further individual role in the discussion, we set $J = 1$ henceforth in this section, and we write $\pi_{\beta,h} = \pi_{\beta,J,h}$.

Let $\deg(u)$ denote the degree of the vertex u in the graph G, and let

$$\Delta = \max\{\deg(u) : u \in V\}$$

be the maximum vertex degree,

2.alexbound **Theorem 2.107 [11].** *Let $\beta \in (0,\infty)$, $p = 1-e^{-\beta}$, $q \in [2,\infty)$, and let R be the random 'red' set of the random-cluster model, governed by the probability measure given in (2.104). Let $\beta' \in (0,\infty)$ and $h' \in (-\infty,\infty)$ be given by*

$$e^{2\beta'} = e^{\beta}\left(\frac{q-2+e^{\beta}}{q-1}\right), \qquad e^{\beta'(\Delta+h')} = \frac{1}{q-1}e^{\beta\Delta}. \tag{2.108}$$

2.alexcond *and let S be the set of vertices having spin $+1$ under the Ising measure $\pi_{\beta',h'}$. We have that*

$$R \leq_{\mathrm{st}} S. \tag{2.109}$$

2.alexcomp

Inequality (2.109) is to be interpreted as

$$\phi_{p,q}(f(R)) \leq \pi_{\beta',h'}(f(S))$$

for all increasing function $f : \{0,1\}^V \to \mathbb{R}$. Its importance lies in the deduction that R is small whenever S is small. The Ising model allows a deeper analysis than do general Potts and random-cluster models (see, for example, Chapter 8). Particularly relevant facts are known for the set of $+$ spins in the Ising model when the external field h' is negative, and thus it becomes important to understand when $h' < 0$ in (2.108). Setting $h' = 0$ and eliminating β' in (2.108), we find that $\beta = \beta_\Delta$ when $h' = 0$, where

$$e^{\beta_\Delta} = \frac{q-2}{(q-1)^{1-(2/\Delta)} - 1}. \tag{2.110}$$

2.alexbd By an elementary argument using monotonicity, we deduce from (2.108) that

$$h' < 0 \quad \text{if and only if} \quad \beta < \beta_\Delta. \tag{2.111}$$

2.alexneg

We make one further note in advance of proving the theorem. By (2.110), $\beta_\Delta \to 0$ as $\Delta \to \infty$; if the maximum vertex degree is large, the domain of validity of the theorem is small. In an important application of the theorem, we shall take G to be a box Λ of the lattice $\mathbb{L}^d$ with so-called 'wired boundary conditions' (see Section 3.2). This amounts to identifying all vertices in the boundary $\partial\Lambda$, and thus to the introduction of a single vertex, w say, having large degree. The method of proof of Theorem 2.107 is valid in this slightly more general setting *under the assumption* that the open cluster containing w is automatically designated red. That is, we let R be the union of the cluster at w together with all other clusters declared red under the above randomization, and we let S be the set of $+$ spins in the Ising model with parameters β', h' and with $\sigma_w = +1$. The conclusion (2.109) of Theorem 2.107 is then valid in the setting, with $\Delta = \max\{\deg(u) : u \in V \setminus \{w\}\}$. An application of this note to the exponential decay of connectivity in two dimensions will be found in Section 5.3.

Proof. We present a direct proof based on the Holley inequality, Theorem 2.7. For $A \subseteq V$ and $u, v \in V$ with $u \neq v$, we write

$$A^u = A \cup \{u\}, \quad A_v = A \setminus \{v\}, \quad A^u_v = (A^u)_v, \quad \text{and so on.}$$

We write μ_1 (respectively, μ_2) for the probability measure governing R (respectively, S), which is to say that

$$\mu_1(A) = \phi_{p,q}(R = A), \quad \mu_2(A) = \pi_{\beta',h'}(S = A), \qquad A \subseteq V.$$

The measures μ_i are strictly positive, and it therefore suffices by Theorem 2.10 to prove that

$$\mu_2(A \cup B)\mu_1(A \cap B) \geq \mu_1(A)\mu_2(B) \qquad \text{if } |A \,\triangle\, B| \leq 2. \tag{2.112}$$

2.alexsufft The proof of (2.112) when $|A \,\triangle\, B| = 1$ follows exactly that when $|A \,\triangle\, B| = 2$, and we therefore restrict ourselves to the latter case.

Suppose that

$$A = C^u, \quad B = C^v, \quad \text{where } C \subseteq V,\ u, v \in V \setminus C,\ u \neq v. \tag{2.113}$$

2.alexass We shall assume for the moment that $u \nsim v$. Let $r = |\{c \in C : a \sim u\}|$, the number of neighbours of u in C. We have from (2.105)–(2.106) that

$$\frac{\mu_2(C^{uv})}{\mu_2(C^v)} = \exp\big(\beta'(2r - \delta) + 2\beta' h'\big) \tag{2.114}$$

2.alex1 where $\delta = \deg(u)$. Also, by (2.104) and Theorem 2.94,

$$\begin{aligned} \frac{\mu_1(C)}{\mu_1(C^u)} &= (1-p)^{2r-\delta} \frac{Z_{\overline{C^u}}(p, q-1)}{Z_{\overline{C}}(p, q-1)} \\ &\geq (1-p)^{2r-\delta}(q-1)\big(1 - p + p(q-1)^{-1}\big)^{\delta - r}. \end{aligned} \tag{2.115}$$

2.alex2 Substituting $p = 1 - e^{-\beta}$ and setting $x = (1-p)/p = e^{\beta}$, we obtain by multiplying (2.114) and (2.115) that

$$\begin{aligned} \frac{\mu_2(A \cup B)\mu_1(A \cap B)}{\mu_1(A)\mu_2(B)} &\geq \exp\big(\beta'(2r-\delta) + 2\beta' h' - \beta(2r - \delta)\big) \\ &\qquad \times (q-1)\big[e^{-\beta} + (1 - e^{-\beta})(q-1)^{-1}\big]^{\delta - r} \\ &= x^{r - \delta/2} \left(\frac{q-2+x}{q-1}\right)^{r-\delta/2} \left(\frac{q-2+x}{x(q-1)}\right)^{-\Delta/2} \frac{x^{\delta - 2r}}{q-1} \\ &\qquad \times (q-1) \left[\frac{q-2+x}{x(q-1)}\right]^{\delta - r} \\ &= \left(\frac{x(q-1)}{q-2+x}\right)^{(\Delta - \delta)/2}, \end{aligned}$$

by (2.108). Now $\delta \leq \Delta$, and $x(q-1)/(q-2+x) \geq 1$ for $x \geq 1$, whence

$$\frac{\mu_2(A \cup B)\mu_1(A \cap B)}{\mu_1(A)\mu_2(B)} \geq 1 \tag{2.116}$$

2.alex3 as required at (2.112).

In the case when $u \sim v$, the quantity r in (2.114) is replaced by $r + 1$, and (2.116) remains valid. We deduce by Theorem 2.10 that $R \leq_{\mathrm{st}} S$ as claimed. $\square$

2.9 Negative association

This chapter closes with a short discussion of negative association when $q \le 1$. Let E be a finite set, and let μ be a probability measure on the sample space $\Omega = \{0, 1\}^E$. There are four relevant concepts of negative association, of which we start at the bottom. The measure μ is said to be *edge-negatively-associated* if

$$\mu(J_e \cap J_f) \le \mu(J_e)\mu(J_f), \quad e, f \in E, \ e \ne f. \tag{2.117}$$

2.ena Recall that $J_e = \{\omega \in \Omega : \omega(e) = 1\}$.

There is a more general notion of negative association, as follows. For $\omega \in \Omega$ and $F \subseteq E$ we define the cylinder event $C(\omega, F)$ generated by ω on F by

$$C(\omega, F) = \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for } e \in F\}.$$

Thus $C(\omega, F)$ contains the set of all $\omega' \in \Omega$ which agree with ω on F. For $E' \subseteq E$ and an event $A \subseteq \Omega$, we say that A is *defined on* E' if, for all $\omega \in \Omega$, $\omega \in A$ if and only if $C(\omega, F) \subseteq A$. We call μ *negatively associated* if

$$\mu(A \cap B) \le \mu(A)\mu(B)$$

for all pairs (A, B) of increasing events with the property that there exists $E' \subseteq E$ such that A is defined on E' and B is defined on its complement $\overline{E'} = E \setminus E'$. For a general account of negative association and its inherent problems, see [**158**].

Our third and fourth concepts of negative dependence involve so-called disjoint occurrence (see [**24**, **92**]). Let A and B be events in Ω. We define $A \Box B$ to be the set of all vectors $\omega \in \Omega$ for which there exists a set $F \subseteq E$ such that $C(\omega, F) \subseteq A$ and $C(\omega, \overline{F}) \subseteq B$, where $\overline{F} = E \setminus F$. Note that the choice of F is allowed to depend on the vector ω.

We say that μ has the *disjoint-occurrence property* if

$$\mu(A \Box B) \le \mu(A)\mu(B) \quad \text{for all } A, B \subseteq \Omega, \tag{2.118}$$

2.disocc and has the *disjoint-occurrence property on increasing events* if (2.118) holds under the additional assumption that A and B are increasing events.

It is evident that:

$$\begin{aligned} &\mu \text{ has the disjoint-occurrence property} \\ &\qquad \Rightarrow \mu \text{ has the disjoint-occurrence property on increasing events} \\ &\qquad \Rightarrow \mu \text{ is negatively associated} \\ &\qquad \Rightarrow \mu \text{ is edge-negatively-associated.} \end{aligned}$$

It was proved by van den Berg and Kesten [**24**] that the product measures $\phi_p = \phi_{p,1}$ on Ω have the disjoint-occurrence property on increasing events, and further by Reimer [**169**] that ϕ_p has the more general disjoint-occurrence property. It is easily seen that the random-cluster measure $\phi_{p,q}$ cannot in general be edge-negatively-associated when $q > 1$. [Example] It may however be conjectured that $\phi_{p,q}$ has some form of negative-association property when $q \le 1$. Such a property would be useful in studying random-cluster measures, particularly in the thermodynamic limit (see Chapter 3), but no such property has yet been proved.

In the absence of an approach to the general case when $q < 1$, we turn next to negative association of weak limits of $\phi_{p,q}$ as $q \downarrow 0$; see Section 1.5, and especially Theorem 1.21. Here is a mild conjecture, as yet unproven.

2.forcs **Conjecture 2.119 [94, 99, 119, 158].** *For any finite graph $G = (V, E)$, the uniform spanning forest measure* USF *and the uniform connected subgraph measure* UCS *are edge-negatively-associated.*

A stronger version of this conjecture is that USF and UCS are negatively associated in the sense described above.

Since USF and UCS are uniform measures, Conjecture 2.119 may be rewritten in the form of two questions concerning subgraph counts. For simplicity we shall consider only graphs with neither loops nor multiple edges. Let $V = \{1, 2, \dots, n\}$, and let K be the set of $N = \binom{n}{2}$ edges of the complete graph on the vertex set V. Let $\mathcal{G}(V)$ denote the set of all graphs on V with neither loops nor multiple edges; since $\mathcal{G}(V)$ is in one–one correspondence with the set of all subsets of K, we shall take $\mathcal{G}(V)$ to be this power set. Thus we speak of subsets of K as being graphs on V. Let $E \subseteq K$. For $X \subseteq E$, let $M^X = M^X(E)$ be the number of subsets E' of E with $E' \supseteq X$ such that the graph (V, E') is connected. Edge-negative-association for connected subgraphs amounts to the inequality

$$M^{\{e,f\}} M^{\varnothing} \le M^e M^f, \quad e, f \in E,\ e \ne f. \tag{2.120}$$

2.eancs Here and later in this context, singleton sets are denoted without their braces, and any empty set is suppressed.

In the second such question, we ask if the same inequality is valid with M^X redefined as the number of subsets E' containing X such that (V, E') is a forest. (See [**119**, **158**].)

With E fixed as above, and with $X, Y \subseteq E$, let $M_Y^X = M_Y^X(E)$ denote the number of subsets E' of E of the required type such that $E' \supseteq X$ and $E' \cap Y = \varnothing$. Inequality (2.120) is easily seen to be equivalent to the inequality

$$M^{\{e,f\}} M_{\{e,f\}} \le M_f^e M_e^f, \quad e, f \in E,\ e \ne f. \tag{2.121}$$

2.eancs2

The corresponding statement for the uniform spanning tree is known.

2.ustKi **Theorem 2.122.** *The uniform spanning tree measure* UST *is edge-negatively-associated.*

The stronger property of negative association of UST has been proved, see [**66**], but we do not discuss this here. See also the discussions in [**21**, **147**].

The best currently known for USF appears to be the following. The proof is by counting using a computer.

2.gw **Theorem 2.123 [99].** *If $G = (V, E)$ has eight or fewer vertices, or has nine vertices and sixteen? or fewer edges, then the uniform forest measure* USF *has the edge-negative-association property.*

Since forests are dual to connected subgraphs for planar graphs, this implies a property of edge-negative-association for the UCS measure on certain planar graphs having fewer than nine faces.

Proof of Theorem 2.122. This is a consequence of the 1847 work of Kirchhoff [**132**] concerning electrical networks. Consider an electrical network on the connected graph G in which each edge corresponds to a unit resistor. The relevant fact from the theory of electrical networks is that, if a unit current flows from a *source* vertex s to a *sink* vertex t ($\neq s$), then the current which flows along the edge $e = \langle x, y \rangle$ in the direction xy equals $N(s, x, y, t)/N$, where N is the number of spanning trees of G and $N(s, u, v, t)$ is the number of spanning trees whose unique path from s to t passes along the edge $\langle u, v \rangle$ in the direction uv.

Let $e = \langle x, y \rangle$, and let μ be the UST measure on G. By the above, $\mu(J_e)$ equals the current flowing along e when a unit current flows through G from source x to sink y. By Ohm's Law, this equals the potential difference between x and y, which in turn equals the effective resistance $R_G(x, y)$ of the network between x and y.

Let $f \in E$, $f \neq e$, and denote by $G.f$ the graph obtained from G by contracting the edge f. There is a one–one correspondence between spanning trees of $G.f$ and spanning trees of G containing f. Therefore, $\mu(J_e \mid J_f)$ equals the effective resistance $R_{G.f}(x, y)$ of the network $G.f$ between x and y.

The so-called Rayleigh principle states that the effective resistance of a network is a non-decreasing function of the individual edge-resistances. It follows that

$$R_{G.f}(x, y) \leq R_G(x, y),$$

as required. $\square$

The usual proof of the Rayleigh principle makes use of the Thomson/Dirichlet variational principle, which in turn asserts that, amongst all unit flows from source to sink, the true current flow of size one is that which minimises the dissipated energy.

A good account of the Kirchhoff theorem may be found in [**36**]. Further accounts of the mathematics of electrical networks include [**61**] and [**147**], the latter containing also much material about the uniform spanning tree.

3

Infinite-Volume Measures

3.1 Infinite graphs

Although there is some interesting theory associated with random-cluster measures on *finite* graphs, the real action, seen from the point of statistical mechanics, takes place in the context of *infinite* graphs. On a finite graph, all functionals are polynomials in p and q, and are therefore smooth, whereas singularities occur when the graph is infinite. These singularities provide most of the mathematical and physical motivation for the study of the random-cluster model.

While one may define random-cluster measures on a broad class of infinite graphs using the methods of this chapter, we shall concentrate in this book on finite-dimensional lattice-graphs. We shall, almost without exception, consider only the case of the (hyper)cubic lattice $\mathbb{L}^d$ in some number d of dimensions. This restriction enables a clear exposition of the theory and open problems without suffering the complications which arise through allowing greater generality. Interesting further questions arise in the non-finite-dimensional setting of non-amenable graphs, and we return to this in Section ?.

There are two ways of defining random-cluster measures on the infinite graph $G = (V, E)$. The first is to consider weak limits of measures on finite subgraphs Λ, in the limit as $\Lambda \to V$. This will be discussed in Section 3.3, following the introduction in Section 3.2 of the notion of boundary conditions. The second way is to restrict oneself to infinite-volume measures whose conditional marginal on a finite domain Λ is the finite-volume random-cluster measure on Λ with the correct boundary condition. This latter route is inspired by work of Dobrushin [**60**] and Lanford–Ruelle [**136**] for Gibbs states, and will be discussed in Section 3.4. In preparation for the required arguments, we summarise next the stochastic ordering and positive association of probability measures on $\mathbb{L}^d$.

Let $\Omega = \{0, 1\}^{\mathbb{E}^d}$, and let $\mathcal{F}$ be the σ-field generated by the cylinder subsets of Ω. Since Ω is a partially ordered set, we may speak of 'increasing' events and random variables. We recall from Section 2.2 the definition of the stochastic ordering of probability measures on $(\Omega, \mathcal{F})$. Given two such probability measures μ_1, μ_2, we write $\mu_1 \le_{\mathrm{st}} \mu_2$ if

$$\mu_1(X) \le \mu_2(X) \quad \text{for all increasing continuous random variables } X. \tag{3.1}$$

old3.1 Note that any increasing random variable X with range $\mathbb{R}$ satisfies $X(0) \le X(\omega) \le X(1)$ for all $\omega \in \Omega$, and is therefore bounded.

One sometimes wishes to apply (3.1) to increasing random variables X which are semicontinuous rather than continuous. An important example of this is the indicator function $X = 1_A$ of an increasing closed event A. This may be done as follows. For $\omega, \xi \in \Omega$ and a box Λ, we write ω_Λ^ξ for the configuration given by

$$\omega_\Lambda^\xi(e) = \begin{cases} \omega(e) & \text{if } e \in \mathbb{E}_\Lambda, \\ \xi(e) & \text{otherwise.} \end{cases} \tag{3.2}$$

old3.2 For any $X : \Omega \to \mathbb{R}$, we define X_Λ^0 and X_Λ^1 by

$$X_\Lambda^b(\omega) = \omega_\Lambda^b \qquad \text{for } b = 0, 1. \tag{3.3}$$

old3.3 Assume that X is increasing. It is easily checked that, as $\Lambda \to \mathbb{Z}^d$,

$$\begin{aligned} X_\Lambda^0 \uparrow X & \quad \text{if and only if } X \text{ is lower semicontinuous,} \\ X_\Lambda^1 \downarrow X & \quad \text{if and only if } X \text{ is upper semicontinuous,} \end{aligned} \tag{3.4}$$

old3.4 where the convergence is pointwise on Ω. The functions X_Λ^0, X_Λ^1 are continuous. Therefore, by the mononone convergence theorem, $\mu_1 \le_{\mathrm{st}} \mu_2$ if and only if

$$\mu_1(X) \le \mu_2(X) \quad \text{for all increasing semicontinuous } X. \tag{3.5}$$

old3.5

It is a useful fact that, when $\mu_1 \le_{\mathrm{st}} \mu_2$, then $\mu_1 = \mu_2$ whenever their marginals are equal. We state this as a theorem for future use, see also [**144**, Section II.2]. Recall that J_e is the event that e is open, that is, $J_e = \{\omega \in \Omega : \omega(e) = 1\}$.

old3.6 **Proposition 3.6.** *Let E be a countable set, let $\Omega = \{0, 1\}^E$, and let $\mathcal{F}$ be the σ-field generated by the cylinder subsets of Ω. Let μ_1, μ_2 be probability measures on $(\Omega, \mathcal{F})$ such that $\mu_1 \le_{\mathrm{st}} \mu_2$. Then $\mu_1 = \mu_2$ if and only if*

$$\mu_1(J_e) = \mu_2(J_e) \qquad \textit{for all } e \in E. \tag{3.7}$$

old3.7

We say that a probability measure μ on $(\Omega, \mathcal{F})$ is *positively associated* if

$$\mu(XY) \ge \mu(X)\mu(Y) \quad \text{for all increasing continuous random variables } X, Y. \tag{3.8}$$

old3.8 Note from the arguments above that μ is positively associated if and only if

$$\mu(XY) \ge \mu(X)\mu(Y) \quad \text{for all increasing semicontinuous } X, Y. \tag{3.9}$$

old3.9

Stochastic inequalities and positive association are conserved by weak convergence, in the following sense.

old3.10 **Proposition 3.10.** *We use the context of Proposition 3.6.*

(a) *Let* $(\mu_{n,i} : n \ge 1)$, $i = 1, 2$, *be sequences of probability measures on* $(\Omega, \mathcal{F})$ *satisfying*: $\mu_{n,i} \Rightarrow \mu_i$ *as* $n \to \infty$, *for* $i = 1, 2$, *and* $\mu_{n,1} \le_{\mathrm{st}} \mu_{n,2}$ *for all* n. *Then* $\mu_1 \le_{\mathrm{st}} \mu_2$.

(b) *Let* $(\mu_n : n \ge 1)$ *be a sequence of probability measures on* $(\Omega, \mathcal{F})$ *satisfying* $\mu_n \Rightarrow \mu$ *as* $n \to \infty$. *If each* μ_n *is positively associated, then so is* μ.

Proof of Proposition 3.6. If $\mu_1 = \mu_2$ then (3.7) holds. Suppose, on the converse, that (3.7) holds. By [**144**, Theorem 2.4] or [**145**, Theorem II.2.4], there exists a 'coupled' measure μ on $(\Omega, \mathcal{F})^2$ with marginals μ_1 and μ_2, and such that

$$\mu\big(\{(\pi, \omega) : \pi \le \omega\}\big) = 1.$$

For any increasing cylinder event A,

$$\begin{aligned} \mu_2(A) - \mu_1(A) &= \mu\big(\{(\pi, \omega) : \pi \notin A,\ \omega \in A\}\big) \\ &\le \sum_{e \in E} \mu\big(\pi(e) = 0,\ \omega(e) = 1\big) \\ &= \sum_{e \in E} \big\{\mu(\omega(e) = 1) - \mu(\pi(e) = 1)\big\} \\ &= \sum_{e \in E} \big\{\mu_2(J_e) - \mu_1(J_e)\big\} = 0. \end{aligned}$$

Since $\mathcal{F}$ is generated by the increasing cylinder events A, the claim is proved. □

Proof of Proposition 3.10. We have that $\mu_{n,1}(X) \le \mu_{n,2}(X)$ for any increasing continuous random variable X, and the conclusion follows by letting $n \to \infty$. Part (b) is proved similarly. □

3.2 Boundary conditions

An important part of statistical mechanics is devoted to understanding the way in which assumptions on the boundary affect what happens in the interior of a region of space. In order to make precise such a discussion for random-cluster models, we introduce next the concept of a 'boundary condition'.

Let Λ be a finite subset of $\mathbb{Z}^d$. We shall later take Λ to be a box, but we retain the extra generality at this point. For $\xi \in \Omega$, we write Ω_Λ^ξ for the (finite) subset of Ω containing all configurations ω satisfying $\omega(e) = \xi(e)$ for $e \in \mathbb{E}^d \setminus \mathbb{E}_\Lambda$; these are the configurations which 'agree with ξ off Λ'. For $\xi \in \Omega$ and values of p, q satisfying $0 \le p \le 1$, $q > 0$, we define $\phi_{\Lambda,p,q}^\xi$ to be the random-cluster measure on the finite graph $(\Lambda, \mathbb{E}_\Lambda)$ 'with boundary condition ξ'; this is the equivalent of a ‘specification’ for Gibbs states (see [**79**]). More precisely, let $\phi^{\xi}_{\Lambda,p,q}$ be the probability measure on the pair $(\Omega, \mathcal{F})$ given by

(3.11)
$$\phi^{\xi}_{\Lambda,p,q}(\omega) = \begin{cases} \dfrac{1}{Z^{\xi}_{\Lambda}(p,q)} \left\{ \displaystyle\prod_{e \in \mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)} & \text{if } \omega \in \Omega^{\xi}_{\Lambda}, \\ 0 & \text{otherwise,} \end{cases}$$

old3.11 where $k(\omega, \Lambda)$ is the number of components of the graph $(\mathbb{Z}^d, \eta(\omega))$ which intersect Λ, and where $Z^{\xi}_{\Lambda}(p,q)$ is the appropriate normalising constant

$$Z^{\xi}_{\Lambda}(p,q) = \sum_{\omega \in \Omega^{\xi}_{\Lambda}} \left\{ \prod_{e \in \mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)}. \tag{3.12}$$

old3.12 Note that $\phi^{\xi}_{\Lambda,p,q}(\Omega^{\xi}_{\Lambda}) = 1$.

The boundary condition ξ influences the measure $\phi^{\xi}_{\Lambda,p,q}$ through the way in which the term $k(\omega, \Lambda)$ in (3.11) counts the number of ω-open clusters of Λ intersecting the boundary $\partial\Lambda$. More precisely, suppose that $x, y \in \partial\Lambda$, and that there exists a path of ξ-open edges of $\mathbb{E}^d \setminus \mathbb{E}_\Lambda$ from x to y. Then any two ω-open clusters of Λ containing x and y, respectively, will contribute only 1 to the count $k(\omega, \Lambda)$.

Random-cluster measures have an important ‘nesting’ property which is best expressed in terms of conditional probabilities. For any finite subset Λ of $\mathbb{Z}^d$, we write as usual $\mathcal{F}_\Lambda$ (respectively, $\mathcal{T}_\Lambda$) for the set of events defined in terms of the states of edges in $\mathbb{E}_\Lambda$ (respectively, $\mathbb{E}^d \setminus \mathbb{E}_\Lambda$).

old3.13 **Lemma 3.13.** *Let $0 \le p \le 1$ and $q > 0$. If Λ, Δ are finite sets of vertices with $\Lambda \subseteq \Delta$, then for every $\xi \in \Omega$ and every event A,*

$$\phi^{\xi}_{\Delta,p,q}(A \mid \mathcal{T}_\Lambda)(\omega) = \phi^{\omega}_{\Lambda,p,q}(A) \qquad \textit{for all } \omega \in \Omega^{\xi}_{\Delta}.$$

Two extremal boundary conditions of special importance are provided by the configurations 0 and 1, comprising ‘all edges closed’ and ‘all edges open’ respectively. One speaks of configurations in Ω^0_Λ as having ‘free’ boundary conditions, and configurations in Ω^1_Λ as having ‘wired’ boundary conditions. The word ‘wired’ refers to the fact that, with boundary condition 1, the set of open clusters of ω restricted to Λ which intersect $\partial\Lambda$ are ‘wired together’ and contribute only 1 in all to the count $k(\omega, \Lambda)$. This terminology originated in the study of electrical networks. ‘Free’ is understood as the converse: such clusters are counted in their actual number when the boundary condition is 0.

In the regime of validity of the FKG inequality, the free and wired boundary conditions provide random-cluster measures which are extremal in the sense of stochastic ordering.

old3.14 **Lemma 3.14.** *Suppose* $0 \le p \le 1$ *and* $q \ge 1$, *and let* Λ *be a finite subset of* $\mathbb{Z}^d$.

(a) *For every* $\xi \in \Omega$, *the probability measure* $\phi^{\xi}_{\Lambda,p,q}$ *is positively associated.*

(b) *If* $\psi, \xi \in \Omega$, *then*

$$\phi^{\psi}_{\Lambda,p,q} \le_{\mathrm{st}} \phi^{\xi}_{\Lambda,p,q} \qquad \textit{whenever } \psi \le \xi.$$

In particular,

$$\phi^{0}_{\Lambda,p,q} \le_{\mathrm{st}} \phi^{\xi}_{\Lambda,p,q} \le_{\mathrm{st}} \phi^{1}_{\Lambda,p,q}.$$

Proof of Lemma 3.13. We apply Theorem 2.46 repeatedly, once for each edge in $\mathbb{E}_\Delta \setminus \mathbb{E}_\Lambda$. □

Proof of Lemma 3.14. The proof is straightforward, if slightly tedious when written out in detail. Since p and q are held constant during this proof, we omit them from future subscripts. Let $q \ge 1$, so that we may use positive association, and let Λ be a finite subset of $\mathbb{Z}^d$. For $\xi \in \Omega$ and for any increasing continuous function $X : \Omega \to \mathbb{R}$, we define the increasing random variable $X^\xi_\Lambda : \Omega \to \mathbb{R}$ by

$$X^\xi_\Lambda(\omega) = X(\omega^\xi_\Lambda)$$

where ω^ξ_Λ is given in (3.2). We may view X^ξ_Λ as an increasing function on $\{0,1\}^{\mathbb{E}_\Lambda}$.

We augment the graph $(\Lambda, \mathbb{E}_\Lambda)$ by adding some extra edges as follows around the boundary $\partial\Lambda$. For every distinct unordered pair $x, y \in \partial\Lambda$, we add a new edge, denoted $[x,y]$, between x and y. If the edge $\langle x,y\rangle$ exists already in Λ, we simply add another in parallel. We write $\mathbb{F}$ for the set of new edges, $\overline{\Omega} = \{0,1\}^{\mathbb{E}_\Lambda \cup \mathbb{F}}$ for the augmented configuration space, and let $\overline{\phi}_\Lambda$ be the random-cluster measure on the augmented graph $(\Lambda, \mathbb{E}_\Lambda \cup \mathbb{F})$.

For $\xi \in \Omega$, let $\overset{\xi}{\sim}$ be the equivalence relation on $\partial\Lambda$ given by $x \overset{\xi}{\sim} y$ if and only if there exists a ξ-open path of $\mathbb{E}^d \setminus \mathbb{E}_\Lambda$ joining x to y. Let $\mathbb{F}^\xi$ be the set of all edges $[x,y] \in \mathbb{F}$ such that $x \overset{\xi}{\sim} y$.

Let X, Y be increasing and continuous on Ω. We have that

$$\begin{aligned}
\phi^\xi_\Lambda(XY) &= \phi^\xi_\Lambda(X^\xi_\Lambda Y^\xi_\Lambda)\\
&= \overline{\phi}_\Lambda\big(X^\xi_\Lambda Y^\xi_\Lambda \,\big|\, \mathbb{F}^\xi \text{ open}, \mathbb{F}\setminus\mathbb{F}^\xi \text{ closed}\big)\\
&\ge \overline{\phi}_\Lambda\big(X^\xi_\Lambda \,\big|\, \mathbb{F}^\xi \text{ open}, \mathbb{F}\setminus\mathbb{F}^\xi \text{ closed}\big)\overline{\phi}_\Lambda\big(Y^\xi_\Lambda \,\big|\, \mathbb{F}^\xi \text{ open}, \mathbb{F}\setminus\mathbb{F}^\xi \text{ closed}\big)\\
&\qquad\qquad \text{by strong positive association}\\
&= \phi^\xi_\Lambda(X^\xi_\Lambda)\phi^\xi_\Lambda(X^\xi_\Lambda) = \phi^\xi_\Lambda(X)\phi^\xi_\Lambda(Y),
\end{aligned}$$

whence ϕ^ξ_Λ is positively associated.

(b) In broad terms, the 'greater' the connections off Λ, the larger is the induced measure within Λ. Let $\psi \le \xi$, whence $\mathbb{F}^\pi \subseteq \mathbb{F}^\xi$. Then

$$\begin{aligned}
\phi_\Lambda^\psi(X) &= \phi_\Lambda^\psi(X_\Lambda^\psi) \\
&= \overline{\phi}_\Lambda\big(X_\Lambda^\psi \,\big|\, \mathbb{F}^\psi \text{ open}, \mathbb{F} \setminus \mathbb{F}^\psi \text{ closed}\big) \\
&\le \overline{\phi}_\Lambda\big(X_\Lambda^\psi \,\big|\, \mathbb{F}^\xi \text{ open}, \mathbb{F} \setminus \mathbb{F}^\xi \text{ closed}\big) \quad \text{by strong positive association} \\
&\le \overline{\phi}_\Lambda\big(X_\Lambda^\xi \,\big|\, \mathbb{F}^\xi \text{ open}, \mathbb{F} \setminus \mathbb{F}^\xi \text{ closed}\big) \quad \text{since } X_\Lambda^\psi \le X_\Lambda^\xi \\
&= \phi_\Lambda^\xi(X_\Lambda^\xi) = \phi_\Lambda^\xi(X).
\end{aligned}$$

The claim therefore holds. □

3.3 Infinite-volume weak limits

We begin with a definition of a weak-limit random-cluster measure on Ω. The use of the letter Λ is restricted throughout this section to boxes of $\mathbb{Z}^d$.

old3.15 **Definition 3.15.** Let $0 \le p \le 1$ and $q > 0$. A probability measure ϕ on $(\Omega, \mathcal{F})$ is called a *limit-random-cluster measure* with parameters p and q if, for some $\xi \in \Omega$, ϕ is an accumulation point of the family $\{\phi_{\Lambda,p,q}^\xi : \Lambda \subseteq \mathbb{Z}^d\}$, that is, there exists a sequence $\boldsymbol{\Lambda} = (\Lambda_n : n \ge 1)$ of boxes satisfying $\Lambda_n \to \mathbb{Z}^d$ as $n \to \infty$ such that

$$\phi_{\Lambda_n,p,q}^\xi \Rightarrow \phi \qquad \text{as } n \to \infty.$$

The set of all such measures ϕ is denoted by $\mathcal{W}_{p,q}$, and the closed convex hull of $\mathcal{W}_{p,q}$ is denoted $\overline{\text{co}\,\mathcal{W}_{p,q}}$.

One might at first sight consider instead the class of all limits of the form

$$\phi = \lim_{n\to\infty} \phi_{\Lambda_n,p,q}^{\xi_n} \tag{3.16}$$

old3.16 for sequences $\boldsymbol{\Lambda}$ of boxes and (ξ_n) of configurations. This provides no extra generality over Definition 3.15 as we explain next, [**90**]. The measure $\phi_{\Lambda,p,q}^\xi$ is influenced by ξ through the connections it provides between points in the boundary $\partial\Lambda$. By arranging for the same connections (and no others) to be provided in a manner which is 'more economical in the use of space' one discovers the following. Let Λ be a box and $\xi \in \Omega$. There exists a box Λ' ($\supseteq \Lambda$) and a configuration ζ such that: $\phi_{\Lambda,p,q}^\zeta(A) = \phi_{\Lambda,p,q}^\psi(A)$ for any event A defined in terms of the states of edges in $\mathbb{E}_\Lambda$, and for any configuration ψ which agrees with ζ on all edges in $\mathbb{E}_{\Lambda'} \setminus \mathbb{E}_\Lambda$. Assume now that (3.16) holds, and define the increasing subsequence $(\Delta_n : n \ge 1)$ of $(\Lambda_n : n \ge 1)$ and the configuration ξ as follows. We set $\Delta_1 = \Lambda_1$ and $\xi(e) = \xi_1(e)$ for $e \in \mathbb{E}_{\Delta_1}$. Having constructed $\Delta_r = \Lambda_{n_r}$ (where

$(n_r : r \geq 1)$ is increasing) and the partial configuration $(\xi(e) : e \in \mathbb{E}_{\Delta_r})$ for $r < R$, we construct Δ_R and $(\xi(e) : e \in \mathbb{E}_{\Delta_R} \setminus \mathbb{E}_{\Delta_{R-1}})$ by the following rule. By the remark above, there exists a box Λ' $(\supseteq \Delta_{R-1})$ and a configuration ζ such that: the measures $\phi^{\xi_{n_{R-1}}}_{\Delta_{R-1},p,q}$ and $\phi^{\psi}_{\Delta_{R-1},p,q}$ are the same on Δ_{R-1} for any configuration ψ which agrees with ζ on $\mathbb{E}_{\Lambda'} \setminus \mathbb{E}_{\Delta_{R-1}}$. We find $m = n_R$ such that $m > n_{R-1}$ and $\Lambda_m \supseteq \Lambda'$, and we set $\Delta_R = \Lambda_m$ and $\xi(e) = \zeta(e)$ for $e \in \mathbb{E}_{\Delta_R} \setminus \mathbb{E}_{\Delta_{R-1}}$. It is now clear that

$$\phi^{\xi}_{\Delta_r,p,q} \Rightarrow \phi \qquad \text{as } r \to \infty.$$

The following claim is standard of its type.

old3.17 **Theorem 3.17 ?[87, 90].** *Suppose* $0 \leq p \leq 1$ *and* $q > 0$.

(a) Existence. *The set* $\mathcal{W}_{p,q}$ *of limit random-cluster measures is non-empty.*

(b) Finite-energy property. *Every* $\phi \in \overline{\text{co}\,\mathcal{W}_{p,q}}$ *satisfies the finite-energy property. More specifically, for* $e \in \mathbb{E}^d$,

$$\min\left\{p, \frac{p}{p+(1-p)q}\right\} \leq \phi(J_e \mid \mathcal{F}_{\mathbb{E}^d\setminus\{e\}}) \leq \max\left\{p, \frac{p}{p+(1-p)q}\right\},$$

ϕ*-almost surely, where* J_e *is the event that e is open, and* $\mathcal{F}_{\mathbb{E}^d\setminus\{e\}}$ *is the* σ*-field generated by* $\{\omega(f) : f \in \mathbb{E}^d \setminus \{e\}\}$.

(c) Positive association. *If, further,* $q \geq 1$, *then any member of* $\mathcal{W}_{p,q}$ *is positively associated.*

Proof. (a) The metric space Ω is the product of discrete spaces, and is therefore compact. It follows that any infinite family of probability measures on Ω is tight, and hence relatively compact (by Prohorov's theorem, see [**28**]), which is to say that any infinite sequence of probability measures contains a weakly convergent subsequence. We apply this to the family $\{\phi^{\xi}_{\Lambda_n,p,q} : n \geq 1\}$ for any given $\xi \in \Omega$ and any given sequence $(\Lambda_n : n \geq 1)$ with $\Lambda_n \to \mathbb{Z}^d$ as $n \to \infty$.

(b) Suppose first that $\phi \in \mathcal{W}_{p,q}$, so that

$$\phi = \lim_{\Lambda\to\mathbb{Z}^d} \phi^{\xi}_{\Lambda,p,q} \tag{3.18}$$

old3.18 for some $\xi \in \Omega$ and some sequence of boxes Λ. By the martingale convergence theorem [**98**, equation (12.3.10)] and weak convergence,

$$\begin{aligned}\phi(J_e \mid \mathcal{F}_{\mathbb{E}^d\setminus\{e\}}) &= \lim_{\Sigma\to\mathbb{Z}^d} \phi(J_e \mid \mathcal{F}_{\Sigma\setminus\{e\}}) && \phi\text{-a.s.}\\ &= \lim_{\Sigma\to\mathbb{Z}^d}\lim_{\Lambda\to\mathbb{Z}^d} \phi^{\xi}_{\Lambda,p,q}(J_e \mid \mathcal{F}_{\Sigma\setminus\{e\}}) && \phi\text{-a.s.}\end{aligned}$$

The claim now follows by Theorem 2.1. It is evident that any convex combination of measures in $\mathcal{W}_{p,q}$ satisfies the same inequalities. A similar argument now yields the claim for weak limits of such combinations.

(c) Let $q \ge 1$. Let $\phi \in \mathcal{W}_{p,q}$ be expressed as in (3.18). By Lemma 3.14(a), each $\phi^{\xi}_{\Lambda,p,q}$ is positively associated, and the claim follows by Proposition 3.10(b). $\square$

Let $\mathbf{\Lambda} = (\Lambda_n : n \ge 1)$ be a sequence of boxes such that $\Lambda_n \to \mathbb{Z}^d$ as $n \to \infty$. When does the limit $\lim_{n\to\infty} \phi^{\xi}_{\Lambda_n,p,q}$ exist, and when does it depend on the choice of the sequence $\mathbf{\Lambda}$ and the boundary condition ξ? Only little is known when $q < 1$. When $q \ge 1$, the FKG inequality holds, and we may thereby answer the question in the extremal cases when $\xi = 0, 1$.

old3.19 **Theorem 3.19 (Thermodynamic limit) [6, 39, 73, 87, 90].** *Suppose* $0 \le p \le 1$ *and* $q \ge 1$.

(a) Existence. *Let* $\mathbf{\Lambda} = (\Lambda_n : n \ge 1)$ *be a sequence of boxes satisfying* $\Lambda_n \to \mathbb{Z}^d$ *as* $n \to \infty$. *The weak limits*

$$\phi^b_{p,q} = \lim_{n\to\infty} \phi^b_{\Lambda_n,p,q}, \qquad \textit{for } b = 0, 1, \tag{3.20}$$

old3.20 *exist and are independent of the choice of* $\mathbf{\Lambda}$.

(b) Translation-invariance. *For* $b = 0, 1$, *the limit measure* $\phi^b_{p,q}$ *is translation-invariant, and each* $\phi^b_{p,q}$ *is extremal in the sense that*

$$\phi^0_{p,q} \le_{\mathrm{st}} \phi \le_{\mathrm{st}} \phi^1_{p,q} \qquad \textit{for all } \phi \in \mathcal{W}_{p,q}. \tag{3.21}$$

old3.21

(c) Ergodicity. *For* $b = 0, 1$, *the measure* $\phi^b_{p,q}$ *is ergodic, in that any translation-invariant random variable is* $\phi^b_{p,q}$*-a.s. constant.*

thermo

Proof of Theorem 3.19. (a) We begin with the case $b = 0$. Let Λ and Δ be two boxes satisfying $\Lambda \subseteq \Delta$, and let A be the event that all edges in $\mathbb{E}_\Delta \setminus \mathbb{E}_\Lambda$ have state 0. By Theorem 2.1(a), $\phi^0_{\Lambda,p,q}$ may be thought of as the marginal measure on $\mathbb{E}_\Lambda$ of $\phi^0_{\Delta,p,q}$ *conditioned* on the event A. Since A is a decreasing event, we have by positive association that

$$\phi^0_{\Lambda,p,q}(B) = \phi^0_{\Delta,p,q}(B \mid A) \le \phi^0_{\Delta,p,q}(B) \tag{3.22}$$

old3.22 for any increasing event B defined in terms of the edges in $\mathbb{E}_\Lambda$. Therefore, the increasing limit

$$\phi^0_{p,q}(B) = \lim_{\Lambda\to\mathbb{Z}^d} \phi^0_{\Lambda,p,q}(B)$$

exists for all increasing cylinder events B, and the value of the limit does not depend on the way that $\Lambda \to \mathbb{Z}^d$. The collection of all such events B is convergence-determining, [**28**, pages 14–19], whence $\phi^0_{p,q}$ exists. For the case $b = 1$, we let A

be the event that all edges in $\mathbb{E}_\Delta \setminus \mathbb{E}_\Lambda$ are open, and we reverse the inequality in (3.22).

(b) To see that $\phi^0_{p,q}$ is translation-invariant, one argues as follows. Let B be an increasing event lying in $\mathcal{F}_F$ for some finite subset F of $\mathbb{E}$. Let τ be a translation of the lattice $\mathbb{L}$, and extend τ to be a shift $\tau : \Omega \to \Omega$ by $\tau\omega(e) = \omega(\tau e)$ for $e \in \mathbb{E}$. For any box Λ containing all endpoints of all edges in F, we have by positive association as in (3.22) that

$$\phi^0_{p,q}(B) \ge \phi^0_{\Lambda,p,q}(B) = \phi^0_{\tau\Lambda,p,q}(\tau B) \to \phi^0_{p,q}(\tau B) \qquad \text{as } \Lambda \to \mathbb{Z}^d.$$

Applying the same argument with τ replaced by τ^{-1}, we find that $\phi^0_{p,q}(B) = \phi^0_{p,q}(\tau B)$. Similar arguments are valid for $\phi^1_{p,q}$.

Turning to inequality (3.21), we note from Lemma 3.14 that

$$\phi^0_{\Lambda,p,q} \le_{\text{st}} \phi^\xi_{\Lambda,p,q} \le_{\text{st}} \phi^1_{\Lambda,p,q},$$

and claim (3.21) follows by Proposition 3.10(a).

(c) We follow the proof of [**21**, **146**] rather than the less elementary approach of [**90**]. Recall that $\mathcal{T}_\Lambda$ denotes the σ-field generated by the random variables $\{\omega(e) : e \notin \mathbb{E}_\Lambda\}$. Let A be an increasing cylinder event defined in terms of the edges in some finite set F, and let Λ be a box containing F. Let T be an event in the tail σ-field $\mathcal{T} = \bigcap_\Lambda \mathcal{T}_\Lambda$. Consider first the case $b = 0$. It suffices to prove that $\phi^0_{p,q}(T) \in \{0, 1\}$. Let $\epsilon > 0$, and let T_ϵ be a cylinder event which depends on the states of edges in some set G disjoint with Λ, and which approximates T in the sense that

$$\phi^0_{p,q}(T_\epsilon) > 0, \quad \phi^0_{p,q}(T_\epsilon \,\triangle\, T) \le \epsilon.$$

Let Δ be a box containing $\mathbb{E}_\Lambda \cup G$. By the FKG inequality,

$$\phi^0_{\Lambda,p,q}(A) \le \phi^0_{\Delta,p,q}(A \mid T_\epsilon), \tag{3.23}$$

old3.23 whence, on taking the limits as $\Delta \to \mathbb{Z}^d$, $\epsilon \downarrow 0$, $\Lambda \to \mathbb{Z}^d$ in that order,

$$\phi^0_{p,q}(A)\phi^0_{p,q}(T) \le \phi^0_{p,q}(A \cap T). \tag{3.24}$$

old3.24 The same argument is valid with T replaced by $\overline{T}$, so that

$$\phi^0_{p,q}(A)\phi^0_{p,q}(\overline{T}) \le \phi^0_{p,q}(A \cap \overline{T}) \tag{3.25}$$

old3.25 The sum of (3.24) and (3.25) holds with equality, whence (3.24) is itself an inequality. Applying it to $A = T_\epsilon$ and letting $\epsilon \downarrow 0$ once again, one obtains that $\phi^0_{p,q}(T)^2 = \phi^0_{p,q}(T)$, so that $\phi^0_{p,q}(T)$ equals either 0 or 1 as required.

The same argument is valid when $b = 1$, with the inequalities in (3.23)–(3.25) reversed. □

We close this section with an account of certain semicontinuity properties of the mean $\phi^b_{p,q}(X)$ of a random variable X, viewed as a function of p, q, for $b \in \{0, 1\}$. For $\omega, \xi \in \Omega$ and a box Λ, we recall from (3.2) the configuration ω^ξ_Λ given by

$$\omega^\xi_\Lambda(e) = \begin{cases} \omega(e) & \text{if } e \in \mathbb{E}_\Lambda, \\ \xi(e) & \text{otherwise.} \end{cases}$$

old3.26 **Proposition 3.26.** *Let* $0 \le p \le 1$ *and* $q \ge 1$.

(a) Comparison inequalities. *For* $b = 0, 1$, *the measures* $\phi^b_{p,q}$ *satisfy the comparison inequalities*:

(3.27)
$$\phi^b_{p_1,q_1} \le_{\mathrm{st}} \phi^b_{p_2,q_2} \quad \textit{if}: \ q_1 \ge q_2 \ge 1, \ \textit{and} \ p_1 \le p_2,$$

(3.28)
$$\phi^b_{p_1,q_1} \ge_{\mathrm{st}} \phi^b_{p_2,q_2} \quad \textit{if}: \ q_1 \ge q_2 \ge 1, \ \textit{and} \ \frac{p_1}{q_1(1-p_1)} \ge \frac{p_2}{q_2(1-p_2)}.$$

old3.28

(b) Upper-semicontinuity. *Let* X *be an increasing upper-semicontinuous random variable. Then* $\phi^1_{p,q}(X)$ *is an upper semicontinuous function of* (p, q), *and is therefore a right-continuous function of* p *and a left-continuous function of* q.

(c) Lower-semicontinuity. *Let* X *be an increasing lower-semicontinuous random variable. Then* $\phi^0_{p,q}(X)$ *is a lower semicontinuous function of* (p, q), *and is therefore a left-continuous function of* p *and a right-continuous function of* q.

Conditions for the semicontinuity of an increasing random varable are given in (3.4). An important class of increasing upper-semicontinuous functions is provided by the indicator functions $X = 1_A$ of increasing *closed* events A. It is easily seen by (3.4) that such an X is indeed upper-semicontinuous, and it follows by part (b) above that $\phi^1_{p,q}(A)$ is right-continuous in p and left-continuous in q (≥ 1). As an important example of such an event A, consider the event $\{0 \leftrightarrow \infty\}$, that there exists an infinite open path in $\mathbb{L}^d$ with endvertex 0.

Similarly, the indicator function of any increasing *open* event A is itself an increasing lower-semicontinuous random variable, and thus part (c) may be applied.

We note that parts (b) and (c) above apply to all increasing continuous random variables, and therefore to any indicator function $X = 1_B$ of an increasing cylinder event B.

Proof of Proposition 3.26. (a) This is a consequence of Theorems 2.46 and 3.10(a). (b) Let $\Lambda_n = [-n, n]^d$. Suppose X satisfies the given condition, and define X^b_n by $X^b_n(\omega) = X(\omega^b_{\Lambda_n})$ for $b = 0, 1$. Using stochastic orderings of measures, and (3.5), we find for $m \le n$ that

$$\begin{aligned} \phi^1_{p,q}(X) \le \phi^1_{\Lambda_n,p,q}(X) \le \phi^1_{\Lambda_n,p,q}(X^1_m) &\quad \text{since } X \le X^1_m \\ \to \phi^1_{p,q}(X^1_m) &\quad \text{as } n \to \infty \\ \to \phi^1_{p,q}(X) &\quad \text{as } m \to \infty, \end{aligned}$$

where we have used the monotone convergence theorem at the last step. Also,

$$\begin{aligned}\phi^1_{\Lambda_n,p,q}(X^1_n) &\ge \phi^1_{\Lambda_{n+1},p,q}(X^1_n) \qquad \text{since } \Lambda_n \subseteq \Lambda_{n+1}\\ &\ge \phi^1_{\Lambda_{n+1},p,q}(X^1_{n+1}) \quad \text{since } X^1_n \ge X^1_{n+1}.\end{aligned}$$

The two sets of inequalities above imply that the sequence $(\phi^1_{\Lambda_n,p,q}(X^1_n) : n \ge 1)$ is decreasing, and has limit $\phi^1_{p,q}(X)$. However, each $\phi^1_{\Lambda_n,p,q}(X^1_n)$ is a continuous function of p and q, whence $\phi^1_{p,q}(X)$ is upper semicontinuous. The claims follow by part (a).

(c) The argument of part (b) is valid with X^1_n replaced by X^0_n, the boundary condition 1 replaced by 0, and with the inequalities reversed. □

3.4 Infinite-volume random-cluster measures

There is a second way to construct infinite-volume measures, this avoids weak limits and works directly with the infinite lattice. The following definition is based upon the well known Dobrushin–Lanford–Ruelle (DLR) definition of a Gibbs state, [**60**, **79**, **136**]. It was introduced in [**87**, **88**], and discussed further in [**39**, **90**]. For any box Λ, we write $\mathcal{T}_\Lambda$ for the σ-field generated by the set $\{\omega(e) : e \in \mathbb{E}^d \setminus \mathbb{E}_\Lambda\}$ of states of edges having at least one endvertex outside Λ.

old3.29 **Definition 3.29.** Let $0 \le p \le 1$ and $q > 0$. A probability measure ϕ on $(\Omega, \mathcal{F})$ is called a *DLR-random-cluster measure* with parameters p and q if

(3.30)
$$\text{for all } A \in \mathcal{F} \text{ and finite boxes } \Lambda, \quad \phi(A \mid \mathcal{T}_\Lambda)(\xi) = \phi^\xi_{\Lambda,p,q}(A) \quad \text{for } \phi\text{-a.e. } \xi.$$

old3.30 The set of such measures is denoted $\mathcal{R}_{p,q}$.

The condition of this definition amounts to the following. Suppose we are given that the configuration off the finite box Λ is that of ξ. Then, for almost every $\xi \in \Omega$, the (conditional) measure on Λ is the finite-volume random-cluster measure $\phi^\xi_{\Lambda,p,q}$. It is not difficult to see, by a calculation of conditional probabilities, that no further generality may be gained by replacing the finite box Λ by a general finite subset of $\mathbb{Z}^d$.

The structure of $\mathcal{R}_{p,q}$ relative to the set $\mathcal{W}_{p,q}$ remains somewhat obscure. It is not known, for example, whether or not $\mathcal{W}_{p,q} \subseteq \mathcal{R}_{p,q}$, and indeed one needs some work even to demonstrate that $\mathcal{R}_{p,q}$ is non-empty. The best result in this direction to date is restricted to probability measures having a certain additional property. For $\omega \in \Omega$, let $I(\omega)$ be the number of infinite open clusters of ω. We say that a probability measure ϕ on $(\Omega, \mathcal{F})$ has the $0/1$-*infinite-cluster property* if $\phi(I \in \{0, 1\}) = 1$.

old3.31 **Theorem 3.31 [90, 91, 94].** *Let* $0 \le p \le 1$ *and* $q > 0$. *If* $\phi \in \overline{\mathrm{co}\, \mathcal{W}_{p,q}}$ *and* ϕ *has the* $0/1$*-infinite-cluster property, then* $\phi \in \mathcal{R}_{p,q}$.

A sufficient condition for the $0/1$-infinite cluster property is provided by the uniqueness theorem of Burton–Keane, [**42**], namely translation-invariance[6] and so-called 'finite energy'. A probability measure ϕ on $(\Omega, \mathcal{F})$ is said to have the 'finite-energy property' if

$$0 < \phi(J_e \mid \mathcal{F}_{\mathbb{E}^d \setminus \{e\}}) < 1 \quad \phi\text{-a.s., for all } e \in \mathbb{E}^d, \tag{3.32}$$

old3.32 where, as before, J_e is the event that e is open. (Recall Theorem 3.17(b).) The meaning of this condition is that, conditional on the state outside any finite region V, every configuration inside V has a probability which lies strictly between 0 and 1. (Explain)

old3.33 **Theorem 3.33 ?[90, 91, 94].** *Let* $0 \le p \le 1$ *and* $q > 0$.

(a) *There exists a translation-invariant probability measure* ϕ *belonging to* $\overline{\mathrm{co}\, \mathcal{W}_{p,q}}$.

(b) *If* $0 < p < 1$, *then every* $\phi \in \overline{\mathrm{co}\, \mathcal{W}_{p,q}}$ *has the finite-energy property.*

(c) *If* $\phi \in \overline{\mathrm{co}\, \mathcal{W}_{p,q}}$ *is translation-invariant, then* ϕ *has the* $0/1$*-infinite cluster property.*

Theorems 3.31 and 3.33 imply jointly that $|\mathcal{R}_{p,q}| \ne \varnothing$ when $0 < p < 1$ and $q > 0$. We now verify some of the basic properties of the set $\mathcal{R}_{p,q}$.

old3.34 **Theorem 3.34.** (a) Existence. *If* $0 \le p \le 1$ *and* $q > 0$, $\mathcal{R}_{p,q}$ *is non-empty and contains at least one translation-invariant member of* $\overline{\mathrm{co}\, \mathcal{W}_{p,q}}$. *Furthermore,* $\mathcal{R}_{p,q}$ *is a convex set of measures.*

(b) Stochastic ordering. *If, further,* $q \ge 1$, *then* $\phi^b_{p,q} \in \mathcal{R}_{p,q}$ *for* $b = 0, 1$, *and*

$$\phi^0_{p,q} \le_{\mathrm{st}} \phi \le_{\mathrm{st}} \phi^1_{p,q} \qquad \textit{for all } \phi \in \mathcal{R}_{p,q}. \tag{3.35}$$

old3.35

It is an important open problem to identify all pairs (p, q) under which $\phi^0_{p,q} = \phi^1_{p,q}$. We have from (3.21) and (3.35) that, when $q \ge 1$,

$$|\mathcal{W}_{p,q}| = |\mathcal{R}_{p,q}| = 1 \quad \text{if and only if} \quad \phi^0_{p,q} = \phi^1_{p,q}, \tag{3.36}$$

old3.36 so this amounts to asking for which pairs (p, q) there exists both a unique DLR-random-cluster measure and a unique limit-random-cluster measure. Various partial answers are known, see Theorems 3.?, and a conjecture appears at (4.31).

[6]The proof in [**42**] assumes ergodicity of the probability measure, rather than just translation-invariance. Further comments about the extension to translation-invariant measures may be found in [**43**], [**81**, page 42].

We make a note concerning the ergodicity of $\phi^b_{p,q}$, $b = 0, 1$, of Theorem 3.19(c). By (3.35), the $\phi^b_{p,q}$ are extremal within the class $\mathcal{R}_{p,q}$, and it follows by a general argument set out in [**79**, Theorem 7.7, Remark 7.13, Proposition 14.9] that they are necessarily ergodic, [**87**, **88**, **90**].

The 0/1-infinite-cluster property is linked to the property of so-called 'almost sure quasilocality', a matter discussed in [**160**]. More...

We note two further properties of DLR-random-cluster-measures, namely the finite-energy property, and positive association when $q \ge 1$. Let $e = \langle x, y\rangle$ be an edge, and let K_e be the event that x and y are joined by an open path not using e. Let $\mathcal{T}_e$ be the σ-field generated by $\{\omega(f) : f \in \mathbb{E}^d \setminus \{e\}\}$.

old3.37 **Proposition 3.37.** *Let $0 \le p \le 1$, $q > 0$, and $\phi \in \mathcal{R}_{p,q}$.*
(a) Finite-energy property. *For ϕ-almost every ω,*

$$\phi(J_e \mid \mathcal{T}_e)(\omega) = \begin{cases} p & \text{if } \omega \in K_e, \\ \dfrac{p}{p + (1-p)q} & \text{if } \omega \notin K_e. \end{cases} \tag{3.38}$$

old3.38

(b) Positive association. *If $q \ge 1$ and the tail σ-field of ϕ is trivial, then ϕ is positively associated.*

Check (3.38).

In the proof of Theorem 3.31, we shall make use of the following technical result.

old3.39 **Lemma 3.39.** *Let $0 \le p \le 1$, $q > 0$, and let $\phi \in \overline{\mathrm{co}\mathcal{W}_{p,q}}$ have the 0/1-infinite cluster property. For any finite box Λ and any cylinder event A defined in terms of the states of edges in $\mathbb{E}_\Lambda$, the random variable $g(\omega) = \phi^\omega_{\Lambda,p,q}(A)$ is ϕ-a.s. continuous, using the product topology on its domain Ω.*

Proof. The random variable $g(\omega) = \phi^\omega_{\Lambda,p,q}(A)$ has a set of discontinuities defined as

$$D(g) = \bigcap_\Delta \left\{\omega : \sup_{\zeta:\zeta=\omega \text{ on } \Delta} |g(\zeta) - g(\omega)| > 0\right\} \tag{3.40}$$

old3.40 where the intersection is over all boxes Δ containing Λ, and we write '$\zeta = \omega$ on Δ' if $\zeta(e) = \omega(e)$ for all $e \in \mathbb{E}_\Delta$. Let $D_{\Lambda,\Delta}$ be the set of all ω ($\in \Omega$) with the property that there exist two points $u, v \in \partial\Lambda$ such that both u and v are joined to $\partial\Delta$ by paths using ω-open edges lying in $\mathbb{E}_\Delta \setminus \mathbb{E}_\Lambda$, but that u is not joined to v by such a path. If $\zeta = \omega$ on Δ, the difference $|g(\zeta) - g(\omega)|$ can be non-zero only if $D_{\Lambda,\Delta}$ occurs. To see this, note that, if $D_{\Lambda,\Delta}$ does not occur, then $k(\zeta, \Lambda) = k(\omega, \Lambda)$ for all $\zeta \in \Omega$ such that $\zeta = \omega$ on Δ, implying that $g(\zeta) = g(\omega)$. It follows that

$$D(g) \subseteq \bigcap_\Delta D_{\Lambda,\Delta},$$

whence

$$\phi(D(g)) \le \phi\Big(\bigcap_{\Delta} D_{\Lambda,\Delta}\Big).$$

However,

$$\bigcap_{\Delta} D_{\Lambda,\Delta} = \{I_\Lambda \ge 2\},$$

where I_Λ is the number of infinite open clusters of $\mathbb{E}^d \setminus \mathbb{E}_\Lambda$.

If $p = 0$ (respectively, $p = 1$), then it is elementary that ϕ is concentrated on the configuration 0 (respectively, 1), and in either case

$$\phi(I_\Lambda \ge 2) = 0. \tag{3.41}$$

old3.41 If $0 < p < 1$ then, by Theorem 3.33(b), there exists a positive number C such that

$$\phi(I \ge 2) \ge C\phi(I_\Lambda \ge 2).$$

Since ϕ has the 0/1-infinite cluster property by assumption, (3.41) follows. In any case, we deduce as required that $\phi(D(g)) = 0$. □

Proof of Theorem 3.31. The proof is straightforward under the stronger hypothesis that $\phi \in \mathcal{W}_{p,q}$, and so we begin with this special case. Suppose then that $\mathbf{\Lambda} = (\Lambda_n : n \ge 1)$, $\xi \in \Omega$, and ϕ ($\in \mathcal{W}_{p,q}$) are such that

$$\phi = \lim_{n\to\infty} \phi^{\xi}_{\Lambda_n,p,q},$$

and assume that ϕ has the 0/1-infinite cluster property. Let Λ be a box and let A be an event defined in terms of the states of edges in $\mathbb{E}_\Lambda$. By Lemma 3.13,

$$\text{if } \Lambda \subseteq \Lambda_n, \quad \phi^{\omega}_{\Lambda,p,q}(A) = \phi^{\xi}_{\Lambda_n,p,q}(A \mid \mathcal{T}_\Lambda)(\omega) \quad \text{for } \phi^{\xi}_{\Lambda_n,p,q}\text{-a.e. } \omega. \tag{3.42}$$

old3.42 Let B be a cylinder event in $\mathcal{T}_\Lambda$. By Lemma 3.39 applied to the measure ϕ, the function $1_B(\omega)\phi^{\omega}_{\Lambda,p,q}(A)$ is ϕ-a.s. continuous, whence

$$\begin{aligned}
\phi\big(1_B(\cdot)\phi^{\cdot}_{\Lambda,p,q}(A)\big) &= \lim_{n\to\infty} \phi^{\xi}_{\Lambda_n,p,q}\big(1_B(\cdot)\phi^{\cdot}_{\Lambda,p,q}(A)\big) \\
&= \lim_{n\to\infty} \phi^{\xi}_{\Lambda_n,p,q}\big(1_B(\cdot)\phi^{\xi}_{\Lambda_n,p,q}(A \mid \mathcal{T}_\Lambda)\big) \qquad \text{by (3.42)3.42)} \\
&= \lim_{n\to\infty} \phi^{\xi}_{\Lambda_n,p,q}(A \cap B) \\
&= \phi(A \cap B).
\end{aligned}$$

Since $\mathcal{T}_\Lambda$ is generated by the collection of all such B, we deduce that

$$\phi(A \mid \mathcal{T}_\Lambda) = \phi^{\cdot}_{\Lambda,p,q}(A) \qquad \phi\text{-a.s.}, \tag{3.43}$$

old3.43 whence $\phi \in \mathcal{R}_{p,q}$ as required for this special case.

We require a further lemma for the general case of Theorem 3.31. Let $X : \Omega \to \mathbb{R}$ be a bounded function, set

$$v(X) = \sup_{\omega,\omega' \in \Omega} |X(\omega) - X(\omega')|,$$

and let the discontinuity set $D(X)$ be given as in (3.40).

old3.44 **Lemma 3.44.** *Let μ_n, μ be probability measures on $(\Omega, \mathcal{F})$ such that $\mu_n \Rightarrow \mu$ as $n \to \infty$. Then, for any bounded measurable function $X : \Omega \to \mathbb{R}$,*

$$\limsup_{n\to\infty} |\mu_n(X) - \mu(X)| \le v(X)\mu(D(X)).$$

Proof. By [**62**, Theorem 11.7.2], there exists a probability space $(\Sigma, \mathcal{G}, \mathbb{P})$ and random variables $\eta_n, \eta : \Sigma \to \Omega$ such that: η_n has law μ_n, η has law μ, and $\eta_n \to \eta$ almost surely. Therefore,

$$X(\eta_n)1_C(\eta) \to X(\eta)1_C(\eta) \qquad \mathbb{P}\text{-a.s.},$$

where $C = \Omega \setminus D(X)$ is the set of points ω at which X is continuous. Now,

$$\begin{aligned}
|\mu_n(X) - \mu(X)| &= \big|\mathbb{P}\big(X(\eta_n) - X(\eta)\big)\big| \\
&\le \mathbb{P}\big|X(\eta_n) - X(\eta)\big| \\
&= \mathbb{P}\big(|X(\eta_n) - X(\eta)|1_C(\eta)\big) + \mathbb{P}\big(|X(\eta_n) - X(\eta)|1_{\overline{C}}(\eta)\big) \\
&\le \mathbb{P}\big(|X(\eta_n) - X(\eta)|1_C(\eta)\big) + v(X)\mathbb{P}(1_{\overline{C}}(\eta)) \\
&\to 0 + v(X)\mu(\overline{C}) = v(X)\mu(D(X)),
\end{aligned}$$

as asserted. □

Let $\phi \in \overline{\text{co}\, \mathcal{W}_{p,q}}$, and write $\phi = \lim_{n\to\infty} \phi_n$ where

$$\phi_n = \frac{1}{K_n} \sum_{1 \le i \le K_n} \phi_{n,i}$$

and each $\phi_{n,i}$ is given as

$$\phi_{n,i} = \lim_{\Delta \to \mathbb{Z}^d} \phi^{\xi_{n,i}}_{\Delta,p,q}. \tag{3.45}$$

old3.45 Equation (3.45) is actually a shorthand, since Δ will in general approach $\mathbb{Z}^d$ along some sequence of boxes which depends on the values of n and i; this will not be important in what follows.

Let Λ be a box, and let $A \in \mathcal{F}_\Lambda$. Let B be a cylinder event in $\mathcal{T}_\Lambda$. Since $\mathcal{F}_\Lambda$ is generated by such cylinder events, it is enough to prove that

$$\phi\big(1_B(\cdot)\phi^{\cdot}_{\Lambda,p,q}(A)\big) = \phi(A \cap B). \tag{3.46}$$

old3.46

Let $D_{\Lambda,\Delta}$ be the event given after (3.40), noting as before that

$$D_{\Lambda,\Delta} \downarrow \{I_\Lambda \ge 2\} \qquad \text{as } \Delta \uparrow \mathbb{Z}^d, \tag{3.47}$$

old3.47 where I_Λ is the number of infinite open clusters contained in $\mathbb{E}^d \setminus \mathbb{E}_\Lambda$.

By (3.45) and Lemma 3.44,

$$\limsup_{\Delta\to\mathbb{Z}^d}\Big|\phi_{n,i}(1_B\phi^{\cdot}_{\Lambda,p,q}(A)) - \phi^{\xi_{n,i}}_{\Delta,p,q}(1_B\phi^{\cdot}_{\Lambda,p,q}(A)\Big| \le 2\phi_{n,i}(I_\Lambda \ge 2). \tag{3.48}$$

old3.48 We have by Lemma 3.13 that

$$\phi^{\xi_{n,i}}_{\Delta,p,q}(1_B\phi^{\cdot}_{\Lambda,p,q}(A)) = \phi^{\xi_{n,i}}_{\Delta,p,q}(A\cap B) \qquad \text{for all large } \Delta,$$

and therefore, on taking the limit as $\Delta \to \mathbb{Z}^d$,

$$\Big|\phi_{n,i}(1_B\phi^{\cdot}_{\Lambda,p,q}(A)) - \phi_{n,i}(A\cap B)\Big| \le 2\phi_{n,i}(I_\Lambda \ge 2). \tag{3.49}$$

old3.49

By (3.45),

$$\phi(1_B\phi^{\cdot}_{\Lambda,p,q}(A)) = \lim_{n\to\infty}\frac{1}{K_n}\sum_i \phi_{n,i}(1_B\phi^{\cdot}_{\Lambda,p,q}(A)),$$
$$\phi(A\cap B) = \lim_{n\to\infty}\frac{1}{K_n}\sum_i \phi_{n,i}(A\cap B),$$

whence

$$\begin{aligned}
&\Big|\phi(1_B\phi^{\cdot}_{\Lambda,p,q}(A)) - \phi(A\cap B)\Big| \\
&\qquad\le \limsup_{n\to\infty}\frac{1}{K_n}\sum_i\Big|\phi_{n,i}(1_B\phi^{\cdot}_{\Lambda,p,q}(A)) - \phi_{n,i}(A\cap B)\Big| \\
&\qquad\le \limsup_{n\to\infty}\frac{1}{K_n}\sum_i 2\phi_{n,i}(I_\Lambda\ge 2) \qquad \text{by (3.47)} \\
&\qquad\le \limsup_{n\to\infty}\frac{1}{K_n}\sum_i 2\phi_{n,i}(D_{\Lambda,\Delta}) \qquad \text{if } \Delta\supseteq\Lambda, \text{ by (3.48)} \\
&\qquad= \limsup_{n\to\infty} 2\phi_n(D_{\Lambda,\Delta}) \\
&\qquad= 2\phi(D_{\Lambda,\Delta}) \\
&\qquad\to 2\phi(I_\Lambda\ge 2) \qquad \text{as } \Delta\to\mathbb{Z}^d.
\end{aligned}$$

The final probability equals 0 as in (3.41), and therefore (3.46) holds. □

Proof of Theorem 3.33. (a) Since $\phi^0_{p,q} \in \mathcal{W}_{p,q}$ when $q \ge 1$, we shall consider the case when $q < 1$ only. By Theorem 3.10(a), we may find $\phi \in \mathcal{W}_{p,q}$. Let

$$\psi_m = \frac{1}{|\Delta_m|}\sum_{x\in\Delta_m}\tau_x\circ\phi \tag{3.50}$$

old3.50 where $\Delta_m = [-m, m]^d$, and $\tau_x \circ \phi$ is the probability measure on $(\Omega, \mathcal{F})$ given by $\tau_x \circ \phi(A) = \phi(\tau_x A)$ for the shift $\tau_x(y) = x + y$ of the lattice. Clearly $\tau_x \circ \phi \in \mathcal{W}_{p,q}$ for all x, whence ψ_m belongs to the convex hull of $\mathcal{W}_{p,q}$. Let ψ be an accumulation point of the family $\{\psi_m : m \geq 1\}$ of measures.

Let $\mathbf{e}$ be a unit vector of $\mathbb{Z}^d$. By (3.50), for any event A,

$$(3.51) \qquad |\psi_m(A) - \tau_{\mathbf{e}} \circ \psi_m(A)| \leq \frac{|\partial \Delta_m|}{|\Delta_m|} \to 0 \qquad \text{as } m \to \infty,$$

old3.51 whence ψ is translation-invariant. Certainly $\psi \in \overline{\mathrm{co}\,\mathcal{W}_{p,q}}$, and the proof of (a) is complete.

(b) This follows from Theorem 3.17(b).

(c) If $p = 0$ (respectively, $p = 1$), then ϕ is concentrated on the configuration 0 (respectively, 1), and the claim holds trivially. If $0 < p < 1$, it follows from (b) and the main theorem of Burton and Keane [**42**]. (See the remark after Theorem 3.31.) □

Proof of Theorem 3.34. (a) By Theorem 3.33, there exists $\phi \in \overline{\mathrm{co}\,\mathcal{W}_{p,q}}$ with the 0/1-infinite cluster property. By Theorem 3.31, $\phi \in \mathcal{R}_{p,q}$, so that $\mathcal{R}_{p,q}$ is non-empty. Its convexity follows immediately from Definition 3.29: if $\phi, \psi \in \mathcal{R}_{p,q}$ and $0 \leq \alpha \leq 1$, then $\alpha\phi + (1-\alpha)\psi$ satisfies the condition of the definition.

(b) Assume $q \geq 1$. We have from Theorem 3.19(b) that the $\phi^b_{p,q}$ are translation-invariant, whence by Theorem 3.33(c) they have the 0/1-infinite cluster property. By Theorem 3.31, each belongs to $\mathcal{R}_{p,q}$. Inequality (3.35) follows from Lemma 3.14(b) and Definition 3.29, on taking the limit in the former as $\Lambda \to \mathbb{Z}^d$. □

Proof of Proposition (3.27). Part (a) is an immediate consequence of Definition 3.29, taken in conjunction with (2.3). Turning to part (b), suppose $q \geq 1$, and let X and Y be continuous increasing random variables on Ω. We have for $\phi \in \mathcal{R}_{p,q}$ that

$$\begin{aligned} \phi(XY) &= \phi\big(\phi(XY \mid \mathcal{T}_\Lambda)\big) \\ &= \phi\big(\phi^{\cdot}_{\Lambda,p,q}(XY)\big) \\ &\geq \phi\big(\phi^{\cdot}_{\Lambda,p,q}(X)\phi^{\cdot}_{\Lambda,p,q}(Y)\big) \quad \text{by positive association} \\ &= \phi\big(\phi(X \mid \mathcal{T}_\Lambda)\phi(Y \mid \mathcal{T}_\Lambda)\big) \\ &\to \phi\big(\phi(X \mid \mathcal{T})\phi(Y \mid \mathcal{T})\big) \quad \text{as } \Lambda \to \mathbb{Z}^d \end{aligned}$$

Gap here. by the backward martingale convergence theorem [**??**], where $\mathcal{T} = \bigcap_\Lambda \mathcal{T}_\Lambda$ is the tail σ-field. If $\mathcal{T}$ is trivial under ϕ, then

$$\phi(X \mid \mathcal{T}) = \phi(X), \quad \phi(Y \mid \mathcal{T}) = \phi(Y) \qquad \phi\text{-a.s.}$$

and the required positive association inequality follows. □

3.5 Uniqueness via convexity

We address next the question of the uniqueness (or not) of limit- and DLR-random-cluster measures on $\mathbb{L}^d$ for given p and q. The main result of this section is the following theorem. Further results concerning the uniqueness of measures may be found at Theorems X and Y. A set is said to be *countable* if it can be put in one–one correspondence with a (possibly empty) subset of the integers.

Monotonicity of P(conn) and $E|C|^{-1}$ when $0 < q < 1$.

old3.52 **Theorem 3.52 [88, 90].** *Let $q \geq 1$. There exists a (possibly empty) countable subset $\mathcal{D}_q$ of $[0,1]$ such that $\phi^0_{p,q} = \phi^1_{p,q}$, and hence there exists a unique random-cluster measure in that $|\mathcal{W}_{p,q}| = |\mathcal{R}_{p,q}| = 1$, if and only if $p \notin \mathcal{D}_q$.*

This will be proved by showing that the asymptotics of the partition function do not depend on the choice of boundary condition, and then by relating the differentiability of the limit function to the uniqueness of measures. Certain convexity properties of the limit function will play a role in studying its differentiability. Rather than working with the usual partition function from (3.12),

$$Z^\xi_\Lambda(p,q) = \sum_{\omega\in\Omega^\xi_\Lambda} \left\{ \prod_{e\in\mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)}, \tag{3.53}$$

old3.53 we shall use the function $Y^\xi_\Lambda : \mathbb{R}^2 \to \mathbb{R}$ given by

$$Y^\xi_\Lambda(\pi,\kappa) = \sum_{\omega\in\Omega^\xi_\Lambda} \exp\bigl\{\pi|\mathbb{E}_\Lambda \cap \eta(\omega)| + \kappa k(\omega,\Lambda)\bigr\}. \tag{3.54}$$

old3.54 This is related to $Z^\xi_\Lambda(p,q)$ as in Section 2.7 by

$$Z^\xi_\Lambda(p,q) = (1-p)^{|\mathbb{E}_\Lambda|} Y^\xi_\Lambda(\pi,\kappa). \tag{3.55}$$

old3.55 where $\pi = \pi(p)$ and $\kappa = \kappa(q)$ are given by

$$\pi(p) = \log\{p/(1-p)\},\ \kappa(q) = \log q, \tag{3.56}$$

old3.56 Note that

$$Z^\xi_\Lambda(p,1) = 1, \qquad Y_\Lambda(\pi,0) = (1-p)^{-|\mathbb{E}_\Lambda|}. \tag{3.57}$$

old3.57

We introduce next a function $G(\pi,\kappa)$ which describes the exponential asymptotics of $Y^\xi_\Lambda(\pi,\kappa)$ as $\Lambda \to \mathbb{Z}^d$. In line with the terminology of statistical mechanics, we call this function the *pressure*. All logarithms will for convenience be natural logarithms.

old3.58 **Theorem 3.58 [83, 88, 90].** *The finite limits*

$$(3.59)\qquad G(\pi,\kappa) = \lim_{\Lambda\to\mathbb{Z}^d}\left\{\frac{1}{|\mathbb{E}_\Lambda|}\log Y_\Lambda^\xi(\pi,\kappa)\right\},\qquad (\pi,\kappa)\in\mathbb{R}^2,$$

old3.59 *exist and are independent of ξ ($\in \Omega$) and of the way in which $\Lambda \to \mathbb{Z}^d$. The 'pressure' function G is a convex function on its domain $\mathbb{R}^2$.*

In the proof, we shall see that G may be approximated from below and above to any required degree of accuracy by smooth functions of (π,κ); see (3.68)–(3.71).

We shall identify the set $\mathcal{D}_q$ of Theorem 3.52 as $\mathcal{D}_q = \mathcal{D}'_{\kappa(q)}$, a set given in the next theorem with $\kappa(q) = \log q$. This turns out to correspond exactly to the points of non-differentiability of the convex function G. Recall that, by convexity, G is differentiable at (π,κ) if and only if G has both its partial derivatives at this point.

Let $\mathcal{D}'$ be the set of all (π,κ) at which G is not differentiable. Since G is convex, $\mathcal{D}'$ has Lebesgue measure 0, and indeed $\mathcal{D}'$ may be covered by a countable collection of rectifiable curves (see [**65**, Theorem 8.18], [**173**, Theorem 2.2.4]). For any line λ of $\mathbb{R}^2$, the restriction of G to λ is convex, whence G restricted to Λ is differentiable along λ except at countably many points. Each point of non-differentiability on λ lies in $\mathcal{D}'$, but the converse is not necessarily true.

The two partial derivatives of G have special physical significance for the random-cluster model, and one may show when $q \ge 1$ (that is, $\kappa \ge 0$), as follows, that G has one partial derivative at any point (π,κ) if and only if it has both.

old3.60 **Theorem 3.60.**

(a) *For each $\kappa \in \mathbb{R}$, there exists a (possibly empty) countable subset $\mathcal{D}'_\kappa$ of $\mathbb{R}$ such that $G(\pi,\kappa)$ is a differentiable function of π if and only if $\pi \notin \mathcal{D}'_\kappa$.*

(b) *For each $\pi \in \mathbb{R}$, there exists a (possibly empty) countable subset $\mathcal{D}''_\pi$ of $\mathbb{R}$ such that $G(\pi,\kappa)$ is a differentiable function of κ if and only if $\kappa \notin \mathcal{D}''_\pi$.*

(c) *Let $\kappa \ge 0$. Then*

$$\mathcal{D}'_\kappa = \mathcal{D}' \cap \big\{\mathbb{R}\times\{\kappa\}\big\},\qquad \mathcal{D}''_\pi = \mathcal{D}' \cap \big\{\{\pi\}\times[0,\infty)\big\},$$

which is to say that, for any given (π,κ), exactly one of the following holds:

(i) *$(\pi,\kappa) \in \mathcal{D}'$, and G has neither partial derivative at (π,κ),*

(ii) *$(\pi,\kappa) \notin \mathcal{D}'$, and G has both partial derivatives at (π,κ).*

Parts (a) and (b) follow from the remarks prior to the theorem. The proof of part (c) is deferred until later in this section.

For given $q > 0$, one thinks of $\mathcal{D}_q = \mathcal{D}'_{\kappa(q)}$ as the set of 'bad' values of p. The situation when $q < 1$ is obscure, but, for $q \ge 1$, the set $\mathcal{D}_q$ is exactly the set of singularities of the random-cluster model in the sense of the next theorem.

Here is some further notation. Suppose that $q \geq 1$. We denote by $h^b(p,q)$ the edge-density under the measure $\phi^b_{p,q}$,

$$h^b(p,q) = \phi^b_{p,q}(\omega(e)=1), \qquad b = 0, 1, \tag{3.61}$$

old3.61 and we note that $h^b(p,q)$ does not depend on the choice of e, by the translation-invariance of the $\phi^b_{p,q}$. We write

$$F(p,q) = G(\pi,\kappa) \tag{3.62}$$

old3.62 where (p,q) and (π,κ) are related by (3.56). We shall use the word 'pressure' for both functions F and G.

old3.63 **Theorem 3.63.** *Suppose that $0 < p < 1$ and $q \geq 1$. The following four statements are equivalent.*

(a) $p \notin \mathcal{D}_q$.

(b) *The edge-density $h^b(x,q)$ is a continuous function of x at the point $x = p$, for $b = 0, 1$.*

(c) *It is the case that $h^0(p,q) = h^1(p,q)$.*

(d) *There is a unique random-cluster measure with parameters p and q, that is, $|\mathcal{W}_{p,q}| = |\mathcal{R}_{p,q}| = 1$.*

What is the set $\mathcal{D}_q$? We shall return to this question in Section ?, but in the meantime we summarise the expected situation. Let $d \geq 2$ be given, and assume $q \geq 1$. It is thought to be the case that $\mathcal{D}_q$ is empty for small values of q (≥ 1), and is a singleton point (i.e., the critical value $p_c(q)$, see Section 4.1) when q is large. It is conjectured that there exists $Q = Q(\mathbb{L}^d)$ such that

$$\mathcal{D}_q = \begin{cases} \varnothing & \text{if } q < Q, \\ \{p_c(q)\} & \text{if } q > Q. \end{cases} \tag{3.64}$$

old3.64 This would imply in particular that $|\mathcal{R}_{p,q}| = 1$ unless $q \geq Q$ and $p = p_c(q)$. A further issue concerns the structure of $\mathcal{R}_{p,q}$ in situations where $|\mathcal{R}_{p,q}| > 1$.

(??What if $q = Q$??)

Proof of Theorem 3.58. Let $0 < p < 1$ and $q > 0$, and let (π,κ) be given by (3.57). We shall use a standard argument of statistical mechanics, namely the near-multiplicativity of $Y^\xi_\Lambda(\pi,\kappa)$ viewed as a function of Λ. The irrelevance to the limit of the boundary condition ξ hinges on the fact that $|\partial\Lambda|/|\Lambda| \to \infty$ as $\Lambda \to \mathbb{Z}^d$.

We show first that (3.59) holds with $\xi = 0$, and shall for the moment suppress explicit reference to the boundary condition. Let $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$, write $|\mathbf{n}| = n_1 n_2 \dots n_d$, and let $\Lambda_{\mathbf{n}}$ be the box $\prod_{i=1}^d [1, n_i]$. By the invariance of $Z_\Lambda(p,q)$ under translations of Λ, we may restrict ourselves to boxes of this type.

Check the following

We fix $\mathbf{k} \in \mathbb{N}^d$, and write

$$(\mathbf{n}, \mathbf{k}) = \left(k_i \left\lfloor \frac{n_i}{k_i} \right\rfloor : 1 \le i \le d\right), \quad \left\lfloor \frac{\mathbf{n}}{\mathbf{k}} \right\rfloor = \prod_{i=1}^{d} \left\lfloor \frac{n_i}{k_i} \right\rfloor.$$

By Theorem 2.91, for $\mathbf{n} \ge \mathbf{k}$,

(3.65)

$$\begin{aligned}
&\left\lfloor \frac{\mathbf{n}}{\mathbf{k}} \right\rfloor \left(\log Z_{\Lambda_{\mathbf{k}}} - d\left(\sum_{i=1}^{d} \frac{\mathbf{k}}{k_i}\right)\log(1 \vee q)\right) + \log Z_{\Lambda_{\mathbf{n}} \setminus \Lambda_{(\mathbf{n},\mathbf{k})}} \\
&\quad \le \log Z_{\Lambda_{\mathbf{n}}} \\
&\quad \le \left\lfloor \frac{\mathbf{n}}{\mathbf{k}} \right\rfloor \left(\log Z_{\Lambda_{\mathbf{k}}} - d\left(\sum_{i=1}^{d} \frac{\mathbf{k}}{k_i}\right)\log(1 \wedge q)\right) + \log Z_{\Lambda_{\mathbf{n}} \setminus \Lambda_{(\mathbf{n},\mathbf{k})}},
\end{aligned}$$

old3.65 Check details here and later.

and furthermore,

$$\begin{aligned}
\big(|\mathbf{n}| - |\mathbf{k}| \cdot \lfloor \mathbf{n}/\mathbf{k} \rfloor\big) \log(1 \wedge q) &\le \log Z_{\Lambda_{\mathbf{n}} \setminus \Lambda_{(\mathbf{n},\mathbf{k})}} \\
&\le \big(|\mathbf{n}| - |\mathbf{k}| \cdot \lfloor \mathbf{n}/\mathbf{k} \rfloor\big) \log(1 \vee q).
\end{aligned}$$

We divide by $|\mathbf{n}|$ and take the limit as the n_i tend to ∞, to find that, for all $\mathbf{k}$,

(3.66)

$$\begin{aligned}
&\frac{1}{|\mathbf{k}|} \log Z_{\Lambda_{\mathbf{k}}} - d\left(\sum_{i=1}^{d} \frac{1}{k_i}\right)\log(1 \vee q) \\
&\qquad \le \liminf_{\mathbf{n}\to\infty} \left\{\frac{1}{|\mathbf{n}|} \log Z_{\Lambda_{\mathbf{n}}}\right\} \le \limsup_{\mathbf{n}\to\infty} \left\{\frac{1}{|\mathbf{n}|} \log Z_{\Lambda_{\mathbf{n}}}\right\} \\
&\qquad \le \frac{1}{|\mathbf{k}|} \log Z_{\Lambda_{\mathbf{k}}} - d\left(\sum_{i=1}^{d} \frac{1}{k_i}\right)\log(1 \wedge q).
\end{aligned}$$

old3.66 Assume that $q \ge 1$, a similar argument is valid when $q < 1$. By the first inequality of (3.66), the limit

$$H(p, q) = \lim_{\mathbf{n}\to\infty} \left\{\frac{1}{|\mathbf{n}|} \log Z_{\Lambda_{\mathbf{n}}}\right\} \tag{3.67}$$

old3.67 exists, and furthermore

$$H(p, q) = \sup_{\mathbf{k}} \left\{\frac{1}{|\mathbf{k}|} \log Z_{\Lambda_{\mathbf{k}}} - d\left(\sum_{i=1}^{d} \frac{1}{k_i}\right)\log(1 \vee q)\right\} \tag{3.68}$$

$$= \inf_{\mathbf{k}} \left\{\frac{1}{|\mathbf{k}|} \log Z_{\Lambda_{\mathbf{k}}} - d\left(\sum_{i=1}^{d} \frac{1}{k_i}\right)\log(1 \wedge q)\right\}. \tag{3.69}$$

old3.69 Since $Z_{\Lambda_{\mathbf{k}}}$ is a continuous function of p and q, these equations imply that $H(p,q)$ may be approximated from below and above to any degree of accuracy by continuous functions, and is therefore continuous. We will obtain greater regularity from the claim of convexity to be proved soon. Evidently, as $\Lambda \to \mathbb{Z}^d$,

$$\frac{1}{\mathbb{E}_\Lambda} \log Z_\Lambda \to \frac{1}{d} H(p,q), \tag{3.70}$$

old3.70 and, by (3.55) and (3.62),

$$\frac{1}{\mathbb{E}_\Lambda} \log Y_\Lambda(\pi,\kappa) \to -\log(1-p) + \frac{1}{d} H(p,q) = F(p,q) = G(\pi,\kappa). \tag{3.71}$$

old3.71

We show next that the same limit is valid with a general boundary condition. Let Λ be a finite box, and let

$$G_\Lambda^\xi(\pi,\kappa) = \frac{1}{|\mathbb{E}_\Lambda|} \log Y_\Lambda^\xi(\pi,\kappa). \tag{3.72}$$

old3.72 For $\omega, \xi \in \Omega$, we define ω_Λ^ξ as in (3.2), noting that $\omega_\Lambda^\xi \in \Omega_\Lambda^\xi$. Clearly,

$$k(\omega_\Lambda^0, \Lambda) - |\partial\Lambda| \le k(\omega_\Lambda^1, \Lambda) \le k(\omega_\Lambda^\xi, \Lambda) \le k(\omega_\Lambda^0, \Lambda),$$

whence

$$Y_\Lambda^0(\pi,\kappa) e^{-\kappa|\partial\Lambda|} \le Y_\Lambda^\xi(\pi,\kappa) \le Y_\Lambda^0(\pi,\kappa) \qquad \text{if } \kappa \ge 0,$$

and with the inequalities reversed when $\kappa < 0$. Therefore,

$$G_\Lambda^0(\pi,\kappa) - \kappa \frac{|\partial\Lambda|}{|\mathbb{E}_\Lambda|} \le G_\Lambda^\xi(\pi,\kappa) \le G_\Lambda^0(\pi,\kappa) \qquad \text{if } \kappa \ge 0,$$

with the inequalities reversed if $\kappa < 0$. Since $|\partial\Lambda|/|\mathbb{E}_\Lambda| \to 0$ as $\Lambda \to \mathbb{Z}^d$, the limit of G_Λ^ξ exists by (3.71), and it is independent of the choice of ξ.

It is clear from its form that $G_\Lambda^\xi(\pi,\kappa)$ is a convex function on its domain $\mathbb{R}^2$. Indeed, Theorem 2.101(b) gives a representation of its second derivative in any given arbitrary direction as the variance of a random variable. We note from Theorem 2.101(a) for later use that

$$\nabla G_\Lambda^\xi(\pi,\kappa) = \frac{1}{|\mathbb{E}_\Lambda|} \Big(\phi_{\Lambda,p,q}^\xi(|\eta(\omega) \cap \mathbb{E}_\Lambda|), \phi_{\Lambda,p,q}^\xi(k(\omega,\Lambda)) \Big). \tag{3.73}$$

old3.73

Since, for any $\xi \in \Omega$, $(G_\Lambda^\xi(\pi,\kappa))_\Lambda$ is a family of convex functions of (π,κ) which converge to the finite limit function $G(\pi,\kappa)$ as $\Lambda \to \mathbb{Z}^d$, it follows that G is itself convex on its domain $\mathbb{R}^2$. $\square$

Proof of Theorem 3.63. This is inspired by a related computation for the Ising model, [**142**]. Let $0 < p < 1$, $q \geq 1$, and let (π, κ) satisfy (3.56). Recall the functions G_Λ^ξ given in (3.72), and note from (3.73) that

$$\frac{dG_\Lambda^\xi}{d\pi} = \frac{1}{|\mathbb{E}_\Lambda|}\phi_{\Lambda,p,q}^\xi\big(|\eta(\omega)\cap\mathbb{E}_\Lambda|\big). \tag{3.74}$$

old3.74 By the convexity of the G_Λ^ξ,

$$\frac{dG_\Lambda^\xi}{d\pi} \to \frac{dG}{d\pi} \qquad \text{as } \Lambda \to \mathbb{Z}^d, \qquad \text{for } \xi \in \Omega \text{ and } p \notin \mathcal{D}_q. \tag{3.75}$$

old3.75 For any box Λ and any edge $e \in \mathbb{E}_\Lambda$,

$$\begin{aligned}\frac{1}{|\mathbb{E}_\Lambda|}\phi_{\Lambda,p,q}^0\big(|\eta(\omega)\cap\mathbb{E}_\Lambda|\big) &\leq \phi_{p,q}^0(J_e)\\ &\leq \phi_{p,q}^1(J_e) \leq \frac{1}{|\mathbb{E}_\Lambda|}\phi_{\Lambda,p,q}^1\big(|\eta(\omega)\cap\mathbb{E}_\Lambda|\big),\end{aligned} \tag{3.76}$$

old3.76 where $J_e = \{\omega(e) = 1\}$, and we have used the translation-invariance of $\phi_{p,q}^0$ and $\phi_{p,q}^1$, together with the stochastic orderings of certain measures. We deduce on passing to the limit as $\Lambda \to \mathbb{Z}^d$ that

$$\frac{dG}{d\pi} = \phi_{p,q}^0(J_e) = \phi_{p,q}^1(J_e) \qquad \text{for } e \in \mathbb{E}^d \text{ and } p \notin \mathcal{D}_q. \tag{3.77}$$

old3.77 In particular, (a) implies (c). By Proposition 3.6, (c) implies that $\phi_{p,q}^0 = \phi_{p,q}^1$, whence by (3.36) part (d) holds.

Since $G(\pi,\kappa)$ is a convex function of π, it has right and left derivatives with respect to π, denoted by $dG/d\pi^\pm$. Furthermore $dG/d\pi^+$ (respectively, $dG/d\pi^-$) is right-continuous (respectively, left-continuous) and non-decreasing. We shall prove that

$$\frac{dG}{d\pi^+} - \frac{dG}{d\pi^-} = \phi_{p,q}^1(J_e) - \phi_{p,q}^0(J_e), \tag{3.78}$$

old3.78 and that

$$\phi_{p,q}^1(J_e) = \lim_{p'\downarrow p}\phi_{p',q}^0(J_e), \quad \phi_{p,q}^0(J_e) = \lim_{p'\uparrow p}\phi_{p',q}^1(J_e). \tag{3.79}$$

old3.79 In advance of proving (3.78) and (3.79), we note the following. If (d) holds then, by (3.78), G is differentiable in π, whence (a) holds. We will thus have proved that (a), (c), and (d) are equivalent. Furthermore, by (3.78)–(3.79), the following three statements are equivalent for any given π:

- $p \notin \mathcal{D}_q$,
- $h^0(x, q)$ is right-continuous at $x = p$,
- $h^1(x, q)$ is left-continuous at $x = p$.

By Proposition 3.26, $h^0(\cdot, q)$ (respectively, $h^1(\cdot, q)$) is left-continuous (respectively, right-continuous), and thus (a) is equivalent to (b).

Finally we prove (3.78) and (3.79), beginning with the latter. We concentrate on the first equation of (3.79); a similar argument is valid for the second. By Proposition 3.26, $h^1(\cdot, q)$ is right-continuous, whence

$$h^1(p, q) = \lim_{p' \downarrow p} h^1(p', q).$$

Now $\mathcal{D}_q$ is countable, whence $\phi^0_{p,q} = \phi^1_{p,q}$, and in particular $h^0(p, q) = h^1(p, q)$ for almost every p. By the monotonicity of $h^0(\cdot, q)$, we deduce that

$$h^1(p, q) = \lim_{p' \downarrow p} h^0(p', q).$$

as required.

Finally, by the semicontinuity of the $dG/d\pi^{\pm}$, (3.77), and Proposition 3.26,

$$\frac{dG}{d\pi^+} = \lim_{\substack{x \downarrow \pi \\ x \notin \mathcal{D}'_\kappa}} \frac{d}{dx} G(x, \kappa) = \phi^1_{p,q}(J_e),$$

$$\frac{dG}{d\pi^-} = \lim_{\substack{x \uparrow \pi \\ x \notin \mathcal{D}'_\kappa}} \frac{d}{dx} G(x, \kappa) = \phi^0_{p,q}(J_e),$$

and (3.78) follows. □

Proof of Theorem 3.52. This is an immediate consequence of Theorem 3.60 with $\kappa = \log q$. □

Proof of Theorem 3.60. Parts (a) and (b) follow by the remarks prior to the statement of the theorem. Turning to part (c), we recall first that G is differentiable at (π, κ), that is $(\pi, \kappa) \notin \mathcal{D}$, if and only if G possesses two partial derivatives at (π, κ). It remains to show therefore that, for $\kappa \geq 0$, $\pi \in \mathcal{D}'_\kappa$ if and only if $\kappa \in \mathcal{D}''_\pi$. Assume $\kappa > 0$ (the case $\kappa = 0$ is trivial). Since, by Theorem 3.58, $\mathcal{D}'_\kappa$ is exactly the set of $\pi = \pi(p)$ such that $\phi^0_{p,q} = \phi^1_{p,q}$, it suffices to show the following.

old3.80 **Lemma 3.80.** *Let* $0 < p < 1$, $q > 1$, *and let* (π, κ) *satisfy (3.56). We have that* $\kappa \in \mathcal{D}'_\pi$ *if and only if* $\phi^0_{p,q} = \phi^1_{p,q}$.

Proof. The function G^ξ_Λ, given in (3.72), is convex in κ, whence

$$\frac{dG^\xi_\Lambda}{d\kappa} \to \frac{dG}{d\kappa} \qquad \text{as } \Lambda \to \mathbb{Z}^d, \qquad \text{for } \xi \in \Omega \text{ and } \kappa \notin \mathcal{D}''_\pi. \tag{3.81}$$

old3.81 as in (3.75). Inequalities (3.76) become

$$
\begin{aligned}
\frac{1}{|\mathbb{E}_\Lambda|}\phi^0_{\Lambda,p,q}(k(\omega,\Lambda)) &\ge \frac{1}{|\mathbb{E}_\Lambda|}\phi^0_{p,q}(k(\omega,\Lambda)) \\
&\ge \frac{1}{|\mathbb{E}_\Lambda|}\phi^1_{p,q}(k(\omega,\Lambda)) \ge \frac{1}{|\mathbb{E}_\Lambda|}\phi^1_{\Lambda,p,q}(k(\omega,\Lambda)),
\end{aligned} \tag{3.82}
$$

old3.82 since $k(\omega,\Lambda)$ is decreasing in ω. Therefore, by 2.101,

$$
\frac{dG^0_\Lambda}{d\kappa} \ge \phi^0_{p,q}\left(\frac{k(\omega,\Lambda)}{|\mathbb{E}_\Lambda|}\right) \ge \phi^1_{p,q}\left(\frac{k(\omega,\Lambda)}{|\mathbb{E}_\Lambda|}\right) \ge \frac{dG^1_\Lambda}{d\kappa} \qquad \text{if } \kappa \notin \mathcal{D}''_\pi. \tag{3.83}
$$

old3.83

For $\omega \in \Omega$ and $x \in \mathbb{Z}^d$, let $C_x(\omega)$ be the ω-open cluster at x, and $|C_x|$ the number of its vertices. We have as in [**92**, Section 4.1] that

$$
k(\omega,\Lambda) = \sum_{x\in\Lambda} \frac{1}{|C_x \cap \Lambda|} \ge \sum_{x\in\Lambda}\frac{1}{|C_x|},
$$

and

$$
\begin{aligned}
k(\omega,\Lambda) - \sum_{x\in\Lambda}\frac{1}{|C_x|} &= \sum_{x\in\Lambda}\left(\frac{1}{|C_x\cap\Lambda|} - \frac{1}{|C_x|}\right) \\
&\le \sum_{\substack{x\in\Lambda \\ x\leftrightarrow\partial\Lambda}} \frac{1}{|C_x\cap\Lambda|} \le |\partial\Lambda|.
\end{aligned}
$$

We apply the ergodic theorem to the family $\{|C_x|^{-1} : x \in \mathbb{Z}^d\}$ of bounded random variables to obtain by the ergodicity of the $\phi^b_{p,q}$ that, for $b = 0, 1$,

$$
\phi^b_{p,q}\left(\frac{k(\omega,\Lambda)}{|\Lambda|}\right) \to \phi^b_{p,q}(|C_0|^{-1}) \qquad \phi^b_{p,q}\text{-a.s. and in } L^1, \text{ as } \Lambda \to \mathbb{Z}^d. \tag{3.84}
$$

old3.84 By (3.81), (3.83), and (3.84),

$$
\frac{dG}{d\kappa} = \frac{1}{d}\phi^0_{p,q}(|C_0|^{-1}) = \frac{1}{d}\phi^1_{p,q}(|C_0|^{-1}) \qquad \text{for } \kappa \notin \mathcal{D}''_\pi. \tag{3.85}
$$

old3.85

By considering a suitable 'coupling' of the measures $\phi^b_{p,q}$, $b = 0, 1$, we deduce that, if $\kappa \notin \mathcal{D}''_\pi$, the laws of the vertex set of C_0 are the same under $\phi^0_{p,q}$ and $\phi^1_{p,q}$. In particular,

$$
\phi^0_{p,q}(x \nleftrightarrow y) = \phi^1_{p,q}(x \nleftrightarrow y) \qquad \text{for } x, y \in \mathbb{Z}^d, \text{ if } \kappa \notin \mathcal{D}''_\pi. \tag{3.86}
$$

old3.86 Let J_e be the event that edge $e = \langle x, y \rangle$ is open, and let K_e be the event that x and y are joined by an open path of $\mathbb{E}^d \setminus \{e\}$. By Proposition 3.37,

$$\begin{aligned}\phi^b_{p,q}(J_e) &= p\phi^b_{p,q}(K_e) + \frac{p}{p+(1-p)q}\phi^b_{p,q}(\overline{K_e}) \\ &= p + \left(\frac{p}{p+(1-p)q} - p\right)\phi^b_{p,q}(\overline{K_e}),\end{aligned}$$

May be wrong. and

$$\phi^b_{p,q}(\overline{K_e}) = \frac{\phi^b_{p,q}(\overline{J_e} \cap \overline{K_e})}{\phi^b_{p,q}(\overline{J_e} \mid \overline{K_e})} = \frac{\phi^b_{p,q}(x \nleftrightarrow y)}{(1-p)q/\{p+(1-p)q\}}.$$

It follows by (3.86) that

$$\phi^0_{p,q}(J_e) = \phi^1_{p,q}(J_e) \qquad \text{for all } e \in \mathbb{E}^d, \text{ if } \kappa \notin \mathcal{D}''_\pi,$$

whence, by Proposition 3.6, $\phi^0_{p,q} = \phi^1_{p,q}$ if $\kappa \notin \mathcal{D}''_\pi$.

Conversely, suppose that $\phi^0_{p,q} = \phi^1_{p,q}$, and let $q' < q < q''$. By Proposition 3.26,

$$\phi^1_{p,q'}(|C|^{-1}) \le \phi^1_{p,q}(|C|^{-1}) = \phi^0_{p,q}(|C|^{-1}) \le \phi^0_{p,q''}(|C|^{-1}),$$

We take the limits as $q' \uparrow q$ and $q'' \downarrow q$ along sequences satisfying $\kappa(q'), \kappa(q'') \notin \mathcal{D}''_\pi$, and we use the monotonicity of these functions to find from (3.85) and Proposition 3.26 again that

$$\frac{dG}{d\kappa^+} - \frac{dG}{d\kappa^-} = \frac{1}{d}\big\{\phi^0_{p,q}(|C|^{-1}) - \phi^1_{p,q}(|C|^{-1})\big\} = 0.$$

Therefore, G has the corresponding partial derivative at (π, κ), which is to say that $\kappa \notin \mathcal{D}''_\pi$ as required. $\square$

4

Phase Transition

4.1 The critical point

The random-cluster model possesses an infinite open cluster if and only if p is sufficiently large. There is a phase transition at the 'critical value' of p separating the regime where all open clusters are finite from the regime in which infinite clusters exist. We explore this phase transition next. *We assume for the entirety of the chapter that* $q \geq 1$, and we shall concentrate on the extremal random-cluster measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$. The quantities of principal interest are the $\phi^b_{p,q}$-*percolation probabilities*, given as

$$\theta^b(p,q) = \phi^b_{p,q}(0 \leftrightarrow \infty), \qquad b = 0,1. \tag{4.1}$$

old4.1 We define the critical points

$$p^b_c(q) = \sup\{p : \theta^b(p,q) = 0\}, \qquad b = 0,1. \tag{4.2}$$

old4.2 It is immediate from Proposition 3.26(a) that the $\theta^b(\cdot, q)$ are non-decreasing functions, and therefore

$$\theta^b(p,q) \begin{cases} = 0 & \text{if } p < p^b_c(q), \\ > 0 & \text{if } p > p^b_c(q). \end{cases} \tag{4.3}$$

old4.3

We have by Theorem 3.52 that $\phi^0_{p,q} = \phi^1_{p,q}$ for almost every p. Therefore, $\theta^0(p,q) = \theta^1(p,q)$ for almost every p, and hence $p^0_c(q) = p^1_c(q)$. Henceforth we use the abbreviated notation

$$p_c(q) = p^0_c(q) = p^1_c(q), \tag{4.4}$$

old4.4 and we refer to $p_c(q)$ as the *critical point* of the random-cluster model.

It is almost trivial to prove that $p_c(q) = 1$ in the very special case when the number d of dimensions satisfies $d = 1$. In contrast, it is fundamental that $0 < p_c(q) < 1$ when $d \geq 2$. Not a great deal is known in general about the way in which $p_c(q)$ behaves when viewed as a function of q. The following basic inequalities are consequences of the comparison inequalities of Theorem 3.26.

old4.5 **Theorem 4.5 [??, 6].** *We have that*

$$\frac{1}{p_c(q)} \leq \frac{1}{p_c(q')} \leq \frac{q/q'}{p_c(q)} - \frac{q}{q'} + 1 \qquad \textit{if } 1 \leq q' \leq q. \tag{4.6}$$

old4.6

From (4.6) we obtain that

$$0 \leq p_c(q) - p_c(q') \leq \frac{(q-q')p_c(q')(1-p_c(q'))}{q' + (q-q')p_c(q')} \qquad \text{for } 1 \leq q' \leq q, \tag{4.7}$$

old4.7 whence, on setting $q' = 1$,

$$0 \leq p_c(q) - p_c(1) \leq \frac{(q-1)p_c(1)(1-p_c(1))}{1+(q-1)p_c(1)} \qquad \text{for } q \geq 1. \tag{4.8}$$

old4.8 Since $0 < p_c(1) < 1$ for $d \geq 2$, [**92**, Theorem 1.10], we deduce the important fact that

$$0 < p_c(q) < 1 \qquad \text{for } q \geq 1. \tag{4.9}$$

old4.9

From (4.7) one obtains that $p_c(q)$ is a continuous non-decreasing function of q, but strict monotonicity requires the further comparison inequality of Theorem 2.49.

old4.10 **Theorem 4.10 [89].** *Assume that $d \geq 2$. When viewed as a function of q, the critical value $p_c(q)$ is Lipschitz-continuous and strictly increasing on the interval $[1, \infty)$.*

Proof of Theorem 4.5. Let $1 \leq q' \leq q$ and let

$$\frac{p'}{q'(1-p')} = \frac{p}{q(1-p)}. \tag{4.11}$$

old4.11 By Theorem 3.26,

$$\theta^1(p', q) \leq \theta^1(p', q') \leq \theta^1(p, q).$$

If $p' < p_c(q')$, then $\theta^1(p', q') = 0$, so that $\theta^1(p', q) = 0$ and therefore $p' \leq p_c(q)$. This implies that $p_c(q') \leq p_c(q)$, the first inequality of (4.6). By a similar argument, if $p < p_c(q)$, then $p' \leq p_c(q')$, whence

$$\frac{p_c(q)}{q(1-p_c(q))} \leq \frac{p_c(q')}{q'(1-p_c(q'))}.$$

This yields the second inequality of (4.6) after a little manipulation. □

In advance of proving Theorem 4.10, we prove an intermediate fact of independent interest.

old4.12 **Proposition 4.12.** *We have for* $0 \le p \le 1$ *and* $q \ge 1$ *that*

$$\phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda) \to \theta^1(p,q) \qquad \text{as } \Lambda \to \mathbb{Z}^d.$$

There is no 'elementary' proof of the corresponding fact for the 0 boundary-condition measure $\phi^0_{\Lambda,p,q}$, and indeed this is unproven for general pairs (p,q).

Proof of Proposition 4.12. It is clear that

$$\phi^1_{p,q}(0 \leftrightarrow \partial\Lambda) \le \phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda) \le \phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Delta) \qquad \text{for } \Delta \subseteq \Lambda,$$

by positive association and the fact that $\{0 \leftrightarrow \partial\Lambda\} \subseteq \{0 \leftrightarrow \partial\Delta\}$ when $\Delta \subseteq \Lambda$. We take the limits as $\Lambda \to \mathbb{Z}^d$ and $\Delta \to \mathbb{Z}^d$ in that order to obtain the result. □

Proof of Theorem 4.10. By (4.7),

$$0 \le \frac{p_c(q) - p_c(q')}{q - q'} \le \frac{1}{q'} \qquad \text{for } 1 \le q' < q,$$

whence $p_c(q)$ is Lipschitz-continuous on the interval $[1, \infty)$. Turning to strict monotonicity, let γ be given as in Theorem 2.49 with $\delta = 2d$, and let $1 \le q_2 < q_1$. Recall that $\gamma(p,q)$ is continuous, and is strictly increasing in p and strictly decreasing in q. We apply Theorem 2.49 to the graph obtained from Λ by identifying all vertices of $\partial\Lambda$, with spanning set $W = \Lambda \setminus \partial\Lambda$ satisfying $\deg(W) = 2d$, and we obtain thereby that

$$\phi^1_{\Lambda,p_1,q_1}(0 \leftrightarrow \partial\Lambda) \le \phi^1_{\Lambda,p_2,q_2}(0 \leftrightarrow \partial\Lambda) \quad \text{if} \quad \gamma(p_1,q_1) \le \gamma(p_2,q_2).$$

We take the limit as $\Lambda \to \mathbb{Z}^d$ and use Proposition 4.12 to deduce that

$$\theta^1(p_1,q_1) \le \theta^1(p_2,q_2) \quad \text{if} \quad \gamma(p_1,q_1) \le \gamma(p_2,q_2). \tag{4.13}$$

old4.13 We claim that

$$\gamma(p_c(q_1), q_1) \ge \gamma(p_c(q_2), q_2). \tag{4.14}$$

old4.14 Suppose on the converse that

$$\gamma(p_c(q_1), q_1) < \gamma(p_c(q_2), q_2).$$

By the continuity of γ, there exist $p_1 > p_c(q_1)$ and $p_2 < p_c(q_2)$ such that $\gamma(p_1,q_1) < \gamma(p_2,q_2)$. By (4.13),

$$\theta^1(p_1,q_1) \le \theta^1(p_2,q_2).$$

However, $\theta^1(p_1,q_1) > 0$ and $\theta^1(p_2,q_2) = 0$, a contradiction, and thus (4.14) holds. If $p_c(q_1) = p_c(q_2)$, then the strict monotonicity of $\gamma(\cdot,\cdot)$ and the fact that $q_2 < q_1$ are in contradiction of (4.14). Therefore $p_c(q_2) < p_c(q_1)$ as claimed. □

4.2 Percolation probabilities

We turn our attention now to certain continuity properties of the percolation probabilities $\theta^b(p,q)$, defined in (4.1) for $b = 0, 1$. Clearly, $\theta^0(p,q) = \theta^1(p,q) = 0$ if $p < p_c(q)$.

old4.15 **Theorem 4.15.** *Let $d \ge 2$ and $q \ge 1$.*

(a) *The function $\theta^0(\cdot, q)$ is left-continuous on* $[0,1] \setminus \{p_c(q)\}$.

(b) *The function $\theta^1(\cdot, q)$ is right-continuous on* $[0,1]$.

(c) $\theta^0(p,q) = \theta^1(p,q)$ *if and only if $p \notin \mathcal{D}_q$, where $\mathcal{D}_q$ is given in Theorem 3.52.*

(d) *Let $p \ne p_c(q)$. The functions $\theta^0(\cdot, q)$ and $\theta^1(\cdot, q)$ are continuous at the point p if and only if $p \notin \mathcal{D}_q$.*

Note first that, by (c), $\mathcal{D}_q \cap [0, p_c(q)) = \varnothing$ for $q \ge 1$. Is θ^b C^∞ for large p?

It is presumably the case that $\theta^0(\cdot, q)$ and $\theta^1(\cdot, q)$ are continuous except possibly at $p = p_c(q)$. In addition it may be conjectured that $\theta^0(\cdot, q)$ is left-continuous everywhere. A verification of this conjecture would include a proof that

$$\theta^0(p_c(q), q) = \lim_{p \uparrow p_c(q)} \theta^0(p,q) = 0,$$

implying in particular that $\theta(p_c(1), 1) = 0$; this last statement is one of the famous open problems of percolation theory (see [**95**, **92**]).

The functions $\theta^0(p,q)$ and $\theta^1(p,q)$ play, respectively, the roles of the magnetisations for Potts measures with free and constant-spin boundary conditions. We state this more fully as a theorem. As in Section 1.3, we write σ_u as the spin at vertex u of a Potts model with q states (where q is now assumed to be integral).

old4.16 **Theorem 4.16.** *Let $d \ge 2$, $0 < p < 1$, and $q \in \{2, 3, \dots\}$. Let π^{f} and π^1, respectively, be the q-state Potts measures on $\mathbb{L}^d$ with free and spin-1 boundary conditions, interaction J, and inverse temperature β satisfying $p = 1 - e^{-\beta J}$. Then*:

$$(1 - q^{-1})\theta^0(p,q)^2 = \lim_{|u| \to \infty} \bigl\{ \pi^{\mathrm{f}}(\sigma_0 = \sigma_u) - q^{-1} \bigr\}, \tag{4.17}$$

$$(1 - q^{-1})\theta^1(p,q) = \pi^1(\sigma_0 = 1) - q^{-1}. \tag{4.18}$$

old4.18

Equation (4.18) is standard (see [**6**, **64**, **88**]). Equation (4.17) is valid also with θ^0 and π^{f} replaced, respectively, by θ^1 and π^1, and the proof is similar.

Proof of Theorem 4.15. We shall prove (a) at the end of Section ?. Part (b) is a consequence of Proposition 3.26(b). Part (d) follows from (a)–(c), on noting that $\theta^b(\cdot, q)$ is non-decreasing for $b = 0, 1$. We turn therefore to the proof of (c).

Certainly $\phi^0_{p,q} = \phi^1_{p,q}$ if $p \notin \mathcal{D}_q$ (by Theorem 3.52), whence $\theta^0(p,q) = \theta^1(p,q)$ for $p \notin \mathcal{D}_q$. Suppose conversely that

$$\theta^0(p,q) = \theta^1(p,q). \tag{4.19}$$

old4.19 We shall now give the main steps in a proof that

$$h^0(p,q) = h^1(p,q); \tag{4.20}$$

old4.20 this will imply that $|\mathcal{R}_{p,q}| = 1$ by Theorem 3.6 (see also Theorem 3.63).

Fix an edge $e = \langle u, v\rangle$, and let $J_e = \{\omega(e) = 1\}$ as usual. For a vertex w, let $I_w = \{w \leftrightarrow \infty\}$, and let H_w be the event that w is in an infinite open path not using e. As in the proof of Proposition 3.6, there exists a probability measure ψ on $(\Omega, \mathcal{F})^2$ with marginals $\phi^0_{p,q}$ and $\phi^1_{p,q}$, and assigning probability 1 to the set of pairs $(\omega_0, \omega_1) \in \Omega^2$ satisfying $\omega_0 \le \omega_1$. Let $F(\omega)$ be the set of vertices which are joined to infinity by open paths of the configuration ω $(\in \Omega)$. We have that

$$0 \le \psi\big(F(\omega_0) \ne F(\omega_1)\big) \le \sum_{w \in \mathbb{Z}^d} \{\phi^1_{p,q}(I_w) - \phi^0_{p,q}(I_w)\} = 0, \tag{4.21}$$

old4.21 by (4.19). Now $J_e \cap I_u \cap I_v$ is an increasing event, whence

$$\phi^0_{p,q}(J_e \cap I_u \cap I_v) \le \phi^1_{p,q}(J_e \cap I_u \cap I_v). \tag{4.22}$$

old4.22 Also,

$$\begin{aligned} \phi^0_{p,q}(\overline{J_e} \cap I_u \cap I_v) &= \phi^0_{p,q}(\overline{J_e} \cap H_u \cap H_v) \\ &= \phi^0_{p,q}(\overline{J_e} \mid H_u \cap H_v)\phi^0_{p,q}(H_u \cap H_v). \end{aligned} \tag{4.23}$$

old4.23 However,

$$\phi^0_{p,q}(\overline{J_e} \mid H_u \cap H_v) = \phi^1_{p,q}(\overline{J_e} \mid H_u \cap H_v)$$

by Proposition 3.37(a) and the fact (Theorem 3.34) that $\phi^0_{p,q}, \phi^1_{p,q} \in \mathcal{R}_{p,q}$. In addition, $\phi^0_{p,q}(H_u \cap H_v) \le \phi^1_{p,q}(H_u \cap H_v)$ since $H_u \cap H_v$ is an increasing event. Therefore (4.23) implies that

$$\begin{aligned} \phi^0_{p,q}(\overline{J_e} \cap I_u \cap I_v) &\le \phi^1_{p,q}(\overline{J_e} \mid H_u \cap H_v)\phi^1_{p,q}(H_u \cap H_v) \\ &= \phi^1_{p,q}(\overline{J_e} \cap H_u \cap H_v) = \phi^1_{p,q}(\overline{J_e} \cap I_u \cap I_v). \end{aligned} \tag{4.24}$$

old4.24 Adding (4.22) and (4.24), we obtain that

$$\phi^0_{p,q}(I_u \cap I_v) \le \phi^1_{p,q}(I_u \cap I_v).$$

Equality holds here by (4.21), and therefore equality holds in (4.22), which is to say that

$$\phi^0_{p,q}(J_e \cap I_u \cap I_v) = \phi^1_{p,q}(J_e \cap I_u \cap I_v). \tag{4.25}$$

old4.25

It is obvious that

$$\phi^0_{p,q}(J_e \cap \overline{I_u} \cap I_v) = \phi^1_{p,q}(J_e \cap \overline{I_u} \cap I_v) \tag{4.26}$$

old4.26 since both sides equal 0; the same equation holds with $\overline{I_u} \cap I_v$ replaced by $I_u \cap \overline{I_v}$.

Finally we prove that

$$\phi^0_{p,q}(J_e \cap \overline{I_u} \cap \overline{I_v}) = \phi^1_{p,q}(J_e \cap \overline{I_u} \cap \overline{I_v}) \tag{4.27}$$

old4.27 which, in conjunction with (4.25), (4.26), and with the subsequent remark, implies (4.20) by addition, as required. Let $\epsilon > 0$. With $A_\Lambda = \{u \nleftrightarrow \partial\Lambda,\ v \nleftrightarrow \partial\Lambda\}$, we have by (4.21) that

$$0 \le \phi^0_{p,q}(A_\Lambda) - \phi^1_{p,q}(A_\Lambda) < \epsilon \quad \text{for all large } \Lambda,$$

and we pick Λ accordingly. The above claim is valid since the central term above converges, as $\Lambda \to \mathbb{Z}^d$, to $\phi^0_{p,q}(\overline{I_u} \cap \overline{I_v}) - \phi^1_{p,q}(\overline{I_u} \cap \overline{I_v})$, which equals 0 by (4.21). The events $\{u \nleftrightarrow \partial\Lambda\}$ and $\{v \nleftrightarrow \partial\Lambda\}$ are cylinder events, whence

$$0 \le \phi^0_{\Delta,p,q}(A_\Lambda) - \phi^1_{\Delta,p,q}(A_\Lambda) < 2\epsilon \quad \text{for all large } \Delta, \tag{4.28}$$

old4.28 and we pick Δ ($\supseteq \Lambda$) accordingly. Let $S = S(\omega) = \{x \in \Delta : x \leftrightarrow \partial\Lambda\}$ and $G = G(\omega) = \Lambda \setminus S$. We now employ a coupling of $\phi^0_{\Delta,p,q}$, $\phi^1_{\Delta,p,q}$ constructed as in [**155**, page 254]. Following this reference, there exists a probability measure ψ_Δ on $\Omega^0_\Delta \times \Omega^1_\Delta$, with marginals $\phi^0_{\Delta,p,q}$ and $\phi^1_{\Delta,p,q}$, which assigns probability 1 to pairs (ω_0, ω_1) satisfying $\omega_0 \le \omega_1$, and with the additional property that, conditional on $G = G(\omega_1)$, both marginals of ψ_Δ on $\mathbb{E}_G$ equal the *free boundary condition* random-cluster measure $\phi^0_{G,p,q}$. Writing $\mathcal{G}$ for the class of all subsets of Λ which contain both u and v, it follows that

$$\begin{aligned}
\phi^1_{\Delta,p,q}(J_e \cap A) &= \sum_{g\in\mathcal{G}} \phi^1_{\Delta,p,q}(J_e,\ G = g) = \sum_{g\in\mathcal{G}} \psi_\Delta\big(\omega_1 \in J_e,\ G(\omega_1) = g\big) \\
&= \sum_{g\in\mathcal{G}} \psi_\Delta\big(\omega_1 \in J_e \,\big|\, G(\omega_1) = g\big)\psi_\Delta\big(G(\omega_1) = g\big) \\
&= \sum_{g\in\mathcal{G}} \psi_\Delta\big(\omega_0 \in J_e \,\big|\, G(\omega_1) = g\big)\psi_\Delta\big(G(\omega_1) = g\big) \\
&= \psi_\Delta(\omega_0 \in J_e,\ \omega_1 \in A),
\end{aligned}$$

and, in addition,

$$\phi^0_{\Delta,p,q}(J_e \cap A) = \psi_\Delta(\omega_0 \in J_e,\ \omega_0 \in A).$$

Therefore,

$$0 \le \phi^0_{\Delta,p,q}(J_e \cap A) - \phi^1_{\Delta,p,q}(J_e \cap A) = \psi_\Delta(\omega_0 \in J_e,\ \omega_0 \in A,\ \omega_1 \notin A),$$

which by (4.28) does not exceed 2ϵ. We take the limits as $\Delta \to \mathbb{Z}^d$, $\Lambda \to \mathbb{Z}^d$, and $\epsilon \downarrow 0$, in that order, to obtain (4.27). $\square$

Proof of Theorem 4.16. By Theorem 1.15,

$$(1-q^{-1})\phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda) = \pi^1_{\Lambda,\beta,J,q}(\sigma_0 = 1) - \frac{1}{q},$$

where $\pi^1_{\Lambda,\beta,J,q}$ is the q-state Potts measure on Λ with parameters β, J, and with '1' boundary condition. Equation (4.18) follows on taking the limit as $\Lambda \to \mathbb{Z}^d$ and recalling Proposition 4.12.

Returning to (4.17), we have similarly to the above that

$$(1-q^{-1})\phi^0_{\Lambda,p,q}(0 \leftrightarrow u) = \pi^{\mathrm{f}}_{\Lambda,\beta,J,q}(\sigma_0 = \sigma_u) - \frac{1}{q}, \qquad u \in \Lambda,$$

so that

$$(1-q^{-1})\phi^0_{p,q}(0 \leftrightarrow u) = \pi^{\mathrm{f}}_{\beta,J,q}(\sigma_0 = \sigma_u) - \frac{1}{q}, \qquad u \in \mathbb{Z}^d,$$

where $\pi^{\mathrm{f}}_{\Lambda,\beta,J,q}$ is the infinite-volume weak limit of the q-state Potts measure on Λ with free boundary condition. The claim is proven once we have shown that

$$\phi^0_{p,q}(0 \leftrightarrow u) \to \theta^0(p,q)^2 \qquad \text{as } |u| \to \infty. \tag{4.29}$$

old4.29 Certainly,

$$\phi^0_{p,q}(0 \leftrightarrow u) \ge \theta^0(p,q)^2,$$

by the positive association of $\phi^0_{p,q}$, Theorem 3.17(c). Conversely, using the fact that $\phi^0_{p,q}$ has the 0/1-infinite cluster property (see Theorems 3.19(c) and 3.33(c)),

$$\phi^0_{p,q}(0 \leftrightarrow u) \le \phi^0_{p,q}(0, u \leftrightarrow \infty) + \phi^0_{p,q}(u \in C,\ |C| < \infty).$$

The last probability tends to zero as $|u| \to \infty$. As for the penultimate term,

$$\begin{aligned}\phi^0_{p,q}(0, u \leftrightarrow \infty) &\le \phi^0_{p,q}(0 \leftrightarrow \partial\Lambda,\ u \leftrightarrow u + \partial\Lambda)\\ &\to \phi^0_{p,q}(0 \leftrightarrow \partial\Lambda)^2 \qquad \text{as } |u| \to \infty,\end{aligned}$$

by the mixing property of $\phi^0_{p,q}$; see [**79**, Theorem 7.7, Proposition 7.9]. Equation (4.29) follows on letting $\Lambda \to \mathbb{Z}^d$. $\square$

4.3 Uniqueness of random-cluster measures

We record in this section some information about the set of values of p at which there exists a unique random-cluster measure.

old4.30 **Theorem 4.30 [6, 90].** *Assume that* $q \geq 1$ *and* $d \geq 2$. *Then* $|\mathcal{W}_{p,q}| = |\mathcal{R}_{p,q}| = 1$ *if any of the following holds*:

(a) $\theta^0(p, q) = \theta^1(p, q)$, *which is to say that* $p \notin \mathcal{D}_q$;

(b) $p > p'$, *where* p' $(= p'(q, d))$ *is a certain real number satisfying* $p_c(q) \leq p' < 1$.

Since $\theta^0(p, q) = \theta^1(p, q) = 0$ when $\theta^1(p, q) = 0$, part (b) implies the uniqueness of random-cluster measures whenever $\theta^1(p, q) = 0$, [**6**, Theorem A.2]. In particular, there is a unique random-cluster measure throughout the subcritical phase, that is, when $0 \leq p < p_c(q)$. It is an important open problem to establish the same conclusion when $p > p_c(q)$. The following conjecture is standard.

uniqconj **Conjecture 4.31.** *Let* $q \geq 1$. *Then* $\phi^0_{p,q} = \phi^1_{p,q}$, *and therefore* $|\mathcal{W}_{p,q}| = |\mathcal{R}_{p,q}| = 1$, *if and only if one of the following holds*:

(i) *either* $p < p_c(q)$ *or* $p > p_c(q)$,

(ii) *we have* $p = p_c(q)$ *and* $\theta^1(p_c(q), q) = 0$.

Proof of Theorem 4.30. Suppose first that (a) holds. By Theorem 4.15(c), $p \notin \mathcal{D}_q$, whence $|\mathcal{W}_{p,q}| = |\mathcal{R}_{p,q}| = 1$ by Theorem 3.52.

Next we sketch a proof that $\phi^0_{p,q} = \phi^1_{p,q}$ if p is sufficiently close to 1. There are certain topological complications in doing this, and we avoid giving all the relevant details, most of which may be found in a closely related passage of [**128**, Sect. 2]. We begin by defining a lattice $\mathcal{L}$, having the same vertex set as $\mathbb{L}^d$ but with edge-relation

$$x \sim y \quad \text{if} \quad |x_i - y_i| \leq 1 \text{ for } 1 \leq i \leq d.$$

For $\omega \in \Omega$, we call a vertex x *white* if $\omega(e) = 1$ for all e incident with x in $\mathbb{L}^d$, and *black* otherwise. For any set V of vertices of $\mathcal{L}$, we define the *black cluster* $B(V)$ as the union of V together with the set of all vertices x_0 of $\mathcal{L}$ for which there exists a path $x_0, e_0, x_1, e_1, \dots, e_{n-1}, x_n$ of alternating vertices and edges of $\mathcal{L}$ such that $x_0, x_1, \dots, x_{n-1} \notin V$, $x_n \in V$, and $x_0, x_1, \dots, x_{n-1}$ are all black. Note that the colours of vertices in V have no effect on $B(V)$, but that $V \subseteq B(V)$. We define

$$\|B(V)\| = \sup\left\{ \sum_{i=1}^{d} |x_i - y_i| : x \in V,\ y \in B(V) \right\}.$$

For any integer n and vertex x, the event $\{\|B(x)\| \geq n\}$ is a decreasing event (we confuse the singleton x with the set $\{x\}$), whence

(4.32)
$$\begin{aligned}\phi^0_{p,q}(\|B(x)\| \ge n) &\le \phi^0_{\Lambda,p,q}(\|B(x)\| \ge n)\\ &\le \phi^0_{\Lambda,\pi,1}(\|B(x)\| \ge n) \quad \text{for any box } \Lambda,\end{aligned}$$

old4.31 where $\pi = p/(p + (1-p)q)$ and we have used the comparison inequalities, Proposition 3.26. Using a Peierls argument (see [**128**, pp. 151–152]) there exists $\alpha(p)$ such that: the percolation (product) measure $\phi_{\pi,1} = \lim_{\Lambda\to\mathbb{Z}^d} \phi^0_{\Lambda,\pi,1}$ satisfies

(4.33)
$$\phi_{\pi,1}(\|B(x)\| \ge n) \le e^{-n\alpha(p)} \quad \text{for all } n,$$

old4.32 and furthermore $\alpha(p) > 0$ if p is sufficiently large, say $p > p'$ for some $p' \in [p_c(q), 1)$.

Let A be an increasing event defined in terms of the edges in the finite subset E of $\mathbb{E}^d$, and let Λ be a box such that $\mathbb{E} \subseteq \mathbb{E}_\Lambda$. Let Δ be a large box satisfying $\Lambda \subseteq \Delta$. For any subset S of $\overline{\Lambda}$ ($= \mathbb{Z}^d \setminus \Lambda$) containing $\partial\Delta$, define the 'interior boundary' $D(S)$ of S to be the set of all vertices x of $\mathcal{L}$ satisfying:

(a) $x \notin S$,

(b) x is adjacent in $\mathcal{L}$ to some vertex of S,

(c) there exists a path of $\mathbb{L}^d$ from x to some vertex in Λ, this path using no vertex of S.

We write $\overline{S} = S \cup D(S)$. Denote by $I(S)$ the set of vertices x_0 for which there exists a path $x_0, e_0, x_1, e_1, \dots, e_{n-1}, x_n$ of $\mathbb{L}^d$ with $x_n \in \Lambda$, $x_i \notin \overline{S}$ for all i. Note that every vertex of $\partial I(S)$ is adjacent to some vertex lying in $D(S)$. We shall concentrate on the case $S = B(\partial\Delta)$.

Let $\epsilon > 0$ and $p > p'$. By (4.32)–(4.33), there exists a box Δ' sufficiently large that

(4.34)
$$\phi^0_{p,q}(K_{\Lambda,\Delta}) \ge 1 - \epsilon \quad \text{if } \Delta \supseteq \Delta',$$

old4.33 where $K_{\Lambda,\Delta} = \left\{\overline{B(\partial\Delta)} \cap \Lambda = \varnothing\right\}$. We pick Δ' accordingly, and let $\Delta \supseteq \Delta'$.

Let us assume that $K_{\Lambda,\Delta}$ occurs, so that $I = I(B(\partial\Delta))$ satisfies $I \supseteq \Lambda$. We note three facts about $B(\partial\Delta)$ and $D(B(\partial\Delta))$:

(a) $D(B(\partial\Delta))$ is $\mathbb{L}^d$-connected in that, for all pairs $x, y \in D(B(\partial\Delta))$, there exists a path of $\mathbb{L}^d$ joining x to y using vertices of $D(B(\partial\Delta))$ only,

(b) every vertex in $D(B(\partial\Delta))$ is white,

(c) $D(B(\partial\Delta))$ is measurable with respect to the colours of vertices in $\mathbb{Z}^d \setminus I$, in the sense that the event $\{B(\partial\Delta) = h,\ D(B(\partial\Delta)) = D(h)\}$ lies in the σ-field generated by the colours of vertices in $\overline{I(h)}$, for any given h satisfying $\overline{h} \subseteq \overline{\Lambda}$.

Claim (a) may be proved by adapting the argument used to prove Lemma 2.23 of [**128**]; claim (b) is a consequence of the definition of $D(B(\partial\Delta))$; claim (c) holds since $D(B(\partial\Delta))$ is part of the ('internal') boundary of the black cluster of $\mathcal{L}$ generated by $\partial\Delta$. We do not include full proofs of (a) and (c) which would be rather long, and which would have much in common with Section 2 of [**128**].

Let $\mathcal{H}_\Lambda$ denote the set of all subsets of $\overline{\Lambda}$, and let h be a subset of $\overline{\Lambda}$ satisfying $\overline{h} \in \mathcal{H}_\Lambda$. The $\phi^0_{p,q}$-probability of A, conditional on $\{B(\partial\Delta) = h\}$, is given by the the wired measure $\phi^1_{I(h),p,q}$. This holds since: (a) every vertex in $\partial I(h)$ is adjacent to some vertex of $D(h)$, and (b) $D(h)$ is $\mathbb{L}^d$-connected and all vertices in $D(h)$ are white. Therefore, by conditional probability and the FKG inequality,

$$\begin{aligned}
\text{(4.35)} \qquad \phi^0_{p,q}(A) &\ge \phi^0_{p,q}\left(\phi^1_{I,p,q}(A)1_{K_{\Lambda,\Delta}}\right) \\
&\ge \phi^0_{p,q}\left(\phi^1_{\Delta,p,q}(A)1_{K_{\Lambda,\Delta}}\right) \qquad \text{since } I \subseteq \Delta \\
&\ge \phi^1_{\Delta,p,q}(A) - \epsilon \qquad \text{by (4.18).}
\end{aligned}$$

old4.34 We take the limits as $\Delta \to \mathbb{Z}^d$ and $\epsilon \downarrow 0$ in that order, thus obtaining $\phi^0_{p,q} \ge_{\text{st}} \phi^1_{p,q}$. Therefore, $\phi^0_{p,q} = \phi^1_{p,q}$. □

4.4 Subcritical phase

Let $q \ge 1$. We consider here the subcritical phase of the random-cluster model on $\mathbb{L}^d$ with $d \ge 2$, that is, when $p < p_c(q)$. We have by Theorem 4.30(a) that $\phi^0_{p,q} = \phi^1_{p,q}$, and hence $|\mathcal{W}_{p,q}| = |\mathcal{R}_{p,q}| = 1$, when $\theta^1(p,q) = 0$. It follows that there is a unique random-cluster measure when $p < p_c(q)$. For the sake of definiteness, we shall work with the measure $\phi^0_{p,q}$, but we note that $\phi_{p,q} = \phi^0_{p,q}$ for all $\phi_{p,q} \in \mathcal{R}_{p,q} \cup \mathcal{W}_{p,q}$.

The subcritical phase is characterised by the (almost sure) absence of an infinite open cluster. Thus all open clusters are almost surely finite, and one seeks estimates on the tail of the size of such a cluster. As described in [**92**, Chapter 6], one may study both the 'radius' $\text{rad}(C)$ and the 'volume' $|C|$ of a cluster C. We concentrate here on the cluster C at the origin, and we define its *radius* by

$$\text{(4.36)} \qquad \text{rad}(C) = \max\{\|y\| : y \in C\} = \max\{\|y\| : x \leftrightarrow y\}.$$

old4.35 We have chosen the distance function $\|\cdot\|$ rather than the $|\cdot|$ used in [**92**]. Note that $\text{rad}(C) \ge n$ if and only if $0 \leftrightarrow \partial\Lambda_n$ where Λ_n is the box $[-n,n]^d$. We note for later use that there exists a positive constant $\beta = \beta(d)$ such that

$$\text{(4.37)} \qquad \beta|C|^{1/d} \le \text{rad}(C) + 1 \le |C|.$$

new4.1

It is believed that the tails of both $\mathrm{rad}(C)$ and $|C|$ decay exponentially when $p < p_c(q)$, but this is currently unproven. That the appropriate limits exist is easy to prove, but the non-triviality of the limiting values remains open. That is, one may see via arguments using subadditivity that the functions

$$\psi(p,q) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n) \right\}, \tag{4.38}$$

$$\zeta(p,q) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \phi^0_{p,q}(|C| \geq n) \right\} \tag{4.39}$$

expvol exist. It is quite another matter to show as expected that

$$\psi(p,q), \zeta(p,q) > 0 \qquad \text{when } p < p_c(q).$$

We confine ourselves in this section to 'soft' arguments concerning exponential decay; 'harder' arguments for radius and volume respectively are to be found in the next two sections. We begin by considering the radius of the cluster at the origin. The existence of the limit in (4.38) relies essentially on positive association. We write $e_n = (n, 0, \dots, 0)$.

old4.36 **Theorem 4.40.** *Let μ be a translation-invariant probability measure on $(\Omega, \mathcal{F})$ which is invariant under permutations of the coordinates of $\mathbb{Z}^d$ and has the positive association property. The limits*

$$\nu(\mu) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \mu(0 \leftrightarrow e_n) \right\},$$
$$\psi(\mu) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \mu(0 \leftrightarrow \partial\Lambda_n) \right\},$$

exist and satisfy

$$\psi(\mu) \leq \nu(\mu),$$
$$\mu(0 \leftrightarrow e_n) \leq e^{-n\nu(\mu)}, \quad \mu(0 \leftrightarrow \partial\Lambda_n) \leq \sigma n^{d-1} e^{-n\psi(\mu)}, \qquad n \geq 1,$$

where $\sigma = 2d^2 6^d$. Note that $0 \leq \nu(\mu), \psi(\mu) \leq \infty$.

If, in addition, μ is invariant under changes of sign of the first coordinate, then $\nu(\mu) = \psi(\mu)$.

old4.36b **Theorem 4.41.** *Fix $q \geq 1$, and suppose that $0 < p \leq 1$. The limit*

$$\psi(p,q) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \phi^0_{p,q}(0 \leftrightarrow e_n) \right\} = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n) \right\}$$

exists and satisfies $0 \leq \psi = \psi(p,q) < \infty$. There exists a constant σ independent of p and q such that

$$\phi^0_{p,q}(0 \leftrightarrow e_n) \leq e^{-n\psi}, \quad \phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n) \leq \sigma n^{d-1} e^{-n\psi}, \qquad n \geq 1. \tag{4.42}$$

old4.37

Proofs of Theorems 4.40 and 4.41. These follow exactly the proofs of the corresponding parts of Theorems 6.10 and 6.44 of [**92**], and make use of the finite-energy property of Theorem 3.17(b). The details are omitted. □

We turn next to the volume $|C|$ of the open cluster at the origin. The existence of the limit in (4.39) relies on the finite-energy property.

new4.2 **Theorem 4.43.** *Let μ be a translation-invariant probability measure on $(\Omega, \mathcal{F})$ satisfying the 'strong finite-energy condition'*

$$\alpha \le \mu(J_e \mid \mathcal{T}_e) \le \beta, \qquad \mu\text{-almost surely,}$$

where α and β are constants satisfying $0 < \alpha \le \beta < 1$. The limit

$$\zeta(\mu) = \lim_{n\to\infty}\left\{-\frac{1}{n}\log\mu(|C| = n)\right\} \tag{4.44}$$

new4.2aa *exists and satisfies*

$$\mu(|C| = n) \le \frac{(1-\beta)^2}{\alpha} n e^{-n\zeta(\mu)}, \qquad n \ge 1. \tag{4.45}$$

new4.2bb *Note that* $0 \le \zeta(\mu) \le \log\{\alpha/(1-\beta)^2\}$.

It is an easy consequence of (4.44)–(4.45) that

$$-\frac{1}{n}\log\mu(n \le |C| < \infty) \to \zeta(\mu) \qquad \text{as } n \to \infty. \tag{4.46}$$

new4.2cc

new4.2b **Theorem 4.47.** *Let $0 < p < 1$. The limit*

$$\zeta(p,q) = \lim_{n\to\infty}\left\{-\frac{1}{n}\log\phi^0_{p,q}(|C| = n)\right\}$$

exists and satisfies

$$\phi^1_{p,q}(|C| = n) \le \frac{(1-p)^2(p+q(1-p))}{p} n e^{-n\zeta(p,q)} \qquad \textit{for all } n \ge 1.$$

Proofs of Theorems 4.43 and 4.47. The proofs follow that of [**92**, Theorem 6.78], making use of the finite-energy property of Theorem 3.17(b). The details are omitted. □

Since $\phi^0_{p,q}(0 \leftrightarrow \infty) > 0$ when $p > p_c(q)$, it is elementary that

$$\psi(p,q) = 0 \qquad \text{when } p > p_c(q). \tag{4.48}$$

old4.38 It is less obvious that

$$\zeta(p,q) = 0 \qquad \text{when } p > p_c(q), \tag{4.49}$$

zetpos a fact proved at Theorem 4.95. It is believed that $\psi(p,q), \zeta(p,q) > 0$ when $p < p_c(q)$, but this is not proved for general values of d, p, q. A partial result in this direction is the following rather weak statement; related results may be obtained via Theorem 2.107 as in Theorem 5.27.

old4.39 **Theorem 4.50.** *Let $q \geq 1$. We have that $\psi(p,q) > 0$ and $\zeta(p,q) > 0$ when $p < p_c(1)$.*

Could something like this be true by comparison when $q < 1$?

Proof. Assume that $q \geq 1$ and $p < p_c(1)$. We have from Proposition 3.26(a) that $\phi^0_{p,q} \leq_{st} \phi_{p,1}$. Thus,

$$\phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n) \leq \phi_{p,1}(0 \leftrightarrow \partial\Lambda_n),$$

whence $\psi(p,q) \geq \psi(p,1)$, and the strict positivity of ψ follows by the corresponding statement for percolation, [**92**, Theorem 6.14].

Similarly, by Theorem 4.40(b),

$$\phi^0_{p,q}(|C| = n) \leq \phi^0_{p,q}(|C| \geq n) \leq \phi_{p,1}(|C| \geq n) \leq \sigma n^{d-1} e^{-n\psi(p,1)},$$

whence, by [**92**, Theorem 6.78], $\zeta(p,q) \geq \zeta(p,1) > 0$. □

It is an important open problem to extend this conclusion to the entire subcritical phase, and we discuss this problem further in the next two sections.

4.5 Exponential decay of radius

Note use of 2.107.

We address next conditions under which one obtains the exponential decay of the radius of an open cluster. In the case of percolation, a useful intermediate step was the proof by Hammersley [**105**] that $\psi(p,1) > 0$ under the hypothesis of summability of the two-point connectivity function,

$$\phi_{p,1}(|C|) = \sum_{x \in \mathbb{Z}^d} \phi_{p,1}(0 \leftrightarrow x) < \infty.$$

Similarly, Simon [**179**] and Lieb [**143**] proved the exponential decay of the two-point function of Ising and other models under a summability assumption on this function, see Chapter 8. Such conclusions provoke the following question in the current context: if $\phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n)$ decays at some polynomial rate as $n \to \infty$, then must it necessarily decay at an exponential rate? An affirmative answer is provided in the discussion which follows.

We shall concentrate on the quantity

$$L(p,q) = \limsup_{n\to\infty} \left\{ n^{d-1} \phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n) \right\}. \tag{4.51}$$

old4.40 It is clear from the comparison inequality, Proposition 3.26, that $L(p,q)$ is non-decreasing in p, and thus

$$L(p,q) \begin{cases} < \infty & \text{if } p < \widetilde{p}_c(q), \\ = \infty & \text{if } p > \widetilde{p}_c(q), \end{cases}$$

where

$$\widetilde{p}_c(q) = \sup\{p : L(p,q) < \infty\}. \tag{4.52}$$

old4.41 Clearly $\widetilde{p}_c(q) \leq p_c(q)$, and equality is believed to hold.

old4.42 **Conjecture 4.53.** *If* $q \geq 1$, *then* $\widetilde{p}_c(q) = p_c(q)$.

Certainly this holds when $q = 1$, see [**92**], and we shall see in Section ? that it holds also when either $q = 2$ or q is sufficiently large.

The condition $L(p, q) < \infty$ amounts to assuming that the radius $R = \mathrm{rad}(C)$ has a tail decaying at least as fast as $n^{-(d-1)}$, and is a weaker assumption than the moment condition $\phi^0_{p,q}(R^{d-1}) < \infty$. In fact, $L(p, q) = 0$ if $\phi^0_{p,q}(R^{d-1}) < \infty$, since

$$n^{d-1}\phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n) = n^{d-1}\phi^0_{p,q}(R \geq n) \leq \sum_{k=n}^{\infty} k^{d-1}\phi^0_{p,q}(R = k).$$

There is a converse statement. If $p < \widetilde{p}_c(q)$ then $L(p, q) < \infty$, implying that

$$n^c\phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n) \to 0 \quad \text{for all } c \text{ satisfying } c < d - 1.$$

This in turn implies that $\phi^0_{p,q}(R^c) < \infty$ for all c satisfying $c < d - 1$ (see [**98**, Exercise 5.6.4]).

We state next the main conclusion of this section.

old4.43 **Theorem 4.54.** *Let* $q \geq 1$ *and* $0 < p < \widetilde{p}_c(q)$. *There exists* $\alpha = \alpha(p, q)$ *satisfying* $\alpha(p, q) > 0$ *such that*

$$\phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n) \leq e^{-n\alpha(p,q)} \quad \textit{for all large } n. \tag{4.55}$$

old4.44 *Thus the function* ψ *in Theorem 4.40 satisfies* $\psi(p, q) > 0$ *whenever* $0 < p < \widetilde{p}_c(q)$.

The proof, which is delayed until later in the section, uses the method of exponential steepness described in Section 2.6. Let A be an event, and recall from Section 2.6 the definition of the random variable H_A,

$$H_A(\omega) = \inf\left\{\sum_e |\omega'(e) - \omega(e)| : \omega' \in A\right\}, \qquad \omega \in \Omega.$$

We shall consider the event $A_n = \{0 \leftrightarrow \partial\Lambda_n\}$, and write H_n for H_{A_n}. The question of ascertaining the asymptotics of H_n may be viewed as a first-passage problem. Imagine you are travelling from 0 to $\partial\Lambda_n$; travel along open edges is instantaneous, but along each closed edge requires time 1. The fastest route requires time H_n, and one is interested in the *time constant* η, defined as $\eta = \lim_{n\to\infty}\{n^{-1}H_n\}$. That η exists is a consequence of a theorem attributed in [**70**, page 748, Erratum] and [**128**, page 259] to Derrienic.

4.derr **Theorem 4.56 (Existence of time constant).** *Let μ be a probability measure on $(\Omega, \mathcal{F})$ which is invariant under permutations of the coordinates of $\mathbb{Z}^d$, and which is translation-invariant and ergodic. The deterministic limit*

$$\eta(\mu) = \lim_{n\to\infty} \left\{ \frac{1}{n} H_n \right\}$$

exists μ-almost surely and in $L^1(\mu)$.

The constant $\eta(\mu)$ is called the *time constant* associated with μ.

Proof. See the comments in [**70**, **128**], and the later paper [**35**]. □

We seek a condition under which the time constant $\eta(\mu)$ is *strictly* positive. As before, C is the vertex set of the open cluster at the origin.

old4.54 **Theorem 4.57 (Positivity of time constant) [70, 97].** *Let μ satisfy the conditions of Theorem 4.56, and in addition be positively associated. If $\mu(|C|^{2d+\epsilon}) < \infty$ for some $\epsilon > 0$, then $\eta(\mu) > 0$.*

We apply Theorem 4.56 to the measure $\mu = \phi^0_{p,q}$ to deduce that the associated time constant

$$\eta(p, q) = \lim_{n\to\infty} \left\{ \frac{1}{n} H_n \right\} \tag{4.58}$$

old4.45 exists $\phi^0_{p,q}$-almost surely. By the comparison inequality, Proposition 3.26, we have that $\eta(p, q)$ is non-increasing in p, and we define

$$p_H(q) = \sup\{p : \eta(p, q) > 0\}. \tag{4.59}$$

old4.46

We define the further critical point

$$p_{\text{corr}}(q) = \sup\{p : \psi(p, q) > 0\}, \tag{4.60}$$

old4.47 where $\psi(p, q)$ is given in Theorem 4.40. The *correlation length* $\xi(p, q)$ is defined by

$$\xi(p, q) = \psi(p, q)^{-1},$$

where we adopt the convention that $0^{-1} = \infty$. Note that $\xi(p, q)$ is non-decreasing in p. Thus $\phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n)$ decays exponentially as $n \to \infty$ if and only if $\xi(p, q) < \infty$.

old4.48 **Theorem 4.61.** *Let $q \geq 1$. It is the case that $p_H(q) = p_{\mathrm{corr}}(q)$.*

By this theorem and the prior observations,

$$p_H(q) = p_{\mathrm{corr}}(q) \leq \widetilde{p}_{\mathrm{c}}(q) \leq p_{\mathrm{c}}(q),$$

and it follows from Theorem 4.54 that $p_{\mathrm{corr}}(q) = \widetilde{p}_{\mathrm{c}}(q)$. Thus,

$$p_H(q) = p_{\mathrm{corr}}(q) = \widetilde{p}_{\mathrm{c}}(q) \leq p_{\mathrm{c}}(q). \tag{4.62}$$

old4.49

In the percolation case, when $q = 1$, the above first-passage problem and the associated time constant $\eta(p, q)$ have been studied in detail; see [**125**, **128**]. Several authors have paid serious attention to a closely related question when $q = 2$ and $d = 2$, namely, the corresponding question for the two-dimensional Ising model, where the 'passage time' H_n is replaced by the minimum number of changes of spin along paths from the origin to $\partial\Lambda_n$; see [**1**, **53**, **70**]. The time constant in the Ising case cannot exceed the corresponding random-cluster time constant $\mu(p, 2)$, since each edge of the Ising model having endpoints with unlike spins gives rise to a closed edge in the associated (coupled) random-cluster process.

We turn now to the proofs of Theorems 4.54 and 4.61. We shall make use of Theorems 2.70 and 2.73 applied to the infinite-volume random-cluster measures. Let A be an increasing (non-empty) cylinder event in the measurable space $(\Omega, \mathcal{F})$, and set $\psi_{p,q} = \phi^0_{\Lambda_m,p,q}$, where m is a positive integer. We apply (2.72) and (2.75) to the random-cluster measure $\psi_{p,q}$ accordingly, noting that

$$\frac{q(1-r)}{s-r} < C < \frac{q}{s-r}.$$

We take the limit as $m \to \infty$, to obtain that, for $0 < r < s < 1$,

$$\phi^0_{r,q}(A) \leq \phi^0_{s,q}(A) \exp\Big\{-4(s-r)\phi^0_{s,q}(H_A)\Big\}, \tag{4.63}$$

$$\phi^0_{r,q}(H_A) \geq \frac{-\log \phi^0_{s,q}(A)}{\log\{q/(s-r)\}} - \frac{C}{C-1}. \tag{4.64}$$

old4.51

We note in passing that inequalities (4.63) and (4.64), with $A = A_n$, imply that the correlation length $\xi(p, q)$ is *strictly* increasing in p whenever it is finite (cf. [**92**, Theorem 6.14]).

Proof of Theorem 4.61. Let $r < s < p_H(q)$. There exists a constant $\gamma = \gamma(s, q)$ (> 0) such that

$$\phi_{s,q}(H_n) \geq n\gamma(s, q) \quad \text{for all } n \geq 1. \tag{4.65}$$

old4.52 Now let $A = A_n = \{0 \leftrightarrow \partial\Lambda_n\}$. In conjunction with (4.65), (4.63) implies the exponential decay of $\phi_{r,q}(A_n)$, whence $r < p_{\mathrm{corr}}(q)$. Therefore $p_H(q) \le p_{\mathrm{corr}}(q)$.

Conversely, suppose that $r < s < p_{\mathrm{corr}}(q)$. There exists $\alpha = \alpha(s,q)$ (> 0) such that $\phi^0_{s,q}(A_n) \le e^{-\alpha n}$ for all n. By (4.64) with $A = A_n$ and some positive $\beta = \beta(r,s,q)$,

$$\phi^0_{r,q}(H_n) \ge \frac{-\log(e^{-\alpha n})}{\log\{q/(s-r)\}} - \beta = \frac{\alpha n}{\log\{q/(s-r)\}} - \beta,$$

whence $r < p_H(q)$. Therefore $p_{\mathrm{corr}}(q) \le p_H(q)$. $\square$

We turn now to the proof of Theorem 4.54, there are two stages. In the first stage, we apply inequalities (4.63) and (4.64) to the event $A = A_n$, and we utilise an iterative scheme in order to prove that $\phi^0_{p,q}(A_n)$ decays 'near-exponentially' when $p < \widetilde{p}_{\mathrm{c}}(q)$. In the second stage, we use Theorems 4.61 and 4.57 to deduce full exponential decay. The conclusion of the first stage may be summarised as follows.

old4.53 **Lemma 4.66.** *Let* $0 < p < 1$ *and* $q \ge 1$, *and suppose that* $p < \widetilde{p}_{\mathrm{c}}(q)$. *There exist constants* $c(p)$, $\Delta(p)$, *satisfying* $c(p) > 0$, $0 < \Delta(p) < 1$, *such that*

$$\phi^0_{p,q}(A_n) \le \exp(-cn^{\Delta}) \quad \textit{for all } n \ge 1.$$

Before embarking on a proof of this, we make two remarks. First, Lemma 4.66 will be proved by an iterative scheme which may be continued further. If this is done, one obtains thereby a proof that $\phi^0_{p,q}(A_n)$ decays at least as fast as $\exp\bigl\{-\alpha_k(p)n/\log_k n\bigr)$ for any $k \ge 1$, where $\alpha_k(p) > 0$ and $\log_k n$ is the kth iterate of logarithm, that is,

$$\log_k x = \log(1 \vee \log_{k-1} x), \qquad k \ge 2.$$

Secondly, the hypothesis of Lemma 4.57 is implied by the conclusion of Lemma 4.66, using (4.37). Therefore these Lemmas imply that $\eta(p,q) > 0$ when $p < \widetilde{p}_{\mathrm{c}}(q)$, whence Theorem 4.54 follows by Theorem 4.61.

Proof of Lemma 4.66. We shall make central use of inequalities (2.72) and (2.75), in an iterative scheme. Rather than using these inequalities in the forms presented for finite graphs, we shall make use of their infinite-volume versions (4.63) and (4.64). In the following, we shall sometimes use *real* quantities when *integers* are required. It will be clear that this notational simplification has no ultimate effect on the validity of the proof. All o(1) and O(1) terms are to be interpreted in the limit $n \to \infty$.

Fix $q \ge 1$. For $p < \widetilde{p}_{\mathrm{c}}(q)$, there exists $c_1(p)$ satisfying $c_1(p) > 0$ such that

$$\phi^0_{p,q}(A_n) \le \frac{c_1(p)}{n^{d-1}} \quad \text{for all } n. \tag{4.67}$$

old4.55 Let $r < s < t < \widetilde{p}_c(q)$. By (4.9),

$$\phi^0_{s,q}(H_n) \geq \frac{-\log \phi^0_{t,q}(A_n)}{\log C} + \mathrm{O}(1) \geq \frac{(d-1)\log n}{\log C} + \mathrm{O}(1)$$

where $1 < C = q/(t-s) < \infty$. Insert this into (4.9) to obtain that

$$\phi^0_{r,q}(A_n) \leq \frac{c_2(r)}{n^{d-1+\Delta_2(r)}} \quad \text{for all } n \tag{4.68}$$

old45.2 for some strictly positive and finite $c_2(r)$ and $\Delta_2(r)$. This holds for all $r < \widetilde{p}_c(q)$, and is an improvement over (5.1).

Next we shall obtain an improvement of (4.68). Let m be a positive integer, and let $R_i = im$ for $0 \leq i \leq K$, where $K = \lfloor n/m \rfloor$. Let L_i be the event $\{\partial\Lambda_{R_i} \leftrightarrow \partial\Lambda_{R_{i+1}}\}$, and let $F_i = H_{L_i}$, the minimal number of extra edges needed for L_i to occur. Clearly,

$$H_n \geq \sum_{i=0}^{K-1} F_i, \tag{4.69}$$

old45.3 since every path from 0 to $\partial\Lambda_n$ traverses each annulus $\Lambda_{R_{i+1}} \setminus \Lambda_{R_i}$. There exists a constant η (≥ 1) (change notation!!) such that $|\partial\Lambda_R| \leq \eta R^{d-1}$ for all R. Therefore, by the translation invariance of $\phi_{p,q}$,

$$\phi_{p,q}(L_i) \leq |\partial\Lambda_{R_i}|\phi_{p,q}(A_m) \leq \eta n^{d-1}\phi_{p,q}(A_m). \tag{4.70}$$

old45.4

Let $r < s < \widetilde{p}_c(q)$, and let $c_2 = c_2(s)$, $\Delta_2 = \Delta_2(s)$ where the functions $c_2(p)$ and $\Delta_2(p)$ are given as in (4.68). It follows from (4.68) and (4.70) that

$$\phi^0_{s,q}(L_i) \leq \eta n^{d-1}\frac{c_2}{m^{d-1+\Delta_2}} \leq \frac{1}{2} \tag{4.71}$$

old45.5 if

$$m = \{(2\eta c_2)n^{d-1}\}^{1/(d-1+\Delta_2)}, \tag{4.72}$$

old45.6 and we choose m accordingly (here and later, we assume that n is large). Now $F_i \geq 1$ if L_i does not occur, whence

$$\phi^0_{s,q}(H_n) \geq \sum_{i=0}^{K-1}\{1 - \phi_{s,q}(L_i)\} \geq \tfrac{1}{2}K \tag{4.73}$$

old45.7 by (4.69) and (4.71). Also,

$$K = \lfloor n/m \rfloor \geq Dn^{\Delta_3} \tag{4.74}$$

old45.8 by (4.72), for appropriate positive constants D, Δ_3 satisfying $D > 0, 0 < \Delta_3 < 1$. In conjunction with (4.9) and (4.73), this lower bound for K implies that

$$\phi^0_{r,q}(A_n) \le \exp\{-c_3 n^{\Delta_3}\} \quad \text{for all } n. \tag{4.75}$$

old45.9 where $c_3 = c_3(r) > 0$, $0 < \Delta_3 = \Delta_3(r) < 1$. This holds for all $r < \widetilde{p}_c(q)$. □

Proof of Theorem 4.57. We prove that $\eta(\mu) > 0$ by an argument to be found in [**70**]. Let Π_n be the set of all paths of $\mathbb{L}^d$ joining the origin to $\partial\Lambda_n$. With $T(\pi)$ denoting the number of closed edges in a path π, we have that

$$T(\pi) + 1 \ge \sum_{x\in\pi} \frac{1}{|C_x \cap \pi|}$$

where the sum is over all vertices x of π, and C_x is the open cluster at x. By Jensen's inequality,

$$\frac{T(\pi)+1}{|\pi|} \ge \frac{1}{|\pi|}\sum_{x\in\pi}\frac{1}{|C_x|} \ge \left\{\frac{1}{|\pi|}\sum_{x\in\pi}|C_x|\right\}^{-1}.$$

Therefore,

$$\frac{H_n+1}{n} \ge \inf_{\pi\in\Pi_n}\left\{\frac{T(\pi)+1}{|\pi|}\right\} \ge K_n^{-1}$$

where

$$K_n = \sup_{\pi\in\Pi_n}\left\{\frac{1}{|\pi|}\sum_{x\in\pi}|C_x|\right\}.$$

Using (4.58), we find that $\eta(\mu) \ge K^{-1}$ μ-a.s., with

$$K = \limsup_{m\to\infty}\left[\sup\left\{\frac{1}{|\pi|}\sum_{x\in\pi}|C_x| : |\pi| = m\right\}\right], \tag{4.76}$$

old45.10 where the (inner) supremum is over all paths from the origin containing m vertices. We propose to show that $K < \infty$ a.s., whence $\eta(\mu) > 0$ as required.

Let $\{\widetilde{C}_x : x \in \mathbb{Z}^d\}$ be a collection of independent subsets of $\mathbb{Z}^d$ with the property that $\widetilde{C}_x$ has the same distribution as C_x. We claim, as in [**70**], that $\{|C_x| : x \in \mathbb{Z}^d\}$ is dominated stochastically by $\{M_x : x \in \mathbb{Z}^d\}$, where

$$M_x = \sup\big\{|\widetilde{C}_y| : y \in \mathbb{Z}^d,\ x \in \widetilde{C}_y\big\}.$$

We prove this inductively. Let $v_1, v_2, \dots$ be a deterministic ordering of $\mathbb{Z}^d$. Given the random variables $\{\widetilde{C}_x : x \in \mathbb{Z}^d\}$, we shall construct a family $\{D_x : x \in \mathbb{Z}^d\}$ having the same joint distributions as $\{C_x : x \in \mathbb{Z}^d\}$ and satisfying (for each x) $D_x \subseteq \widetilde{C}_y$ for some y depending on x. First, we set $D_{v_1} = \widetilde{C}_{v_1}$. Given

$D_{v_1}, D_{v_2}, \dots, D_{v_n}$, we define $E = \bigcup_{i=1}^{n} D_{v_i}$. If $v_{n+1} \in E$, we set $D_{v_{n+1}} = D_{v_j}$ for some j such that $v_{n+1} \in D_{v_j}$. If $v_{n+1} \notin E$, we argue as follows. Let ΔE be the set of edges of $\mathbb{Z}^d$ having exactly one endpoint in E. We may find a (random) subset F of $\widetilde{C}_{v_{n+1}}$ such that F has the conditional distribution of $C_{v_{n+1}}$ *given that* all edges in ΔE are closed; we now set $D_{v_{n+1}} = F$. This can be done since μ is positively associated. We obtain the required stochastic domination accordingly.

Reference Strassen

It follows by (4.76) that

$$K \le \limsup_{m\to\infty} \left[\sup \left\{ \frac{1}{|\pi|} \sum_{x\in\pi} M_x : |\pi| = m \right\} \right] \quad \text{a.s.}$$

By Lemma 2 of [**70**, page 760],

$$K \le 2 \limsup_{m\to\infty} \left[\sup \left\{ \frac{1}{|\Gamma|} \sum_{x\in\Gamma} |\widetilde{C}_x|^2 : |\Gamma| = m \right\} \right] \quad \text{a.s.}$$

where the (inner) supremum is over all animals Γ of $\mathbb{L}^d$ having m vertices and containing the origin. Using the result of [**57**], the right side is almost surely finite so long as $|\widetilde{C}_x|^2$ has finite $(d+\epsilon)$th moment for some $\epsilon > 0$. The required conclusion follows. □

4.6 Exponential decay of volume

For percolation, there is a beautiful proof of the exponential decay of volume, and this uses only that of radius. This proof hinges on the independence of the states of different edges, and may therefore not be extended to general random-cluster models, see [**92**, Theorem 6.78]. We shall instead make use here of the block arguments of [**126**], obtaining thereby the exponential decay of volume subject to a condition on p which is believed to hold throughout the subcritical phase. This condition differs from that of the last section, namely that $p < \widetilde{p}_c(q)$, in that it involves the decay rate of certain *finite-volume* probabilities.

Let $a \ge 1$, and let

$$L^a(p,q) = \limsup_{n\to\infty} \left\{ n^{d-1} \phi^1_{\Lambda_{an},p,q}(0 \leftrightarrow \partial\Lambda_n) \right\}. \tag{4.77}$$

new4.40 As at (4.51), $L^a(p,q)$ is non-decreasing in p, and thus

$$L^a(p,q) \begin{cases} = 0 & \text{if } p < \widetilde{p}^a_c(q), \\ \in (0,\infty] & \text{if } p > \widetilde{p}^a_c(q), \end{cases}$$

where

$$\widetilde{p}^a_c(q) = \sup\{p : L^a(p,q) = 0\}. \tag{4.78}$$

new4.41 Clearly $\widetilde{p}_{\mathrm{c}}^{a}(q)$ is non-decreasing in a, and furthermore $\widetilde{p}_{\mathrm{c}}^{a}(q) \le \widetilde{p}_{\mathrm{c}}(q)$ for all $a \ge 1$. We set

$$\widetilde{p}_{\mathrm{c}}^{\infty}(q) = \lim_{a\to\infty} \widetilde{p}_{\mathrm{c}}^{a}(q).$$

It is elementary that $\widetilde{p}_{\mathrm{c}}^{\infty}(q) \le \widetilde{p}_{\mathrm{c}}(q)$, and we conjecture that equality holds.

Conjecture 4.79. *If $q \ge 1$, then $\widetilde{p}_{\mathrm{c}}^{\infty}(q) = \widetilde{p}_{\mathrm{c}}(q)$.*

4.conj2

Here is the main result of this section.

expvolthm **Theorem 4.80.** *Let $q \ge 1$. There exists $\rho(p,q)$, satisfying $\rho(p,q) > 0$ when $p < \widetilde{p}_{\mathrm{c}}^{\infty}(q)$, such that*

$$\phi_{p,q}^{0}(|C| \ge n) \le e^{-n\rho(p,q)} \qquad \textit{for all } n \ge 1.$$

The hypothesis of this theorem is stronger than that of Theorem 4.54, and so is its conclusion, since

$$\phi_{p,q}^{0}(\mathrm{rad}(C) \ge n) \le \phi_{p,q}^{0}(|C| \ge n) = \phi_{p,q}^{1}(|C| \ge n)$$

when $p < p_{\mathrm{c}}(q)$.

Proof. We adapt the arguments of [**126**, Section 2]. For $N \ge 1$ and $1 \le i \le d$, we define the box

$$T_N(i) = [0,3N]^{i-1} \times [0,N] \times [0,3N]^{d-i}.$$

For $\omega \in \Omega$, an *i-crossing* of $T_N(i)$ is an open path $x_0, e_0, x_1, e_1, \dots, e_m$ of alternating vertices and edges of $T_n(i)$ such that the ith coordinate of x_0 (respectively, x_m) is 0 (respectively, N). Thus crossings are in the short direction. For $b \ge 3$, we define

$$\tau_N^b = \phi_{bN,p,q}^{1}(T_N(i) \text{ has an } i\text{-crossing}), \tag{4.81}$$

4.taudefn noting by symmetry that this does not depend on the value of i.

Let N be a fixed positive integer. From $\mathbb{L}^d$ we construct a new lattice $\mathcal{L}$ as follows. First, $\mathcal{L}$ has vertex set $\mathbb{Z}^d$. Two vertices $\mathbf{x}, \mathbf{y}$ of $\mathcal{L}$ are deemed adjacent in $\mathcal{L}$ if and only if $|x_i - y_i| \le 3$ for all $1 \le i \le d$. The better to distinguish vertices of $\mathbb{L}^d$ and $\mathcal{L}$, we shall use bold letters to indicate the latter. Let $\omega \in \Omega$. Vertex $\mathbf{x}$ of $\mathcal{L}$ is coloured *white* if there exists $i \in \{1,2,\dots,d\}$ such that $N\mathbf{x} + T_N(i)$ has an i-crossing, and is coloured *black* otherwise.

The following lemma relates the size of the open cluster C of origin 0 of $\mathbb{L}^d$ and the sizes of white clusters of $\mathcal{L}$. For $\mathbf{x} \in \mathbb{Z}^d$, we write $W_{\mathbf{x}}$ for the connected cluster of white vertices of $\mathcal{L}$ containing $\mathbf{x}$.

Lemma 4.82 [126]. *Let $\omega \in \Omega$. Assume that C contains some vertex v with*

$$v \in \prod_{j=1}^{d} \big[x_j N, (x_j + 1)N - 1\big],$$

for some $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$ satisfying

$$|x_j| \geq 2 \quad \textit{for some} \quad 1 \leq j \leq d. \tag{4.83}$$

4.bndx *Then the origin $\mathbf{0}$ of $\mathcal{L}$ possesses some neighbour $\mathbf{y}$ such that*

$$|W_{\mathbf{y}}| \geq 7^{-2d} \frac{|C| - (4N)^d}{N^d}. \tag{4.84}$$

4.whtopen

Proof. This is a simplification of that of [**126**, Lemma 2]. □

Since $\mathbf{0}$ has fewer than 7^d neighbours on $\mathcal{L}$, it follows(?) by the lemma that

$$\phi^0_{p,q}(|C| \geq n) \leq 7^d \phi^0_{p,q}(|W_{\mathbf{0}}| \geq An - 1), \tag{4.85}$$

4.bnd1 where $A = 7^{-2d} N^d$. Thus

$$\begin{aligned} &\phi^0_{p,q}(|C| \geq n) \\ &\qquad \leq 7^d \sum_{m \geq An-1} a_m \max\big\{\phi^0_{p,q}(\text{all vertices in } w \text{ are white}) : |w| = m\big\}, \end{aligned} \tag{4.86}$$

4.bnd2 where a_m is the number of connected sets w with m vertices of $\mathcal{L}$ containing $\mathbf{0}$, and w represents a typical such set. We have from the final display of the proof of [**126**, Lemma 3] that

$$a_m \leq 7^{2d} (7^d e)^m, \tag{4.87}$$

4.bnd3 and it remains to bound $\phi^0_{p,q}$(all vertices in w are white).

Fix $b > 3$, we shall choose b later. Let w be as above, containing m vertices. There exists a constant $c = c(b)$ such that w contains at least t vertices $\mathbf{y}(1), \mathbf{y}(2), \dots, \mathbf{y}(t)$ such that $t \geq cm$ and the boxes $\mathbf{y}(r) + \Lambda_{bN}$, $1 \leq r \leq t$, of $\mathbb{L}^d$ are disjoint. We may choose such a set $\{\mathbf{y}(r) : 1 \leq r \leq t\}$ in a way which depends only on the set w. We have now that

$$\phi^0_{p,q}(\text{all vertices in } w \text{ are white}) \leq \phi^0_{p,q}(\mathbf{y}(r) \text{ is white, for } 1 \leq r \leq t). \tag{4.88}$$

4.bnd6 The events $\{\mathbf{y}(r) \text{ is white}\}$, $1 \le r \le t$, are dependent under $\phi^0_{p,q}$. However, by positive association,

(4.89)
$$\phi^0_{p,q}(\mathbf{y}(r) \text{ is white, for } 1 \le r \le t) \le \phi^0_{p,q}(\mathbf{y}(r) \text{ is white, for } 1 \le r \le t \mid E),$$

4.bnd5 where E is the event that every edge e having both endpoints in $\mathbf{y}(r) + \partial\Lambda_{bN}$ for some $1 \le r \le t$ is open. Now, conditional on E, the events $\{\mathbf{y}(r) \text{ is white}\}$, $1 \le r \le t$, are independent (give reference), whence by symmetry

$$\phi^0_{p,q}(\mathbf{y}(r) \text{ is white, for } 1 \le r \le t) \le \{\phi^1_{\Lambda_{bN},p,q}(\mathbf{0} \text{ is white})\}^t \le (d\tau^b_N)^t. \tag{4.90}$$

4.bnd7 Combining (4.86)–(4.90) we find that

$$\phi^0_{p,q}(|C| \ge n) \le 7^d \sum_{m \ge An-1} 7^{2d}(7^d e)^m (d\tau^b_N)^{cm}, \tag{4.91}$$

4.bnd8 when $d\tau^b_N < 1$.

Let $a > 1$ and choose $b > 3 + a$, noting that $\mathbf{x} + \Lambda_{aN} \subseteq \Lambda_{bN}$ for all $\mathbf{x} \in \mathbb{Z}^d$ lying in the region $R = [0, 3N]^{d-1} \times \{0\}$. If T^d_N has a d-crossing, then there exists $\mathbf{x} \in R$ such that $\mathbf{x} \leftrightarrow \mathbf{x} + \partial\Lambda_N$. By further use of positive association,

(4.92)
$$\begin{aligned}\tau^b_N &\le \sum_{\mathbf{x} \in R} \phi^1_{\Lambda_{bN},p,q}(\mathbf{x} \leftrightarrow \mathbf{x} + \partial\Lambda_N)\\ &\le |R|\phi^1_{\Lambda_{aN},p,q}(\mathbf{0} \leftrightarrow \partial\Lambda_N) = (3N+1)^{d-1}\phi^1_{\Lambda_{aN},p,q}(\mathbf{0} \leftrightarrow \partial\Lambda_N).\end{aligned}$$

4.bnd9

Suppose now that $p < \widetilde{p}^{\,\infty}_{\mathrm{c}}(q)$, and choose $a > 1$ such that $p < \widetilde{p}^{\,a}_{\mathrm{c}}(q)$. With $b > 3 + a$, the right side of (4.92) may be made as small as required by a suitable choice of N large, and we choose N in such a way that

$$7^d e(d\tau^b_N)^c < 1.$$

Inequality (4.91) provides the required exponential bound for $\phi^0_{p,q}(|C| \ge n)$. □

4.7 Supercritical phase

In studying the supercritical phase, we shall assume that $d \ge 3$. The case $d = 2$ is simpler, see Chapter 5. Graphical duality (otherwise known as 'Whitney duality') for planar graphs leads to a duality theory for two-dimensional random-cluster measures in which the dual of a supercritical process is subcritical. Such duality permits the study of the supercritical phase via the theory of the subcritical phase.

Assume then that $d \geq 3$, The first property of note is the almost-sure uniqueness of the infinite open cluster, Theorem 3.33(c): every member ϕ of the closed convex hull of $\mathcal{W}_{p,q}$ has the 0/1-infinite cluster property in that the number I of infinite open clusters satisfies $\phi(I \in \{0, 1\}) = 1$. Applying this to the ergodic measures $\phi^b_{p,q}$ when $q \geq 1$, we find for $b = 0, 1$ that

$$\phi^b_{p,q}(I = 1) = 1 \quad \text{when} \quad \theta^b(p, q) > 0. \tag{4.93}$$

47.1

Suppose now that $q \geq 1$ and $p > p_c(q)$. The theory of percolation, [**92**, Chapter 8], provides a wish-list for the more general random-cluster model, but only parts of this list have been verified rigorously. We shall work in this section with the wired measure $\phi^1_{p,q}$. The following conjecture is outstanding.

finradconj **Conjecture 4.94.** *There exists $\sigma = \sigma(p, q)$, satisfying $\sigma(p, q) > 0$ when $p > p_c(q)$, such that*

$$\phi^1_{p,q}(n \leq \text{rad}(C) < \infty) \leq e^{-n\sigma(p,q)} \qquad \textit{for all } n \geq 1.$$

Ideas please

Much attention has been paid to the sizes and shapes of clusters C formed in models of statistical mechanics. When C is infinite with a strictly positive probability, but is constrained to be of some large *finite* size n, then C is said to form a large 'droplet'. The asymptotic shape of such a droplet, in the limit of large n, is prescribed in general terms by the theory of so-called Wulff shapes, named after the author of [**196**]. The picture for a random-cluster model is not complete, but techniques have emerged through the work of [**48**, **49**, **164**] which may be expected to reveal in due course the Wulff theory of large clusters in the random-cluster model. An full account of this work would be too lengthy for inclusion here, and we content ourseves with a brief summary beginning with some more elementary estimates.

subexpbnd **Theorem 4.95.** *There exist $\gamma = \gamma(p, q)$, $\gamma' = \gamma'(p, q)$ satisfying*

$$\begin{aligned} &\gamma(p, q) > 0 && \textit{for } p > p_c(q), \\ &\gamma'(p, q) < \infty && \textit{for } p \textit{ sufficiently close to } 1, \end{aligned}$$

such that

$$\exp(-\gamma n^{(d-1)/d}) \leq \phi^1_{p,q}(|C| = n) \leq \exp(-\gamma' n^{(d-1)/d}), \qquad n \geq 1.$$

Proof. The first inequality may be proved as was [**92**, Theorem 8.61], with use of the finite-energy property of Theorem 3.17(b). The second was proved in [**164**], and we omit the long proof. $\square$

Check latest on γ' for smaller p

We ask now whether such results may be refined to give exact asympotics, and whether such asymptotics are valid under the weaker hypothesis that $p > p_c(q)$. These matters are addressed in some detail in [**48**], where the asymptotic shape of droplets is studied in the special case of the Ising model, when $q = 2$. The proofs to date rely on two assumptions on the value of p, namely that p is such that $\phi^0_{p,q} = \phi^1_{p,q}$, cf. Conjecture 4.31, and secondly that p exceeds a certain 'slab critical point' $\widehat{p}_c(q)$ defined as follows. Fix $q \ge 1$, and let $d \ge 3$. Let $S(n, L) = [-n, n]^{d-1} \times [-L, L]$. Let $\psi^{n,L}_{p,q} = \phi^0_{S(n,L),p,q}$ be the random-cluster measure on $S(n, L)$ with parameters p, q, and with free boundary conditions. We denote by $\Pi(p, L)$ the property that:

$$\text{there exists } \alpha > 0 \text{ such that, for all } x \in S(n, L) \text{ and all } n,\ \psi^{n,L}_{p,q}(0 \leftrightarrow x) > \alpha.$$

It is not hard to see that $\Pi(p, L) \Rightarrow \Pi(p', L')$ if $p \le p'$ and $L \le L'$. It is thus natural to define the quantities

$$\widehat{p}_c(q, L) = \inf\{p : \Pi(p, L) \text{ occurs}\}, \qquad \widehat{p}_c(q) = \lim_{L\to\infty} \widehat{p}_c(q, L),$$

and it is clear that $\widehat{p}_c(q) \ge p_c(q)$. It is believed that equality holds here, and it is a major open problem to prove this.

old45.11 **Conjecture 4.96 [164].** *Let $q \ge 1$ and $d \ge 3$. We have that $\widehat{p}_c(q) = p_c(q)$.*

Subject to a verification of this conjecture, and of a positive answer to the question of the uniqueness of random-cluster measures when $p > p_c(q)$, the block arguments of [**48**, **164**] may be expected to result in a fairly complete picture of the supercritical phase of random-cluster models with $q \ge 1$; see [**49**] also.

The case $q = 1$ is special, percolation enjoys a spatial independence not shared with general random-cluster models. This additional property has been used in the formulation of a type of 'dynamic renormalisation', which has in turn yielded a proof that $\widehat{p}_c(1) = p_c(1)$ for percolation in three or more dimensions, [**92**, Chapter 7], [**95**]. Such arguments do not to date have a random-cluster counterpart. Instead in the random-cluster setting, one exploits what might be termed 'static renormalisation' methods, or 'block arguments' [**48**, **164**]. One divides space into blocks, constructs events of an appropriate nature on such blocks, having large probabilities, and then allows these events to combine across space. There have been substantial successes using this technique, of which the most striking is the resolution, subject to certain side conditions, of the so-called Wulff construction for the asymptotic shape of large Ising droplets.

Rather than discussing the physical background of the Wulff construction, we mention instead its impact on random-cluster models. Let $\Lambda_n = [-n, n]^d$, and consider the wired random-cluster measure $\phi^1_{\Lambda_n,p,q}$ with $p > p_c(q)$. The larger is an open cluster, the more likely it is to be joined to the boundary $\partial\Lambda_n$. Suppose that we condition on the event that there exists in Λ_n an open cluster C which does not touch $\partial\Lambda_n$ and which has volume of the order of the volume n^d of the box. What can be said about the shape of C? Since $p > p_c(q)$, there is little cost in having large *volume*, and the price is spent around its *boundary*. Indeed, the price may be expressed as a surface integral of an appropriate function termed 'surface tension'. This 'surface tension' may be specified as the exponential rate of decay of a certain probability. The Wulff prediction for the shape of C is that, when re-scaled in the limit of large n, its shape converges to the solution of a certain variational problem, that is, the limit shape is obtained by minimising a certain surface integral subject to a condition on its volume.

No proof of this general picture for random-cluster models has appeared in the literature, although it is believed that the methods of [**48**, **49**, **164**] enable such a proof. The authors of [**48**] have instead concentrated on using random-cluster technology to solve the corresponding question for the asymptotic shape of large droplets in the Ising model. The outcome is an important 'large deviation' theorem which utilises block arguments and yields a full solution to the Ising problem whenever the corresponding random-cluster model (which has $q = 2$) has parameter p satisfying $p > \widehat{p}_c(2)$ and $\phi^0_{p,2} = \phi^1_{p,2}$. Here, $\widehat{p}_c(2)$ is the limit of a certain decreasing sequence of critical points defined on slabs in $\mathbb{Z}^d$, and is conjectured to be equal to the critical point $p_c(2)$. [Closely related results have been obtained in [**32**]. Fluctuations in droplet shape for *two-dimensional* random-cluster models have been studied in [**13**, **14**].]

5

In Two Dimensions

5.1 Planar duality

The duality theory of planar graphs provides a technique for studying random-cluster models in two dimensions. We shall see that, for a dual pair (G, G_d) of finite planar graphs, the measures $\phi_{G,p,q}$ and $\phi_{G_d,p_d,q}$ are dual measures in a certain sense to be explained soon, where p and p_d are related by $p_d/(1-p_d) = q(1-p)/p$. Such a duality survives the passage to a thermodynamic limit, and may thus be applied also to infinite planar graphs such as the square lattice $\mathbb{L}^2$. The square lattice has the further property of being isomorphic to its (infinite) dual, and this observation leads to many results of significance for the square lattice. We begin with an account of planar duality.

A graph is called *planar* if it may be imbedded in $\mathbb{R}^2$ in such a way that two edges intersect only at a shared endvertex. Let $G = (V, E)$ be a simple planar graph imbedded in $\mathbb{R}^2$; G may be either finite or infinite. We obtain its dual graph $G_d = (V_d, E_d)$ as follows (the roman letter 'd' denotes 'dual' rather than number of dimensions). We place a dual vertex within each face of G, including any infinite face of G if such exist. For each $e \in E$ we place a dual edge $e_d = \langle x_d, y_d \rangle$ joining the two dual vertices lying in the two faces of G abutting e; if these two faces are the same, then $x_d = y_d$ and e_d is a loop. Thus, V_d is in one–one correspondence with the set of faces of G, and E_d is in one–one correspondence with E. It is easy to see as in Figure 5.1 that the dual $\mathbb{L}^2_d$ of the square lattice $\mathbb{L}^2$ is isomorphic to $\mathbb{L}^2$. What is the relevance of graphical duality to random-cluster measures on G?

Suppose that G is finite. A configuration $\omega \in \Omega$ $(= \{0, 1\}^E)$ gives rise to a dual configuration ω_d lying in the space $\Omega_d = \{0, 1\}^{E_d}$ and defined by $\omega_d(e_d) = 1 - \omega(e)$. That is, e_d is declared open if and only if e is closed. As before, to each configuration ω_d corresponds the set $\eta(\omega_d) = \{e_d \in E_d : \omega_d(e_d) = 1\}$ of its 'open edges', so that $\eta(\omega_d)$ is in one–one correspondence with $E \setminus \eta(\omega)$. Let $f(\omega_d)$ be the number of faces of the graph $(V_d, \eta(\omega_d))$, including the unique infinite face. By drawing a picture, one may easily be convinced (see Figure 5.2) that the faces

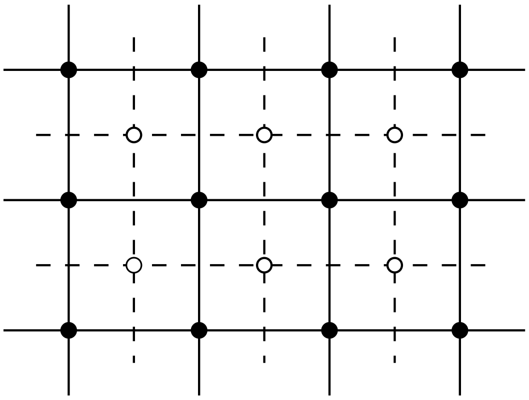


Figure 5.1. The planar dual of the square lattice $\mathbb{L}^2$ is isomorphic to the square lattice.

5.planardual

of $(V_{\mathrm{d}}, \eta(\omega_{\mathrm{d}}))$ are in one–one correspondence with the components of $(V, \eta(\omega))$, and therefore

$$f(\omega_{\mathrm{d}}) = k(\omega), \tag{5.1}$$

5.facecomp in the obvious notation. We shall make use of Euler's formula (see [**191**]), namely

$$k(\omega) = |V| - |\eta(\omega)| + f(\omega) - 1, \qquad \omega \in \Omega, \tag{5.2}$$

5.euler and we note also for later use that

$$|\eta(\omega)| + |\eta(\omega_{\mathrm{d}})| = |E|. \tag{5.3}$$

5.edge

The random-cluster measure on G is given by

$$\phi_{G,p,q}(\omega) \propto \left(\frac{p}{1-p}\right)^{|\eta(\omega)|} q^{k(\omega)}, \qquad \omega \in \Omega,$$

where the constant of proportionality depends on G, p, and q. Therefore,

(5.4)

$$\begin{aligned}
\phi_{G,p,q}(\omega) &\propto \left(\frac{p}{1-p}\right)^{-|\eta(\omega_{\mathrm{d}})|} q^{f(\omega_{\mathrm{d}})} && \text{by (5.1) and (5.3)} \\
&\propto \left(\frac{q(1-p)}{p}\right)^{|\eta(\omega_{\mathrm{d}})|} q^{k(\omega_{\mathrm{d}})} && \text{by (5.2) applied to } \omega_{\mathrm{d}} \\
&\propto \phi_{G_{\mathrm{d}},p_{\mathrm{d}},q}(\omega_{\mathrm{d}}),
\end{aligned}$$

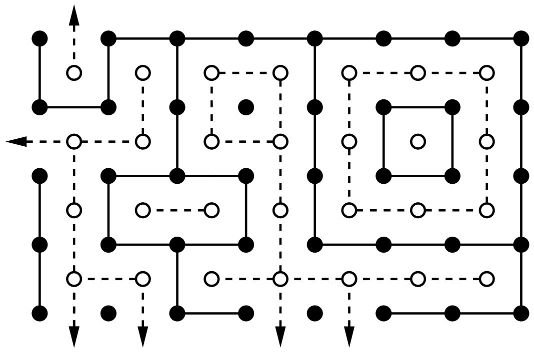


Figure 5.2. A primal configuration ω (with solid lines and vertices) and its dual configuration ω_{d} (with dashed lines and hollow vertices). The arrows join the given vertices of the dual to a dual vertex in the infinite face. Note that each face of the dual graph corresponds to a unique component of the primal graph lying 'just within'.

oldfig5.1

5.primdual where the dual parameter p_{d} is given according to

$$\frac{p_{\mathrm{d}}}{1-p_{\mathrm{d}}} = \frac{q(1-p)}{p}. \tag{5.5}$$

5.dualpar Note that the dual value of p_{d} satisfies $(p_{\mathrm{d}})_{\mathrm{d}} = p$. Since (5.4) involves probability measures, we deduce that

$$\phi_{G,p,q}(\omega) = \phi_{G_{\mathrm{d}},p_{\mathrm{d}},q}(\omega_{\mathrm{d}}) \qquad \text{for } \omega \in \Omega. \tag{5.6}$$

5.primaldual

It will later be convenient to work with the edge-parameter[7]

$$x = \frac{q^{-\frac{1}{2}}p}{1-p}, \tag{5.7}$$

5.ptox and then the primal/dual transformation (5.5) becomes

$$xx_{\mathrm{d}} = 1. \tag{5.8}$$

5.xdual

The unique fixed point of the mapping $p \mapsto p_{\mathrm{d}}$ is easily seen from (5.5) to be given by $p = p_{\mathrm{sd}}(q)$ where

$$p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{5.9}$$

[7]Later in this chapter we shall instead work with the parameter $y = p/(1-p)$, for which the primal/dual relation becomes $yy_{\mathrm{d}} = q$.

5.sddef We note at this point that

$$\phi_{G,p_{\rm sd}(q),q}(\omega) \propto q^{\frac{1}{2}|\eta(\omega)|+k(\omega)} \propto q^{\frac{1}{2}(k(\omega_{\rm d})+k(\omega))},$$

by Euler's formula. This representation of the random-cluster measure at the 'self-dual point' $p_{\rm sd}(q)$ highlights the duality of measures.

When we keep track of the constants of proportionality in (5.4), we find that the partition function

$$Z_{G,p,q} = \sum_{\omega\in\Omega} p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|} q^{k(\omega)}$$

satisfies the duality relation

$$Z_{G,p,q} = q^{|V|-1}\left(\frac{1-p}{p_{\rm d}}\right)^{|E|} Z_{G_{\rm d},p_{\rm d},q} \tag{5.10}$$

5.1334 which, when $p = p_{\rm d} = p_{\rm sd}(q)$, becomes

$$Z_{G,p_{\rm sd}(q),q} = q^{|V|-1-\frac{1}{2}|E|} Z_{G_{\rm d},p_{\rm sd}(q),q}. \tag{5.11}$$

5.1335

We turn now to the square lattice $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$. Let $\Lambda(n) = [-n,n]^2$, viewed as a subgraph of $\mathbb{L}^2$, and note that its dual graph $\Lambda(n)_{\rm d}$ may be obtained from the box $[-n-1,n]^2 + (\frac{1}{2},\frac{1}{2})$ by identifying all boundary vertices. See Figure 5.3. By the above, and with a small 'fix' on the boundary of $\Lambda(n)_{\rm d}$,

$$\phi^0_{\Lambda(n),p,q}(\omega) = \phi^1_{\Lambda(n)_{\rm d},p_{\rm d},q}(\omega_{\rm d}) \tag{5.12}$$

5.primdual3 for configurations ω on Λ_n. Let A be a cylinder event of $\Omega = \{0,1\}^{\mathbb{E}^2}$, and write $A_{\rm d}$ for the dual event of $\Omega_{\rm d} = \{0,1\}^{\mathbb{E}^2_{\rm d}}$, that is, $A_{\rm d} = \{\omega_{\rm d} : \omega \in A\}$. On letting $n \to \infty$ in (5.12), we obtain by Theorem 3.19(a) that

$$\phi^0_{p,q}(A) = \phi^1_{p_{\rm d},q}(A_{\rm d}),$$

and by a similar argument that

$$\phi^1_{p,q}(A) = \phi^0_{p_{\rm d},q}(A_{\rm d}).$$

In writing thus, the measures $\phi^b_{p_{\rm d},q}$ are to be interpreted as measures on $\Omega_{\rm d}$. We summarise this in a theorem.

5.freewired **Theorem 5.13.** *Consider the square lattice $\mathbb{L}^2$, and let $q \ge 1$. Then*

$$\phi^b_{p,q}(A) = \phi^{1-b}_{p_{\rm d},q}(A_{\rm d}) \quad \textit{for } b = 0, 1, \textit{ and for all cylinder events } A,$$

where $A_{\rm d} = \{\omega_{\rm d} : \omega \in A\}$.

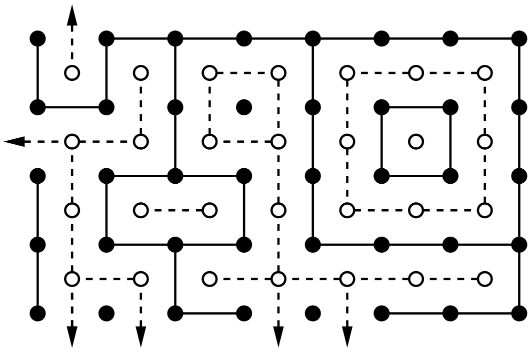


Figure 5.3. The dual of the box $\Lambda_n = [-n, n]^2$ is obtained from the box $[-n-1, n] + (\frac{1}{2}, \frac{1}{2})$ by identifying all vertices in its boundary.

oldfig5.2

5.2 Value of the critical point

Consider the random-cluster model on the two-dimensional lattice $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$, with parameters p and q satisfying $q \geq 1$. It is a remarkable conjecture that the critical point and the self-dual point are equal.

5.exactpc **Conjecture 5.14.** *Let $q \geq 1$. The critical value $p_c(q)$ of $\mathbb{L}^2$ is given by*

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}} \qquad \textit{for } q \geq 1. \tag{5.15}$$

5.conjsd

This conjecture is known to hold when $q = 1$, $q = 2$, and for $q \geq 25.72$. The $q = 1$ case was answered by Kesten, [**124**], in his famous proof that the critical probability of bond percolation on $\mathbb{L}^2$ is $\frac{1}{2}$. For $q = 2$, the value of $p_c(2)$ given above agrees with the celebrated calculation by Onsager, [**157**], of the critical temperature of the Ising model on $\mathbb{Z}^2$, and is implied by probabilistic results in the modern vernacular, see [**3**] and Chapter 8. The formula (5.15) for $p_c(q)$ has been established rigorously in [**134**, **135**] for sufficiently large (real) values of q, specifically $q \geq 25.72$ (see also [**91**]). This is explored further in Section 5.4, see Theorem 5.32.

Several other remarkable conjectures about the phase transition on $\mathbb{L}^2$ may be found in the physics literature (see [**18**]), as consequences of 'exact' but non-rigorous arguments involving ice-type models. These include exact formulae for the asymptotic behaviour of the partition function $\lim_{\Lambda\uparrow\mathbb{Z}^2}\{Z_{\Lambda,p,q}\}^{1/|\Lambda|}$, and also for the edge-densities at the self-dual point $p_{sd}(q)$, namely $h^b(q) = \phi^b_{p_{sd}(q),q}(e \text{ is open})$, for $b = 0, 1$.

Conjecture 5.14 asserts that $p_c(q) = p_{sd}(q)$ for $q \geq 1$. One part of this equality is known. Recall that $\theta^0(p, q) = \phi^0_{p,q}(0 \leftrightarrow \infty)$.

5.pclower **Theorem 5.16 [90, 187].** *Consider the square lattice $\mathbb{L}^2$, and let $q \ge 1$.*

(a) *We have that $\theta^0(p_{\mathrm{sd}}(q), q) = 0$, whence $p_{\mathrm{c}}(q) \ge p_{\mathrm{sd}}(q)$ for $q \ge 1$.*

(b) *There exists a unique random-cluster measure if $p \ne p_{\mathrm{sd}}(q)$, that is,*

$$|\mathcal{R}_{p,q}| = |\mathcal{W}_{p,q}| = 1 \quad \text{if } p \ne \frac{\sqrt{q}}{1+\sqrt{q}}.$$

The complementary inequality $p_{\mathrm{c}}(q) \le p_{\mathrm{sd}}(q)$ has eluded mathematicians despite progress by physicists, [**111**]. Here is an intuitive argument to justify the latter inequality. Suppose on the contrary that $p_{\mathrm{c}}(q) > p_{\mathrm{sd}}(q)$, so that $p_{\mathrm{c}}(q)^{\mathrm{d}} < p_{\mathrm{sd}}(q)$. For $p \in (p_{\mathrm{c}}(q)^{\mathrm{d}}, p_{\mathrm{c}}(q))$ we have also that $p^{\mathrm{d}} \in (p_{\mathrm{c}}(q)^{\mathrm{d}}, p_{\mathrm{c}}(q))$. Therefore, for $p \in (p_{\mathrm{c}}(q)^{\mathrm{d}}, p_{\mathrm{c}}(q))$, both primal and dual processes comprise, almost surely, the union of *finite* open clusters. This contradicts the intuitive picture, supported for $p \ne p_{\mathrm{c}}(q)$ by our knowledge of percolation, of finite open clusters of one process floating in an infinite open ocean of the other process.

The conjectured exact value of $p_{\mathrm{c}}(q)$ would be proven if one could show that, for all $p < p_{\mathrm{c}}(q)$, there exists $A = A(p) < \infty$ such that

$$\phi^0_{p,q}(0 \leftrightarrow \partial\Lambda_n) \le \frac{A}{n} \qquad \text{for all } n. \tag{5.17}$$

5.necconj This would, by Theorem 4.54, imply the exponential decay of connections throughout the subcritical phase. Under the hypothesis that $p_{\mathrm{sd}}(q) < p_{\mathrm{c}}(q)$, this contradicts the self-duality of the process at the self-dual point $p_{\mathrm{sd}}(q)$. (MORE DETAILS NEEDED)

Rigorous numerical upper bounds of impressive accuracy have been achieved for the square lattice and other two-dimensional lattices.

5.pcupper **Theorem 5.18 [11].** *Consider the square lattice $\mathbb{L}^2$, and let $q \ge 2$. Then*

$$p_{\mathrm{c}}(q) \le \frac{\sqrt{q}}{\sqrt{1-q^{-1}}+\sqrt{q}}. \tag{5.19}$$

5.alex1

For example, when $q = 10$, we have that $0.760 \le p_{\mathrm{c}}(10) \le 0.769$, to be compared with the conjecture that $p_{\mathrm{c}}(10) = \sqrt{10}/(1+\sqrt{10}) \simeq 0.760$. The upper bound in (5.19) is the dual value of $p_{\mathrm{sd}}(q-1)$. See also Theorem 5.27.

Exact values for the critical points of the triangular and hexagonal lattices may be conjectured similarly, using graphical duality together with the star–triangle transformation; see Section 5.6.

Proof of Theorem 5.16. (a) There are at least two ways of proving this. One way is to use the circuit-construction argument pioneered by Harris, [**109**], and developed further in [**78**]; we shall follow this route in the forthcoming Theorem

5.44. In the present case, we shall instead adapt an argument of [**92**, **197**] using the 0/1-infinite cluster property. Let $p = p_{\mathrm{sd}}(q)$, so that $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are dual measures in the sense of Theorem 5.13.

For each positive integer n, let $A^{\mathrm{l}}(n)$ (respectively $A^{\mathrm{r}}(n)$, $A^{\mathrm{t}}(n)$, $A^{\mathrm{b}}(n)$) be the event that some vertex on the left (respectively right, top, bottom) side of the square $T(n) = [0,n]^2$ lies in an infinite open path of $\mathbb{L}^2$ which uses no other vertex of $T(n)$. Clearly $A^{\mathrm{l}}(n)$, $A^{\mathrm{r}}(n)$, $A^{\mathrm{t}}(n)$, and $A^{\mathrm{b}}(n)$ are increasing events whose union is the event that some vertex in $T(n)$ lies in an infinite open cluster. Furthermore, by the rotation-invariance of $\mathbb{L}^2$,

$$\text{for each } b = 0, 1 \text{ and } n \geq 1, \quad \phi^b_{p,q}(A^u(n)) \text{ is constant for } u = \mathrm{l}, \mathrm{r}, \mathrm{t}, \mathrm{b}. \tag{5.20}$$

5.rotinv

Suppose that $\theta^0(p,q) > 0$, whence by stochastic ordering we have also that $\theta^1(p,q) > 0$. For $b = 0, 1$, there exists an infinite open cluster with $\phi^b_{p,q}$-probability 1, whence

$$\phi^b_{p,q}\big(A^{\mathrm{l}}(n) \cup A^{\mathrm{r}}(n) \cup A^{\mathrm{t}}(n) \cup A^{\mathrm{b}}(n)\big) \to 1 \qquad \text{as } n \to \infty.$$

By the FKG inequality,

$$\phi^b_{p,q}(T(n) \nleftrightarrow \infty) \geq \phi^b_{p,q}\big(\overline{A^{\mathrm{l}}(n)}\big)\phi^b_{p,q}\big(\overline{A^{\mathrm{r}}(n)}\big)\phi^b_{p,q}\big(\overline{A^{\mathrm{t}}(n)}\big)\phi^b_{p,q}\big(\overline{A^{\mathrm{b}}(n)}\big),$$

whence, by (5.20),

$$\phi^b_{p,q}(A^u(n)) \to 1 \qquad \text{as } n \to \infty, \text{ for } u = \mathrm{l}, \mathrm{r}, \mathrm{t}, \mathrm{b}. \tag{5.21}$$

5.tcn15 We choose N such that

$$\phi^b_{p,q}(A^u(N)) > \tfrac{7}{8} \qquad \text{for } u = \mathrm{l}, \mathrm{r}, \mathrm{t}, \mathrm{b}, \text{ and } b = 0, 1. \tag{5.22}$$

5.tcn16

Moving to the dual lattice, we define the dual box

$$T(n)_{\mathrm{d}} = \{x + (\tfrac{1}{2}, \tfrac{1}{2}) : 0 \leq x_1, x_2 \leq n\}. \tag{5.23}$$

5.tcn17 Let $A^{\mathrm{l}}_{\mathrm{d}}(n)$ (respectively, $A^{\mathrm{r}}_{\mathrm{d}}(n)$, $A^{\mathrm{t}}_{\mathrm{d}}(n)$, $A^{\mathrm{b}}_{\mathrm{d}}(n)$) be the event that some vertex on the left (respectively right, top, bottom) side of $T(n)_{\mathrm{d}}$ lies in an infinite closed path of the dual lattice $\mathbb{L}^2_{\mathrm{d}}$ which uses no other vertex of $T(n)_{\mathrm{d}}$.

Let $A^u(N,m)$ (respectively, $A^u_{\mathrm{d}}(N,m)$) be the event that some vertex in the side of $T(n)$ (respectively, $T(n)_{\mathrm{d}}$) labelled u is joined to $\partial\Lambda_m$ (respectively, $\partial\Lambda_m +$

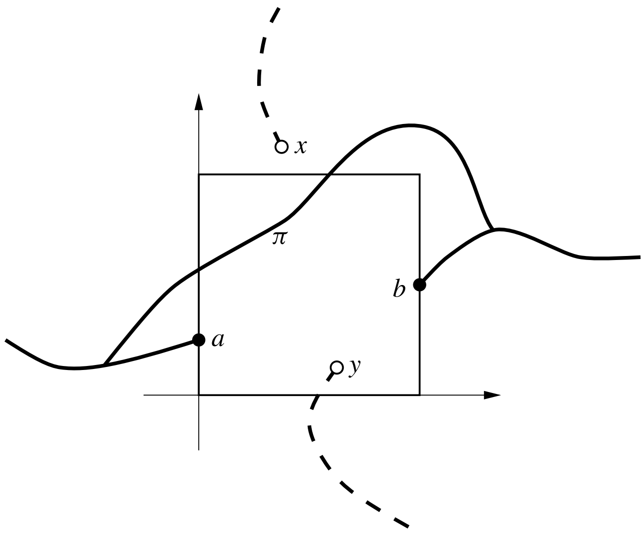


Figure 5.4. Vertices a and b lie in infinite open clusters of $\mathbb{L}^2 \setminus T(N)$, and vertices x and y lie in infinite closed clusters of $\mathbb{L}^2_{\mathrm{d}} \setminus T(N)_{\mathrm{d}}$. If there exists a unique infinite open cluster, then there exists an open path π joining a to b, and thus the infinite closed clusters at x and y are disjoint.

oldfig11-7

$(\frac{1}{2}, \frac{1}{2}))$ by an open path (respectively, open dual path) which uses no other vertex of $T(n)$ (respectively, $T(n)_{\mathrm{d}}$). By Theorem 5.13, and by (5.22),

$$\begin{aligned}
\phi^0_{p,q}(A^u_{\mathrm{d}}(N)) &= \lim_{m\to\infty} \phi^0_{p,q}(A_{\mathrm{d}}(N,m)) \\
&= \lim_{m\to\infty} \phi^1_{p,q}(A^u(N,m)) \\
&= \phi^1_{p,q}(A^u(N)) \\
&> \tfrac{7}{8} \qquad \text{for } u = \mathrm{l, r, t, b}.
\end{aligned} \tag{5.24}$$

5.tcn18

Consider now the event $A = A^{\mathrm{l}}(N) \cap A^{\mathrm{r}}(N) \cap A^{\mathrm{t}}_{\mathrm{d}}(N) \cap A^{\mathrm{b}}_{\mathrm{d}}(N)$, that there exist infinite open paths of $\mathbb{L}^2 \setminus T(N)$ touching the left and right sides of $T(N)$, and also infinite closed paths of $\mathbb{L}^2_{\mathrm{d}} \setminus T(N)_{\mathrm{d}}$ touching the top and bottom sides of $T(N)_{\mathrm{d}}$. This event is illustrated in Figure 5.4.

The probability that A does not occur satisfies

$$\begin{aligned}
\phi^0_{p,q}(\overline{A}) &\le \phi^0_{p,q}\big(\overline{A^{\mathrm{l}}(N)}\big) + \phi^0_{p,q}\big(\overline{A^{\mathrm{r}}(N)}\big) + \phi^1_{p,q}\big(\overline{A^{\mathrm{t}}_{\mathrm{d}}(N)}\big) + \phi^1_{p,q}\big(\overline{A^{\mathrm{b}}_{\mathrm{d}}(N)}\big) \\
&\le \tfrac{1}{2} \qquad \text{by (5.22) and (5.24),}
\end{aligned}$$

giving that $\phi^b_{p,q}(A) > \frac{1}{2}$.

We now use the fact that every random-cluster measure $\phi^b_{p,q}$ has the 0/1-infinite cluster property, see Theorem 3.33(c). If A occurs, then $\mathbb{L}^2 \setminus T(N)$ contains two disjoint infinite open clusters, since the clusters in questions are separated physically by infinite closed paths of the dual; any open path of $\mathbb{L}^2 \setminus T(N)$ joining these two clusters would contain an edge which crosses a closed edge of the dual, and no such edge can exist. Similarly, on A, the graph $\mathbb{L}^2_{\mathrm{d}} \setminus T(N)_{\mathrm{d}}$ contains two disjoint infinite closed clusters, separated physically by infinite open paths of $\mathbb{L}^2 \setminus T(N)$. Now, the whole lattice $\mathbb{L}^2$ contains (almost surely) a *unique* infinite open cluster, and it follows that there exists (almost surely on A) an open connection of the lattice between the fore-mentioned infinite open clusters. By the geometry of the situation (see Figure 5.4 again), this connection forms a barrier to possible closed connections of the dual joining the two infinite closed clusters. Hence, almost surely on A, the dual lattice contains two or more infinite closed clusters. Since the latter event has probability 0, it follows that $\phi^b_{p,q}(A) = 0$ in contradiction of the deduction that $\phi^b_{p,q}(A) > \frac{1}{2}$. The initial hypothesis that $\theta^0(p,q) > 0$ is therefore incorrect, and the proof is complete.

(b) It follows from Theorem 5.16(a) that $\theta^1(p,q) = 0$ for $p < p_{\mathrm{sd}}(q)$, whence, by Theorem 4.30(a), $|\mathcal{R}_{p,q}| = |\mathcal{W}_{p,q}| = 1$ if $p < p_{\mathrm{sd}}(q)$.

Suppose now that $p > p_{\mathrm{sd}}(q)$, so that p^{d}, given by (5.5), satisfies $p^{\mathrm{d}} < p_{\mathrm{sd}}(q)$. The claim now follows by Theorem 5.13. Here is one way to complete the proof. By part (a) above,

$$\phi^0_{p^{\mathrm{d}},q}(e \text{ is closed}) = \phi^1_{p^{\mathrm{d}},q}(e \text{ is closed}),$$

and by Theorem 5.13,

$$\phi^0_{p,q}(e \text{ is open}) = \phi^1_{p^{\mathrm{d}},q}(e \text{ is closed}), \quad \phi^1_{p,q}(e \text{ is open}) = \phi^0_{p^{\mathrm{d}},q}(e \text{ is closed}),$$

whence $\phi^0_{p,q}(e \text{ is open}) = \phi^1_{p,q}(e \text{ is open})$. By Proposition 3.6, $\phi^0_{p,q} = \phi^1_{p,q}$, so that $|\mathcal{R}_{p,q}| = |\mathcal{W}_{p,q}| = 1$ by (3.36). □

We precede the proof of Theorem 5.18 with a lemma.

5.pcupperdual **Lemma 5.25.** *Let p and p_{d} be related by (5.5), and let $q \ge 1$. With C the open cluster at the origin of the dual lattice, and $b \in \{0,1\}$,*

$$\text{if} \quad \phi^b_{p_{\mathrm{d}},q}(\mathrm{rad}(C)) < \infty \quad \text{then} \quad \theta^{1-b}(p,q) > 0.$$

Proof. Let $\Lambda(n) = [-n,n]^2$, and let $r < t$. By Theorem 5.13,

$$\phi^0_{p,q}\big(\partial\Lambda(r) \leftrightarrow \partial\Lambda(t)\big) = \phi^1_{p_{\mathrm{d}},q}\bigg(\bigcup_{r \le s < t} A_s(r,t)\bigg),$$

where $A_s(r,t)$ is the event that $(s+\frac{1}{2},\frac{1}{2})$ belongs to an open circuit of the dual, lying in $\Lambda(t)_{\mathrm{d}}\setminus\Lambda(r)_{\mathrm{d}}$ and having $\Lambda(r)$ in its interior. By the translation-invariance of random-cluster measures, Theorem 3.19(b),

$$\phi^0_{p,q}\big(\partial\Lambda(r) \nleftrightarrow \partial\Lambda(t)\big) \le \sum_{s=r}^{t-1} \phi^1_{p_{\mathrm{d}},q}\big(\mathrm{rad}(C) \ge r+s\big),$$

whence, on passing to the limit $t\to\infty$,

$$\phi^0_{p,q}\big(\partial\Lambda(r) \nleftrightarrow \infty\big) \le \sum_{s=r}^{\infty} \phi^1_{p_{\mathrm{d}},q}\big(\mathrm{rad}(C) \ge r+s\big). \tag{5.26}$$

5.radsum Suppose that $\phi^b_{p_{\mathrm{d}},q}(\mathrm{rad}(C)) < \infty$, and pick R such that

$$\sum_{s=R}^{\infty} \phi^1_{p_{\mathrm{d}},q}\big(\mathrm{rad}(C) \ge r+s\big) < 1.$$

We have from (5.26) that $\phi^0_{p,q}(\partial\Lambda(r) \nleftrightarrow \infty) > 0$, whence $\theta^0(p,q) > 0$ as required. □

Proof of Theorem 5.18. Let $q \ge 2$ and $b \in \{0,1\}$. We have from the forthcoming Theorem 5.27 that $\phi^1_{p_{\mathrm{d}},q}(\mathrm{rad}(C)) < \infty$ when $p_{\mathrm{d}} < p_{\mathrm{sd}}(q-1)$. Therefore, by Lemma 5.25, $\theta^0(p,q) > 0$ whenever

$$p > \big(p_{\mathrm{sd}}(q-1)\big)_{\mathrm{d}} = \frac{\sqrt{q}}{\sqrt{1-q^{-1}}+\sqrt{q}},$$

and the claim follows. □

5.3 Exponential decay

A valuable consequence of the comparison methods developed in [**11**] is the exponential decay of connectivity functions when $q \ge 2$ and p is such that

$$p < p_{\mathrm{sd}}(q-1) = \frac{\sqrt{q-1}}{1+\sqrt{q-1}}.$$

The next theorem is a stronger statement than this.

5.expdecay **Theorem 5.27 (Exponential decay) [11].** *Let $q \ge 2$, and consider the random-cluster model on $\mathbb{L}^2$. There exists $\alpha = \alpha(p,q)$ satisfying $\alpha(p,q) > 0$ when $p < p_{\mathrm{sd}}(q-1)$ such that*

$$\phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda) \le e^{-\alpha\,\mathrm{rad}(\Lambda)} \qquad \textit{for all boxes } \Lambda.$$

Let $q \geq 2$ and $\Lambda_n = [-n, n]^2$. We have from this theorem and the positive association of random-cluster measures that

$$\phi^1_{\Lambda_n, p, q}(0 \leftrightarrow \partial\Lambda_m) \leq \phi^1_{\Lambda_m, p, q}(0 \leftrightarrow \partial\Lambda_m), \qquad m \leq n,$$

and therefore, on taking the limit as $n \to \infty$,

$$\phi^1_{p,q}(0 \leftrightarrow \partial\Lambda_m) \leq e^{-\alpha m} \qquad \text{when } p < p_{\mathrm{sd}}(q-1).$$

This is the exponential decay for the infinite-volume measures claimed above.

Proof. We use the comparison of the random-cluster model with the Ising model with external field, as described in Section 2.8. Consider the wired random-cluster measure on Λ. By Theorem 2.107 and the note following, the set of vertices which are joined to $\partial\Lambda$ by open paths is stochastically smaller than the set of $+$ spins in the Ising model on Λ with $+$ boundary conditions and parameters β', h' satisfying (2.108). The maximum vertex degree of $\mathbb{L}^2$ is $\Delta = 4$ and, by (2.110),

$$e^{\beta_4} = \frac{q-2}{\sqrt{q-1}-1} = 1 + \sqrt{q-1},$$

so that

$$1 - e^{-\beta_4} = \frac{\sqrt{q-1}}{1+\sqrt{q-1}} = p_{\mathrm{sd}}(q-1).$$

Let $p = 1 - e^{-\beta} < p_{\mathrm{sd}}(q-1)$. It follows by (2.111) that $h' < 0$. By stochastic domination,

$$\phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda) \leq \pi^+_{\Lambda,\beta',h'}(0 \leftrightarrow^+ \partial\Lambda), \tag{5.28}$$

5.rcisingbnd where $\{0 \leftrightarrow^+ \partial\Lambda\}$ is the event that there exists a path of Λ joining 0 to some point of $\partial\Lambda$ all of whose vertices have spin $+1$. By results of [**52**, **110**] (see the discussion in [**11**, page 438]), the right side of (5.28) decays exponentially in the radius of Λ. □

5.4 First-order phase transition

The $q = 1$ case of the random-cluster measure is the percolation model, with associated product measure $\phi_p = \phi_{p,1}$. One of the outstanding problems for percolation is to prove the continuity in general dimensions of the percolation probability $\theta(p) = \phi_p(0 \leftrightarrow \infty)$ at the critical point $p = p_{\mathrm{c}} = p_{\mathrm{c}}(1)$, see [**92**, Section 8.3]. By a standard arguments of semi-continuity, this amounts to proving that $\theta(p_{\mathrm{c}}) = 0$, which is to say that there exists almost surely no infinite open cluster *at* the critical point. The situation for general q is quite different. It turns out that $\theta^1(p_{\mathrm{c}}(q), q) > 0$ for all large q, and indeed more is expected.

5.firstorder **Conjecture 5.29.** *Consider the d-dimensional lattice $\mathbb{L}^d$ where $d \geq 2$. There exists $Q = Q(d) \in (1, \infty)$ such that*

$$\theta^1(p_c(q), q) \begin{cases} = 0 & \text{if } q < Q, \\ > 0 & \text{if } q > Q. \end{cases}$$

We speak of the phase transition as being of *second order* if $\theta^1(p_c(q), q) = 0$, and of *first order* otherwise; this is the vernacular of statistical physics. Thus the random-cluster transition is expected to be of first order if and only if q is sufficiently large. There are two remaining problems here: to prove the existence of a 'sharp transition in q', and to calculate the 'critical value' $Q(d)$ of q. The first problem is stangely difficult. It is natural to seek some monotonicity, perhaps of the function $f(q) = \theta^1(p_c(q), q)$, but this has proved elusive even in two dimensions. As for the value of $Q(d)$, it is believed (give ref) that $Q(d)$ is non-increasing in d and satisfies

$$Q(d) = \begin{cases} 4 & \text{if } d = 2, \\ 2 & \text{if } d \geq 6. \end{cases} \tag{5.30}$$

5.qvalue

A first-order transition is characterised by the fact that the order-parameter $\theta^1(p, q)$ is discontinuous. Two further indicators of first-order transition are: discontinuity in the edge-densities $h^b(p, q) = \phi^b_{p,q}(\omega(e) = 1)$, $b = 0, 1$; and the existence of a so-called 'mass gap'. The edge-densities are sometimes termed the 'energy' functions, since they arise thus in the Potts model.

The term 'mass gap' arises in the study of the exponential decay of correlations in the subcritical phase, particularly in the limit as $p \uparrow p_c(q)$. Of the various ways of expressing this, we choose to work with the probability $\phi^0_{p,q}(0 \leftrightarrow \partial[-n, n]^d)$. Recall from Theorem 4.40 that there exists a function $\alpha(p, q)$ such that

$$\phi^0_{p,q}\big(0 \leftrightarrow \partial[-n, n]^d\big) \approx e^{-n\alpha(p,q)} \qquad \text{as } n \to \infty.$$

Clearly, $\alpha(p, q)$ is a non-increasing function of p, and $\alpha(p, q) = 0$ if $\theta^0(p, q) > 0$. It is believed that $\alpha(p, q) > 0$ if $p < p_c(q)$. We speak of the limit

$$\mu(q) = \lim_{p \uparrow p_c(q)} \alpha(p, q)$$

Perhaps add more here. as the *mass gap*.

5.massgap **Conjecture 5.31.** *Consider the d-dimensional lattice $\mathbb{L}^d$ where $d \geq 2$. Then*

$$\mu(q) \begin{cases} = 0 & \text{if } q < Q(d), \\ > 0 & \text{if } q > Q(d), \end{cases}$$

where $Q(d)$ is given in Conjecture 5.29.

There is a special argument discovered first for Potts models, [**134**], which may be used to show that the phase transition is of first order when q is sufficiently large. Such arguments are best formulated in the context of the random-cluster model, and they take a very simple form in the special case $d = 2$, as shown in the this section.

We begin with a reminder concerning the number a_n of self-avoiding walks on $\mathbb{L}^2$ beginning at the origin. It is standard, [**148**], that $a_n^{1/n} \to \mu$ as $n \to \infty$, for some constant μ called the *connective constant* of the lattice. Let

$$Q = \left\{ \tfrac{1}{2}\left(\mu + \sqrt{\mu^2 - 4} \right) \right\}^4.$$

We have that $2.620 < \mu < 2.696$, see [**180**], whence $21.61 < Q < 25.72$. We set

$$\psi(q) = \frac{1}{24} \log \left\{ \frac{(1+\sqrt{q})^4}{q\mu^4} \right\},$$

noting that $\psi(q) > 0$ if and only if $q > Q$. We write $\Lambda(n) = [-n, n]^2$.

5.first2d **Theorem 5.32 (First-order transition in two dimensions) [91, 134].** *Consider the square lattice $\mathbb{L}^2$, and let $q > Q$.*

(a) Critical point. *The critical point is given by $p_c(q) = \sqrt{q}/(1+\sqrt{q})$.*

(b) Discontinuous transition. *We have that $\theta^1(p_c(q), q) > 0$.*

(c) Non-vanishing mass gap. *For any $\psi < \psi(q)$ and all large n,*

$$\phi^0_{p_c(q),q}\big(0 \leftrightarrow \partial\Lambda(n)\big) \le e^{-n\psi}.$$

Hence, in particular, $\theta^0(p_c(q), q) = 0$.

Perhaps add discty of energy.

Similar conclusions may be obtained for general d (≥ 2) when q is sufficiently large ($q > Q(d)$ for suitable $Q(d)$), as shown in [**135**] using so-called Pirogov–Sinai theory. Whereas, in the case $d = 2$, planar duality provides an especially simple proof, the proof for general d utilises nested sequences of surfaces of $\mathbb{R}^d$ and requires a control of the effective boundary conditions within the surfaces. See Section 6.?.

The idea of the proof of the above theorem is as follows. There is a partial order on circuits Γ of $\mathbb{L}^2$ given by $\Gamma \le \Gamma'$ if the bounded component of $\mathbb{R}^2 \setminus \Gamma$ is a subset of the bounded component of $\mathbb{R}^2 \setminus \Gamma'$. We work at the self-dual point $p = p_{\mathrm{sd}}(q)$, and with the box $\Lambda(n)$ with wired boundary conditions. Roughly speaking, an 'outer contour' is defined to be a circuit Γ of the dual graph $\Lambda(n)_{\mathrm{d}}$ all of whose edges are open in the dual (that is, they traverse closed edges in the primal graph $\Lambda(n)$), and which is maximal with this property. Using self-duality, one may show that

$$\phi^1_{\Lambda(n),p_{\mathrm{sd}}(q),q}(\Gamma \text{ is an outer circuit}) \le \frac{1}{q}\left(\frac{q}{(1+\sqrt{q})^4}\right)^{|\Gamma|/4},$$

for any given circuit Γ of $\Lambda(n)_{\mathrm{d}}$. Combined with a circuit-counting argument of Peierls-type involving the connective constant, this estimate implies after a little work the claims of Theorem 5.32. The idea of the proof appeared in [**134**] in the context of Potts models, and the random-cluster formulation may be found in [**91**]; see also Section 6.? of the current work.

Proof of Theorem 5.32. We fix $n \ge 1$. Let $\Lambda = \Lambda(n) = [-n,n]^2$, and let $\Lambda_{\mathrm{d}} = [-n,n-1]^2 + (\frac{1}{2},\frac{1}{2})$ be those vertices of the dual of Λ which lie inside Λ (that is, we omit the vertex in the infinite face of Λ). We shall work with 'wired' boundary conditions on Λ, and we let ω be a configuration on the edges of Λ. The *exterior* (respectively, *interior*) of a given circuit Γ of either $\mathbb{L}^2$ or its dual $\mathbb{L}^2_{\mathrm{d}}$ is defined to be the unbounded (respectively, bounded) component of $\mathbb{R}^2 \setminus \Gamma$. A circuit Γ of Λ_{d} is called an *outer circuit* of a configuration ω if the following properties hold:

(a) all edges of Γ are open in the dual configuration ω_{d}, which is to say that they traverse closed edges of Λ,
(b) the origin of $\mathbb{L}^2$ is in the interior of Γ,
(c) every vertex of Λ lying in the exterior of Γ, but within distance of $1/\sqrt{2}$ of some vertex of Γ, belongs to the same component of ω.

See Figure 5.5 for an illustration of the meaning of 'outer circuit'.

Each circuit Γ of Λ_{d} partitions the set $\mathbb{E}_\Lambda$ of edges of Λ into three sets, being

$$\begin{aligned} E &= \{\text{edges of } \Lambda \text{ exterior to } \Gamma\}, \\ I &= \{\text{edges of } \Lambda \text{ interior to } \Gamma\}, \\ \Gamma' &= \{\text{edges of } \Lambda \text{ crossing } \Gamma\}. \end{aligned}$$

The edges I form a connected subgraph of Λ. We write G_Γ ($\subseteq \{0,1\}^E$) for the set of configurations of edges in E satisfying property (c) above.

Here is some further notation. Let F be a finite subset of $\mathbb{E}_\Lambda$, and write V_F for the set of all vertices incident to members of F. For $\omega \in \{0,1\}^F$, we write

$$\pi_F(\omega) = p^{|\eta(\omega)|}(1-p)^{|F\setminus\eta(\omega)|}q^{k(\omega)}$$

where $k(\omega)$ is the number of clusters of $(V_F, \eta(\omega))$. We shall sometimes impose a boundary condition on F as follows. Let $\partial_{\mathrm{ext}}F$ be the set of vertices in V_F which belong to infinite paths of $\mathbb{L}^2$ using no other vertex of V_F. We write

$$\pi^1_F(\omega) = p^{|\eta(\omega)|}(1-p)^{|F\setminus\eta(\omega)|}q^{k^1(\omega)}$$

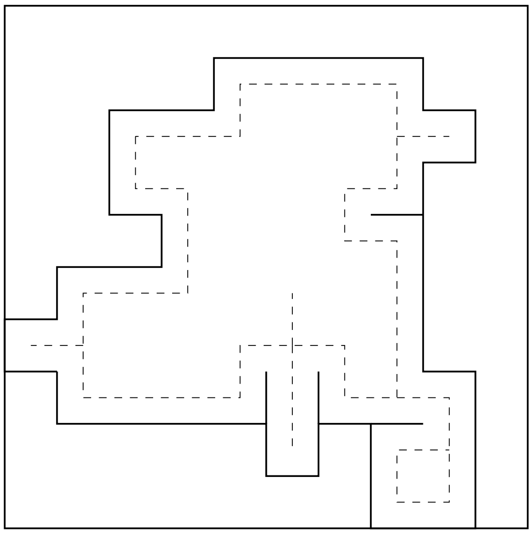


Figure 5.5. The dashed lines include an outer circuit Γ of the dual Λ_{d}.

oldfig13.2

where $k^1(\omega)$ is the number of clusters of $(V_F, \eta(\omega))$ with the convention that clusters which intersect $\partial_{\mathrm{ext}} F$ are counted only as one in total.

Our target is to obtain an upper bound for the probability that a given circuit Γ is an outer circuit. This we shall do by examining certain partition functions. Since no open component of ω contains points lying in both the exterior and interior of an outer circuit, the event $\mathrm{OC}(\Gamma) = \{\Gamma \text{ is an outer circuit}\}$ satisfies, for any dual circuit Γ having 0 in its interior,

$$\begin{aligned} \phi^1_{\Lambda,p,q}(\mathrm{OC}(\Gamma)) &= \frac{1}{Z^1_{\Lambda,p,q}} \sum_{\omega} 1_{\mathrm{OC}(\Gamma)}(\omega)\pi^1_{\Lambda}(\omega) \\ &= \frac{1}{Z^1_{\Lambda,p,q}} (1-p)^{|\Gamma|} Z^1_E(\Gamma) Z_I \end{aligned} \tag{5.33}$$

5.1341 where

$$Z^1_E(\Gamma) = \sum_{\omega' \in G_\Gamma} \pi^1_E(\omega'), \qquad Z_I = \sum_{\omega'' \in \{0,1\}^I} \pi_I(\omega'').$$

Next we use duality. Let I_{d} be the set of dual edges which cross the primal edges I, and let m be the number of vertices of Λ inside Γ. By (5.10),

$$Z_I = q^{m-1} \left(\frac{1-p}{p^{\mathrm{d}}} \right)^{|I|} Z^1_{I_{\mathrm{d}}, p_{\mathrm{d}}, q} \tag{5.34}$$

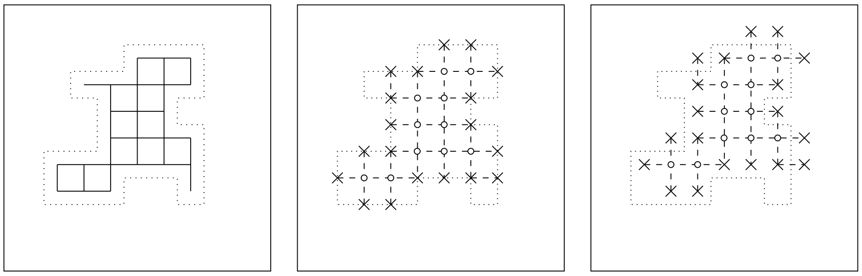


Figure 5.6. The interior edges I of Γ are marked in the leftmost picture, and the dual I_{d} in the centre picture (the vertices marked with a cross are identified as a single vertex). The shifted set $I^* = I_{\mathrm{d}} + (\frac{1}{2}, \frac{1}{2})$ is drawn in the rightmost picture. Note that $I^* \subseteq I \cup \Gamma'$.

oldfig13.3

5.1342 where p_{d} satisfies (5.5), and where $Z^1_{I_{\mathrm{d}}, p_{\mathrm{d}}, q}$ is the partition function for dual configurations, having wired boundary conditions, on the set V_{d} of vertices incident to I_{d} (that is, all vertices of V_{d} on its boundary are identified, as indicated in Figure 5.6).

Now, $Z_{\cdot, p, q}$ has the following property of supermultiplicativity when $q \ge 1$. For any dual circuit Γ with the origin in its interior,

$$\begin{aligned} Z^1_{\Lambda,p,q} &\ge \sum_{\omega \in G_\Gamma \times \{0,1\}^{I \cup \Gamma'}} \pi^1_\Lambda(\omega) \\ &= \sum_{\omega' \in G_\Gamma} \pi^1_E(\omega') \sum_{\omega'' \in \{0,1\}^{I \cup \Gamma'}} \pi^1_{I \cup \Gamma'}(\omega'') \\ &= Z^1_E(\Gamma) Z^1_{I \cup \Gamma', p, q}. \end{aligned} \tag{5.35}$$

5.firstorder2

Let $I^* = I_{\mathrm{d}} + (\frac{1}{2}, \frac{1}{2})$, where I_{d} is thought of as a subset of $\mathbb{R}^2$. Note from Figure 5.6 that $I^* \subseteq I \cup \Gamma'$, and therefore

$$Z^1_{I \cup \Gamma', p, q} \ge Z^1_{I^*, p, q} \tag{5.36}$$

5.firstorder3 by Lemma 2.97 and inequality (2.98). By (5.35)–(5.36),

$$Z^1_{\Lambda,p,q} \ge Z^1_E(\Gamma) Z^1_{I^*,p,q} = Z^1_E(\Gamma) Z^1_{I_{\mathrm{d}},p,q}. \tag{5.37}$$

5.1343

Now assume that $p = \sqrt{q}/(1 + \sqrt{q})$, so that $p = p_{\mathrm{d}}$. Then, by (5.33)–(5.37) and (5.11),

$$\text{(5.38)} \qquad \begin{aligned}\phi^1_{\Lambda,p,q}(\mathrm{OC}(\Gamma)) &= (1-p)^{|\Gamma|}\frac{Z^1_E(\Gamma)Z_I}{Z^1_{\Lambda,p,q}} \\ &= (1-p)^{|\Gamma|}q^{m-1-\frac{1}{2}|I|}\frac{Z^1_E(\Gamma)Z^1_{I_{\mathrm{d}},p,q}}{Z^1_{\Lambda,p,q}} \\ &\le (1-p)^{|\Gamma|}q^{m-1-\frac{1}{2}|I|}.\end{aligned}$$

5.1344 Since each vertex of Λ (inside Γ) has degree 4, we have that

$$4m = 2|I| + |\Gamma|,$$

whence

$$\text{(5.39)} \qquad \phi^1_{\Lambda,p,q}(\mathrm{OC}(\Gamma)) \le (1-p)^{|\Gamma|}q^{\frac{1}{4}|\Gamma|-1} = \frac{1}{q}\left(\frac{q}{(1+\sqrt{q})^4}\right)^{|\Gamma|/4}.$$

5.1345

The number of dual circuits of Λ having length l and containing the origin in their interior is no greater than la_l, where a_l is the number of self-avoiding walks of $\mathbb{L}^2$ beginning at the origin and having length l. Therefore

$$\sum_\Gamma \phi^1_{\Lambda,p,q}(\mathrm{OC}(\Gamma)) \le \sum_{l=4}^{\infty}\frac{1}{q}\left(\frac{q}{(1+\sqrt{q})^4}\right)^{l/4} la_l.$$

Now $l^{-1}\log a_l \to \mu$ as $l \to \infty$, where μ is the connective constant of $\mathbb{L}^2$. Suppose now that $q > Q$, so that $q\mu^4 < (1+\sqrt{q})^4$. It follows that there exists $A(q)$ $(< \infty)$ such that

$$\sum_\Gamma \phi^1_{\Lambda,p,q}(\mathrm{OC}(\Gamma)) < A(q) \quad \text{for all } n.$$

If $A(q) < 1$ (which holds for sufficiently large q), then

$$\begin{aligned}\phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda) &= \phi^1_{\Lambda,p,q}(\mathrm{OC}(\Gamma) \text{ occurs for no } \Gamma) \\ &\ge 1 - A(q) > 0.\end{aligned}$$

(We have used the assumption of wired boundary conditions here.) This implies, by taking the limit $n \to \infty$, that $\theta^1(p,q) > 0$ when $p = \sqrt{q}/(1+\sqrt{q})$. Using Theorem 5.16(a), this implies parts (a) and (b) of the theorem, when q is sufficiently large.

For general $q > Q$, we have only that $A(q) < \infty$. In this case, we find N $(< n)$ such that

$$\sum_{\Gamma \text{ outside } \Lambda(N)} \phi^1_{\Lambda,p,q}(\mathrm{OC}(\Gamma)) < \tfrac{1}{2}$$

where Γ is said to be outside $\Lambda(N)$ if it contains $\Lambda(N)$ in its interior. This implies that

$$\phi^1_{\Lambda,p,q}(\Lambda(N) \leftrightarrow \partial\Lambda) \geq \tfrac{1}{2}.$$

Let $n \to \infty$, and deduce that $\phi^1_{p,q}(\Lambda(N) \leftrightarrow \infty) \geq \frac{1}{2}$, implying that $\theta^1(p,q) > 0$ as required.

Turning to part (c), let $p = p_{\mathrm{d}} = \sqrt{q}/(1+\sqrt{q})$ and $n \leq r$. Let A_n be the (cylinder) event that the point $(\frac{1}{2}, \frac{1}{2})$ lies in the interior of an open circuit of length at least n, this circuit having the property that its interior is contained in the interior of *no* open circuit having length strictly less than n. We have from Theorem 5.13 and (5.39) that

$$\phi^0_{\Lambda(r),p,q}(A_n) \leq \sum_{m=n}^{\infty} \frac{m a_m}{q} \left(\frac{q}{(1+\sqrt{q})^4} \right)^{m/4} \qquad \text{for all large } r. \tag{5.40}$$

5.1346 [Here, we use the observation that, if A_n occurs in $\Lambda(r)$, then there exists a maximal open circuit Γ of $\Lambda(r)$ containing $(\frac{1}{2}, \frac{1}{2})$. In the dual of $\Lambda(r)$, Γ constitutes an outer circuit.]

We write LR_n for the event that there is an open crossing of the rectangle $R_n = [0,n] \times [0,2n]$ from its left to its right side, and we set $\lambda_n = \phi^0_{p,q}(\mathrm{LR}_n)$. We may find a point x on the left side of R_n and a point y on the right side such that

$$\phi^0_{p,q}(x \leftrightarrow y \text{ in } R_n) \geq \frac{\lambda_n}{(2n+1)^2}.$$

By placing six of these rectangles side by side (as in Figure 5.7), we find by the FKG inequality that

$$\phi^0_{p,q}\Big(x \leftrightarrow x + (6n, 0) \text{ in } [0,6n] \times [0,2n]\Big) \geq \left(\frac{\lambda_n}{(2n+1)^2} \right)^6. \tag{5.41}$$

5.1347 We now use four copies of the rectangle $[0,6n] \times [0,2n]$ to construct an annulus around the origin (see Figure 5.8). If each of these copies contains an open crossing, then the annulus contains a circuit. Using the FKG inequality again, we deduce that

$$\phi^0_{p,q}(A_{4n}) \geq \left(\frac{\lambda_n}{(2n+1)^2} \right)^{24}. \tag{5.42}$$

5.1348

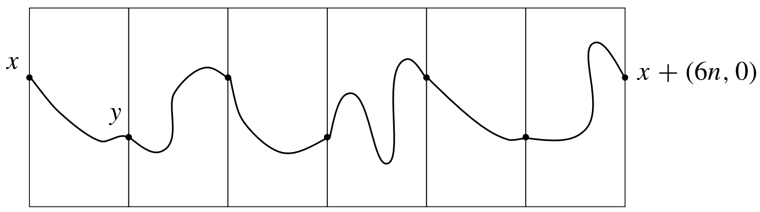


Figure 5.7. Six copies of a rectangle having width n and height $2n$ may be put together to make a rectangle with size $6n$ by $2n$. If each is crossed by an open path joining the images of x and y, then the larger rectangle is crossed between its shorter sides.

oldfig13.4

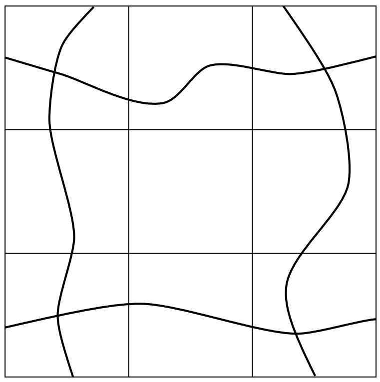


Figure 5.8. If each of four rectangles having dimensions $6n$ by $2n$ is crossed by an open path between its shorter sides, then the annulus contains an open circuit having the origin in its interior.

oldfig13.5

Finally, if $0 \leftrightarrow \partial\Lambda(n)$, then one of the four rectangles $[0,n] \times [-n,n]$, $[-n,n] \times [0,n]$, $[-n,0] \times [-n,n]$, $[-n,n] \times [-n,0]$ is traversed by an open path betwen its two longer sides. This implies that

$$\phi^0_{p,q}\big(0 \leftrightarrow \partial\Lambda(n)\big) \le 4\lambda_n. \tag{5.43}$$

5.1349

Combining (5.40)–(5.43), we obtain that

$$\begin{aligned}\phi^0_{p,q}\big(0 \leftrightarrow \partial\Lambda(n)\big) &\le 4\Big\{(2n+1)^2\phi^0_{p,q}(A_{4n})\Big\}^{1/24}\\ &\le 4\left\{(2n+1)^2 \sum_{m=4n}^{\infty} \frac{ma_m}{q}\left(\frac{q}{(1+\sqrt{q})^4}\right)^{m/4}\right\}^{1/24}.\end{aligned}$$

As before, $m^{-1}\log a_m \to \mu$ as $m \to \infty$, whence part (c) follows. □

5.5 General lattices in two dimensions

Planar duality provides a technique of great value for the study of interacting systems on a two-dimensional lattice $\mathcal{L}$, but it is no panacea. It may be summarised in the two statements: the external boundary of a bounded connected subgraph of $\mathcal{L}$ is topologically one-dimensional, and the statistical mechanics of the boundary may be studied via an appropriate stochastic model on a certain dual lattice $\mathcal{L}_{\mathrm{d}}$. Thus duality provides a relation between a primal model on $\mathcal{L}$ and a dual model on $\mathcal{L}_{\mathrm{d}}$. In situations where the dual model is related to the primal, or to some other known system, one may sometimes obtain exact results. The exact calulations of critical probabilities of percolation models on the square, triangular, and hexagonal lattices are examples of this (see [**92**, Chapter 11]). Individuals less burdened by the drive for mathematical rigour have exploited duality to obtain exact but non-rigorous predictions for two-dimensional processes (see, for example, [**18**]), of which a major example is Conjecture 5.14, namely that $p_{\mathrm{c}}(q) = \sqrt{q}/(1+\sqrt{q})$ for the random-cluster model on the square lattice $\mathbb{L}^2$. Such predictions are often beautiful and always provocative to mathematicians.

We shall not explore duality in general here, noting only in passing the existence of many open problems of significance in extending known results for, say, the square lattice to general primal/dual pairs. Instead we discuss two specific issues relating, in turn, to the critical points of a general primal/dual pair, and in the next section to exact calculations for the triangular and hexagonal lattices.

Here is our definition of a lattice, [**92**, Section 12.1]. A *lattice in d dimensions* is a connected, loopless graph $\mathcal{L}$, with bounded vertex degrees, which is embedded in $\mathbb{R}^d$ in such a way that:

(a) the translations $x \to x + e$ are automorphisms of $\mathcal{L}$ for each unit vector e parallel to a coordinate axis;

(b) all edges are of finite length; and

(c) every compact subset of $\mathbb{R}^d$ intersects only finitely many edges.

Let $\mathcal{L} = (\mathbb{V}, \mathbb{E})$ be a planar two-dimensional lattice, and let $\mathcal{L}_{\mathrm{d}}$ be its dual lattice, as defined in Section 5.1. We shall require some further symmetries of $\mathcal{L}$, namely that:

(d) the reflection mappings $\rho_{\mathrm{h}}, \rho_{\mathrm{v}} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\rho_{\mathrm{h}}(x, y) = (-x, y), \quad \rho_{\mathrm{v}}(x, y) = (x, -y), \quad (x, y) \in \mathbb{R}^2,$$

are automorphisms of $\mathcal{L}$.

Let $0 \le p \le 1$ and $q \ge 1$. Under the above conditions, the random-cluster measures $\phi^0_{p,q} = \phi^b_{\mathcal{L},p,q}$, $b = 0, 1$, exist and are invariant under horizontal and vertical translations, and under horizontal and vertical axis reflection. They are in addition ergodic with respect to horizontal and vertical translation (separately), and they are positively associated. Such facts may be proved in exactly the same manner as were the corresponding statements for the hypercubic lattice $\mathbb{L}^d$ in
Check Chapter 3, and we shall not dwell on the details.

We recall the definition (5.5) of the dual point p_{d} to p, and we write $p_{\mathrm{c}}(q, \mathcal{L})$ for the critical value of $\mathcal{L}$.

5.gprimdual **Theorem 5.44.** *The critical points $p_{\mathrm{c}}(q, \mathcal{L})$, $p_{\mathrm{c}}(q, \mathcal{L}_{\mathrm{d}})$ satisfy the inequality*

$$p_{\mathrm{c}}(q, \mathcal{L}) \geq \big(p_{\mathrm{c}}(q, \mathcal{L}_{\mathrm{d}})\big)_{\mathrm{d}}. \tag{5.45}$$

5.gprimdual2

Since p_{d} is strictly decreasing in p, (5.45) implies the complementary inequality

$$p_{\mathrm{c}}(q, \mathcal{L}_{\mathrm{d}}) \geq \big(p_{\mathrm{c}}(q, \mathcal{L})\big)_{\mathrm{d}}. \tag{5.46}$$

5.gprimdual3 If $p_{\mathrm{c}}(q, \mathcal{L}) = p_{\mathrm{c}}(q, \mathcal{L}_{\mathrm{d}})$, then as in Theorem 5.16 we obtain that $p_{\mathrm{c}}(q, \mathcal{L}) \geq p_{\mathrm{sd}}(q)$.

Proof of Theorem 5.44. Let $p > p_{\mathrm{c}}(q, \mathcal{L})$, so that $\phi^1_{\mathcal{L},p,q}(0 \leftrightarrow \infty) > 0$. The arguments leading to the main result of [**78**] may be adapted to the current setting[8] to show that all open clusters in the dual lattice $\mathcal{L}_{\mathrm{d}}$ are almost surely finite. Therefore, $p_{\mathrm{d}} \leq p_{\mathrm{c}}(q, \mathcal{L}_{\mathrm{d}})$, whence $p \geq \big(p_{\mathrm{c}}(q, \mathcal{L}_{\mathrm{d}})\big)_{\mathrm{d}}$ as required. □

It may be conjectured that equality holds in (5.45). In a situation where $\mathcal{L}_{\mathrm{d}}$ is isomorphic to $\mathcal{L}$, this would imply as in Conjecture 5.14 that $p_{\mathrm{c}}(q, \mathcal{L}) = p_{\mathrm{sd}}(q)$.

Theorem 5.44 may be used to prove the uniqueness of random-cluster measures for $p \neq p_{\mathrm{c}}(q)$. Some further notation must first be introduced to deal with case when $\mathcal{L}$ is not edge-transitive. Let $S = [0, 1)^2$ ($\subseteq \mathbb{R}^2$). Let I_S be the set of all edges of $\mathcal{L}$ which have both endvertices in S, and let E_S be the number of edges of $\mathcal{L}$ having exactly one endvertex in S. For $\omega \in \Omega = \{0, 1\}^{\mathbb{E}}$, we write

$$N_S(\omega) = \sum_{e \in I_S} 1_{\{\omega(e)=1\}} + \sum_{e \in E_S} \tfrac{1}{2} 1_{\{\omega(e)=1\}}, \tag{5.47}$$

5.edgecount and we define the *edge-density* by

$$h^b_{\mathcal{L}}(p, q) = \frac{1}{N_S(1)} \phi^b_{\mathcal{L},p,q}(N_S), \qquad b \in \{0, 1\}. \tag{5.48}$$

5.edgedens If $\mathcal{L}$ is edge-transitive (that is, for $e, f \in \mathbb{E}$, there exists an automorphism of $\mathcal{L}$ which maps e onto f), it may be seen that $h^b_{\mathcal{L}}(p, q)$ is simply the probability under
Check. $\phi^1_{\mathcal{L},p,q}$ that a given edge is open.

[8]Paper [**78**] treats site models on $\mathbb{Z}^2$ governed by measures with certain translation-invariance, ergodicity, and positive association properties. The arguments are however quite general and apply also to edge models on planar graphs with corresponding properties.

5.uniquedual **Corollary 5.49.** *Let $\mathcal{L}$, $\mathcal{L}_\mathrm{d}$ be a primal/dual pair of lattices in two dimensions and suppose $\mathcal{L}$ satisfies* (a)–(d) *above. Let $0 \le p \le 1$ and $q \ge 1$, and assume that $p \ne p_\mathrm{c}(q, \mathcal{L})$.*

(a) *The edge-density $h^b_{\mathcal{L}}(x, q)$ is a continuous function of x at the point $x = p$, for $b = 0, 1$.*

(b) *It is the case that $h^0_{\mathcal{L}}(p, q) = h^1_{\mathcal{L}}(p, q)$.*

(c) *There is a unique random-cluster measure on $\mathcal{L}$ with parameters p and q, that is, $|\mathcal{W}_{p,q}(\mathcal{L})| = |\mathcal{R}_{p,q}(\mathcal{L})| = 1$, in the natural notation.*

In the notation of Theorem 3.63, we have that $\mathcal{D}_q \subseteq \{p_\mathrm{c}(q, \mathcal{L})\}$. In particular, if there is a first-order phase transition, then it can take place only at the critical point $p = p_\mathrm{c}(q, \mathcal{L})$. Note from Theorem 4.15 that the percolation probabilities $\theta^b_{\mathcal{L}}(\cdot, q) = \phi^b_{\mathcal{L},p,q}(0 \leftrightarrow \infty)$, $b = 0, 1$, are continuous except possibly at the value $p = p_\mathrm{c}(q)$. [This may also be proved directy for a primal/dual pair, using the arguments of Theorems 4.30 and 5.44.] Using Theorem 5.16(b), if the square lattice has a first-order phase transition for some given value of q, then it is necessarily the case that $p_\mathrm{c}(q) = p_\mathrm{sd}(q)$ for this value. Related matters for Potts models are discussed in [**25**].

Proof. When $p < p_\mathrm{c}(q)$, this follows from Theorems 3.63 and 4.30(a). When $p > p_\mathrm{c}(q, \mathcal{L})$, we have from (5.45) that $p_\mathrm{d} < p_\mathrm{c}(q, \mathcal{L}_\mathrm{d})$. By (5.3) and Theorem 5.13,

$$h^b_{\mathcal{L}}(p, q) + h^{1-b}_{\mathcal{L}_\mathrm{d}}(p_\mathrm{d}, q) = 1,$$

whence by the first part applied to $\mathcal{L}_\mathrm{d}$, each $h^b_{\mathcal{L}}(x, q)$ is continuous at the point $x = p$. Part (b) follows as in Theorem 3.63, and part (c) by the argument used in
Check the proof of Theorem 5.16(b). □

5.6 Square, triangular, and hexagonal lattices

There is a host of exact but non-rigorous 'results' for two-dimensional models which, while widely accepted by physicists, continue to be subjected to mathematical investigations. Some of these claims have been made rigorous and, in so doing, mathematicians have discovered new structures of beauty and complexity. The best contemporary example of new structure provoked by the physics is the theory of stochastic Löwner evolutions (SLE) and their impact on percolation, Brownian motion, and on other systems with a property of conformal invariance; see Section 5.7 for a short account of SLE in the random-cluster context.

Amongst 'exact' but non-rigorous results for the random-cluster model is the claim that, for the square lattice with $q \ge 1$,

$$p_\mathrm{c}(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

Baxter's 1982 book [**18**] remains a good source for this and related statements, usually in the context of Potts models but extendable to random-cluster models with $q \in [1, \infty)$. Such statements are achieved typically by following a sequence of transformations between models, arriving thus at a 'soluble ice-type model' on a new graph termed the 'medial graph'. It has proved difficult to ascertain whether such methods are entirely rigorous, since they involve chains of argument which may seem innocuous but in fact omit analytical details. We attempt here no more than brief accounts of some of the conclusions together with a partial mathematical commentary.

Let us begin with the square lattice $\mathbb{L}^2$. Instead of working with a single edge-parameter p, we allow greater generality by associating to each horizontal (respectively, vertical) edge the parameter p_{h} (respectively, p_{v}), and we write $\mathbf{p} = (p_{\mathrm{h}}, p_{\mathrm{v}})$. It will be convenient as in (5.7)–(5.8) to work instead with the parameters $\mathbf{x} = (x_{\mathrm{h}}, x_{\mathrm{v}})$ given by

$$x_{\mathrm{h}} = \frac{q^{-\frac{1}{2}} p_{\mathrm{h}}}{1 - p_{\mathrm{h}}}, \qquad x_{\mathrm{v}} = \frac{q^{-\frac{1}{2}} p_{\mathrm{v}}}{1 - p_{\mathrm{v}}},$$

and their dual values $x_{\mathrm{h,d}}$, $x_{\mathrm{v,d}}$ given by

$$x_{\mathrm{h}} x_{\mathrm{h,d}} = 1, \qquad x_{\mathrm{v}} x_{\mathrm{v,d}} = 1.$$

We abuse notation by writing $\phi^b_{G,\mathbf{x},q}$ for a random-cluster measure on a graph G, and furthermore we write

$$\phi^b_{\mathbf{x},q} = \lim_{\Lambda \uparrow \mathbb{L}^2} \phi^b_{\Lambda,\mathbf{x},q}, \qquad \theta^b(\mathbf{x}, q) = \phi^b_{\mathbf{x},q}(0 \leftrightarrow \infty).$$

The duality map of Section 5.1 maps a random-cluster model on $\mathbb{L}^2$ with parameters $\mathbf{x} = (x_{\mathrm{h}}, x_{\mathrm{v}})$ to the random-cluster model on $\mathbb{L}^2_{\mathrm{d}}$ with parameters $\mathbf{x}_{\mathrm{d}} = (x_{\mathrm{v,d}}, x_{\mathrm{h,d}})$. The primal and dual models have the same parameters whenever $x_{\mathrm{h}} = x_{\mathrm{v,d}}$ and $x_{\mathrm{v}} = x_{\mathrm{h,d}}$, which is to say that

$$x_{\mathrm{h}} x_{\mathrm{v}} = 1, \tag{5.50}$$

5.xdual2 and thus we refer to the system as 'self-dual' if (5.50) holds. The following conjecture generalises Conjecture (5.15).

5.conjsd2 **Conjecture 5.51.** *Let* $x_{\mathrm{h}}, x_{\mathrm{v}} \in (0, \infty)$ *and* $q \geq 1$. *We have for* $b = 0, 1$ *that*

$$\theta^b(\mathbf{x}, q) \begin{cases} = 0 & \text{if } x_{\mathrm{h}} x_{\mathrm{v}} < 1, \\ > 0 & \text{if } x_{\mathrm{h}} x_{\mathrm{v}} > 1. \end{cases}$$

The proof in the case of percolation, when $q = 1$, may be found at [**92**, Theorem 11.115]. Partial progress in the direction of the general conjecture is provided by the next theorem.

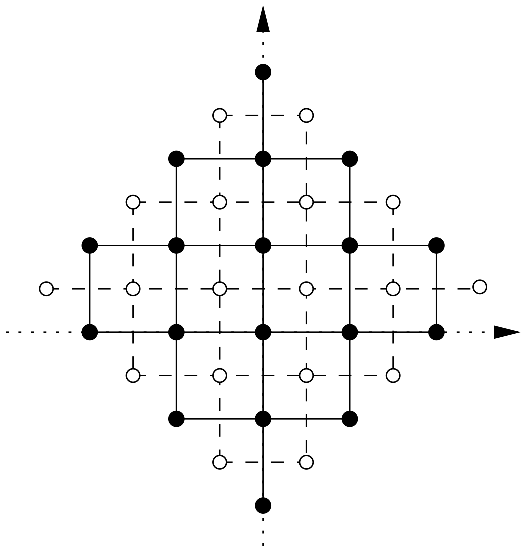


Figure 5.9. The subgraph $D(n)$ of the square lattice $\mathbb{L}^2$ when $n = 2$, and the associated 'dual' graph $D(n)_{\mathrm{d}}$ of the dual lattice $\mathbb{L}^2_{\mathrm{d}}$.

5.inhomsquare

5.xnoperc **Theorem 5.52.** *Let* $x_{\mathrm{h}}, x_{\mathrm{v}} \in (0, \infty)$ *and* $q \geq 1$. *We have that*

$$\theta^0(\mathbf{x}, q) = 0 \quad \textit{if} \quad x_{\mathrm{h}} x_{\mathrm{v}} \leq 1.$$

Proof.

□

Here are two exact but non-rigorous claims for this process. We recall from Theorem 3.58 the 'pressure' function G given in the current context as

$$G(\mathbf{x}, q) = \lim_{\Lambda \to \mathbb{Z}^d} \left\{ \frac{1}{|\mathbb{E}_\Lambda|} \log Y_\Lambda(\mathbf{p}, q) \right\}, \qquad (\mathbf{x}, q) \in (0, \infty)^2 \times [1, \infty),$$

where

$$\begin{aligned} Y_\Lambda(\mathbf{p}, q) &= (1 - p_{\mathrm{h}})^{-\frac{1}{2}|\mathbb{E}_\Lambda|}(1 - p_{\mathrm{v}})^{-\frac{1}{2}|\mathbb{E}_\Lambda|} Z_\Lambda(\mathbf{p}, q) \\ &= \sum_{\omega \in \Omega_\Lambda} (x_{\mathrm{h}}\sqrt{q})^{|\eta_{\mathrm{h}}(\omega)|}(x_{\mathrm{v}}\sqrt{q})^{|\eta_{\mathrm{v}}(\omega)|} q^{k(\omega)}, \end{aligned}$$

and $\eta_{\mathrm{h}}(\omega)$ (respectively, $\eta_{\mathrm{v}}(\omega)$) is the set of open horizontal (respectively, vertical) edges of the configuration ω ($\in \Omega_\Lambda$). Using duality as in Section 5.1 (see (5.4)), one obtains via Theorem 3.58 that

$$G(\mathbf{x}, q) = G(\mathbf{x}_{\mathrm{d}}, q) + \tfrac{1}{2}\log(x_{\mathrm{h}} x_{\mathrm{v}}), \qquad \mathbf{x} \in (0, \infty)^2. \tag{5.53}$$

5.freeen By mapping the random-cluster model onto an ice-type model as in [**18**, Section 12.5], one obtains the following exact computation,

$$G(\mathbf{x}, q) = \tfrac{1}{2}\psi(x_{\mathrm{h}}) + \tfrac{1}{2}\psi(x_{\mathrm{v}}) + \tfrac{1}{4}\log q, \qquad \text{when } x_{\mathrm{h}}x_{\mathrm{v}} = 1,$$

where the function $\psi : (0, \infty) \to \mathbb{R}$ is given as follows:

(i) when $0 < q < 4$, choose $\mu \in (0, \frac{1}{2}\pi)$ and $\gamma \in (0, \mu)$ by

$$2\cos\mu = \sqrt{q}, \qquad x = \frac{\sin\gamma}{\sin(\mu - \gamma)},$$

and then

$$\psi(x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sinh((\pi - \mu)t)\sinh(2\gamma t)}{t\sinh(\pi t)\cosh(\mu t)}\,dt;$$

(ii) when $q = 4$, let $\tau = x/(1 + x)$, and then

$$\psi(x) = \int_0^{\infty} \frac{e^{-y}}{y}\operatorname{sech} y \sinh(2\tau y)\,dy;$$

(iii) when $q > 4$, choose $\lambda > 0$ and $\beta \in (0, \lambda)$ by

$$2\cosh\lambda = \sqrt{q}, \qquad x = \frac{\sinh\beta}{\sinh(\lambda - \beta)},$$

and then

$$\psi(x) = \beta + \sum_{n=1}^{\infty} \frac{e^{-n\lambda}}{n}\operatorname{sech}(n\lambda)\sinh(2n\beta).$$

Our second exact relation concerns the asymptotic probability that a given edge is open. Let

$$h^b(\mathbf{p}, q) = \lim_{\Lambda \uparrow \mathbb{Z}^2} \phi^b_{\Lambda,\mathbf{p},q}(|\eta|), \qquad b \in \{0, 1\},$$

be the mean density of open edges, as in Theorem 3.63??. By the translation-invariance of the infinite-volume measures, the mean numbers of open horizontal and vertical edges satisfy

$$\begin{aligned} &\frac{2}{|\mathbb{E}_\Lambda|}\phi^b_{\Lambda,\mathbf{p},q}(|\eta_{\mathrm{h}}|) \to h^b_{\mathrm{h}}(\mathbf{p}, q) = \phi^b_{\mathbf{p},q}(e_{\mathrm{h}} \text{ is open}), \\ &\frac{2}{|\mathbb{E}_\Lambda|}\phi^b_{\Lambda,\mathbf{p},q}(|\eta_{\mathrm{v}}|) \to h^b_{\mathrm{v}}(\mathbf{p}, q) = \phi^b_{\mathbf{p},q}(e_{\mathrm{v}} \text{ is open}), \end{aligned} \tag{5.54}$$

5.meanhv as $\Lambda \uparrow \mathbb{Z}^2$, where e_{h} (respectively, e_{v}) is a representative horizontal (respectively, vertical) edge of $\mathbb{L}^2$. Thus,

$$h^b(\mathbf{p}, q) = \tfrac{1}{2}\big(h^b_{\mathrm{h}}(\mathbf{p}, q) + h^b_{\mathrm{v}}(\mathbf{p}, q)\big).$$

We have as in Theorem 5.49 that, except possibly on the self-dual curve $x_h x_v = 1$, the functions $h_h^b(\cdot, q)$, $h_v^b(\cdot, q)$ are continuous and $h_{h/v}^0(\mathbf{p}, q) = h_{h/v}^1(\mathbf{p}, q)$. [We write h/v to indicate that either possibility, chosen consistently through a given equation, is valid.] It is believed that the $h_{h/v}^0(\cdot, q)$ are continuous from below, and the $h_{h/v}^1(\cdot, q)$ continuous from above, on the self-dual curve, whence

$$h_{h/v}^0(\mathbf{p}, q) = \lim_{\mathbf{p}'\uparrow\mathbf{p}} h_{h/v}^0(\mathbf{p}', q), \quad h_{h/v}^1(\mathbf{p}, q) = \lim_{\mathbf{p}'\downarrow\mathbf{p}} h_{h/v}^1(\mathbf{p}', q), \qquad \mathbf{p} \in (0,1)^2.$$

By duality as in Theorem 5.13,

$$h_{h/v}^0(\mathbf{p}, q) + h_{v/h}^1(\mathbf{p}_d, q) = 1, \qquad \mathbf{p} \in (0,1)^2. \tag{5.55}$$

5.dualdens

It is believed when $1 \le q < 4$ that the transition is of second order, and thus in particular that the $h_{h/v}^b(\mathbf{p}, q)$ are continuous *on* the self-dual curve, whence

$$h_{h/v}^0(\mathbf{p}, q) = h_{h/v}^1(\mathbf{p}, q), \qquad \mathbf{p} \in (0,1)^2, \quad 1 \le q < 4.$$

It follows from this and (5.55) that one should have

$$h^b(\mathbf{p}, q) = \tfrac{1}{2}\big(h_h^b(\mathbf{p}, q) + h_v^b(\mathbf{p}, q)\big) = \tfrac{1}{2} \quad \text{for } b \in \{0,1\}, \text{ when } x_h x_v = 1,\ 1 \le q < 4. \tag{5.56}$$

5.edgedens2

The transition is expected to be of first order when $q > 4$, and the exact computations reported in [**18**, Section 12.5] then yield when $x_h x_v = 1$ that

$$\begin{aligned} h_{h/v}^0(\mathbf{p}, q) &= x_{h/v}[\psi'(x_{h/v}) - \zeta(x_{h/v}) P_0], \\ h_{h/v}^1(\mathbf{p}, q) &= x_{h/v}[\psi'(x_{h/v}) + \zeta(x_{h/v}) P_0], \end{aligned} \tag{5.57}$$

5.exactdens where λ is given as in case (iii) above, and

$$\zeta(x) = \frac{\sinh\lambda}{1 + x^2 + 2x\cosh\lambda}, \qquad P_0 = \prod_{m=1}^{\infty} (\tanh(m\lambda))^2.$$

Thus one expects when $x_h x_v = 1$ and $q > 4$ that, as in (5.56),

$$h^0(\mathbf{p}, q) + h^1(\mathbf{p}, q) = x_h \psi'(x_h) + x_v \psi'(x_v) = 1.$$

Note from the definition of λ that

$$\zeta(x) = \frac{\frac{1}{2}\sqrt{q-4}}{1 + x^2 + x\sqrt{q}},$$

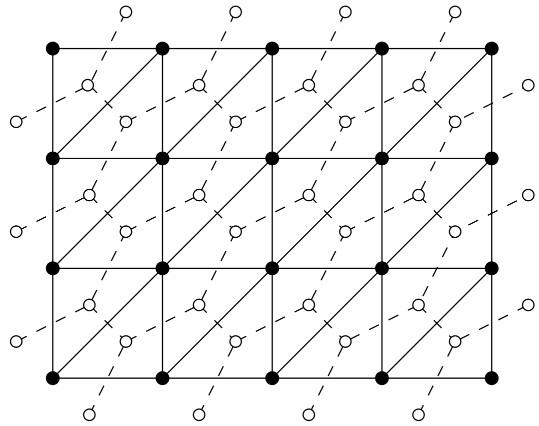


Figure 5.10. The triangular lattice $\mathbb{T}$ and its dual (hexagonal) lattice $\mathbb{H}$. The above embedding of these lattices in $\mathbb{R}^2$ is unconventional but convenient.

5.hexdual

thus underlining the relevance of the condition $q > 4$. In the symmetric case $x_h = x_v = 1$, we deduce when $q > 4$ that the discontinuity of the edge-density at the critical point equals

$$h^1(\mathbf{p}, q) - h^0(\mathbf{p}, q) = \frac{\sqrt{q-4}}{2+\sqrt{q}} \prod_{m=1}^{\infty} (\tanh(m\lambda))^2.$$

We turn now from speculation concerning the (self-dual) square lattice towards rigorous mathematics concerning the triangular/hexagonal pair of lattices, denoted $\mathbb{T}$ and $\mathbb{H}$ respectively. It is elementary that $\mathbb{T}$ is the dual of $\mathbb{H}$, and vice versa (see Figure 5.10). This fact permits a relation as in Theorem 5.13 between the random-cluster model on $\mathbb{T}$ with parameters p, q and that on $\mathbb{H}$ with parameters p_d, q. There is a second transformation of $\mathbb{T}$ into $\mathbb{H}$ called the star–triangle transformation and illustrated in Figure 5.11. As shown in that figure, one replaces alternate triangles of $\mathbb{T}$ by a star, and the resulting graph is isomorphic to $\mathbb{H}$. We shall henceforth refer to the shaded triangles in Figure 5.11 as *grey triangles*.

We explain next the relevance of the star–triangle transformation[9] to the random-cluster model on $\mathbb{T}$. This we do with the extra generality allowed by assigning to each edge e the individual edge-parameter p_e; rather than working with the p_e, we shall work with the variables

$$y_e = \frac{p_e}{1-p_e};$$

[9]References to the star–triangle transformation in the context of the Potts model may be found in [**18**, **195**].

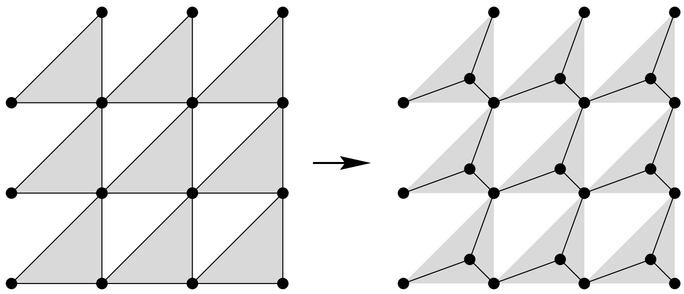


Figure 5.11. In the star–triangle transformation, alternate triangles are replaced as shown by stars. The triangular lattice $\mathbb{T}$ is thereby transformed into a copy of the hexagonal lattice $\mathbb{H}$. The shaded triangles are referred to henceforth as *grey* triangles.

5.startri

see the footnote at (5.7). Thus, for any finite subgraph $G = (V, E)$ of $\mathbb{T}$,

$$\phi_{G,p,q}(\omega) = \frac{1}{Z_G(p,q)} \left\{ \prod_{e \in E} y_e^{\omega(e)} \right\} q^{k(\omega)}, \qquad \omega \in \{0,1\}^E,$$

where we note that the normalising constant $Z_G(p, q)$ is not the same as before. Suppose that G contains some grey triangle $T = ABC$ with edge-set $E_T = \{e_1, e_2, e_3\}$, drawn on the left side of Figure 5.12. We propose to replace T by the star S on the right side, adding thus an extra vertex, and with the parameter values y_1', y_2', y_3' as shown. We shall see that that, under certain conditions on the y_i and y_i', the probabilities of a large family of 'connection events' are not altered by this transformation. The conditions in question are as follows:

$$\psi_{\mathbb{T}}(y_1, y_2, y_3) = 0, \tag{5.58}$$

$$y_i y_i' = q \quad \text{for} \quad i = 1, 2, 3, \tag{5.59}$$

5.startri3 where

$$\psi_{\mathbb{T}}(y_1, y_2, y_3) = y_1y_2y_3 + y_1y_2 + y_2y_3 + y_3y_1 - q. \tag{5.60}$$

5.defphi We note for later use that $\psi_{\mathbb{H}}(y_1', y_2', y_3') = 0$ under (5.58)–(5.59), where

$$\psi_{\mathbb{H}}(y_1, y_2, y_3) = y_1y_2y_3 - q(y_1 + y_2 + y_3) - q^2. \tag{5.61}$$

5.defphip

Let $\omega \in \{0, 1\}^E$, and define the equivalence relation $\leftrightarrow_\omega$ on V in the usual way, thus $u \leftrightarrow v$ if and only if there exists a open path of ω from u to v. We think of $\leftrightarrow_\omega$ as a random equivalence relation. Write $G^S = (V^S, E^S)$ for the graph obtained from G after the replacement of T by S; note that V^S is obtained from

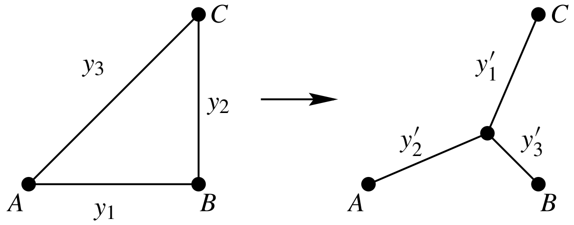


Figure 5.12. Bond parameters in the star–triangle transformation. The 'grey' triangle T on the left is replaced by the star S on the right, and the corresponding parameters are as marked.

5.startriprob

V by the addition of a vertex in the centre of T. Each $\omega^S \in \{0,1\}^{E^S}$ gives rise similarly to an equivalence relation on V which we denote as $\leftrightarrow_{\omega^S}$.

5.startri4 **Lemma 5.62.** *Let $G = (V, E)$ be a finite subgraph of $\mathbb{T}$ and let $T = ABC$ be a grey triangle of G as above. We have that*

$$\phi_{G,\mathbf{p},q}(\leftrightarrow_\omega) = \phi_{G^S,\mathbf{p}^S,q}(\leftrightarrow_{\omega^S}),$$

where $\mathbf{p} = (p_e : e \in E)$ and $\mathbf{p}^S = (p_e^S : e \in E^S)$ are such that $p_e^S = p_e$ for $e \in E \setminus E_T$, and on T and S the corresponding parameters y_i, y_i' satisfy (5.58)–(5.59).

Is the display for all ω?

Proof. Let e_1, e_2, e_3 (respectively, e_i') be the edges of T (respectively, S), and write y_i (respectively, y_i') for the corresponding parameters as in Figure 5.12. Let $(\omega(e) : e \in E \setminus \{e_1, e_2, e_3\})$ be given, and consider the conditional random-cluster measures P_ω^T (respectively, P_ω^S) on the e_i (respectively, on the e_i'). There are three disjoint classes of configuration ω which must be considered, depending on which of the following holds:

(a) A, B, C are in distinct open clusters of ω restricted to $E \setminus \{e_1, e_2, e_3\}$,

(b) two members of $\{A, B, C\}$ are in the same such cluster, and the third is not,

(c) A, B, and C are all in the same such cluster.

In each case, we propose to show that, under (5.58)–(5.59), the connections between A, B, C have the same conditional probabilities under both P_ω^T and P_ω^S. The required calculations are simple in principle, and we shall omit many details. In particular, we shall verify the claim under case (b) only, the other two cases being similar. Assume then that (b) holds, and suppose for definiteness that the configuration ω is such that: A and B are joined off T, but that C is joined off T to neither A nor B. By Theorem 2.1, the probabilities of connections internal to

T are given as follows:

$$
(5.63)\qquad
\begin{aligned}
P_\omega^T\big(A \leftrightarrow B \text{ and } B \nleftrightarrow C \text{ in } T\big) &= \frac{1}{Z} y_1 q^2,\\
P_\omega^T\big(A \nleftrightarrow B \text{ and } B \leftrightarrow C \text{ in } T\big) &= \frac{1}{Z} y_2 q,\\
P_\omega^T\big(A \nleftrightarrow B \text{ and } A \leftrightarrow C \text{ in } T\big) &= \frac{1}{Z} y_3 q,\\
P_\omega^T\big(A \leftrightarrow B \leftrightarrow C \text{ in } T\big) &= \frac{1}{Z}(y_1y_2y_3 + y_1y_2 + y_2y_3 + y_3y_1)q,
\end{aligned}
$$

5.tricon where

$$
(5.64)\qquad Z = (y_1y_2y_3 + y_1y_2 + y_2y_3 + y_3y_1 + y_2 + y_3)q + (1 + y_1)q^2.
$$

5.ztri Note that the events in question concern the existence (or not) of open paths within T only. The remaining term $P_\omega^T(A \nleftrightarrow B \nleftrightarrow C \text{ in } T)$ is given by the fact that the sum of all such probabilities on T equals 1.

The corresponding probabilities for connections internal to S are

$$
(5.65)\qquad
\begin{aligned}
P_\omega^S\big(A \leftrightarrow B \text{ and } B \nleftrightarrow C \text{ in } S\big) &= \frac{1}{Z'} y_2' y_3' q^2,\\
P_\omega^S\big(A \nleftrightarrow B \text{ and } B \leftrightarrow C \text{ in } S\big) &= \frac{1}{Z'} y_1' y_3' q,\\
P_\omega^S\big(A \nleftrightarrow B \text{ and } A \leftrightarrow C \text{ in } S\big) &= \frac{1}{Z'} y_1' y_2' q,\\
P_\omega^S\big(A \leftrightarrow B \leftrightarrow C \text{ in } S\big) &= \frac{1}{Z'} y_1' y_2' y_3' q,
\end{aligned}
$$

5.starcon where

$$
(5.66)\qquad Z' = (y_1'y_2'y_3' + y_1'y_2' + y_1'y_3')q + (y_2'y_3' + y_1' + y_2' + y_3')q^2 + q^3.
$$

5.zstar

It is left to the reader to check that, under (5.58)–(5.59), the probabilities in (5.63) and (5.65) are equal[10]. Similar computations are valid in cases (a) and (c) also, and it follows that, in loose terms, the replacement of T by S is 'invisible' to connections elsewhere in the graph G. □

Lemma 5.62 allows us to replace one grey triangle of G by a star. This process may be iterated until every grey triangle of G has been thus replaced. If G is itself a union of grey triangles, then the resulting graph is a subgraph of the

[10] It suffices for the purpose of the lemma to prove slightly fewer equalities than the four of (5.63) and (5.65).

hexagonal lattice $\mathbb{H}$. By working on a square region Λ of $\mathbb{T}$ and passing to the limit as $\Lambda \uparrow \mathbb{T}$, we find in particular that connections on $\mathbb{T}$ have the same probabilities as connections on $\mathbb{H}$ so long as the edge-parameters on $\mathbb{T}$ satisfy (5.58) and the corresponding parameters on $\mathbb{H}$ satisfy (5.59). In particular the percolation probabilities are the same. We shall make the last statement more specific.

Let $\mathbf{p} = (p_e : e \in \mathbb{E}) \in (0,1)^{\mathbb{E}}$, and let $y_e = p_e/(1-p_e)$. We speak of $\mathbf{p}$ as being of *type* γ if, for every grey triangle T, the three parameters y_1, y_2, y_3 of the edges of T satisfy $\psi_{\mathbb{T}}(y_1, y_2, y_3) = \gamma$. Suppose now that $\mathbf{p}$ is of type 0, as in (5.58). Applying the star–triangle transformation to every grey triangle of $\mathbb{T}$, we obtain a copy $\mathbb{H}$ of the hexagonal lattice, and we choose the parameters $\mathbf{p}' = (p'_e : e \in \mathbb{E}_{\mathbb{H}})$ of edges of this lattice in such a way that (5.59) holds. By the above discussion, the percolation probabilities $\theta^b_{\mathbb{T}}$ and $\theta^b_{\mathbb{H}}$ satisfy

(5.67) $$\theta^b_{\mathbb{T}}(\mathbf{p}, q) = \theta^b_{\mathbb{H}}(\mathbf{p}', q), \qquad b = 0, 1,$$

5.trihexperc whenever $q \geq 1$.

Let $\mathcal{L}$ be a lattice, and let $\mathbf{p}$ be a vector of reals indexed by the edge-set of $\mathcal{L}$; we call the pair $(\mathcal{L}, \mathbf{p})$ a *labelled lattice*. An *automorphism* of a labelled lattice $(\mathcal{L}, \mathbf{p})$ is a graph automorphism τ of $\mathcal{L}$ such that $p_{\tau(e)} = p_e$ for every edge e.

Equation (5.67) leads to a proposal for the so-called 'critical surfaces' of the triangular and hexagonal lattices. The crude argument is as follows. Suppose that $\mathbf{p}, \mathbf{p}'$ are as above. If $\theta^0_{\mathbb{T}}(\mathbf{p}, q) > 0$ then, by (5.67), $\theta^0_{\mathbb{H}}(\mathbf{p}', q) > 0$ also. Using duality as in Theorem 5.13, if we accept a picture of an infinite primal ocean of $\mathbb{H}$ surrounding bounded islands of its dual, then $\theta^1_{\mathbb{H}_\mathrm{d}}((\mathbf{p}')_\mathrm{d}, q) = 0$. If the initial labelled lattice $(\mathbb{T}, \mathbf{p})$ has a sufficiently large automorphism group then it may, by (5.59), be the case that

$$0 = \theta^1_{\mathbb{H}_\mathrm{d}}((\mathbf{p}')_\mathrm{d}, q) = \theta^1_{\mathbb{T}}(\mathbf{p}, q).$$

From this contradiction we may then deduce that $\theta^0_{\mathbb{T}}(\mathbf{p}, q) = 0$ whenever $\mathbf{p}$ is of type 0.

On the other hand, one may convince oneself that there should exist no non-empty interval $(\alpha, \beta) \subseteq \mathbb{R}$ such that: neither $\mathbb{T}$ nor its dual lattice possesses an infinite cluster whenever the type of $\mathbf{p}$ lies in (α, β). It is a consequence of these non-rigorous arguments that

(5.68) $$\theta^0_{\mathbb{T}}(\mathbf{p}, q) \begin{cases} = 0 & \text{if } \mathbf{p} \text{ is of non-positive type,} \\ > 0 & \text{if } \mathbf{p} \text{ is of positive type,} \end{cases}$$

5.gendual with a similar conjecture for the hexagonal lattice.

Such arguments may be used to justify the assertions that the critical surfaces of $\mathbb{T}$ and $\mathbb{H}$ are given respectively by $\psi_{\mathbb{T}}(\mathbf{y}) = 0$ and $\psi_{\mathbb{H}}(\mathbf{y}') = 0$, which is to say that

(5.69) $$\begin{array}{ll} \mathbb{T} \text{ has critical surface} & y_1y_2y_3 + y_1y_2 + y_2y_3 + y_3y_1 - q = 0, \\ \mathbb{H} \text{ has critical surface} & y'_1y'_2y'_3 - q(y'_1 + y'_2 + y'_3) - q^2 = 0. \end{array}$$

5.critsurfs It is not known how to make (5.69) rigorous, neither is it even accepted that the statements are true in complete generality since no assumption is made above about the automorphism groups of the labelled lattices.

We move now to the special case of the homogeneous random-cluster model on $\mathbb{T}$, having constant edge-parameter $p_e = p$ for every edge e. One part of the discussion may be made rigorous as follows.

5.tricrit **Theorem 5.70.** *Let $q \ge 1$.*

(a) *Consider the random-cluster model on the triangular lattice $\mathbb{T}$, and let p be such that $y = p/(1-p)$ satisfies $y^3 + 3y^2 - q = 0$. Then $\theta^0_{\mathbb{T}}(p, q) = 0$, and therefore $p_c(q) \ge p$.*

(b) *Consider the random-cluster model on the hexagonal lattice $\mathbb{H}$, and let p' be such that $y = p'/(1-p')$ satisfies $y^3 - 3qy - q^2 = 0$. Then $\theta^0_{\mathbb{H}}(p', q) = 0$, and therefore $p_c(q) \ge p'$.*

Proof. This may be proved either by an adaptation of the argument used to prove Theorems 5.16(a) and 5.52, or by following the proof of Theorem 5.44. The former approach utilises the $0/1$-infinite cluster property, and the latter approach makes use of the circuit-generation procedure pioneered in [**109**] and extended in [**78**]. Under either method, it is important that the labelled lattices are invariant under translations and possesses axes of mirror symmetry. $\square$

It is generally believed that the critical values of $\mathbb{T}$ and $\mathbb{H}$ are the values given in Theorem 5.70. In order to prove this it would be sufficient as in (5.17) to show that, for all $p < p_c(q, \mathbb{T})$, there exists $A < \infty$ such that

$$\phi^0_{\mathbb{T},p,q}(0 \leftrightarrow \partial\Lambda_n) < \frac{A}{n} \qquad \text{for all } n. \tag{5.71}$$

Check 5.triconj

We close this section with an open problem. Arguably the simplest system on the triangular lattice which possesses insufficent symmetry to allow the argument of the above proof is that in which every horizontal (respectively, vertical, diagonal) edge of $\mathbb{T}$ has edge-parameter p_h (respectively, p_v, p_d). The ensuing labelled lattice has properties of translation-invariance but has no axis of mirror-symmetry. Instead, it is symmetric under reflections in the origin. We conjecture that the equivalent of Theorem 5.70 holds for this process, namely that

$$\theta^0_{\mathbb{T}}(\mathbf{p}, q) = 0 \quad \text{if} \quad \psi_{\mathbb{T}}(p_h, p_v, p_d) = 0. \tag{5.72}$$

5.kesten Indeed one expects also that the critical surface is given by $\psi_{\mathbb{T}}(p_h, p_v, p_d) = 0$. The proof in the case of percolation, when $q = 1$, may be found at [**92**, Theorem 11.116].

5.7 Stochastic Löwner evolutions

Many exact calculations are 'known' for critical processes in two dimensions, but the physical arguments involved have sometimes appeared in varying degrees magical or revelationary to mathematicians. The new technology of stochastic Löwner evolutions (SLE), discovered by Schramm [**176**], promises to provide a rigorous underpinning of many such arguments in a manner most consonant with modern probability theory. Roughly speaking, the theory of SLE informs us of the correct weak limit of a critical process in the limit of large spatial scales, and in addition provides a mechanism for performing calculations for the limit process.

Let $\mathbb{H} = (-\infty, \infty) \times (0, \infty)$ be the upper half-plane of $\mathbb{R}^2$, with closure $\overline{\mathbb{H}}$. We view $\mathbb{H}$ and $\overline{\mathbb{H}}$ as subsets of the complex plane. Consider the ordinary differential equation

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - B_{\kappa t}}, \qquad z \in \overline{\mathbb{H}} \setminus \{0\},$$

subject to the boundary condition $g_0(z) = z$, where $t \in [0, \infty)$, κ is a positive constant, and $(B_t : t \geq 0)$ is a standard Brownian motion. The solution exists when $g_t(z)$ is bounded away from $B_{\kappa t}$. More specifically, for $z \in \overline{\mathbb{H}}$, let τ_z be the infimum of all times τ such that 0 is a limit point of $g_s(z) - B_{\kappa s}$ in the limit as $s \uparrow \tau$. We let

$$H_t = \{z \in \mathbb{H} : \tau_z > t\}, \qquad K_t = \{z \in \overline{\mathbb{H}} : \tau_z \leq t\},$$

so that H_t is open, and K_t is compact. It may now be seen that g_t is a conformal homeomorphism from H_t to $\mathbb{H}$.

We call $(g_t : t \geq 0)$ a *stochastic Löwner evolution* (SLE) with parameter κ, written SLE_κ, and we call the K_t the *hulls* of the process. There is good reason to believe that the family $K = (K_t : t \geq 0)$ provides the correct scaling limit of a variety of random spatial processes, the value of κ being chosen according to the process in question. General properties of SLE_κ, viewed as a function of κ have been studied in [**170**, **189**], and a beautiful theory has emerged. For example, the hulls K form almost surely a simple path if and only if $\kappa \leq 4$. If $\kappa > 8$, then SLE_κ generates almost surely a space-filling curve.

Schramm [**176**] has identified the relevant value of κ for several different processes, and has indicated that percolation has scaling limit SLE_6, but full rigorous proofs are incomplete. In the case of percolation, Smirnov [**181**, **182**] has proved the very remarkable result that, *for site percolation on the triangular lattice*, the scaling limit exists and is SLE_6 (this last statement is illustrated and partly explained in Figure 5.13), but the existence of the limit is open for other lattices and for bond percolation.

It is possible to perform calculations on stochastic Löwner evolutions, and in particular to confirm, [**140**, **183**], the values of many critical exponents associated with percolation (or, at least, site percolation on the triangular lattice). The consequences are in agreement with predictions of mathematical physicists previously

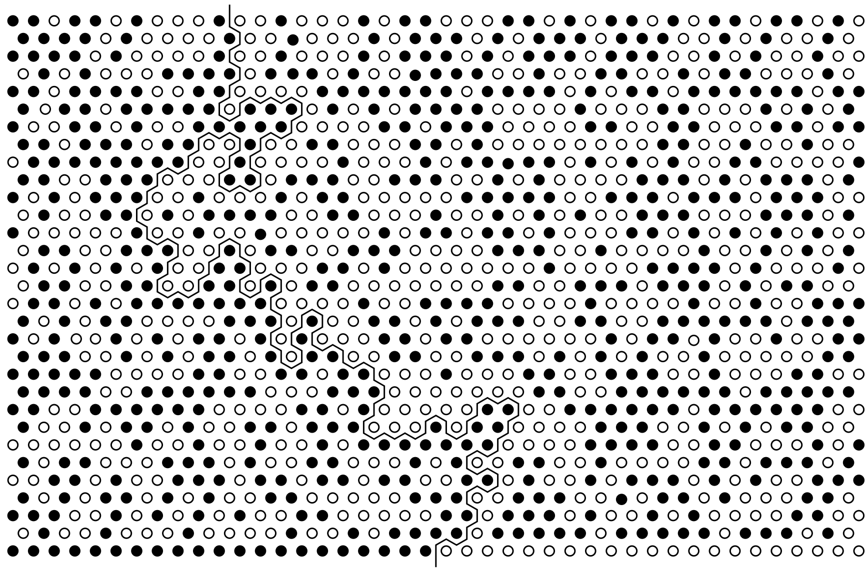


Figure 5.13. Site percolation on the triangular lattice with p equal to the critical point $\frac{1}{2}$, and with a mixed boundary condition along the lower side. The interface traces the boundary between the white and the black clusters touching the boundary, and is termed the 'exploration process'. In the limit of small lattice-spacing, the interface converges in a certain manner to the graph of a function which satisfies the Löwner differential equation driven by a Brownian motion with variance parameter $\kappa = 6$.

fig5.sle

considered near-miraculous (see [**92**], Chapter 9). In addition, SLE_6 satisfies the appropriate version of Cardy's formula, [**46**, **137**].

The technology of SLE is a major piece of contemporary mathematics which promises to explain phase transitions in an important class of two-dimensional disordered systems, and to help bridge the gap between probability theory and conformal field theory. It has already provided complete explanations of conjectures, by mathematicians and physicists, associated with two-dimensional Brownian motions and specifically their intersection exponents and fractionality of frontier, [**138**, **139**].

Extra work is needed in order to prove the validity of the limiting operation for other percolation models and random processes. In another remarkable recent paper [**141**], Lawler, Schramm, and Werner have verified the existence of the scaling limit for loop-erased random walk and for the uniform spanning tree Peano curve, and have shown them to be SLE_2 and SLE_8 respectively. It is believed that self-avoiding walk on $\mathbb{L}^2$, [**148**], has scaling limit $\text{SLE}_{8/3}$.

We turn now to the random-cluster model on $\mathbb{L}^2$ with parameters p and q. For $1 \le q < 4$, it is believed as in Conjectures 5.14 and 5.29 that the percolation probability $\theta(p, q)$, viewed as a function of p, is continuous at the critical point

$p_c(q)$, and furthermore that $p_c(q) = \sqrt{q}/(1 + \sqrt{q})$. It seems likely that, when re-scaled in the manner similar to that of percolation (illustrated in Figure 5.13), the exploration process of the model converges to a limit process of SLE type. It then remains only to specify the parameter κ of the limit in terms of q. It has been conjectured in [**170**] that κ satisfies

$$\cos(4\pi/\kappa) = -\tfrac{1}{2}\sqrt{q}, \qquad \kappa \in (4, 8).$$

This value is consistent with Smirnov's theorem [**182**], and also with the finding of [**141**] that the scaling limit of the uniform spanning tree Peano curve is SLE_8, on recalling from Theorem 1.21 that the uniform spanning tree measure is obtainable as a limit of the random-cluster measure as $p, q \downarrow 0$.

There are uncertainties over how this programme will develop. For a start, the theory of random-cluster models is not as complete as that of percolation and of the uniform spanning tree. Secondly, the existence of spatial limits is currently known only in certain special cases. The programme is however ambitious and full of promise, and should ultimately yield a full picture of the critical behaviour — including values of exponents — of random-cluster models, and hence of Ising/Potts models, with $q \le 4$.

7

Dynamics of Random-Cluster Models

7.1 How/why, finite/infinite volume

Three reasons: physical interest, simulation, and to obtain an additional method for studying the invariant measure, often through a monotone coupling.

7.2 Glauber dynamics

Let $G = (V, E)$ be a finite graph, with $\Omega = \{0, 1\}^E$ as usual. Let $0 < p < 1$ and $q > 0$. We shall construct a reversible Markov chain in continuous time having as unique invariant measure the random-cluster measure $\phi_{p,q}$ on Ω. A feature of the 'Glauber dynamics' of this section is that the set of permissible jumps comprises exactly those in which the state of a single edge, e say, changes. To this end, we recall first the notation prior to (2.12). For $\omega \in \Omega$ and $e \in E$, let ω^e and ω_e be the configurations obtained by 'switching e on' and 'switching e off', respectively, that is

$$\omega^e(f) = \begin{cases} 1 & \text{if } f = e, \\ \omega(f) & \text{if } f \neq e, \end{cases} \qquad \omega_e(f) = \begin{cases} 0 & \text{if } f = e, \\ \omega(f) & \text{if } f \neq e. \end{cases}$$

Let $X = (X_t : t \geq 0)$ be a Markov chain, [**98**, Chapter 6], on the state space Ω with generator $Q = (q_{\omega,\omega'} : \omega, \omega' \in \Omega)$ satisfying

$$\text{(7.1)} \qquad q_{\omega_e,\omega^e} = p, \quad q_{\omega^e,\omega_e} = (1-p)q^{D(e,\omega_e)}, \qquad \text{for } \omega \in \Omega,\ e \in E,$$

7.1 where $D(e, \xi)$ is the indicator function of the event that the endpoints of e are joined by no open path of ξ. Equations (7.1) specify the rates at which single edges are acquired or lost by the current configuration. We set $q_{\omega,\xi} = 0$ if ω and

ξ differ on two or more edges, and we choose the diagonal elements $q_{\omega,\omega}$ in such a way that Q, when viewed as a matrix, has row sums zero, that is,

$$q_{\omega,\omega} = -\sum_{\xi:\xi\neq\omega} q_{\omega,\xi}.$$

Note that X proceeds by moves in which single edges change their states, it is not permissible for two or more edge-states to change simultaneously. We say in this regard that X proceeds by 'local moves'.

It is elementary that the so-called 'detailed balance equations'

$$\phi_{p,q}(\omega)q_{\omega,\omega'} = \phi_{p,q}(\omega')q_{\omega',\omega}, \quad \omega, \omega' \in \Omega, \tag{7.2}$$

7.2 hold, whence X is reversible with respect to the random-cluster measure $\phi_{p,q}$. It is easily seen that the chain is irreducible, and therefore $\phi_{p,q}$ is the unique invariant measure of the chain and, in particular,

$$X_t \Rightarrow \phi_{p,q} \qquad \text{as } t \to \infty,$$

where '$\Rightarrow$' denotes weak convergence. There are of course many Markov chains with generators satisfying the detailed balance equations (7.2). It is important only that the ratio $q_{\omega,\omega'}/q_{\omega',\omega}$ satisfies

$$\frac{q_{\omega,\omega'}}{q_{\omega',\omega}} = \frac{\phi_{p,q}(\omega')}{\phi_{p,q}(\omega)}, \quad \omega, \omega' \in \Omega. \tag{7.3}$$

7.3 We call a Markov chain on Ω a *Glauber process* if it proceeds by local moves and has a generator Q satisfying (7.3), cf. [**144**, page 191]. We have concentrated here on continuous-time processes, but Glauber processes may be constructed in discrete time also.

Two extensions of this dynamical structure which have proved useful are as follows. The evolution may be specified in terms of a 'graphical representation' constructed via a family of independent Poisson processes. This allows a natural coupling of the measures $\phi_{p,q}$ for different p and q. Such couplings are monotone in p when $q \geq 1$. One may similarly couple the unconditional measure $\phi_{p,q}(\cdot)$ and the conditioned measure $\phi_{p,q}(\cdot \mid A)$. Such couplings permit probabilistic interpretations of differences of the form $\phi_{p',q}(B \mid A) - \phi_{p,q}(B)$ when $q \geq 1$, $p \leq p'$, and A and B are increasing, and this can be useful in particular calculations (see [**25**, **89**, **90**]).

One needs to be slightly more careful when G is an *infinite* graph. Such a situation may be studied by constructing a Glauber process on a finite subgraph H of G, and then passing to the thermodynamic limit as $H \to G$. Such a limit may be justified when $q \geq 1$, using the positive association of random-cluster measures, [**90**]. We shall discuss such limits in Section 7.8, in the more general context of 'coupled dynamics'. For a reason which will emerge later, we shall give the details for the Gibbs sampler of Section 7.3, rather than for the Glauber process of (7.1), which may however be treated in an essentially identical manner.

Note that the generator (7.1) of the Markov chain given above depends on the random variable $D(e, \omega_e)$, and that this random variable is a 'non-local' function of the configuration ω in the sense that there is no absolute bound on the distance from e of edges whose states may be relevant to its value. It is this feature of non-locality which leads to interesting complications when the graph is infinite, and these complications are linked in part to the $0/1$-infinite-cluster property introduced before Theorem 3.31. Further discussion may be found in [**90**, **160**].

7.3 Gibbs sampler

Once again we take $G = (V, E)$ to be a finite graph, and we let $0 < p < 1$ and $q > 0$. We consider in this section a special Glauber process, termed the *Gibbs sampler* (or *heat bath algorithm*). As before, this is a Markov chain X on the state space $\Omega = \{0, 1\}^E$ which proceeds by local moves. Its basic rule is as follows. We choose an edge e at random, and we set the state of e according to the conditional measure of $\omega(e)$ given the current states of the other edges. This may be done in either discrete or continuous time, we give the details for continuous time here, and shall return to the case of discrete time in Section 7.4.

Let $X = (X_t : t \geq 0)$ be the Markov chain on the state space Ω with generator $Q = (q_{\omega,\omega'}; \omega, \omega' \in \Omega)$ given by

$$\begin{aligned} q_{\omega_e,\omega^e} &= \frac{\phi_{p,q}(\omega^e)}{\phi_{p,q}(\omega^e) + \phi_{p,q}(\omega_e)}, \\ q_{\omega^e,\omega_e} &= \frac{\phi_{p,q}(\omega_e)}{\phi_{p,q}(\omega^e) + \phi_{p,q}(\omega_e)}, \end{aligned} \tag{7.4}$$

7.4 for $\omega \in \Omega$ and $e \in E$. Thus each edge is selected at rate 1, and the state of that edge is changed according to the correct conditional measure. It is evident that the detailed balance equations (7.2) hold as before, whence X is reversible with respect to $\phi_{p,q}$. Using irreducibility, $\phi_{p,q}$ is the unique invariant measure of the chain, and thus, in particular, $X_t \Rightarrow \phi_{p,q}$ as $t \to \infty$.

There is a nice way of thinking about the dynamic described in (7.4). To each edge e is associated an exponential alarm clock which rings at times of a Poisson process with intensity 1. Suppose that the alarm clock at e rings at time T, and let U be a random variable with the uniform distribution on the interval $[0, 1]$. Let $X_{T-} = \omega$ denote the current state of the process. The state of e jumps to the value $X_T(e)$ given as follows:

$$\begin{aligned} &\text{when } \omega(e) = 1, \quad \text{we set } X_T(e) = 0 \quad \text{if} \quad U \leq \frac{\phi_{p,q}(\omega_e)}{\phi_{p,q}(\omega^e) + \phi_{p,q}(\omega_e)}, \\ &\text{when } \omega(e) = 0, \quad \text{we set } X_T(e) = 1 \quad \text{if} \quad U > \frac{\phi_{p,q}(\omega_e)}{\phi_{p,q}(\omega^e) + \phi_{p,q}(\omega_e)}. \end{aligned} \tag{7.5}$$

7.5 It is easily seen that this rule generates a Markov chain which satisfies (7.4) and proceeds by local moves. This version of the chain X has two benefits. First, it is a neat way of implementing the Gibbs sampler in practice, since it requires only two random mechanisms: one which samples edges at random, and another which produces uniformly distributed random variables. A secondary benefit is that it provides directly a coupling of a variety of such Markov chains with different values of p and q, and with different initial states. Such a coupling is easily to seen to be monotone in p (and q?) and the initial state.

Explain. May need $q \geq 1$.

7.4 Coupling from the past

In running Monte Carlo experiments, one requires the ability to sample from the probability measure $\phi_{p,q}$. The Glauber processes of the last two sections certainly converge weakly to $\phi_{p,q}$ as $t \to \infty$, but this is not as good as having in the hand a sample with the *exact* distribution. The Propp–Wilson approach to sampling termed 'coupling from the past', [**168**], provides a mechanism for obtaining samples with the correct distribution, and is in addition especially well suited to the random-cluster model when $q \geq 1$. We describe this here. Some illustrations of the method in practice may be found in [**116**].

Let $G = (V, E)$ be a finite graph and let $0 < p < 1$ and $q > 0$. We shall later restrict ourselves to the case $q \geq 1$, since this will be important in the subsequent analysis of the algorithm. We provide ourselves first with a discrete-time reversible Markov chain $Z = (Z_n : n \geq 0)$ with state space Ω and having unique invariant measure $\phi_{p,q}$. The discrete-time Gibbs sampler provides a suitable example of such a chain, and proceeds as follows, see [**104**]. At each stage, we pick a random edge e, chosen uniformly from E and independently of all earlier choices, and we make e open with the correct conditional probability, given the configuration on the other edges. The corresponding transition matrix is given by $\Pi = (\pi_{\omega,\omega'} : \omega, \omega' \in \Omega)$ with

$$\pi_{\omega_e,\omega^e} = \frac{1}{|E|} \cdot \frac{\phi_{p,q}(\omega^e)}{\phi_{p,q}(\omega^e) + \phi_{p,q}(\omega_e)}, \quad \pi_{\omega_e,\omega_e} + \pi_{\omega_e,\omega^e} = \frac{1}{|E|},$$
$$\pi_{\omega^e,\omega_e} = \frac{1}{|E|} \cdot \frac{\phi_{p,q}(\omega_e)}{\phi_{p,q}(\omega^e) + \phi_{p,q}(\omega_e)}, \quad \pi_{\omega^e,\omega^e} + \pi_{\omega^e,\omega_e} = \frac{1}{|E|},$$

for $e \in E$ and $\omega \in \Omega$. A neat way to implement this is to follow the route given in the last section. Suppose that $Z_n = \omega$. Let e_n be a random edge of E, and let U_n be uniformly distributed on the interval $[0, 1]$, these variables being chosen independently of all earlier choices. We obtain Z_{n+1} from ω by retaining the states of all edges except possibly that of e_n. We set

$$(7.6) \qquad Z_{n+1}(e_n) = 0 \quad \text{if and only if} \quad U_n \leq \frac{\phi_{p,q}(\omega_{e_n})}{\phi_{p,q}(\omega^{e_n}) + \phi_{p,q}(\omega_{e_n})}.$$

7.6 Thus the evolution of the chain is determined by the sequences e_n, U_n, and the initial state Z_0. One may make this construction explicit by writing

$$Z_{n+1} = \psi(Z_n, e_n, U_n)$$

for some deterministic function $\psi : \Omega \times E \times [0,1] \to \Omega$.

We highlight a certain property of monotonicity of the function ψ when $q \geq 1$. Fix $e \in E$ and $u \in [0,1]$. We have that $\psi(\cdot, e, u)$ takes values in $\{0,1\}$ with

$$\psi(\omega, e, u) = 0 \quad \text{if and only if} \quad u \leq \frac{\phi_{p,q}(\omega_e)}{\phi_{p,q}(\omega^e) + \phi_{p,q}(\omega_e)}, \quad \omega \in \Omega.$$

We have as in ?? that

$$\frac{\phi_{p,q}(\omega_e)}{\phi_{p,q}(\omega^e) + \phi_{p,q}(\omega_e)} \leq \frac{\phi_{p,q}(\xi_e)}{\phi_{p,q}(\xi^e) + \phi_{p,q}(\xi_e)} \quad \text{if } \xi \leq \omega,$$

when $q \geq 1$, implying that

$$\psi(\xi, e, u) \leq \psi(\omega, e, u) \quad \text{for } \xi \leq \omega. \tag{7.7}$$

7.7 Now let $Z^\nu = (Z_n^\nu : n \geq 0)$ be the Markov chain constructed via (7.6) with initial state $Z_0 = \nu$. It follows by (7.7) that

$$Z_n^\xi \leq Z_n^\omega \text{ for all } n, \quad \text{if} \quad \xi \leq \omega \text{ and } q \geq 1, \tag{7.8}$$

7.coup1 which is to say that the coupling is monotone in the initial state: if one such chain
monotonicity in p? starts below another, then it remains below it for all time.

Instead of running the chain Z 'forwards' in time in order to approximate the invariant measure $\phi_{p,q}$, we shall run it 'backwards' in time in a special manner which results in a sample having the exact distribution. We let $W = (W(\omega) : \omega \in \Omega)$ be a vector of random variables such that each $W(\omega)$ has the distribution of Z_1 conditional on $Z_0 = \omega$,

$$\mathbb{P}(W(\omega) = \xi) = \pi_{\omega,\xi}, \quad \omega, \xi \in \Omega.$$

Following the scheme described above, we may take $W(\omega) = \psi(\omega, e, U)$ where e and U are chosen at random with the correct distributions. Let W_{-m}, $m \geq 1$, be independent random vectors distributed as W, that is, $W_{-m}(\cdot) = \psi(\cdot, e_m, U_m)$ where the set $\{(e_m, U_m) : m \geq 1\}$ comprises independent pairs of independent random variables, each e_i being uniform on E, and each U_i being uniform on $[0,1]$. We now construct a sequence Y_{-n}, $n \geq 1$, of random maps from Ω to Ω by the following inductive procedure. First, for $\omega \in \Omega$, we set $Y_{-1}(\omega) = W_{-1}(\omega)$. Having found $Y_{-1}, Y_{-2}, \dots, Y_{-m}$, we define

$$Y_{-m-1}(\omega) = Y_{-m}(W_{-m-1}(\omega)).$$

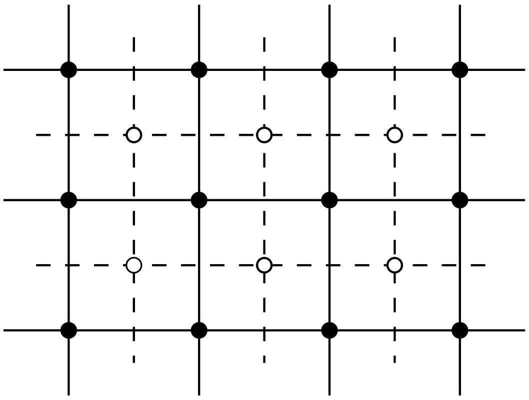


Figure 7.1. An illustration of the method of coupling from the past.

7.fig1

That is, $Y_{-m-1}(\omega)$ is obtained from ω by passing in one step to $W_{-m-1}(\omega)$, and then applying Y_{-m} to this new state. The exact dependence structure of this scheme is an important ingredient of what follows.

We terminate this process at the earliest value of m at which 'coalescence has occurred', that is, at the moment M given by

$$M = \min\{m : Y_{-m}(\cdot) \text{ is the constant function}\}. \tag{7.9}$$

7.propstop By the definition of M, the value $Y_{-M} = Y_{-M}(\omega)$ does not depend on the choice of ω. The process of coalescence is illustrated in Figure 7.1.

We prove next that Y_{-M} is governed as required by $\phi_{p,q}$.

Theorem 7.10 [168]. *The random variable M is almost surely finite, and*

$$\mathbb{P}(Y_{-M} = \xi) = \phi_{p,q}(\xi), \quad \xi \in \Omega.$$

7.propp

This procedure may seem unwieldy in practice, since Ω will often be large, and it seems necessary to keep track in (7.9) of the $Y_{-m}(\omega)$ *for every* $\omega \in \Omega$. The reality is simpler at least when $q \geq 1$, which we henceforth assume. By the monotonicity (7.8) of the coupling when $q \geq 1$, it suffices to follow the trajectories of the 'smallest' and 'largest' configurations, namely those beginning, respectively, with *every edge closed* and with *every edge open*. The processes starting at intermediate configurations remain sandwiched between these extremal processes, for all times m. Thus one may define M instead by

$$M = \min\{m : Y_{-m}(0) = Y_{-m}(1)\},$$

where 0 and 1 denote the vectors of zeros and ones as before. This is a great advantage in computational terms, since one is required to calculate only the $Y_{-m}(b)$ for $b = 0, 1$, and to find the earliest m at which they are equal.

We note two final points. It is an important problem to obtain bounds on the length of time required to achieve such coalescence, see [**??**]. Secondly, the implementation of the method is greatly simplified by the monotonicity. Even in situations where there is no such monotonicity, progress may be possible, see [**??**].

Proof of Theorem 7.10. □

7.5 Swendsen–Wang dynamics

It is a major target of statistical physics to understand the time-evolution of disordered systems, and a prime example lies in the study of the Ising model. A multiplicity of types of dynamics have been proposed. The majority of these share a quality of 'locality' in the sense that the evolution involves changes to the states of vertices in close proximity to one another, perhaps single spin-flips, or spin-exchanges. The state space is generally large, of size 2^N where N is the number of vertices, and the Hamiltonian has complicated structure. When subjected to 'local dynamics', the process may approach equilibrium very slowly (see [**149**, **174**] for accounts of recent work of relevance). 'Non-local dynamics', on the other hand, have the potential to approach equilibrium faster, since they permit large jumps around the state space, relatively unconstrained by neighbourly relations. The random-cluster model has played a role in the development of a simple but attractive such system, namely that proposed by Swendsen and Wang [**185**] and described as follows for the Potts model with q states.

As usual, $G = (V, E)$ is a finite graph, typically a large box in $\mathbb{Z}^d$, and we let $q \in \{2, 3, \dots\}$. Consider a q-state Potts model on G with parameters β, J satisfying $0 < \beta J < \infty$. This Potts model has state space $\Sigma = \{1, 2, \dots, q\}^V$; the corresponding random-cluster model has state space $\Omega = \{0, 1\}^E$ and parameter $p = 1 - e^{-\beta J}$. Suppose that, at some time n, we have obtained a configuration σ_n ($\in \Sigma$). We construct σ_{n+1} as follows.

I. We let $\omega_n \in \Omega$ be given as follows. For $e = \langle x, y\rangle \in E$,

$$\text{if } \sigma_n(x) \neq \sigma_n(y), \quad \text{let } \omega_n(e) = 0,$$
$$\text{if } \sigma_n(x) = \sigma_n(y), \quad \text{let } \omega_n(e) = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{otherwise,} \end{cases}$$

different edges receiving independent states. The edge configuration ω_n is carried forward to the next stage.

II. To each cluster C of the graph $(V, \eta(\omega_n))$ we assign an integer chosen uniformly at random from the set $\{1, 2, \dots, q\}$, different clusters receiving independent labels. We let $\sigma_{n+1}(x)$ be the value thus assigned to the cluster containing the vertex x.

Theorem 7.11 [185]. *The Markov chain* $\sigma = (\sigma_n : n \geq 0)$ *has as unique invariant measure the Potts measure on* Σ *with parameters* β *and* J.

Proof. Use Theorem 1.12. Expand. □

The Swendsen–Wang algorithm leads to samples which generally converge to equilibrium faster than those defined via local dynamics. This is especially evident in the 'high β' (or 'low temperature') phase, for the following reason. Consider for example the simulation of an Ising model on a finite box with free boundary conditions, and suppose that the initial state is $+1$ at all vertices. If β is large, then local dynamics result in samples which remain close to the '$+$ phase' for a very long time. Only after a long wait will the process achieve an average magnetisation close to 0. Swendsen–Wang dynamics, on the other hand, can achieve large jumps in average magnetisation even in a single step, since the spin allocated to a given large cluster of the corresponding random-cluster model is equally likely to be either of the two possibilities. A rigorous analysis of rates of convergence is however incomplete. It turns out that, *at* the critical point, Swendsen–Wang dynamics approach equilibrium only slowly, [**40**]. A further discussion is available in [**81**].

Algorithms of Swendsen–Wang type have been described for other statistical mechanical models having graphical representations of random-cluster-type; see [**55**, **56**]. Related work may be found in [**193**].

7.6 Coupled dynamics on a finite graph

Let $G = (V, E)$ be a graph, possible infinite. Associated with G there is a family $\phi_{G,p,q}$ of random-cluster measures indexed by the parameters $p \in [0, 1]$ and $q \in (0, \infty)$ [we omit a discussion of boundary conditions until the next section]. It has proved fruitful to couple these measures by finding a family $(Z(e) : e \in E)$ of random variables taking values in $[0, 1]$ whose 'level-sets' are governed by the $\phi_{G,p,q}$. It might be the case, for example, that for any fixed $0 < p < 1$, the configuration $(Z_p(e) : e \in E)$ given by

$$Z_p(e) = \begin{cases} 1 & \text{if } Z(e) \leq p, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies

$$\phi_{G,p,q}(A) = \mathbb{P}(Z_p \in A) \quad \text{for any event } A.$$

Such a coupling has been especially valuable in the study of percolation theory, where one may simply take the $Z(e)$ to be independent and uniformly distributed on the interval $[0, 1]$, see [**92**, **106**]. The picture for random-cluster measures is more complex owing to the dependence structure of the process, and we explore this next. Such a coupling has been explored in detail in [**90**] but we choose here to follow a minor variant which might be termed a 'coupled Gibbs sampler'. We shall assume for the moment that G is finite, returning in the next two sections to the case of infinite G.

Assume that $G = (V, E)$ is a finite graph, and let $q \geq 1$. Let $X = [0, 1]^E$, and let $\mathcal{B}$ be the Borel σ-field of subsets of X. We shall construct a Markov process $Z = (Z_t : t \geq 0)$ on the state space X, and we do this via a so-called graphical construction. We shall select edges at rate 1, and to this end we provide ourselves with a family of independent Poisson processes termed 'alarm clocks'. For each arrival-time of these processes, we shall require a uniformly distributed random variable.

(a) For each edge $e \in E$, let $A(e) = (A_n(e) : n \geq 1)$ be the (increasing) sequence of arrival times of a Poisson process with intensity 1.

(b) Let $(\alpha_n(e) : n \geq 1,\ e \in E)$ be a family of independent random variables having the uniform distribution on the interval $[0, 1]$.

We write $\mathbb{P}$ for the appropriate probability measure.

Let $e = \langle x, y \rangle$, and let $\mathcal{P}_e$ be the set of paths in G having endvertices x and y but not using the edge e. We define the function $F : E \times X \to [0, 1]$ by

$$F(e, \nu) = \inf_{\pi \in \mathcal{P}_e} \max_{f \in \pi} \nu(f), \quad e \in E,\ \nu \in X. \tag{7.12}$$

7.9 The maximum is taken over all edges f lying in the path π, and the infimum is taken over the finite set $\mathcal{P}_e$; we shall later consider situations in which $\mathcal{P}_e$ is infinite.

The state of the edge e may jump only at the times $A_n(e)$. When it jumps, it takes a new value which depends on the states of the other edges, and also on the value of the corresponding $\alpha_n(e)$. We describe next the value to which it will jump.

Suppose that the Poisson alarm clock at edge e rings at time $T = A_N(e)$, with corresponding uniform random variable $\alpha = \alpha_N(e)$. Let the current state of the process be $Z_{T-} = \nu$, and write $F = F(e, \nu)$. We define Z_T by

$$Z_T(f) = \begin{cases} \nu(f) & \text{if } f \neq e, \\ \rho(e) & \text{if } f = e, \end{cases} \tag{7.13}$$

rc6.1 where the new value $\rho(e)$ is given by

$$\rho(e) = \begin{cases} \alpha & \text{if } \alpha > F, \\ F & \text{if } \dfrac{F}{F + q(1-F)} < \alpha \leq F, \\ \dfrac{q\alpha}{1 - \alpha + q\alpha} & \text{if } \alpha \leq \dfrac{F}{F + q(1-F)}. \end{cases} \tag{7.14}$$

rc6.2 We note that

$$\frac{F}{F+q(1-F)} \le F,$$

since $q \ge 1$. If T is an arrival time of two or more Poisson processes indexed by different edges, then we update the process on these edges according to the rules above. There is probability zero that such a time T ever occurs.

Here is a more formal definition of the process. Let $C(X)$ be the space of continuous real-valued functions on X. The generator G of the process is the mapping $G : C(X) \to C(X)$ given by

$$Gf(\nu) = \sum_{e \in E} \int_0^1 \big(f(\nu_e^u) - f(\nu)\big)\, dH_\nu(u), \quad f \in C(X),\ \nu \in X. \tag{7.15}$$

7.10 Here, the configuration $\nu_e^u \in X$ is given by

$$\nu_e^u(f) = \begin{cases} \nu(f) & \text{if } f \ne e, \\ u & \text{if } f = e, \end{cases}$$

and the distribution function $H_\nu(\cdot) : [0,1] \to [0,1]$ may be calculated from (7.14) as

$$H_\nu(u) = \begin{cases} u & \text{if } F < u \le 1, \\ F & \text{if } u = F, \\ \dfrac{u}{u+q(1-u)} & \text{if } 0 \le u < F. \end{cases} \tag{7.16}$$

7.11 Equation (7.15) is to be interpreted as follows. Suppose the alarm clock at e rings at time T, and the current state is ν. The local state at e jumps to a new value $\rho(e)$ which (given ν) does not depend on its previous value and has distribution function H_ν.

There follows the main theorem of this section, the proof of which follows exactly the corresponding proof of [**90**] and is deferred until later in the section. Indeed, the only essential difference between Z and the process considered in [**90**] is that Z is reversible.

7.thm1 **Theorem 7.17.** *There exists a probability measure μ on $(X, \mathcal{B})$ such that the process $Z = (Z_t : t \ge 0)$ has unique invariant measure μ and is reversible with respect to μ.*

The purpose of the above construction is to achieve a level-set representation of evolving random-cluster processes on G. Let $0 < p < 1$, and recall that $\Omega = \{0,1\}^E$. For $\nu \in X$, we define two 'projected elements' $\pi^p\nu$ and $\pi_p\nu$ of Ω by

$$\pi^p\nu(e) = \begin{cases} 1 & \text{if } \nu(e) \le p, \\ 0 & \text{if } \nu(e) > p, \end{cases} \quad e \in E, \tag{7.18}$$

rc6.4 and

$$\pi_p \nu(e) = \begin{cases} 1 & \text{if } \nu(e) < p, \\ 0 & \text{if } \nu(e) \geq p, \end{cases} \qquad e \in E. \tag{7.19}$$

rc6.5 We point out that

$$\pi_p \nu \leq \pi^p \nu \quad \text{for all } p \text{ and } \nu, \tag{7.20}$$

rc6.8 and that

$$\pi_{p_1} \nu_1 \leq \pi_{p_2} \nu_2, \quad \pi^{p_1} \nu_1 \leq \pi^{p_2} \nu_2, \quad \text{for } p_1 \leq p_2 \text{ and } \nu_1 \leq \nu_2. \tag{7.21}$$

rc6.9 In writing $\nu_1 \leq \nu_2$ here, we are using the partial order '$\leq$' on X given by $\nu_1 \leq \nu_2$ if and only if $\nu_1(e) \leq \nu_2(e)$ for all $e \in E$.

We concentrate next on the projected processes $\pi^p Z = (\pi^p Z_t : t \geq 0)$ and $\pi_p Z = (\pi_p Z_t : t \geq 0)$. An important difference between these two processes will become clear in the next section when we introduce boundary conditions.

7.thm2 **Theorem 7.22 [90].** *Let* $0 < p < 1$.
(a) *The process* $\pi^p Z = (\pi^p Z_t : t \geq 0)$ *is a Markov chain on the state space* Ω *having unique invariant distribution* $\phi_{p,q}$*, and it is reversible with respect to* $\phi_{p,q}$*. Furthermore,*

$$\pi^{p_1} Z_t \leq \pi^{p_2} Z_t \quad \textit{for all } t, \quad \textit{if } p_1 \leq p_2. \tag{7.23}$$

rc6.10

(b) *The same conclusions are valid with* π^p *replaced throughout* (a) *by* π_p.

Proof of Theorem 7.22. (a) We begin with a calculation involving the function F defined in (7.12). Let $e \in E$, $\nu \in X$, and let $\gamma = \pi^p \nu \in \Omega$. We claim that

$$F(e, \nu) > p \quad \text{if and only if} \quad \gamma = \pi^p \nu \in D_e, \tag{7.24}$$

7.12 where D_e ($\subseteq \Omega$) is the set of all configurations such that the endvertices of e are not joined by an open path of $E \setminus \{e\}$. This may be seen from (7.12) by noting that: $F(e, \nu) > p$ if and only if, for every $\pi \in \mathcal{P}_e$, there exists an edge $f \in \pi$ such that $\nu(f) > p$.

The projected process $\pi^p Z$ changes its value only when Z changes its value. Assume that $Z_t = \nu$ and $\pi^p Z_t = \pi^p \nu = \gamma$. Let $\gamma' \in \Omega$. The rate at which $\pi^p Z$ jumps subsequently to the new state γ' depends only on: (i) the arrival-times of the Poisson processes $A(e)$ subsequent to t, (ii) the associated values of the random variables α, and (iii) the set of edges $F_\nu = \{e \in E : F(e, \nu) > p\}$. By (7.24), $F_\nu = \{e \in E : \gamma \in D_e\}$, which depends on γ only and not further on ν. It follows that $\pi^p Z = (\pi^p Z_t : t \geq 0)$ is a time-homogeneous Markov chain on Ω.
Further explanation desirable This argument is expanded in the following computation of the jump rates.

Let $Q = (q_{\gamma,\omega} : \gamma, \omega \in \Omega)$ denote the generator of the process $\pi^p Z$. Since Z proceeds (almost surely) by local moves, we have that

$$q_{\gamma,\omega} = 0 \quad \text{if} \quad \sum_e |\gamma(e) - \omega(e)| \geq 2,$$

and it remains to calculate the terms q_{γ_e,γ^e} and q_{γ^e,γ_e} for $\gamma \in \Omega$ and $e \in E$. Consider first q_{γ_e,γ^e}. By (7.15),

$$\mathbb{P}(\pi^p Z_{t+h} = \gamma^e \mid \pi^p Z_t = \gamma_e) = h H_\nu(p) + \mathrm{o}(h), \quad \text{as } h \downarrow 0,$$

whence, by (7.16) and (7.24),

$$q_{\gamma_e,\gamma^e} = H_\nu(p) = \begin{cases} \dfrac{p}{p + q(1-p)} & \text{if } \gamma \in D_e, \\ p & \text{if } \gamma \notin D_e. \end{cases} \tag{7.25}$$

7.14 By a similar argument,

$$q_{\gamma^e,\gamma_e} = 1 - H_\nu(p) = \begin{cases} \dfrac{q(1-p)}{p + q(1-p)} & \text{if } \gamma \in D_e, \\ 1 - p & \text{if } \gamma \notin D_e. \end{cases} \tag{7.26}$$

7.15 Therefore,

$$\begin{aligned} \frac{q_{\gamma_e,\gamma^e}}{q_{\gamma^e,\gamma_e}} = \frac{H_\nu(p)}{1 - H_\nu(p)} &= \begin{cases} \dfrac{p}{q(1-p)} & \text{if } \gamma \in D_e, \\ \dfrac{p}{1-p} & \text{if } \gamma \notin D_e, \end{cases} \\ &= \frac{\phi_{p,q}(\gamma^e)}{\phi_{p,q}(\gamma_e)}. \end{aligned} \tag{7.27}$$

7.16 It follows as in (7.3) that the detailed balance equations hold, whence the process $\pi^p Z$ is reversible with respect to $\phi_{p,q}$. That $\phi_{p,q}$ is the unique invariant measure is a consequence of the irreducibility of the chain.

Inequality (7.38) follows from (7.21).

(b) A similar argument is valid with π_p in place of π^p, on replacing (7.24) by the statement

$$F(e, \nu) \geq p \quad \text{if and only if} \quad \gamma = \pi_p \nu \in D_e, \tag{7.28}$$

7.13 and by replacing $H_\nu(p)$ by

$$H_\nu(p-) = \lim_{u \uparrow p} H_\nu(u)$$

in the calculations (7.25)–(7.27). □

We turn now to the proof of Theorem 7.17, which is preceeded by a lemma. The product space $X = [0, 1]^E$ is equipped with the Borel σ-field $\mathcal{B}$. An event $A \in \mathcal{B}$ is called *increasing* if $\nu' \in A$ whenever there exists $\nu \in A$ such that $\nu \leq \nu'$, and it is called *decreasing* if its complement is increasing. For $\zeta \in X$, we write $Z^\zeta = (Z_t^\zeta : t \geq 0)$ for the Markov process given above and having initial state $Z_0 = \zeta$.

7.lem1 **Lemma 7.29.**

(a) *If* $\zeta \le \nu$ *then* $Z_t^\zeta \le Z_t^\nu$ *for all* t.

(b) *Let* E *be an increasing event in* $\mathcal{B}$. *The function*

$$g^b(t) = \mathbb{P}(Z_t^b \in E)$$

is non-decreasing in t *if* $b = 0$, *and is non-increasing if* $b = 1$.

Proof of Lemma 7.29. (a) This follows from the transition rules (7.13)–(7.14) together with the fact that $F(e, \nu)$ is non-decreasing in ν.

(b) By conditional expectation,

$$g^b(s+t) = \mathbb{P}\big\{\mathbb{P}(Z_{s+t}^b \in E \mid Z_s^b)\big\}, \quad b = 0, 1.$$

Using the time-homogeneity of the processes (A, α), and the fact that

$$0 \le Z_s^b \le 1 \quad \text{for } b = 0, 1 \text{ and } s \ge 0,$$

we deduce by part (a) that

$$g^b(s+t) \begin{cases} \ge g^b(t) & \text{if } b = 0, \\ \le g^b(t) & \text{if } b = 1, \end{cases} \tag{7.30}$$

7.17 as required. □

Proof of Theorem 7.17. We prove first that there exists a unique measure μ which is invariant for Z. To this end shall prove that Z_t converges weakly as $t \to \infty$, and we shall write μ for the weak limit. We have from Lemma 7.29(a) that

$$Z_t^0 \le Z_t^\nu \le Z_t^1 \quad \text{for all } t \text{ and } \nu. \tag{7.31}$$

7.18 By Lemma 7.29(b), Z_t^b is stochastically increasing in t if $b = 0$, and stochastically decreasing if $b = 1$. It therefore suffices to show that

$$Z_t^1 - Z_t^0 \Rightarrow 0 \quad \text{as} \quad t \to \infty. \tag{7.32}$$

7.19 Let $\epsilon > 0$, and write $\mathcal{E} = \{N^{-1}, 2N^{-1}, \dots, (N-1)N^{-1}\}$ where N is a positive integer satisfying $N^{-1} < \epsilon$. Then

$$\mathbb{P}\Big(|Z_t^1(e) - Z_t^0(e)| > \epsilon \text{ for some } e \in E\Big) \le \sum_{e \in E} \sum_{p \in \mathcal{E}} \mathbb{P}\big(Z_t^0(e) < p < Z_t^1(e)\big).$$

Now,

$$\begin{aligned} \mathbb{P}\big(Z_t^0(e) < p < Z_t^1(e)\big) &\le \mathbb{P}(\pi^p Z_t^0(e) = 1) - \mathbb{P}(\pi^p Z_t^1(e) = 1) \\ &\to 0 \qquad \text{as } t \to \infty \end{aligned}$$

by the ergodicity of the Markov chain $\pi^p Z$, see Theorem 7.22.

We show finally that Z is reversible with respect to its weak limit μ. Let $e \in E$, and write $\mathcal{T}_e$ for the sub-σ-field of $\mathcal{B}$ generated by the states of edges other than e. By the argument in the proof of Theorem 7.22, the random state ν taking values in X satisfies

(7.33)
$$\begin{aligned}\mu\big(\nu(e) \le p \,\big|\, \mathcal{T}_e\big)(\nu) &= \phi_{p,q}\big(\pi^p\nu(e) = 1 \,\big|\, \pi^p\nu(f) \text{ for } f \ne e\big)\\ &= H_\nu(p).\end{aligned}$$

7.20 By (7.15),

$$fGg(\nu) = \sum_{e \in E} \int_0^1 \big(f(\nu)g(\nu_e^u) - f(\nu)g(\nu)\big)\, dH_\nu(u), \qquad f, g \in C(X).$$

Let κ_e denote the conditional distribution of the state of e given $\mathcal{T}_e$, evaluated at ν. By conditional expectation,

$$\begin{aligned}\mu(fGg - gGf) &\\ &= \mu\left(\sum_{e\in E} \kappa_e\left(\int_0^1 \big[f(\nu)g(\nu_e^u) - f(\nu_e^u)g(\nu)\big]\, dH_\nu(u)\right)\right)\\ &= \mu\left(\sum_{e\in E}\left(\kappa_e(f)\int_0^1 g(\nu_e^u)\, dH_\nu(u) - \kappa_e(g)\int_0^1 f(\nu_e^u)\, dH_\nu(u)\right)\right)\\ &= \mu\left(\sum_{e\in E}\big[\kappa_e(f)\kappa_e(g) - \kappa_e(g)\kappa_e(f)\big]\right) \qquad \text{by (7.33)}\\ &= 0,\end{aligned}$$

as required. □

7.7 Lattice dynamics

In the last section, we constructed a Markov process Z on the state space $X = [0,1]^E$ for a *finite* edge-set E. In moving to an infinite graph, we shall require a discussion of boundary conditions, and we embark on this next. Since our target is to study the lattice $\mathbb{L}^d$, we shall assume for convenience that our graphs are boxes in this lattice. Consider then a box Λ of $\mathbb{L}^d$, and write $X = [0,1]^{\mathbb{E}}$. For $\zeta \in X$, we write

(7.34)
$$X_\Lambda^\zeta = \big\{\nu \in X : \nu(e) = \zeta(e) \text{ for } e \notin \mathbb{E}_\Lambda\big\},$$

7.box the subset of X containing configurations which agree with ζ off Λ. We define

$$F(e,\nu) = \inf_{\pi\in\mathcal{P}_e} \max_{f\in\pi} \nu(f), \quad e \in \mathbb{E},\ \nu \in X. \tag{7.35}$$

7.9new as in (7.12).

We provide ourselves with a family of independent Poisson processes $A(e) = (A_n(e) : n \geq 1)$, $e \in \mathbb{E}$, and an associated collection $(\alpha_n(e) : n \geq 1,\ e \in \mathbb{E})$ of random variables uniformly distributed on the interval $[0,1]$. Let Λ be a box and let $\zeta \in X$. The above variables may be used as in the last section to construct a family of coupled Markov processes $Z^\zeta_\Lambda = (Z^\zeta_{\Lambda,t} : t \geq 0)$ taking values in X^ζ_Λ and indexed by the pair Λ, ζ.

For $\nu, \zeta \in X$, and a box Λ, we denote by (ν,ζ) $[= (\nu,\zeta)_\Lambda]$ the configuration which agrees with ν on $\mathbb{E}_\Lambda$ and with ζ off $\mathbb{E}_\Lambda$. We sometimes suppress the subscript Λ when using this notation. For example, the expression $Z^{(\nu,\zeta)}_{\Delta,t}$ denotes the value of the process on the box Δ at time t, with initial value $(\nu,\zeta)_\Delta$. Finally, with π^p, π_p given as in (7.18)–(7.19), we write Υ^p_Λ for the set of all ζ $(\in X)$ with the property that $\pi^p[(0,\zeta)_\Lambda]$ has at most one infinite cluster.

rc6.3 **Theorem 7.36.** *For $\zeta \in X$ and a box Λ, there exists a probability measure μ^ζ_Λ on $(X, \mathcal{B})$, satisfying $\mu^\zeta_\Lambda(X^\zeta_\Lambda) = 1$, such that*

$$Z^{(\nu,\zeta)}_{\Lambda,t} \Rightarrow \mu^\zeta_\Lambda \quad \textit{as } t \to \infty, \quad \textit{for all } \nu.$$

The process Z^ζ_Λ is reversible with respect to μ^ζ_Λ.

A complication arises in studying the projected process $\pi^p Z^\zeta_\Lambda$, depending on whether or not $\zeta \in \Upsilon^p_\Lambda$.

rc6.1thm **Theorem 7.37.** (a) *The process $\pi_p Z^\zeta_\Lambda = (\pi_p Z^\zeta_{\Lambda,t} : t \geq 0)$ is a Markov chain on the state space $\pi_p X^\zeta_\Lambda$ having unique stationary distribution $\phi^{\pi_p\zeta}_{\Lambda,p,q}$, and it is reversible with respect to this measure. Furthermore,*

$$\pi_{p_1} Z^\zeta_{\Lambda,t} \leq \pi_{p_2} Z^\zeta_{\Lambda,t} \quad \textit{for all } t, \textit{ if } p_1 \leq p_2. \tag{7.38}$$

rc6.10

(b) *Statement (a) is valid with the operator π_p replaced throughout by π^p, so long as $\zeta \in \Upsilon^p_\Lambda$.*

We note two further facts for future use. First, there is a sample-path monotonicity of the graphical representation, which will enable us to take the process-limit as $\Lambda \uparrow \mathbb{Z}^d$. Secondly, if ν and ζ are members of X which are close to one another, then then so are $Z^{(\nu,b)}_{\Lambda,t}$ and $Z^{(\zeta,b)}_{\Lambda,t}$, for $b \in \{0,1\}$.

rcthm6.4 **Lemma 7.39.** (a) *Let* Λ *and* Δ *be boxes satisfying* $\Lambda \subseteq \Delta$. *Then*

$$Z_{\Lambda,t}^{(\zeta,0)} \le Z_{\Delta,t}^{(\zeta,0)} \quad \textit{for all } \zeta \textit{ and } t, \tag{7.40}$$

rc6.11 *and*

$$Z_{\Lambda,t}^{(\zeta,1)} \ge Z_{\Delta,t}^{(\zeta,1)} \quad \textit{for all } \zeta \textit{ and } t. \tag{7.41}$$

rc6.12 (b) *Let* Λ *be a box, let* $b \in \{0,1\}$, *and let* $\nu, \zeta \in X$. *We have for all* $t \ge 0$ *and all* $e \in \mathbb{E}$ *that*

$$\big|Z_{\Lambda,t}^{(\nu,b)}(e) - Z_{\Lambda,t}^{(\zeta,b)}(e)\big| \le \max_{f \in \mathbb{E}_\Lambda} \big\{|\nu(f) - \zeta(f)|\big\}. \tag{7.42}$$

rc6.13

Before moving to the proofs, we make two notes concerning the value of q. First, the above construction may be extended in order to couple together random-cluster processes with different values of p *and* different values of q (satisfying $q \ge 1$); this is achieved by a suitable coupling of the processes $\{C(e) : e \in \mathbb{E}\}$ for different q. Secondly, some of the arguments of this section may be recast in the 'non-FKG' case when $q < 1$. When $q < 1$, we alter the definitions of the processes $A(e)$, $B(e)$, $C(e)$ so that $B(e)$ has rate q and $C(e)$ has rate $1-q$. With minor changes elsewhere, this enables the construction to proceed, but unfortunately with the loss of Lemmas 6.2 and 6.4.

7.8 Coupled dynamics on the infinite lattice

Pass to limit $\Lambda \uparrow \mathbb{Z}^d$. Explore reversibility of the infinite-space process.

Rewrite–

In this section we study certain Markov processes on the state space $X = [0,1]^{\mathbb{E}}$. We show the existence of two different transition semigroups with the same (unique) invariant measure. The first of these semigroups gives rise to a 'level-set representation' of *free boundary condition* random-cluster processes, and the second of *wired boundary condition* processes.

We arrive at such Markov processes by studying the limit of the finite-volume process $Z_{\Lambda,t}^{\zeta}$, defined in the last section, as $\Lambda \uparrow \mathbb{Z}^d$. The two 'extreme' boundary conditions ζ are $\zeta = 0, 1$, and we define accordingly the following monotone limits:

$$Z_t^{(\zeta,0)} = \lim_{\Lambda \uparrow \mathbb{Z}^d} Z_{\Lambda,t}^{(\zeta,0)}, \quad Z_t^{(\zeta,1)} = \lim_{\Lambda \uparrow \mathbb{Z}^d} Z_{\Lambda,t}^{(\zeta,1)}, \tag{7.1}$$

which limits exist by virtue of Lemma 6.4(a). In particular we write

$$Z_t^0 = Z_t^{(0,0)}, \quad Z_t^1 = Z_t^{(1,1)}. \tag{7.2}$$

We shall show that the processes $(Z_t^b : t \geq 0)$, for $b = 0, 1$, are Markovian, and we shall explore their properties in the limit as $t \to \infty$.

A possible alternative to the methodology of this section might employ the 'martingale method' described in [**113**, **144**]. For general accounts of the theory of Markov processes, consult the books [**31**, **144**, **178**].

The state space $X = [0, 1]^{\mathbb{E}}$ is a compact metric space equipped with the Borel σ-field $\mathcal{B}$. Let $D(X)$ be the set of functions $G : \mathbb{R} \to X$ which are right-continuous with left limits. For $s \in [0, \infty)$, let e_s be the evaluation mapping defined by $e_s(G) = G(s)$. Let $\mathcal{H}$ be the smallest σ-field of subsets of $D(X)$ with respect to which each e_s is measurable, and let $\mathcal{H}_t$ be the smallest such σ-field defined in terms of $\{e_s : s \leq t\}$. We write $B(X)$ for the space of bounded measurable functions from X to $\mathbb{R}$, and $C(X)$ for the space of continuous functions.

We now introduce two transition functions and semigroups, as follows. For $b \in \{0, 1\}$ and $t \geq 0$, let

$$P_t^b(\zeta, A) = P(Z_t^{(\zeta,b)} \in A), \qquad \zeta \in X,\ A \in \mathcal{B}, \tag{7.3}$$

and let $S_t^b : B(X) \to B(X)$ be given by

$$S_t^b f(\zeta) = P(f(Z_t^{(\zeta,b)})), \qquad \zeta \in X,\ f \in B(X). \tag{7.4}$$

Theorem 7.1. *Let $b \in \{0, 1\}$. The process $(Z_t^b : t \geq 0)$ is a Markov process with sample paths in $D(X)$ and Markov transition function $(P_t^b : t \geq 0)$.*

Theorem 7.2. *There exists a translation-invariant probability measure μ on $(X, \mathcal{B})$ such that*

$$Z_t^b \Rightarrow \mu \quad \textit{as} \quad t \to \infty, \quad \textit{for } b = 0, 1.$$

Note that the weak limit in the latter theorem is identical for the two processes Z_t^0 and Z_t^1. It follows by monotonicity that, as $t \to \infty$,

$$Z_t^{(\zeta,b)} \Rightarrow \mu \quad \text{for} \quad \zeta \in X \text{ and } b = 0, 1; \tag{7.5}$$

recall Lemma 6.2(a) and (7.1).

We turn attention now to the 'level-set processes' of Z_t^0 and Z_t^1. Fix $p \in (0, 1)$, and write

$$L_{p,t}^0 = \pi_p Z_t^0, \quad L_{p,t}^1 = \pi^p Z_t^1, \qquad t \geq 0; \tag{7.6}$$

here, π^p and π_p are defined in (6.4) and (6.5).

Theorem 7.3. (a) *The processes $(L_{p,t}^b : t \geq 0)$, $b = 0, 1$, are Markov processes on the state space $\Omega = \{0, 1\}^{\mathbb{E}}$, with weak limits given by*

$$L_{p,t}^b \Rightarrow \phi_{p,q}^b \quad \textit{as } t \to \infty, \tag{7.7}$$

where $\phi^b_{p,q}$ is the random-cluster measure defined in (3.5) for $b = 0, 1$. The measure $\phi^b_{p,q}$ is reversible for the process $L^b_{p,t}$.

(b) *The measures $\phi^b_{p,q}$, for $b = 0, 1$, are 'level-set' measures of μ, in that*

$$\phi^0_{p,q}(A) = \mu\big(\{\zeta : \pi_p\zeta \in A\}\big), \quad \phi^1_{p,q}(A) = \mu\big(\{\zeta : \pi^p\zeta \in A\}\big), \tag{7.8}$$

for all $A \in \mathcal{F}$.

We make several remarks before proving the above theorems. First, the two weak limits $\phi^0_{p,q}$ and $\phi^1_{p,q}$ in Theorem 7.3 are identical if and only if $p \notin \mathcal{D}_q$, where $\mathcal{D}_q$ is given in Theorem 4.1.

Next, let μ be the limit measure of Theorem 7.2, let $e \in \mathbb{E}$, and define the marginal 'atomic' function

$$J(x) = \mu\big(\{\zeta \in X : \zeta(e) = x\}\big) \quad \text{for } 0 \le x \le 1;$$

since μ is translation-invariant, J does not depend on the choice of the edge e.

Proposition 7.4. *We have that*

$$h^1(p,q) - h^0(p,q) = J(1-p)$$

where $h^b(p,q) = \phi^b_{p,q}(\omega(e) = 1)$.

In the light of Theorem 4.2, this implies that $p \in \mathcal{D}_q$ if and only if $J(1-p) \neq 0$, thereby providing a representation of $\mathcal{D}_q$ in terms of atoms of the weak limit μ of the stochastic random-cluster processes Z^0_t and Z^1_t. It is this representation that we employ at the end of this section in order to prove the left-continuity of the percolation probability $\theta^0(\cdot, q)$ (cf. Theorem 5.2(a) and [**23**]).

As discussed already after Theorem 4.2, it is believed that there exists $Q = Q(d)$ such that

$$\mathcal{D}_q = \begin{cases} \varnothing & \text{if } q < Q, \\ \{p_c(q)\} & \text{if } q > Q, \end{cases}$$

and it is a first rate challenge to prove this. The above results provide a probabilistic (but incomplete) justification for this claim, as follows. The set $\mathcal{D}_q$ is exactly the set of atoms of the μ-measure of the random variable $1 - \zeta(e)$, for $\zeta \in X$. These atoms presumably arise through an accumulation of edges e having the same value $Z^b_t(e)$. Such coalescences occur only at the times of firing of the processes $C(e)$; see (6.2). These Poisson processes have rate $q - 1$, indicating that coalescences are more frequent for larger q.

Next we make some remarks about uniqueness of infinite clusters. The Burton–Keane [**42**] result implies (see Theorem 3.3) the $\phi^b_{p,q}$-a.s. uniqueness of the infinite cluster, for $b \in \{0, 1\}$ and $0 \le p \le 1$. It is another matter to obtain such uniqueness *simultaneously for all values of p*. That is, we may ask whether or not

$$\mu(I^b_p = 1 \text{ for all } p \text{ and } b = 0, 1) = 1,$$

where $I_p^0(\zeta)$ (resp. $I_p^1(\zeta)$) is the number of infinite open clusters of $\pi_p\zeta$ (resp. $\pi^p\zeta$). Such matters have been considered by Alexander [**9**].

Finally, we describe the transition rules of the projected processes $L_{p,t}^0$ and $L_{p,t}^1$; it turns out that the transition mechanisms of these two chains differ in an interesting (but ultimately unimportant) regard. It is convenient to summarise the following discussion by writing down directly the infinitesimal generators of the two processes, and we do this next.

We begin with some notation. Let $e = \langle x, y\rangle \in \mathbb{E}$, and let $\mathcal{P}_e$ be (as after (6.3)) the set of all paths of Ł which join x to y but do not use the edge e. Let $\mathcal{Q}_e$ be the set of all pairs $\alpha = (\alpha_1, \alpha_2, \dots)$, $\beta = (\beta_1, \beta_2, \dots)$ of vertex-disjoint semi-infinite paths (where α_i and β_j are the vertices of these paths) with $\alpha_1 = x$ and $\beta_1 = y$; we require $\alpha_i \neq \beta_j$ for all i, j. Thus $\mathcal{Q}_e$ comprises pairs (α, β) of paths; we call an element (α, β) of $\mathcal{Q}_e$ *open* if all the edges of α and β are open.

For $b = 0, 1$, let G^b be the linear operator, with domain a suitable subset of $C(\Omega)$, given by

$$(7.9)\quad G^b f(\omega) = \sum_{e\in\mathbb{E}} \Big\{ p\big(f(\omega^e) - f(\omega)\big) + h^b(e,\omega)\big(f(\omega_e) - f(\omega)\big)\Big\}, \quad \omega \in \Omega,$$

where ω^e and ω_e are given in (6.15); here, $h^b(e, \omega)$ is defined by

$$(7.10)\qquad h^b(e,\omega) = (1-p)\Big\{1 + (q-1)1_{D^b(e)}(\omega)\Big\}, \quad \omega \in \Omega,$$

where

$$(7.11)\qquad D^0(e) = \{\text{no path in } \mathcal{P}_e \text{ is open}\},$$

$$(7.12)\qquad D^1(e) = \{\text{no element in } \mathcal{P}_e \cup \mathcal{Q}_e \text{ is open}\}.$$

Note that $G^b f$ is well defined for all cylinder functions f, since the infinite sum in (7.9) may then be written as a finite sum. However, $G^b f$ is not generally continuous when $q > 1$, even for cylinder functions f. For example, suppose $q > 1$, let f be the indicator function of the event that a given edge e is open, and let ω be a configuration satisfying

(a) $\omega(e) = 1$,

(b) no path in $\mathcal{P}_e$ is open, under ω,

(c) some pair (α, β) in $\mathcal{Q}_e$ is open, under ω.

Then

$$G^b f(\omega) = -h^b(e, \omega).$$

However, $h^b(e, \cdot)$ is discontinuous at ω for $b = 0, 1$, since, for $b \in \{0, 1\}$ and for every finite box Λ, there exists $\omega' \in \Omega$ agreeing with ω on $\mathbb{E}_\Lambda$ such that $h^b(e, \omega') \neq h^b(e, \omega)$. Perhaps such difficulties may be avoided by restricting the space Ω of configurations. With a little further care, one may see that the Markov transition functions of $L^0_{p,t}$ and $L^1_{p,t}$ are not Feller; see the notes at the end of this section.

In describing the transition rules of the processes $L^0_{p,t}$ and $L^1_{p,t}$, we shall make use of the following lemma, which is of use also in the proofs of Theorems 7.1 and 7.3. Recall the function $F(e,\nu)$ defined on $\mathbb{E}\times X$ by (6.3).

Lemma 7.5. *Let $e\in\mathbb{E}$, $\nu\in X$, and let $(\nu_\Lambda)_\Lambda$ be a family of elements of X indexed by finite boxes Λ.*
(a) *If $\nu_\Lambda\uparrow\nu$ as $\Lambda\to\mathbb{Z}^d$, then*

$$F(e,\nu_\Lambda)\uparrow F(e,\nu). \tag{7.13}$$

(b) *If $\nu_\Lambda\in X^1_\Lambda$ and $\nu_\Lambda\downarrow\nu$ as $\Lambda\to\mathbb{Z}^d$, then*

$$F(e,\nu_\Lambda)\downarrow G(e,\nu) \tag{7.14}$$

where

$$G(e,\nu)=\sup_{\pi\in\mathcal{P}_e\cup\mathcal{Q}_e}\inf_{f\in\pi}\nu(f). \tag{7.15}$$

Note that, in the definition (7.15) of $G(e,\nu)$, $\mathcal{P}_e$ contains certain paths π, and $\mathcal{Q}_e$ contains certain pairs $\pi=(\alpha,\beta)$ of paths; for $\pi=(\alpha,\beta)\in\mathcal{Q}_e$, the infimum in (7.15) is over all edges f lying either in α or in β.

Consider the process Z^0_t. Since Z^0_t is the increasing limit of $Z^{(0,0)}_{\Lambda,t}$ as $\Lambda\to\mathbb{Z}^d$, we have from the definition (6.5) of π_p that

$$L^0_{p,t}=\lim_{\Lambda\uparrow\mathbb{Z}^d}\pi_p Z^{(0,0)}_{\Lambda,t}. \tag{7.16}$$

Assume that $Z^{(0,0)}_{\Lambda,t}=\zeta_\Lambda$ for each Λ, and $\zeta=\lim_{\Lambda\uparrow\mathbb{Z}^d}\zeta_\Lambda$, so that

$$L^0_{p,t}=\pi_p\zeta=\lim_{\Lambda\uparrow\mathbb{Z}^d}\pi_p\zeta_\Lambda. \tag{7.17}$$

Fix an edge $e\in\mathbb{E}$, and assume first that ζ is such that $\pi_p\zeta(e)=0$. At what rate does the state of e change from 0 to 1? Examining the transitions of the process $Z^0_{\Lambda,\cdot}$ (see (6.1)–(6.3)), we see that this occurs at the next firing of the process $A(e)$ that results in an associated α_m satisfying $\alpha_m>1-p$; the intensity of this transition is p, as in the proof of Theorem 6.1. Assume next that ζ is such that $\pi_p\zeta(e)=1$, and consider the intensity at which e assumes the state 0. Returning to (6.2), we see as in the proof of Theorem 6.1 that there are two independent sources of such a transition, namely the two processes $B(e)$ and $C(e)$. The process $B(e)$ fires at rate 1, and produces such a transition with probability

$P(\beta_m \le 1-p) = 1-p$; the associated effective intensity is $1-p$. The process $C(e)$ fires at rate $q-1$, and produces such a transition with probability

$$\begin{cases} 0 & \text{if } \lim_{\Lambda} F(e, \nu_\Lambda) > 1-p, \\ 1-p & \text{if } \lim_{\Lambda} F(e, \nu_\Lambda) \le 1-p, \end{cases} \tag{7.18}$$

where $\nu_\Lambda = Z^0_{\Lambda,T-}$ and T is the time of the firing in question of $C(e)$. Now $\nu = \lim_{\Lambda \to \mathbb{Z}^d} \nu_\Lambda$ is an increasing limit, whereby $F(e, \nu_\Lambda) \uparrow F(e, \nu)$ by Lemma 7.5(a). We have therefore that $\lim_\Lambda F(e, \nu_\Lambda) \le 1-p$ if and only if $F(e, \nu) \le 1-p$, which is equivalent to the statement $\pi_p \nu \in D^0(e)$, by (6.3), (6.5), and (7.11). In conclusion, the state of e flips from 1 to 0 at rate

$$\begin{cases} (1-p) & \text{if } \nu \notin D^0(e) \\ (1-p) + (q-1)(1-p) & \text{if } \nu \in D^0(e), \end{cases} \tag{7.19}$$

in agreement with (7.9) with $b = 0$.

We turn next to the process $L^1_{p,t}$. This time, Z^1_t is the decreasing limit of $Z^1_{\Lambda,t}$ as $\Lambda \to \mathbb{Z}^d$, and

$$L^1_{p,t} = \lim_{\Lambda \to \mathbb{Z}^d} \pi^p Z^1_{\Lambda,t} \tag{7.20}$$

as in (7.16); we have used the definition (6.4) of π^p here, noting that the corresponding statement with π_p (in place of π^p) fails in general. We now follow the above argument step by step, noting that increasing limits are replaced by decreasing limits, π_p by π^p, $F(e, \nu)$ by $G(e, \nu)$ (defined in (7.15)), and $D^0(e)$ by $D^1(e)$. Our conclusion is in agreement with (7.9) with $b = 1$.

Next appear the proofs, beginning with Lemma 7.5.

Proof of Lemma 7.5. (a) Suppose $\nu_\Lambda \uparrow \nu$. Certainly $F(e, \nu_\Lambda)$ is non-decreasing in Λ, whence the limit

$$\lambda = \lim_{\Lambda \to \mathbb{Z}^d} F(e, \nu_\Lambda)$$

exists and satisfies $\lambda \le F(e, \nu)$. Now, for $x \in (0, 1)$, we have that $\lambda \le x$ if and only if $F(e, \nu_\Lambda) \le x$ for all Λ. By (6.3), this occurs if and only if

$$\forall \pi \in \mathcal{P}_e,\ \forall \Lambda,\ \exists f \in \pi \text{ with } \nu_\Lambda(f) \le x.$$

Since all paths in $\mathcal{P}_e$ are finite, this implies

$$\forall \pi \in \mathcal{P}_e,\ \exists f \in \pi \text{ with } \nu(f) \le x,$$

which implies in turn that $F(e, \nu) \le x$. Therefore $F(e, \nu) \le \lambda$.

(b) Suppose $\nu_\Lambda \in X^1_\Lambda$ and $\nu_\Lambda \downarrow \nu$. First we prove that the decreasing limit $\lambda = \lim_\Lambda F(e, \nu_\Lambda)$ satisfies

$$\lambda \le G(e, \nu). \tag{7.21}$$

Let $x \in (0, 1)$, and suppose $G(e, \nu) < x$; we shall deduce that $\lambda < x$, thus obtaining (7.21). Write $e = \langle u, v \rangle$, and call a finite set S of edges of Ł a *cutset* (for e) if

(i) $e \notin S$,

(ii) every path in $\mathcal{P}_e$ contains at least one edge of S,

(iii) S is minimal with the two properties above, in the sense that no strict subset of S satisfies (i) and (ii).

We write $G(e, \nu) = \max\{A, B\}$ where

$$A = \sup_{\pi \in \mathcal{P}_e} \min_{f \in \pi} \nu(f), \qquad B = \sup_{\pi \in \mathcal{Q}_e} \inf_{f \in \pi} \nu(f).$$

Since $G(e, \nu) < x$, we have that $A, B < x$, which implies that there exists a cutset S with $\nu(f) < x$ for all $f \in S$. To see this, argue as follows. For $w \in \mathbb{Z}^d$, let $C_w(\nu)$ be the set of vertices of Ł that are connected to the vertex w by paths π of Ł satisfying: π does not contain the edge e, and every edge f of π satisfies $\nu(f) \geq x$. If $u \in C_v(\nu)$, then there exists $\pi \in \mathcal{P}_e$ with $\nu(f) \geq x$ for all $f \in \pi$, which contradicts the fact that $A < x$. Therefore $u \notin C_v(\nu)$. Furthermore either $C_u(\nu)$ or $C_v(\nu)$ (or both) is finite, since if both were infinite then there would exist $\pi = (\alpha, \beta) \in \mathcal{Q}_e$ with $\nu(f) \geq x$ for all f in α and β, thereby contradicting the fact that $B < x$. Suppose without loss of generality that $C_u(\nu)$ is finite, and let R be the subset of $\mathbb{E} \backslash \{e\}$ containing all edges g having exactly one vertex in $C_u(\nu)$. Certainly $\nu(g) < x$ for all $g \in R$, and additionally every path in $\mathcal{P}_e$ contains some edge of R. However R may fail to be minimal with this property, in which case we replace R by a subset S which is minimal; S is the required cutset.

We have that $\nu(f) < x$ for all $f \in S$, implying (since S is finite) that

$$\text{for all large } \Lambda,\ \nu_\Lambda(f) < x \text{ for all } f \in S,$$

and therefore (using the finiteness of S again)

$$\text{for all large } \Lambda,\ F(e, \nu_\Lambda) < x,$$

implying that $\lambda < x$ as required for (7.21).

Finally we prove that

$$\lambda \geq G(e, \nu), \tag{7.22}$$

and we achieve this by proving that $\lambda \geq A$ and $\lambda \geq B$, separately. That $\lambda \geq A$ is an immediate consequence of the fact that $\nu_\Lambda \geq \nu$, so we turn towards the inequality $\lambda \geq B$. For $\pi = (\alpha, \beta) \in \mathcal{Q}_e$, where α has endpoint u, and β has endpoint v, let α_Λ (respectively β_Λ) denote the initial segment of α (resp. β) joining u (resp. v) to the earliest vertex w_1 of α (resp. w_2, of β) lying in $\partial\Lambda$. Since $w_1, w_2 \in \partial\Lambda$ and $w_1 \neq w_2$, there exists a path γ joining w_1 to w_2 and using no other vertex of Λ. We denote by π' the path comprising α_Λ, followed by γ, followed by β_Λ taken in reverse order; note that $\pi' \in \mathcal{P}_e$, and denote by $\mathcal{P}_{e,\Lambda}$ the set of all $\pi' \in \mathcal{P}_e$ obtainable in this way from any $\pi = (\alpha, \beta) \in \mathcal{Q}_e$. Now

$$\begin{aligned}
F(e, \nu_\Lambda) &\geq \sup_{\pi' \in \mathcal{P}_{e,\Lambda}} \min_{f \in \pi'} \nu_\Lambda(f) && \text{since } \mathcal{P}_{e,\Lambda} \subseteq \mathcal{P}_e \\
&= \sup_{\pi' \in \mathcal{P}_{e,\Lambda}} \min_{f \in \pi' \cap \mathbb{E}_\Lambda} \nu_\Lambda(f) && \text{since } \nu_\Lambda(f) = 1 \text{ for } f \notin \mathbb{E}_\Lambda \\
&\geq \sup_{\pi' \in \mathcal{P}_{e,\Lambda}} \min_{f \in \pi' \cap \mathbb{E}_\Lambda} \nu(f) && \text{since } \nu_\Lambda \geq \nu \\
&= \sup_{\pi \in \mathcal{Q}_e} \min_{f \in \pi \cap \mathbb{E}_\Lambda} \nu(f) \\
&\geq \sup_{\pi \in \mathcal{Q}_e} \inf_{f \in \pi} \nu(f) = B,
\end{aligned}$$

where we have used the fact that every $\pi' \in \mathcal{P}_{e,\Lambda}$ arises from some $\pi \in \mathcal{Q}_e$. Inequality (7.22) follows. □

Proof of Theorem 7.1. The transitions of the process $(Z_t^b : t \geq 0)$ are given in terms of families of independent doubly-stochastic Poisson processes. In order that Z_t^b be a Markov process, it suffices therefore to prove the following:

(a) sample paths lie in $D(X)$,

(b) the distribution of $(Z_{s+t}^b : t \geq 0)$, given $(Z_u^b : 0 \leq u \leq s)$, depends only on Z_s^b.

First we prove (a). Let F be a finite subset of $\mathbb{E}$, let $t > 0$, and let

$$\begin{aligned}
S &= \sup\{A_m(e), B_r(e), C_s(e) < t : e \in F,\ m, r, s \geq 1\}, \\
T &= \inf\{A_m(e), B_r(e), C_s(e) \geq t : e \in F,\ m, r, s \geq 1\}.
\end{aligned}$$

Since, for each edge e, the processes $A(e)$, $B(e)$, $C(e)$ have only finitely many arrivals in any finite time-interval, we have that $S < t \leq T$. Now

$$Z_{\Lambda,s}^b(e) = Z_{\Lambda,S}^b(e) \quad \text{for} \quad S \leq s < T,\ e \in F. \tag{7.23}$$

Therefore $Z_s^b(e) = Z_S^b(e)$ for $s \in [S, t)$, whence the limit $Z_{t-}^b(e)$ exists for $e \in F$.

If $T > t$, then $Z_t^b(e) = Z_{t+}^b(e)$ for $e \in F$ by (7.23), whence $Z_\cdot^b$ is right-continuous at t. If $T = t$, then $Z_{\Lambda,s}^b(e) = Z_{\Lambda,t}^b(e)$ for $e \in F$ and $t \leq s < U$ where

$$U = \inf\{A_m(e), B_r(e), C_s(e) > t : e \in F,\ m, r, s \geq 1\},$$

implying right-continuity as before.

Next we prove (b). We have that $Z_{s+t}^b = \lim_{\Lambda \to \mathbb{Z}^d} Z_{\Lambda,s+t}^b$, where the processes $Z_{\Lambda,s+t}^b$ are given in terms of a graphical representation of compound Poisson processes. Therefore, conditional on $(Z_{\Lambda,u}^b,\ Z_u^b : 0 \leq u \leq s,\ \Lambda \subseteq \mathbb{Z}^d)$, the process $(Z_{s+t}^b : t \geq 0)$ has law which depends only on the family $(Z_{\Lambda,s}^b : \Lambda \subseteq \mathbb{Z}^d)$

indexed by finite boxes Λ. Write $\zeta_\Lambda = Z^b_{\Lambda,s}$ and $\zeta = \lim_{\Lambda\to\mathbb{Z}^d} \zeta_\Lambda = Z^b_s$. We need to show that the (conditional) law of Z^b_{s+t} does not depend on the family (ζ_Λ) but only on its limit ζ. To achieve this, we shall use Lemma 6.4(b).

First we introduce one more piece of notation. Let $s, t \geq 0$ and $\nu \in X$. Denote by $Y^{(\nu,b)}_{\Lambda,s+t}$ the state (in X^b_Λ) at time $s+t$ obtained from the evolution rules (6.1)–(6.3), starting at time s in state $(\nu, b) = (\nu, b)_\Lambda$.

Suppose that $b = 0$, so that $\zeta_\Lambda \uparrow \zeta$ as $\Lambda \to \mathbb{Z}^d$. Let $\epsilon > 0$ and let Δ be a finite box. There exists a box Λ' such that $\Lambda' \supseteq \Delta$ and

$$\zeta(e) - \epsilon \leq \zeta_\Lambda(e) \leq \zeta(e) \quad \text{for all } e \in \mathbb{E}_\Delta, \text{ if } \Lambda \supseteq \Lambda'.$$

It follows by Lemma 6.4(b) that

$$Y^{(\zeta,b)}_{\Delta,s+t} - \epsilon \leq Y^{(\zeta_\Lambda,b)}_{\Delta,s+t} \leq Y^{(\zeta_\Lambda,b)}_{\Lambda,s+t} \leq Y^{(\zeta,b)}_{\Lambda,s+t}, \quad \text{if } \Lambda \supseteq \Lambda'.$$

Use the fact that $Y^{(\zeta_\Lambda,b)}_{\Lambda,s+t} = Z^b_{\Lambda,s+t}$, and pass to the limits as $\Lambda \to \mathbb{Z}^d$, $\Delta \to \mathbb{Z}^d$, $\epsilon \downarrow 0$, to obtain that

$$\lim_{\Lambda\to\mathbb{Z}^d} Y^{(\zeta,b)}_{\Lambda,s+t} = Z^b_{s+t}, \tag{7.24}$$

implying as required that Z^b_{s+t} depends on ζ but not further on the family (ζ_Λ). The same argument is valid when $b = 1$, with the above inequalities reversed and the sign of ϵ changed.

The Markov transition function associated with the process Z^b_t is the family $(Q^b_{s,t} : 0 \leq s \leq t)$ given by

$$Q^b_{s,t}(\zeta, A) = P\big(Z^b_{s+t} \in A \mid Z^b_s = \zeta\big), \quad \zeta \in X,\ A \in \mathcal{B}.$$

In the light of the remarks above and particularly (7.24), we have that

$$Q^b_{s,t}(\zeta, A) = Q^b_{0,t-s}(\zeta, A),$$

and that

$$Q^b_{0,t-s}(\zeta, A) = P(Z^{(\zeta,b)}_{t-s} \in A) = P^b_{t-s}(\zeta, A)$$

as required. □

Proof of Theorem 7.2. We have from Lemma 6.2 that the limits ψ^b, given by

$$\psi^b(A) = \lim_{t\to\infty} P(Z^b_t \in A), \quad b = 0, 1,$$

exist for any increasing event A. Therefore Z^0_t and Z^1_t converge weakly as $t \to \infty$. It therefore suffices to show that

$$Z^1_t - Z^0_t \Rightarrow 0 \quad \text{as} \quad t \to \infty.$$

Since we are working with the product topology on X, it will be enough to show that, for all $\epsilon > 0$ and all finite subsets F of $\mathbb{E}$,

$$P\Big(|Z_t^1(f) - Z_t^0(f)| > \epsilon \text{ for some } f \in F\Big) \to 0 \quad \text{as } t \to \infty. \tag{7.25}$$

Let $\mathcal{D} = \mathcal{D}_q$ be as in Theorem 4.1, and let $\epsilon > 0$. Pick a finite subset $\mathcal{E}$ of $\overline{\mathcal{D}} = (0,1) \setminus \mathcal{D}$ such that every interval of the form $(x, x+\epsilon)$ contains some point of $\mathcal{E}$, as x ranges over $[0, 1-\epsilon)$; recall from Theorem 4.2 that

$$\phi_{p,q}^0 = \phi_{p,q}^1 \quad \text{if} \quad p \in \mathcal{E}. \tag{7.26}$$

We have that, for $f \in \mathbb{E}$,

$$\begin{aligned}
P(|Z_t^1(f) - Z_t^0(f)| > \epsilon) &\le \sum_{p \in \mathcal{E}} P\big(Z_t^0(f) < 1 - p \le Z_t^1(f)\big) \\
&\le \sum_{p \in \mathcal{E}} P\big(Z_{\Lambda,t}^0(f) < 1 - p \le Z_{\Lambda,t}^1(f)\big) && \text{for all boxes } \Lambda \\
&\to \sum_{p \in \mathcal{E}} \big\{\phi_{\Lambda,p,q}^1(J_f) - \phi_{\Lambda,p,q}^0(J_f)\big\} && \text{as } t \to \infty \\
&\to \sum_{p \in \mathcal{E}} \big\{\phi_{p,q}^1(J_f) - \phi_{p,q}^0(J_f)\big\} && \text{as } \Lambda \to \mathbb{Z}^d \\
&= 0 && \text{by (7.26)}, \quad \blacksquare
\end{aligned}$$

where $J_f = \{\omega(f) = 1\}$. Equation (7.25) follows since F is finite.

That the limit measure μ is translation-invariant is a consequence of (for example) Theorem 7.3 and the fact that $\phi_{p,q}^0$ and $\phi_{p,q}^1$ are translation-invariant (see Theorem 3.1). □

Proof of Theorem 7.3. (a) That the projected processes $(L_{p,t}^b : t \ge 0)$, $b = 0, 1$, are Markovian follows from Theorem 7.1 and the discussion after Lemma 7.5.

Let A be an increasing event in $\mathcal{F}$. Using Lemma 6.2, we have that the limits

$$\psi_p^b(A) = \lim_{t \to \infty} P(L_{p,t}^b \in A)$$

exist for $b = 0, 1$. Since $L_{p,t}^0 \le L_{p,t}^1$, it follows that

$$\psi_p^0(A) \le \psi_p^1(A) \quad \text{for increasing } A \in \mathcal{F}. \tag{7.27}$$

Assume now that A is an increasing event defined in terms of the edges in the finite subset F of $\mathbf{E}$. Then

$$\psi_p^0(A) = \lim_{t \to \infty} P(L_{p,t}^0 \in A) \tag{7.28}$$

$$\begin{aligned} &\geq \lim_{t\to\infty} P(\pi_p Z^0_{\Lambda,t} \in A) && \text{since } L^0_{p,t} \geq \pi_p Z^0_{\Lambda,t} \\ &= \phi^0_{\Lambda,p,q}(A) && \text{by Theorem 6.1} \\ &\to \phi^0_{p,q}(A) && \text{as } \Lambda \to \mathbb{Z}^d, \end{aligned}$$

and similarly

$$\psi^1_p(A) \leq \phi^1_{p,q}(A). \tag{7.29}$$

Combining (7.27)–(7.29), we deduce that

$$\phi^0_{p,q}(A) = \psi^0_p(A) = \psi^1_p(A) = \phi^1_{p,q}(A) \quad \text{if } p \notin \mathcal{D}_q,$$

where $\mathcal{D}_q$ is given in Theorem 4.1 (see also Theorem 4.2). This proves (7.7) whenever $p \notin \mathcal{D}_q$, since $\mathcal{F}$ is generated by the increasing finite-dimensional cylinders.

In order to show that

$$\phi^0_{p,q}(A) = \psi^0_p(A), \quad \phi^1_{p,q}(A) = \psi^1_p(A),$$

for *all* p and any such event A, it suffices to show that $\psi^0_p(A)$ is left-continuous in p, and $\psi^1_p(A)$ is right-continuous (the conclusion will then follow by Proposition 4.4). We confine ourselves here to the case of $\psi^0_p(A)$, since the other case is exactly similar.

Fix $p \in (0,1)$, and let A be an increasing finite-dimensional event of $\mathcal{F}$, defined in terms of the edges in the finite set F. Let

$$B_p = \{\zeta \in X : \pi_p\zeta \in A\}, \quad C_p = \{\zeta \in X : \pi^p\zeta \in A\}$$

be the corresponding events in $\mathcal{B}$, and note, from (6.4)–(6.5), that B_p is increasing and open, and that C_p is increasing and closed. Furthermore, $C_{p-\epsilon} \subseteq B_p$ if $\epsilon > 0$, and $B_p \setminus C_{p-\epsilon} \to \varnothing$ as $\epsilon \downarrow 0$. We have by stochastic monotonicity that $\lim_{t\to\infty} P(Z^0_t \in B_p)$ exists, and by weak convergence (see Theorem 7.2) that

$$\lim_{t\to\infty} P(Z^0_t \in B_p) \geq \mu(B_p).$$

We claim further that $P(Z^0_t \in B_p) \leq \mu(B_p)$ for all t, whence

$$P(Z^0_t \in B_p) \to \mu(B_p) \quad \text{as } t \to \infty. \tag{7.30}$$

To see the claim, suppose $P(Z^0_T \in B_p) > \mu(B_p) + \eta$ for some T and $\eta > 0$. Then $P(Z^0_t \in C_{p-\epsilon}) > \mu(C_{p-\epsilon}) + \frac{1}{2}\eta$ for some $\epsilon > 0$ and for all $t \geq T$. This contradicts the fact that $Z^0_t \Rightarrow \mu$, since $C_{p-\epsilon}$ is closed.

Now, for $h > 0$,

$$\begin{aligned}\psi_p^0(A) - \psi_{p-h}^0(A) &= \lim_{t\to\infty} \Big\{ P(Z_t^0 \in B_p) - P(Z_t^0 \in B_{p-h}) \Big\} \\ &= \mu(B_p \backslash B_{p-h}) \qquad \text{by (7.30).}\end{aligned}$$

However $B_p \backslash B_{p-h} \to \varnothing$ as $h \downarrow 0$ since B_p and B_{p-h} are open; hence $\psi_{p-h}^0(A) \to \psi_p^0(A)$ as $h \downarrow 0$.

In the corresponding argument for $\psi_p^1(A)$, the set B_p is replaced by the increasing closed event C_p, and the difference $B_p \backslash B_{p-h}$ is replaced by $C_{p+h} \backslash C_p$.

Finally we prove that $L_{p,t}^0$ is reversible; the argument is similar for $L_{p,t}^1$. Let f and g be increasing cylinder functions mapping Ω to $\mathbb{R}$, and let $U_{\Lambda,t}^0$ (resp. U_t^0) be the transition semigroup of the process $\pi_p Z_{\Lambda,t}^0$ (resp. $L_t^0 = \pi_p Z_t^0$). If $\Lambda \subseteq \Delta$ then

$$f(\eta) U_{\Lambda,t}^0 g(\eta) \le f(\eta) U_{\Delta,t}^0 g(\eta) \le f(\eta) U_t^0 g(\eta), \quad \eta \in \Omega,$$

by Lemmas 6.2 and 6.4. Therefore

$$\phi_{\Delta,p,q}^0\Big(f(\eta) U_{\Lambda,t}^0 g(\eta)\Big) \le \phi_{\Delta,p,q}^0\Big(f(\eta) U_{\Delta,t}^0 g(\eta)\Big) \le \phi_{p,q}^0\Big(f(\eta) U_t^0 g(\eta)\Big) \quad \text{if } \Lambda \subseteq \Delta,$$

since $\phi_{\Delta,p,q}^0 \le \phi_{p,q}^0$. Take the limits as $\Delta \to \mathbb{Z}^d$ and $\Lambda \to \mathbb{Z}^d$, and use the monotone convergence theorem to deduce that

$$\text{(7.31)} \qquad \phi_{\Delta,p,q}^0\Big(f(\eta) U_{\Delta,t}^0 g(\eta)\Big) \to \phi_{p,q}^0\big(f(\eta) U_t^0 g(\eta)\big) \quad \text{as } \Delta \to \mathbb{Z}^d.$$

The left side of (7.31) is unchanged when f and g are exchanged, by the reversibility of $\pi_p Z_{\Delta,t}^0$ (see Theorem 6.1). Therefore the right side of (7.31) is unchanged by this exchange, implying the required reversibility (see [**144**, p. 91]).

(b) It suffices to prove (7.8) for increasing finite-dimensional events A, since such events generate $\mathcal{F}$. For such A, (7.8) follows from (7.30) in the case of $\phi_{p,q}^0$, and similarly for $\phi_{p,q}^1$. □

Proof of Proposition 7.4. This is an immediate consequence of Theorem 7.3(b). □

Proof of Theorem 5.2(a). This was deferred from Section 5. We follow the argument of [**23**] as reported in [**92**]. For $p \in (0, 1)$ and $\zeta \in X$, we call an edge e *p-open* if $\pi_p \zeta(e) = 1$, which is to say that $\zeta(e) > 1 - p$. Let $C_p = C_p(\zeta)$ be the p-open cluster of Ł containing the origin, and note that $C_{p'} \subseteq C_p$ if $p' \le p$.

The function θ^0 is defined by (5.1) in terms of the measure $\phi_{p,q}^0$. In the light of Theorem 7.3(b), we have that

$$\theta^0(p, q) = \mu(|C_p| = \infty),$$

where μ is given in Theorem 7.2. Therefore

(7.32)
$$\begin{aligned}\theta^0(p,q) - \theta^0(p-,q) &= \lim_{p'\uparrow p} \mu\big(|C_p| = \infty,\ |C_{p'}| < \infty\big)\\ &= \mu\big(|C_p| = \infty,\ |C_{p'}| < \infty \text{ for all } p' < p\big).\end{aligned}$$

Let $p > p_c(q)$ and suppose $|C_p| = \infty$. If $p_c(q) < \alpha < p$, there exists a.s. an α-open infinite cluster I_α, and furthermore I_α is a.s. a subgraph of C_p, since otherwise there would exist at least two infinite p-open clusters (an event having probability 0, by Theorem 3.3). It follows that there exists a p-open path π joining the origin to some vertex of I_α. Such a path π has finite length and each edge e in π satisfies $\zeta(e) > 1 - p$; therefore $\beta = \min\{\zeta(e) : e \in \pi\}$ satisfies $\beta > 1 - p$. If p' satisfies $p' \ge \alpha$ and $1 - \beta < p' < p$ then there exists a p'-open path joining the origin to some vertex of I_α, so that $|C_{p'}| = \infty$. However $p' < p$, implying that the event on the right-hand side of (7.32) has probability zero, as required. □

Proof of non-Feller property. Finally we show (as promised before Lemma 7.5) that the processes $L^b_{p,t}$ are not Feller. For simplicity we take $d = 2$ and $b = 0$; a similar argument is valid for $d > 2$ and/or $b = 1$. Take e to be the edge with endpoints $(0,0)$ and $(1,0)$, and let f be the indicator function of the event that the edge e is open. We shall show that the function $U^0_s f$ is not continuous for sufficiently small positive values of s, where U^0_s is the transition semigroup associated with $L^0_{p,t}$. Let V be the set of vertices $x = (x_1, x_2)$ satisfying

$$\text{either } x_1 \ge |x_2| + 1, \quad \text{or } -x_1 \ge |x_2|,$$

and let $\mathbb{E}_V$ be the set of edges having both endpoints in V; note that $e \in \mathbb{E}_V$. Fix a positive integer n, and let Δ be the box $[-n,n]^2$. Let $\omega^0,\ \omega^1\ (\in \Omega)$ be the configurations given by

$$\omega^b(f) = \begin{cases} 1 & \text{if } f \in \mathbb{E}_\Delta \cap \mathbb{E}_V,\\ 0 & \text{if } f \in \mathbb{E}_\Delta \setminus \mathbb{E}_V,\\ b & \text{otherwise,}\end{cases} \qquad \text{where } b = 0, 1.$$

Note that ω^0 and ω^1 depend on n, and also that $\omega^1 \notin D^0(e)$ but $\omega^0 \in D^0(e)$. Taking ω^0 and ω^1 as initial configurations, we claim that this property persists with strictly positive probability for a non-zero time-interval, under the evolution according to the appropriate semigroup U^0_s.

For $b = 0, 1$, let $K^b_{\Lambda,t}$ be the process $\pi_p Z^{(\zeta^b,0)}_{\Lambda,t}$ for some ζ^b satisfying $\omega^b = \pi_p \zeta^b$; the value of ζ^b is otherwise immaterial. We write $K^b_t = \lim_{\Lambda\to\mathbb{Z}^d} K^b_{\Lambda,t}$, which limit exists by the usual monotonicity. We claim that there exist $\epsilon,\ \eta\ (> 0)$, not depending on the value of n, such that

(7.33)
$$P\big(K^1_\eta(e) = 1,\ K^0_\eta(e) = 0\big) > \epsilon.$$

This implies that

$$P(K_\eta^1(e) = 1) - P(K_\eta^0(e) = 1) > \epsilon,$$

irrespective of the value of n, and therefore that the semigroup U_s^0 is not Feller. In order to prove (7.33), we use a percolation argument. Let $\eta > 0$. For each edge f, we set $X_f = 0$ if none of the processes $A(f)$, $B(f)$, $C(f)$ have fired during the time-interval $[0, \eta]$, and $X_f = 1$ otherwise. Since the sum of the intensities of these three processes is $q+1$, we have that $\{X_f : f \in \mathbb{E}\}$ is a family of independent Bernoulli variables with common parameter $1 - e^{-(q+1)\eta}$. Choose η sufficiently small such that

$$1 - e^{-(q+1)\eta} < \tfrac{1}{4},$$

noting that $\frac{1}{4}$ is less than the critical probability of bond percolation on the square lattice (see [**92**]). Routine percolation arguments may now be used to obtain that there exists $\epsilon' > 0$ such that

$$P\Big(K_{\Lambda,\eta}^1 \notin D^0(e),\ K_{\Lambda,\eta}^0 \in D^0(e) \text{ for all } t \in [0, \eta] \,\Big|\, X_e = 0\Big) > \epsilon',$$

for all Λ containing $[-2n, 2n]^2$. Suppose that $A(e)$ and $B(e)$ do not fire during $[0, \eta]$, but that $C(e)$ does indeed fire once, with an associated value σ satisfying $\sigma < 1 - p$. At this time T of firing, the edge e is removed from the lower process $K_{\Lambda,T}^0$ but not from the upper process $K_{\Lambda,T}^1$, for all large Λ. Therefore

$$P\Big(K_{\Lambda,\eta}^1(e) = 1,\ K_{\Lambda,\eta}^0(e) = 0\Big) > \epsilon, \quad \text{for all } \Lambda \text{ containing } [-2n, 2n]^2,$$

with $\epsilon = \epsilon'(1-p)e^{-2\eta}\{(q-1)\eta e^{-(q-1)\eta}\}$. Now take the limit as $\Lambda \to \mathbb{Z}^d$ to obtain (7.33). $\square$

7.8 Simultaneous uniqueness

8

Flows in Poisson Graphs

8.1 Potts models and flows

For suitable values of the two variables of which the Tutte polynomial is a function, one obtains counts of colourings, forests, and flows, together with other combinatorial quantities, in addition to the random-cluster and Potts partition functions. The algebra of the Tutte polynomial may be used to obtain representations of the Potts correlation functions, which have in turn the potential to explain the decay of correlations in the two phases of an infinite-volume Potts measure. It is thus that many beautiful results have been derived in the Ising case, when $q = 2$, see [**2**, **3**, **7**]. The cases $q \in \{3, 4, \dots\}$, and more generally $q \in (1, \infty)$, remain largely unexplained. We summarise this methodology in this chapter, beginning with the definition of a flow on an directed graph.

Let $H = (W, F)$ be a finite graph with vertex set W and edge set F, and let q be an integer satisfying $2 \le q < \infty$. We permit H to have multiple edges, but the presence of loops is immaterial and we therefore exclude this. To each $e \in F$ we allocate a direction, turning H thus into a directed graph which we denote $\vec{H} = (W, \vec{F})$. When the edge $e = \langle u, v \rangle \in F$ receives the direction from u to v, we write $\vec{e} = [u, v\rangle$ for the corresponding directed edge[11], and we speak of u as the *tail* and v as the *head* of $\vec{e}$. It will turn out that the choices of directions are immaterial to the principal conclusions which follow. A function $f : \vec{F} \to \{0, 1, 2, \dots, q-1\}$ is called a *mod-q flow* on $\vec{H}$ if

$$\sum_{\substack{\vec{e} \in \vec{F}: \\ \vec{e} \text{ has head } w}} f(\vec{e}) - \sum_{\substack{\vec{e} \in \vec{F}: \\ \vec{e} \text{ has tail } w}} f(\vec{e}) = 0 \qquad \text{modulo } q, \quad \text{for all } w \in W, \tag{8.1}$$

8.flowdef which is to say that flow is conserved at every vertex. A mod-q flow f is called *non-zero* if $f(\vec{e}) \ne 0$ for all $\vec{e} \in \vec{F}$. We write $C_H(q)$ for the number of non-zero

[11] This is not a good notation since $\vec{H}$ may have multiple edges

mod-q flows on $\vec{H}$. It is the case that $C_H(q)$ does not depend on the directions allocated to the edges of H, [**186**]. The function $C_H(q)$, viewed as a function of q, is called the *flow polynomial* of H.

The flow polynomial of the graph $H = (W, F)$ is an evaluation of its Tutte polynomial. Recall from Section 2.7 the (Whitney) rank-generating function and the Tutte polynomial,

$$W_H(u, v) = \sum_{F' \subseteq F} u^{r(H')} v^{c(H')}, \qquad u, v \in \mathbb{R}, \tag{8.2}$$

$$T_H(u, v) = (u-1)^{|W|-1} W_H\big((u-1)^{-1}, v-1\big), \tag{8.3}$$

8.Tuttedef where $r(H') = |W| - k(H')$ is the *rank* of the subgraph $H' = (W, F')$, $c(H') = |F'| - |W| + k(H')$ is its *co-rank*, and $k(H')$ is the number of its connected components. Note that

$$W_H(u, v) = (u/v)^{|W|} \sum_{F' \subseteq F} v^{|F'|} (v/u)^{k(H')}, \qquad u, v \neq 0. \tag{8.4}$$

8.rank2 The flow polynomial of H is obtained from the two polynomials above by the relation

$$\begin{aligned} C_H(q) &= (-1)^{|F|} W_H(-1, -q) \\ &= (-1)^{|F|-|W|+1} T_H(0, 1-q), \qquad q \in \{2, 3, \dots\}. \end{aligned} \tag{8.5}$$

8.flowpodef See [**26**, **186**]. It is easily seen that $C_H(q)$ is indeed a polynomial in q. When the need for a different notation arises, we shall write $C(H; q)$ for $C_H(q)$, and similarly for other polynomials.

We return now to the random-cluster and Potts models on the finite graph $G = (V, E)$. It is convenient to allow a separate parameter for each edge of G, and thus we let $J = (J_e : e \in E)$ be a vector of non-negative numbers, and we let $\beta > 0$. We set

$$p_e = 1 - e^{-\beta J_e q}, \qquad e \in E. \tag{8.6}$$

8.pj

For $q \in \{2, 3, \dots\}$, the q-state Potts measure on the configuration space $\Sigma = \{1, 2, \dots, q\}^V$ is written as

$$\pi_{\beta J, q}(\sigma) = \frac{1}{Z_{\mathrm{P}}} \exp\bigg(\sum_{e \in E} \beta J_e (q \delta_e(\sigma) - 1) \bigg), \qquad \sigma \in \Sigma, \tag{8.7}$$

8.Pottsdef where, for $e = \langle x, y \rangle \in E$,

$$\delta_e(\sigma) = \delta_{\sigma_x, \sigma_y} = \begin{cases} 1 & \text{if } \sigma_x = \sigma_y, \\ 0 & \text{otherwise,} \end{cases}$$

and Z_{P} is the partition function

$$Z_{\mathrm{P}} = \sum_{\sigma\in\Sigma} \exp\biggl(\sum_{e\in E} \beta J_e(q\delta_e(\sigma) - 1)\biggr). \tag{8.8}$$

8.pottspart This differs slightly from (1.5)–(1.6) in that different edges e may have different interactions J_e, and these interactions have been 're-parametrised' by the factor q. The reasons for defining $\pi_{\beta J,q}$ thus will emerge in the calculations which follow.

The corresponding two-point correlation function is given as in (1.13) by

$$\tau_{\beta J,q}(x, y) = \pi_{\beta J,q}(\sigma_x = \sigma_y) - \frac{1}{q}, \qquad x, y \in V. \tag{8.9}$$

8.taudef We shall work often with the quantity $q\tau_{\beta J,q}(x, y) = \pi_{\beta J,q}(q\delta_{\sigma_x,\sigma_y} - 1)$, and, for ease of notation in the following, we write

$$\sigma(x, y) = q\tau_{\beta J,q}(x, y), \qquad x, y \in V, \tag{8.10}$$

8.sigdef thereby suppressing reference to the parameters βJ and q. Note that, in the Ising case when $q = 2$, $\sigma(x, y)$ is simply the mean of the product $\sigma_x\sigma_y$ of the Ising spins at x and at y; see (1.7).

From the graph $G = (V, E)$ we construct next a certain random graph. Let $\lambda = (\lambda_e : e \in E)$ be a family of numbers such that $0 \le \lambda_e < \infty$ for all e. Suppose that $\pi = (\pi(e) : e \in E)$ is a family of independent random variables such that $\pi(e)$ has the Poisson distribution with parameter λ_e. We now construct a random graph $G_\pi = (V, E_\pi)$ having vertex set V and, for each $e \in E$, having exactly $\pi(e)$ edges in parallel joining the endvertices of the edge e [the original edge e is itself removed]. We call G_π a *Poisson graph with intensity* λ, and we write $\mathbb{P}_\lambda$ and $\mathbb{E}_\lambda$ for the appropriate probability measure and expectation operator.

For $x, y \in V$, $x \ne y$, we denote by $G_\pi^{x,y}$ the graph obtained from G_π by adding an edge with endvertices x, y. [If x and y are already adjacent in G_π, we add exactly one further edge between them.] Potts-correlations and flows are related by the following theorem.

8.flowcorr **Theorem 8.11 [84].** *Let* $q \in \{2, 3, \dots\}$ *and* $\lambda_e = \beta J_e$. *Then*

$$\sigma(x, y) = \frac{\mathbb{E}_\lambda(C(G_\pi^{x,y}; q))}{\mathbb{E}_\lambda(C(G_\pi; q))} \qquad \textit{for all } x, y \in V,\ x \ne y. \tag{8.12}$$

8.flowcorr2

This formula takes an especially simple form when $q = 2$, since non-zero mod-2 flows necessarily take only the value 1. A finite graph $H = (W, F)$ is called *even* if the degree of every vertex w ($\in W$) is even. By (8.12), for any graph G',

$$\mathbb{E}_\lambda(C(G'; 2)) = \mathbb{P}_\lambda(G' \text{ is even}), \tag{8.13}$$

8.flowcorr3 see [**2**]. Observations of this sort have led when $q = 2$ to the so-called 'random-current' expansion for Ising models, thereby after some work [**2**, **3**, **7**] leading to proofs amongst other things of the exponential decay of correlations in the high-temperature regime. We shall return to the case $q = 2$ in Section 8.2.

Whereas Theorem 8.11 concerns Potts models only, there is a random-cluster generalisation. We restrict ourselves here to the situation in which every edge has the same parameter p, but we note that the result is easily generalised to allowing different parameters for each edge.

The final statement of the next theorem is a curiosity. Recall that $\phi_{G,p}$ denotes product measure on $\Omega = \{0, 1\}^E$ with density p.

8.flowconn **Theorem 8.14 [84].** *Let $0 \le p < 1$ and $q \in (0, \infty)$. Let $\lambda_e = \lambda$ for all $e \in E$, where $p = 1 - e^{-\lambda q}$. Then, for $x, y \in V$, $x \neq y$,*

(8.15)
$$\begin{aligned}(q-1)\phi_{G,p,q}(x \leftrightarrow y) &= \frac{\mathbb{E}_\lambda\big((-1)^{1+|E_\pi|}T(G_\pi^{x,y}; 0, 1-q)\big)}{\mathbb{E}_\lambda\big((-1)^{|E_\pi|}T(G_\pi; 0, 1-q)\big)} \\ &= \frac{\mathbb{E}_\lambda(C(G_\pi^{x,y}; q))}{\mathbb{E}_\lambda(C(G_\pi; q))} \qquad \textit{if } q \in \{2, 3, \dots\}.\end{aligned}$$

8.flowconn2 *Furthermore, if $q \in \{2, 3, \dots\}$,*

$$\phi_{G,p}(q^{k(\omega)}) = (1-p)^{|E|(q-2)/q} q^{|V|} \mathbb{E}_\lambda(C(G_\pi; q)). \tag{8.16}$$

8.compflow

When $q = 2$, equation (8.16) reduces to

$$\phi_{G,p}(2^{k(\omega)}) = 2^{|V|}\mathbb{P}_\lambda(G_\pi \text{ is even}). \tag{8.17}$$

8.compflow2 This may be simplified further. Let $\omega'(e)$ be $\pi(e)$ reduced modulo 2. It is easily seen that G_π is even if and only if $G_{\omega'}$ is even, and that the $\omega'(e)$, $e \in E$, are independent Bernoulli variables with

$$\mathbb{P}_\lambda(\omega'(e) = 1) = \tfrac{1}{2}(1 - e^{-2\lambda}) = \tfrac{1}{2}p.$$

Equation (8.16) becomes thus

$$\phi_{G,p}(2^{k(\omega)}) = 2^{|V|}\phi_{G,p/2}(\partial\omega' = \varnothing) \tag{8.18}$$

8.star where $\omega' \in \{0, 1\}^E$ is selected according to $\phi_{G,p/2}$, the set of *sources* $\partial\omega'$ is given by

$$\partial\omega' = \{x \in V : \deg_{\omega'}(x) \text{ is odd}\},$$

and $\deg_{\omega'}(x)$ is the number of open edges (in ω') which are incident to the vertex x.

We recall from the definition of E_π that

$$|E_\pi| = \sum_{e\in E} \pi(e).$$

why this? Too many recalls

Recalling (8.5) and Theorem 1.15, we may view Theorem 8.14 as a generalisation of Theorem 8.11 for constant J_e. Theorem 8.14 may be proved directly, but we shall derive it here from Theorem 8.11.

Proof of Theorem 8.11. Since the parameter β appears always with the factor J_e, we may without loss of generality take $\beta = 1$. Recall that $\pi = \{\pi(e) : e \in E\}$ is a collection of independent Poisson variables, used to construct the random graph $G_\pi = (V, E_\pi)$.

We begin with a calculation involving the Potts partition function Z_{P} given in (8.8). We write $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and shall consider vectors $m = (m_e : e \in E) \in \mathbb{Z}_+^E$. By Taylor's theorem,

$$\begin{aligned}
\text{(8.19)}\qquad \exp\Bigl(-\sum_{e\in E} J_e\Bigr) Z_{\mathrm{P}} &= \sum_{m\in\mathbb{Z}_+^E} \Biggl(\prod_{e\in E} \frac{J_e^{m_e}}{m_e!} e^{-J_e}\Biggr) \partial^m Z_{\mathrm{P}}\Big|_{J=0} \\
&= \mathbb{E}_\lambda \left(\partial^\pi Z_{\mathrm{P}}\Big|_{J=0}\right)
\end{aligned}$$

8.taylor where

$$\partial^m Z_{\mathrm{P}} = \Biggl(\prod_{e\in E} \frac{\partial^{m_e}}{\partial J_e^{m_e}}\Biggr) Z_{\mathrm{P}}, \qquad m \in \mathbb{Z}_+^E.$$

By (8.8), and similarly to the proof of Theorem 1.9(a),

(8.20)

$$\begin{aligned}
\partial^m Z_{\mathrm{P}}\Big|_{J=0} &= \sum_{\sigma\in\Sigma} \prod_{e\in E} (q\delta_e(\sigma) - 1)^{m_e} \\
&= \sum_{\sigma\in\Sigma} \prod_{e\in E_m} (q\delta_e(\sigma) - 1) \\
&= \sum_{\sigma\in\Sigma} \prod_{e\in E_m} \sum_{n_e\in\{0,1\}} [-\delta_{n_e,0} + \delta_{n_e,1} q\delta_e(\sigma)] \\
&= \sum_{n\in\{0,1\}^{E_m}} \sum_{\sigma\in\Sigma} (-1)^{|\{e:n_e=0\}|} q^{|\{e:n_e=1\}|} \Biggl(\prod_{e\in E_m} \delta_e(\sigma)^{n_e}\Biggr) \\
&= \sum_{n\in\{0,1\}^{E_m}} (-1)^{|\{e:n_e=0\}|} q^{|\{e:n_e=1\}|} q^{k(m,n)},
\end{aligned}$$

8.zpart where $k(m,n)$ is the number of connected components of the graph obtained from G_m after deletion of all edges e with $n_e = 0$. Thus, by (8.4),

$$\partial^m Z_{\mathrm{P}}\Big|_{J=0} = (-1)^{|E_m|} \sum_{n\in\{0,1\}^{E_m}} (-q)^{|\{e:n_e=1\}|} q^{k(m,n)} \tag{8.21}$$
$$= (-1)^{|E_m|} q^{|V|} W_{G_m}(-1,-q)$$
$$= q^{|V|} C(G_m; q).$$

8.zpart2 [Perhaps go directly to flows without using W_H.] Combining (8.19)–(8.21), we conclude that

$$\exp\Big(-\sum_{e\in E} J_e\Big) Z_{\mathrm{P}} = q^{|V|}\mathbb{E}_\lambda(C(G_\pi; q)). \tag{8.22}$$

8.partflow Equation (8.16) follows as in (1.11), recall (8.8) and (2.100).

Let $x, y \in V$ and $x \neq y$. We define the unordered pair $f = (x, y)$, and write $\delta_f(\sigma) = \delta_{\sigma_x,\sigma_y}$ for $\sigma \in \Sigma$. We have that

(8.23)
$$\sigma(x,y) = \pi_{\beta J,q}\big(q\delta_f(\sigma) - 1\big)$$
$$= \frac{1}{Z_{\mathrm{P}}} \sum_{\sigma\in\Sigma} (q\delta_f(\sigma) - 1) \exp\Big(\sum_{e\in E} \beta J_e(q\delta_e(\sigma) - 1)\Big).$$

8.partflow3 By an analysis parallel to (8.19)–(8.22), we find that

(8.24)
$$\exp\Big(-\sum_{e\in E} J_e\Big) \sum_{\sigma\in\Sigma} (q\delta_f(\sigma) - 1) \exp\Big(\sum_{e\in E} \beta J_e(q\delta_e(\sigma) - 1)\Big)$$
$$= q^{|V|}\mathbb{E}_\lambda(C(G_\pi^{x,y}; q)),$$

8.partflow2 and the claim follows by (8.22) and (8.23). □

Proof of Theorem 8.14. By (8.5) and Theorems 1.15 and 8.11, (8.15) holds for $q \in \{2, 3, \dots\}$. Since both sides are the ratios of polynomials in q of finite order, (8.15) is an identity in real q. □

8.2 Flows for the Ising model

Henceforth in this chapter *we assume that* $q = 2$. Let $H = (W, F)$ be a finite graph, and let $\deg_F(w)$ denote the degree of the vertex w ($\in W$). We call H an *even* graph if $\deg_F(w)$ is even for every $w \in W$. Let $\vec{H} = (V, \vec{F})$ be a directed graph obtained from H by assigning a direction to each edge in E. Each non-zero mod-2 flow on $\vec{H}$ may by definition take only the value 1. Since $1 = -1$ modulo 2, the number $C_H(2)$ of such flows is given exactly by

$$C_H(2) = \begin{cases} 1 & \text{if } H \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{8.25}$$

8.floweven Thus by Theorem 8.11, in the case of the Ising model on a finite graph $G = (V, E)$,

$$\sigma(x, y) = 2\tau_{\lambda,2}(x, y) = \frac{\mathbb{P}_\lambda(G_\pi^{x,y} \text{ is even})}{\mathbb{P}_\lambda(G_\pi \text{ is even})}, \qquad 0 \le \lambda < \infty. \tag{8.26}$$

8.isingeven The value of such a representation will become clear during the following discussion, whose development is based on material in [**2**, **143**, **179**]. In advance of this, we make a remark concerning (8.26). In deciding whether G_π or $G_\pi^{x,y}$ is an even graph, we need only know the numbers $\pi(e)$ when reduced modulo 2. That is, we can work with $\omega' \in \Omega = \{0, 1\}^E$ given by $\omega'(e) = \pi(e)$ mod 2. Since $\pi(e)$ has the Poisson distribution with parameter λ, $\omega'(e)$ has the Bernoulli distribution with parameter

$$p' = \mathbb{P}_\lambda(\pi(e) \text{ is odd}) = \tfrac{1}{2}(1 - e^{-2\lambda}).$$

We obtain thus from (8.26) that

$$\sigma(x, y) = \frac{\phi_{p'}(\partial\omega' = \{x, y\})}{\phi_{p'}(\partial\omega' = \varnothing)}, \tag{8.27}$$

8.isingperc where as before $\phi_{p'}$ denotes product measure on Ω with density p'. See also (8.17) and (8.18).

Let $G = (V, E)$ be as above. Let $M = (M_e : e \in E)$ be a sequence of disjoint finite sets indexed by E, and let $m_e = |M_e|$. As noted in the last section, the vector M may be used to construct a multigraph $G_m = (V, E_m)$ in which each $e \in E$ is replaced by m_e edges in parallel; we may take M_e to be the set of such edges. For $x, y \in V$, we write '$x \leftrightarrow y$ in m' if x and y lie in the same component of G_m. We define the set ∂M of *sources* of M by

$$\partial M = \left\{ v \in V : \sum_{e \sim v} m_e \text{ is odd} \right\}, \tag{8.28}$$

8.source1 where the sum is over all edges e incident to v. Thus, for example, G_m is even if and only if ∂M is empty. From the vector M we construct a vector $N = (N_e : e \in E)$

by deleting each member of each M_e with probability $\frac{1}{2}$, independently of all other elements. That is, we let B_i, $i \in \bigcup_e M_e$, be independent Bernoulli random variables with parameter $\frac{1}{2}$, and we set

$$N_e = \{i \in M_e : B_i = 1\}, \qquad e \in E.$$

We write $\mathbb{P}^M$ for the appropriate probability measure.

The following lemma is pivotal for the computations which follow.

8.switching **Theorem 8.29.** *Let M and m be as above. If $x, y \in V$ are such that $x \neq y$ and $x \leftrightarrow y$ in m then, for $A \subseteq V$,*

$$\mathbb{P}^M\big(\partial N = A,\ \partial(M \setminus N) = \{x, y\}\big) = \mathbb{P}^M\big(\partial N = A \vartriangle \{x, y\},\ \partial(M \setminus N) = \varnothing\big).$$

Proof. Take M_e to be the set of edges of G_m parallel to e, and assume that $x \leftrightarrow y$ in m. Fix $A \subseteq V$. Let $\mathcal{M}$ be the set of all vectors $n = (n_e : e \in E)$ with $n_e \subseteq M_e$ for all e. Let p be a path of G_m with endpoints x, y, and consider the map $\zeta : \mathcal{M} \to \mathcal{M}$ given by

$$\zeta(n) = n \vartriangle p.$$

The map ζ is one–one, and maps $\{n \in \mathcal{M} : \partial n = A,\ \partial(M \setminus n) = \{x, y\}\}$ to $\{n' \in \mathcal{M} : \partial n' = A \vartriangle \{x, y\},\ \partial(M \setminus n') = \varnothing\}$. Each member of $\mathcal{M}$ is equiprobable under $\mathbb{P}^M$, and the claim follows. □

Let $\lambda = (\lambda_e : e \in E)$ be a vector of non-negative reals. Recall from the last section the definition of a Poisson graph with parameter λ. The following is a fairly immediate corollary of the last theorem. Let $M = (M_e : e \in E)$ and $M' = (M'_e : e \in E)$ be vectors of mutually disjoint finite sets, and suppose that the random variables $m_e = |M_e|$, $m'_e = |M'_e|$, $e \in E$ are independent and such that, for each $e \in E$, m_e and m'_e have the Poisson distribution with parameter λ_e. We write $M \cup M' = (M_e \cup M'_e : e \in E)$, and $\mathbb{P}$ for the appropriate probability measure. The following lemma is based on the so-called switching lemma of [**2**].

8.switch2 **Corollary 8.30 (Switching lemma).** *If $x, y \in V$ are such that $x \neq y$ and $x \leftrightarrow y$ in $m + m'$ then, for $A \subseteq V$,*

$$\begin{aligned}\mathbb{P}\big(\partial M = \{x, y\},\ \partial M' = A \,\big|\, M \cup M'\big)& \\ = \mathbb{P}\big(\partial M = \varnothing,\ \partial M' = A \vartriangle \{x, y\} \,\big|\, M \cup M'\big)& \qquad \mathbb{P}\text{-a.s.}\end{aligned}$$

Proof. Conditional on the sets $M_e \cup M'_e$ for $e \in E$, the sets M_e are selected by the independent removal of each element with probability $\frac{1}{2}$. The claim now follows from Theorem 8.29. [Give more details.] □

We present two applications of Theorem 8.30 to the Ising model, as in [**2**]. For $m = (m_e : e \in E) \in \mathbb{Z}_+^E$, we define

$$\partial m = \Big\{v \in V : \sum_{e \sim v} m_e \text{ is odd}\Big\}, \tag{8.31}$$

8.source2 as in (8.28). In our study of the correlation functions $\tau_{\lambda,2}(x,y)$, we shall as before write

$$\sigma(x,y) = 2\tau_{\lambda,2}(x,y) = \pi_{\lambda,2}(2\delta_{\sigma_x,\sigma_y} - 1), \qquad x, y \in V,$$

thereby suppressing reference to λ. We have from (8.26) that

$$\sigma(x,y) = \frac{\mathbb{P}_\lambda(\partial\pi = \{x,y\})}{\mathbb{P}_\lambda(\partial\pi = \varnothing)}. \tag{8.32}$$

8.tausource Let $\mathbb{Q}_A$ denote the probability measure associated with π *conditional* on the event $\{\partial\pi = A\}$, that is,

$$\mathbb{Q}_A(E) = \mathbb{P}_\lambda(\pi \in E \mid \partial\pi = A).$$

We have need of two independent copies π_1, π_2 of π with potentially different conditionings, and thus we write $\mathbb{Q}_{A;B} = \mathbb{Q}_A \times \mathbb{Q}_B$.

8.appls **Theorem 8.33 [2].** *Let $x, y, z \in V$ be distinct vertices.*

(a) *We have that*

$$\sigma(x,y)^2 = \mathbb{Q}_{\varnothing;\varnothing}\big(x \leftrightarrow y \text{ in } \pi_1 + \pi_2\big).$$

(b) *We have that*

$$\sigma(x,y)\sigma(y,z) = \sigma(x,z)\mathbb{Q}_{\{x,z\};\varnothing}\big(x \leftrightarrow y \text{ in } \pi_1 + \pi_2\big).$$

Proof. (a) By (8.32) and Theorem 8.30,

$$\begin{aligned}
\sigma(x,y)^2 &= \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda(\partial\pi_1 = \{x,y\},\ \partial\pi_2 = \{x,y\})}{\mathbb{P}_\lambda(\partial\pi = \varnothing)^2} \\
&= \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda(\partial\pi_1 = \partial\pi_2 = \varnothing,\ x \leftrightarrow y \text{ in } \pi_1 + \pi_2)}{\mathbb{P}_\lambda(\partial\pi = \varnothing)^2} \\
&= \mathbb{Q}_{\varnothing;\varnothing}\big(x \leftrightarrow y \text{ in } \pi_1 + \pi_2\big).
\end{aligned}$$

(b) Similarly,

$$\begin{aligned}
&\sigma(x,y)\sigma(y,z) \\
&\quad = \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda\big(\partial\pi_1 = \{x,y\},\ \partial\pi_2 = \{y,z\}\big)}{\mathbb{P}_\lambda(\partial\pi = \varnothing)^2} \\
&\quad = \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda\big(\partial\pi_1 = \{x,z\},\ \partial\pi_2 = \varnothing,\ x \leftrightarrow y \text{ in } \pi_1 + \pi_2\big)}{\mathbb{P}_\lambda(\partial\pi = \varnothing)^2} \\
&\quad = \frac{\mathbb{P}_\lambda(\partial\pi_1 = \{x,z\})}{\mathbb{P}_\lambda(\partial\pi = \varnothing)} \cdot \mathbb{P}_\lambda \times \mathbb{P}_\lambda\Big(x \leftrightarrow y \text{ in } \pi_1 + \pi_2 \,\Big|\, \partial\pi_1 = \{x,z\},\ \partial\pi_2 = \varnothing\Big) \\
&\quad = \sigma(x,z)\mathbb{Q}_{\{x,z\};\varnothing}\big(x \leftrightarrow y \text{ in } \pi_1 + \pi_2\big),
\end{aligned}$$

as required. □

Theorem 8.33(b) leads to the important correlation inequality known as the Simon inequality. Let $x, z \in V$ be distinct vertices. A subset $W \subseteq V$ is said to *separate* x and z if $x, z \notin W$ and every path from x to z contains some vertex of W.

8.simon **Theorem 8.34 (Simon inequality[12]) [179].** *Let $x, z \in V$ be distinct vertices, and let W separate x and z. Then*

$$\sigma(x,z) \le \sum_{y \in W} \sigma(x,y)\sigma(y,z).$$

Proof. By Theorem 8.33(b),

$$\begin{aligned}\sum_{y \in W} \frac{\sigma(x,y)\sigma(y,z)}{\sigma(x,z)} &= \sum_{y \in W} \mathbb{Q}_{\{x,z\};\varnothing}\big(x \leftrightarrow y \text{ in } \pi_1 + \pi_2\big) \\ &= \mathbb{Q}_{\{x,z\};\varnothing}\big(\big|\{y \in W : x \leftrightarrow y \text{ in } \pi_1 + \pi_2\}\big|\big).\end{aligned}$$

Assume that the event $\partial\pi_1 = \{x, z\}$ occurs. On this event, $x \leftrightarrow z$ in $\pi_1 + \pi_2$ almost surely. Since W separates x and z, the set $\{y \in W : x \leftrightarrow y \text{ in } \pi_1 + \pi_2\}$ is almost surely non-empty on this event. Thus its (conditional) mean cardinality is at least one, and the result follows. □

The Ising model on $G = (V, E)$ corresponds in the manner described in Chapter 1 to a random-cluster measure $\phi_{G,p,q}$ with $q = 2$. We have from Theorem 1.9 that

$$\sigma(x,y) = 2\tau_{\lambda,2}(x,y) = \phi_{G,p,q}(x \leftrightarrow y),$$

where $p = 1 - e^{-\lambda q}$ and $q = 2$. Therefore the Simon inequality may be written in the form

$$\phi_{G,p,q}(x \leftrightarrow z) \le \sum_{y \in W} \phi_{G,p,q}(x \leftrightarrow y)\phi_{G,p,q}(y \leftrightarrow z) \tag{8.35}$$

8.simon2 whenever W separates x and z. It is a curious fact that this inequality holds also when $q = 1$, as was noticed by Hammersley [**105**]; see [**92**, Chapter 6]. It may be conjectured that it holds whenever $1 \le q \le 2$.

The Simon inequality is a special case of the Boel–Kasteleyn inequalities, [**33**, **34**]. It has an important consequence for Potts/random-cluster models on infinite lattices, namely that the two-point correlation function decays exponentially whenever it is summable. This fact is closely related to the corresponding percolation theorem of Hammersley, see [**92**, Theorem 6.1], [**105**]. Recall the notation of the related discussion in Sections 4.4 and 4.5. Consider the random-cluster measure $\phi_{p,q}$ on $\mathbb{L}^d$ where $d \ge 2$. We shall consider only the case $p < p_c(q)$ and, as explained at the beginning of Section 4.4, it is therefore unnecessary to consider the question of boundary conditions.

[12] In association with related inequalities of Hammersley [**105**] and Lieb [**143**], see Theorem 8.41, this is sometimes termed the Hammersley–Simon–Lieb inequality.

8.expdecay **Corollary 8.36 [179].** *Assume that $q = 2$ and*

$$\sum_{x \in \mathbb{Z}^d} \phi_{p,q}(0 \leftrightarrow x) < \infty. \tag{8.37}$$

8.summ *There exists $\gamma = \gamma(p,q) \in (0,\infty)$ such that*

$$\phi_{p,q}(0 \leftrightarrow z) \le e^{-\|z\|\gamma(p,q)}, \qquad z \in \mathbb{Z}^d.$$

It is evident that (8.37) is both necessary and sufficient for exponential decay.

Proof. One uses the Simon inequality in the form (8.35) as in [**105**, **179**]. As before, we set $\Lambda_n = [-n,n]^d$ and $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$. By (8.37), we may find $c \in (0,1)$ and $N \ge 1$ such that

$$\sum_{x \in \partial\Lambda_N} \phi_{p,q}(0 \leftrightarrow x) < c.$$

For any integer $k > 1$ and any vertex $z \in \partial\Lambda_{kN}$, we have by progressive use of Theorem 8.34 and the translation invariance of $\phi_{p,q}$ (see ??) that

$$\begin{aligned}
&\phi_{p,q}(0 \leftrightarrow z)\\
&\quad\le \sum_{\substack{x_1:\\ \|x_1\|=N}} \phi_{p,q}(0 \leftrightarrow x_1)\phi_{p,q}(x_1 \leftrightarrow z)\\
&\quad\le \sum_{\substack{x_1:\\ \|x_1\|=N}} \sum_{\substack{x_2:\\ \|x_2-x_1\|=N}} \phi_{p,q}(0 \leftrightarrow x_1)\phi_{p,q}(x_1 \leftrightarrow x_2)\phi_{p,q}(x_2 \leftrightarrow z)\\
&\quad\le \sum_{\substack{x_1:\\ \|x_1\|=N}} \cdots \sum_{\substack{x_k:\\ \|x_k-x_{k-1}\|=N}} \phi_{p,q}(0 \leftrightarrow x_1)\cdots\phi_{p,q}(x_{k-1} \leftrightarrow x_k)\phi_{p,q}(x_k \leftrightarrow z)\\
&\quad\le c^k.
\end{aligned}$$

Therefore there exists $g > 0$ such that

$$\phi_{p,q}(0 \leftrightarrow z) \le e^{-\|z\|g} \qquad \text{if } \|z\| \text{ is a multiple of } N.$$

More generally, take $z \in \mathbb{Z}^d$ and write $\|z\| = kN + l$ where $0 \le l < N$. By Theorem 8.34,

$$\phi_{p,q}(0 \leftrightarrow z) \le \sum_{\substack{x:\\ \|x\|=kN}} \phi_{p,q}(0 \leftrightarrow x)\phi_{p,q}(x \leftrightarrow z) \le e^{-kNg}.$$

Furthermore, $\phi_{p,q}(0 \leftrightarrow z) < 1$ for all $z \ne 0$, and it follows that there exists $\gamma > 0$ such that $\phi_{p,q}(0 \leftrightarrow z) \le e^{-\|z\|\gamma}$ for all $z \in \mathbb{Z}^d$. □

We close this section with an improvement of the Simon inequality due to Lieb [**143**]. This improvement may seem at first sight to be slender, but it leads to a significant conclusion termed the 'vanishing of the mass gap'.

First we re-visit the switching lemma, Theorem 8.29. As usual, $G = (V, E)$ is a finite graph, and we partition E as $E = F \cup H$; thus $F \cap H = \varnothing$. Let $M = (M_e : e \in E)$ be a vector of disjoint finite sets, with cardinalities $m_e = |M_e|$. We write $M^F = (M_e : e \in F)$ and define the vector m^F by

$$m_e^F = \begin{cases} m_e & \text{if } e \in F, \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for M^H and m^H. It is elementary that $m = m^F + m^H$, and that the sets of sources of M^F and M^H are related by

$$\partial M^F \,\triangle\, \partial M^H = \partial M. \tag{8.38}$$

8.dissource As discussed prior to Theorem 8.29, we select subsets N_e from the M_e by deleting each member independently at random with probability $\frac{1}{2}$.

8.switch3 **Theorem 8.39.** *Let F, H, M, and m be as above. If $x, y \in V$ are such that $x \neq y$ and $x \leftrightarrow y$ in m^F then, for $A \subseteq V$,*

$$\begin{aligned} &\mathbb{P}^M\big(\partial N^F = \{x, y\},\ \partial N^H = \varnothing,\ \partial(M \setminus N) = A\big) \\ &\qquad = \mathbb{P}^M\big(\partial N^F = \varnothing,\ \partial N^H = \varnothing,\ \partial(M \setminus N) = A \,\triangle\, \{x, y\}\big). \end{aligned}$$

Proof. This follows that of Theorem 8.29. Let p be a path of G_{m^F} with endpoints x and y, and consider the map $\zeta(n) = n \,\triangle\, p$ for $n \in \mathcal{M}$. This map is a one–one correspondence between the two subsets of $\mathcal{M}$ corresponding to the two events in question. □

We obtain as in Theorem 8.30 the following corollary involving the two independent random vectors M and M', each being such that $m_e = |M_e|$ and $m'_e = |M'_e|$ have the Poisson distribution with parameter λ. The proof follows that of Theorem 8.30, which is itself retrieved on setting $H = \varnothing$.

8.switch4 **Corollary 8.40.** *If $x, y \in V$ are such that $x \neq y$ and $x \leftrightarrow y$ in $m + m'$ then, for $A \subseteq V$,*

$$\begin{aligned} &\mathbb{P}\big(\partial M^F = \{x, y\},\ \partial M^H = \varnothing,\ \partial M' = A \,\big|\, M \cup M'\big) \\ &\qquad = \mathbb{P}\big(\partial M^F = \varnothing,\ \partial M^H = \varnothing,\ \partial M' = A \,\triangle\, \{x, y\} \,\big|\, M \cup M'\big) \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

Before stating the key step in the Lieb improvement of the Simon inequality, we introduce some further notation. Let π_1 and π_2 be independent copies of the Poisson field π, with intensity λ. We write $\mathbb{Q}_{A,B;C}$ for the probability measure governing the pair π_1, π_2 *conditional on the event* $\{\partial\pi_1^F = A\} \cap \{\partial\pi_1^H = B\} \cap \{\partial\pi_2 = C\}$. Recall from (8.10) that $\sigma(x, y)$ denotes a certain correlation function associated with the graph $G = (V, E)$. We write $\sigma_F(x, y)$ for the quantity defined similarly on the smaller graph (V, F).

8.liebsim **Theorem 8.41.** *Let $x, y, z \in V$ be distinct vertices.*

(a) *We have that*

$$\sigma_F(x,y)\sigma(y,z) = \sigma(x,z)\mathbb{Q}_{\varnothing,\varnothing;\{x,z\}}\big(x \leftrightarrow z \text{ in } \pi_1^F + \pi_2^F\big).$$

(b) **Lieb inequality [143].** *Let W separate x and z, and let F be the subset of E containing all edges having at least one endpoint not separated by W from x. Then*

$$\sigma(x,z) \le \sum_{y \in W} \sigma_F(x,y)\sigma(y,z).$$

Since $\sigma_F(x,y) \le \sigma(x,y)$ by ??, the Lieb inequality is an improvement of the Simon inequality.

Proof. (a) Since F and H are disjoint, we have that π_1^F and π_1^H are independent vectors. As in the proof of Theorem 8.33(b), by Theorem 8.40,

$$\begin{aligned}
&\sigma_F(x,y)\sigma(y,z)\\
&\quad= \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda\big(\partial\pi_1^F = \{x,y\},\ \partial\pi_1^G = \varnothing,\ \partial\pi_2 = \{y,z\}\big)}{\mathbb{P}_\lambda(\partial\pi_1^F = \varnothing)\mathbb{P}_\lambda(\partial\pi_1^G = \varnothing)\mathbb{P}_\lambda(\partial\pi_2 = \varnothing)}\\
&\quad= \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda\big(\partial\pi_1^F = \partial\pi_1^G = \varnothing,\ \partial\pi_2 = \{x,z\},\ x \leftrightarrow y \text{ in } \pi_1^F + \pi_2^F\big)}{\mathbb{P}_\lambda(\partial\pi_1^F = \varnothing)\mathbb{P}_\lambda(\partial\pi_1^G = \varnothing)\mathbb{P}_\lambda(\partial\pi_2 = \varnothing)}\\
&\quad= \sigma(x,z)\cdot\frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda\big(\partial\pi_1^F = \partial\pi_1^G = \varnothing,\ \partial\pi_2 = \{x,z\},\ x \leftrightarrow y \text{ in } \pi_1^F + \pi_2^F\big)}{\mathbb{P}_\lambda(\partial\pi^F = \varnothing)\mathbb{P}_\lambda(\partial\pi^G = \varnothing)\mathbb{P}_\lambda(\partial\pi_2 = \{x,z\})}\\
&\quad= \sigma(x,z)\mathbb{Q}_{\varnothing,\varnothing;\{x,z\}}\big(x \leftrightarrow y \text{ in } \pi_1^F + \pi_2^F\big),
\end{aligned}$$

(b) Evidently,

$$\begin{aligned}
\sum_{y \in W} \frac{\sigma_F(x,y)\sigma(y,z)}{\sigma(x,z)} &= \sum_{y \in W} \mathbb{Q}_{\varnothing,\varnothing;\{x,z\}}\big(x \leftrightarrow y \text{ in } \pi_1^F + \pi_2^F\big)\\
&= \mathbb{Q}_{\varnothing,\varnothing;\{x,z\}}\big(\big|\{y \in W : x \leftrightarrow y \text{ in } \pi_1^F + \pi_2^F\}\big|\big)\\
&\ge 1,
\end{aligned}$$

since, conditional on $\partial\pi_2 = \{x,z\}$, π_2 contains almost surely a path from x to z, and any such path necessarily intersects W. □

We turn now to the question of exponential decay, which we formulate in the context of the random-cluster model on $\mathbb{Z}^d$ with $q = 2$. It follows by Theorem 8.41 with $W = \partial\Lambda_k$, just as in Theorem 8.36, that $\phi_{p,q}(0 \leftrightarrow z)$ decays exponentially as $\|z\| \to \infty$ if and only if

$$\sum_{x \in \partial\Lambda_k} \phi_{\Lambda_k,p,q}(0 \leftrightarrow x) < 1 \qquad \text{for some } k \ge 1, \tag{8.42}$$

8.liebnec where $q = 2$. Condition (8.42) is a 'finite-volume condition' in that it uses probability measures on *finite* graphs only. We arrive thus at the following result, sometimes termed the 'vanishing of the mass gap'.

8.expdeclieb **Theorem 8.43 [143].** *Let $\psi(p,q)$ be given as in Theorem 4.40(b), and set $q = 2$. Then $\psi(p,q)$ decreases continuously to* 0 *as p increases.*

Proof. We shall in the following consider only values of p satisfying $\epsilon < p < 1-\epsilon$ where $\epsilon > 0$ is a fixed small quantity. By Theorem 2.36, with $q = 2$ throughout,

$$\begin{aligned}
0 &\le \frac{d}{dp} \sum_{x\in\partial\Lambda_k} \phi_{\Lambda_k,p,q}(0 \leftrightarrow x) \\
&\le \sum_{x\in\partial\Lambda_k} \frac{1}{\epsilon(1-\epsilon)} \mathrm{cov}_{\Lambda_k,p,q}(1_{\{0\leftrightarrow x\}}, |\eta|) \\
&\le \sum_{x\in\partial\Lambda_k} \frac{1}{\epsilon(1-\epsilon)} \sqrt{\phi_{\Lambda_k,p,q}(|\eta|^2)} \qquad \text{by Cauchy–Schwarz} \\
&\le C_1 k^{2d-1},
\end{aligned}$$

for some constant $C_1 = C_1(\epsilon)$. Therefore, for $\epsilon < p < p' < 1-\epsilon$,

$$\sum_{x\in\partial\Lambda_k} \phi_{\Lambda_k,p',q}(0 \leftrightarrow x) \le C_1 k^{2d-1}(p'-p) + \sum_{x\in\partial\Lambda_k} \phi_{\Lambda_k,p,q}(0 \leftrightarrow x).$$

It follows that, if (8.42) holds for some $p \in (\epsilon, 1-\epsilon)$, then it holds for some $p' > p$. That is, if $\phi_{p,q}(0 \leftrightarrow z)$ decays exponentially as $\|z\| \to \infty$, then the same holds for some p' satisfying $p' > p$.

More details are necessary because ψ is defined slightly differently in Chapter 4 from that needed here. □

Could some of the results of this section be valid for more general values of q than just $q = 2$? It is known that the mass gap vanishes when $q = 1$, [**92**, Theorem 6.14], and *does not vanish* for sufficiently large values of q (and any $d \ge 2$), see [**135**] and Chapter ?. Therefore, Theorem 8.41 cannot hold for large q, with $\sigma(x,y)$ given by (8.10). It seems possible that the result holds for sufficiently small q, but this is unproven.

One may ask whether the weaker Simon inequality, Theorem 8.34, could hold for more general values of q. The following example would need to be assimilated in such a result.

8.jansonex **Example 8.44**[13]**.** Let $G = (V, E)$ be a graph comprising a cycle of length m, so that $|E| = m$. See Figure 8.1. We shall work with its 'reduced' partition function

$$Y = \sum_{\omega\in\Omega} \left(\frac{p}{1-p}\right)^{|\eta(\omega)|} q^{k(\omega)}, \tag{8.45}$$

[13] Calculations by S. Janson, on 11 March 2003 at Melbourn.

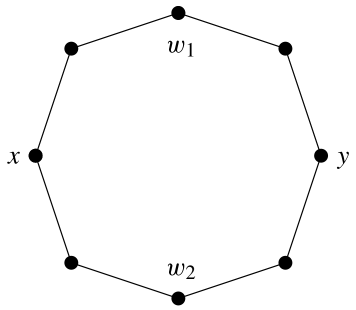


Figure 8.1. A cycle of length 8 with four marked vertices.

8.cycle

8.redpart where $\Omega = \{0,1\}^E$ as usual. Since

$$k(\omega) = \begin{cases} 1 & \text{if } \eta(\omega) = E, \\ m - |\eta(\omega)| & \text{otherwise,} \end{cases}$$

we have that

(8.46)
$$\begin{aligned} Y &= \sum_{j=0}^{m-1} \binom{m}{j} \alpha^j q^{m-j} + \alpha^m q = (\alpha + q)^m - \alpha^m + \alpha^m q \\ &= Q^m + (q-1)\alpha^m. \end{aligned}$$

8.redpart2 where

$$\alpha = \frac{p}{1-p}, \qquad Q = q + \alpha.$$

Let $x, y \in V$, let P_1, P_2 be the two paths joining them, and let k and l be their respective lengths. Configurations which contain P_1 but not P_2 contribute

$$Y_1 = \sum_{j=0}^{l-1} \binom{l}{j} \alpha^{k+j} q^{m-k-j} = \alpha^k \sum_{j=0}^{l-1} \binom{l}{j} \alpha^j q^{l-j} = \alpha^k (Q^l - \alpha^l)$$

to the summation (8.45) defining Y, with a similar contribution Y_2 from configurations containing P_2 but not P_1. The single configuration containing both P_1 and P_2 contributes $Y_{12} = q\alpha^m$ to the summation. Therefore,

$$\sigma(x,y) = \frac{Y_1 + Y_2 + Y_{12}}{Y} = \frac{(\alpha/Q)^k + (\alpha/Q)^l + (q-2)(\alpha/Q)^m}{1 + (q-1)(\alpha/Q)^m}. \tag{8.47}$$

8.jancorr

We consider now a specific example, illustrated in Figure 8.1. Take $m = 8$, let x, y be opposite to one another, and let w_1, w_2 be the intermediate vertices indicated in the figure. For fixed q and small α, we have by (8.47) that

$$\sigma(x, y) = 2\left(\frac{\alpha}{Q}\right)^4 + (q-2)\left(\frac{\alpha}{Q}\right)^8 + \mathrm{O}(\alpha^{12}). \tag{8.48}$$

8.jancorr2 [A similar expression is valid for fixed α and large q.] Similarly,

$$\sigma(x, w_j) = \sigma(y, w_j) = \left(\frac{\alpha}{Q}\right)^2 + \left(\frac{\alpha}{Q}\right)^6 + \mathrm{O}(\alpha^8), \qquad j = 1, 2.$$

Hence,

$$\begin{aligned}\sum_{j=1}^{2} \sigma(x, w_j)\sigma(y, w_j) &= 2\left(\left(\frac{\alpha}{Q}\right)^2 + \left(\frac{\alpha}{Q}\right)^6 + \mathrm{O}(\alpha^8)\right)^2 \\ &= 2\left(\frac{\alpha}{Q}\right)^4 + 4\left(\frac{\alpha}{Q}\right)^8 + \mathrm{O}(\alpha^{10}).\end{aligned} \tag{8.49}$$

8.jancorr3

Comparing (8.48)–(8.49), we see that

$$\sigma(x, y) > \sum_{j=1}^{2} \sigma(x, w_j)\sigma(y, w_j)$$

if $q > 6$ and α is sufficiently small. This may be compared with the conclusion of Theorem 8.34.

9

On Other Graphs

9.1 On complete graphs

The Potts model is one of the standard processes of statistical physics. When the underlying graph is complete, the ensuing process is termed 'mean-field'; unlike the corresponding lattice models, the mean-field Potts model is exactly soluble (see Wu (1982), Kesten and Schonmann (1990)). Intimately related to Potts models are random-cluster models, whose discovery was reported in a series of papers by Fortuin and Kasteleyn around 1970 (see Grimmett (1994a, b) for a survey and historical account). The subject of the current paper is a 'complete' analysis of random-cluster models on complete graphs, i.e., the mean-field random-cluster model.

The model in question is as follows. We shall consider graphs on a set V of n vertices, and so the edge-sets of our graphs will be subsets of $V^{(2)}$, the set of $\binom{n}{2}$ pairs of elements of V. Let $0 < p < 1$, $q > 0$, and for $E \subseteq V^{(2)}$ define

(1.1)

$$\widetilde{P}(E; n, p, q) = p^{|E|}(1-p)^{\binom{n}{2}-|E|}q^{c(V,E)} = (1-p)^{\binom{n}{2}}\left(\frac{p}{1-p}\right)^{|E|} q^{c(V,E)},$$

where $c(V, E)$ denotes the number of components of the graph (V, E). Furthermore, let

$$Z(n, p, q) = \sum_{E} \widetilde{P}(E; n, p, q) \tag{1.2}$$

and define the probability

$$\mathbb{P}(E; n, p, q) = \frac{\widetilde{P}(E; n, p, q)}{Z(n, p, q)}. \tag{1.3}$$

Thus, for every n, p, q, $\mathbb{P}(E; n, p, q)$ is a probability measure on the set of possible edge-sets E; hence it describes the distribution of a random graph with n

vertices, which will be denoted by $G(n, p, q)$. We sometimes write $\mathbb{P}(E; n, p, q)$ as $\mathbb{P}(E; V, p, q)$; as is customary, V is usually taken to be $\{1, 2, \ldots, n\}$.

The random-cluster measure given in (1.3) has two parameters, p and q. When $q = 1$, we recover the usual Erdős–Rényi model $G(n, p)$; see Bollobás (1985) for the general theory of such a random graph. When $q \in \{2, 3, \ldots\}$, the random-cluster model corresponds to a Potts model on the complete graph K_n with q states and with strength of interaction $J = -\log(1 - p)$. Indeed the phase structures of the two models are closely linked; see Edwards and Sokal (1988) and Grimmett (1994b). Whereas Potts models are defined for integer values only of q, the random-cluster model is well defined for all real positive q. In this sense, random-cluster models generalize Potts models. Professor Kasteleyn has pointed out that they are more general than Potts models in a further regard: there are questions which may be asked about random-cluster models for which there is no corresponding question for Potts models (see Grimmett (1994a)). Material related to the current paper may be found in Whittle (1986, 1994).

The principal technique for analysing the mean-field Potts model relies upon the assumption that q is an integer. This technique is invalid in its basic form for general real values of q, and therefore one needs extra methods in order to understand random-cluster models. Our principal extra technique, described in Section 3, is a method whereby properties of $G(n, p, q)$ may be studied via corresponding properties of the usual random graph $G(n, p)$. It is striking that our results are valid for all $q > 0$, including the regime $0 < q < 1$; when $0 < q < 1$, the FKG inequality is no longer valid, and consequently there is a problem in studying the phase transition in lattice systems (see Grimmett (1994b)).

Our principal results concern the process $G(n, p, q)$ defined above. There are complementary results for a 'fixed edge-number' model, and we sketch these later in this section; see also Section 11.

As in the Erdős–Rényi theory of the giant component in the case $q = 1$, we set $p = \lambda/n$ where λ is a positive constant, and shall study the size of the largest component of the ensuing $G(n, \lambda/n, q)$ in the limit as $n \to \infty$. It turns out that there is a critical value of λ, depending on the value of q, which marks the arrival of a 'giant component' of the graph. This critical value is given by

$$\lambda_c(q) = \begin{cases} q & \text{if } 0 < q \le 2 \\ 2\left(\dfrac{q-1}{q-2}\right)\log(q-1) & \text{if } q > 2. \end{cases} \tag{1.4}$$

Another quantity having a central role is the function $\theta(\lambda, q)$ given as follows. We define

$$\theta(\lambda, q) = \begin{cases} 0 & \text{if } \lambda < \lambda_c(q) \\ \theta_{\max} & \text{if } \lambda \ge \lambda_c(q) \end{cases} \tag{1.5}$$

where $\theta_{\max}$ is the largest root of the equation

$$e^{-\lambda\theta} = \frac{1-\theta}{1+(q-1)\theta}. \tag{1.6}$$

It may be seen that $\theta(\lambda, q) > 0$ if and only if either (a) $\lambda > \lambda_c(q)$, or (b) $\lambda = \lambda_c(q)$ and $q > 2$ (a proof of this is given after equation (6.22)). Furthermore $\theta(\lambda, q)$ is nondecreasing in λ, and it follows that $\theta(\cdot, q)$ is continuous if $0 < q \le 2$, and has a unique (jump) discontinuity at $\lambda = \lambda_c(q)$ if $q > 2$. This jump discontinuity corresponds to the 'first-order phase transition' of statistical physics.

We shall prove the following (and more) in Sections 2, 4, and 9.

(a) If $0 < \lambda < \lambda_c(q)$ and $q > 0$, then a.e. $G(n, p, q)$ has largest component having order $\log n$.

(b) If $\lambda > \lambda_c(q)$ and $q > 0$, then a.e. $G(n, p, q)$ consists of a 'giant component' of order $\theta(\lambda, q)n$, together with other components of order $\log n$ or smaller.

(c) If $\lambda = \lambda_c(q)$ and $0 < q \le 2$, then a.e. $G(n, p, q)$ has largest component of order $n^{2/3}$.

(d) If $\lambda = \lambda_c(q)$ and $q > 2$, then a.e. $G(n, p, q)$ is either as in (a) or as in (b).

In the above, we say that 'a.e. $G(n, p, q)$ satisfies property Π', for a given sequence $p = p(n)$ and a fixed q, if $\mathbb{P}(G(n, p, q) \text{ has } \Pi) \to 1$ as $n \to \infty$. Care must be taken in interpreting the above remarks concerning the orders of components. See Theorems 2.1–2.3 for more precise statements.

There are two main steps in establishing the above facts. The first step is to establish the relation (1.6) by studying the size of the largest component of $G(n, \lambda/n, q)$. When $q > 2$, the relation (1.6) has three solutions for large λ; in order to decide which of these is the density of the largest component, we shall study the number of edges in $G(n, \lambda/n, q)$. That is to say, we shall find the function $\psi(\lambda, q)$ such that a.e. $G(n, \lambda/n, q)$ has (order) $\psi(\lambda, q)n$ edges. It is interesting to note that, at the critical point of a first-order phase transition, the function $\psi(\cdot, q)$ is discontinuous.

We present in Theorem 2.6 the asymptotic behaviour of $n^{-1} \log Z(n, \lambda/n, q)$ as $n \to \infty$; this amounts, in physical terms, to a calculation of the 'free energy'. As a consequence of this calculation, we obtain large-deviation results for the number of components of $G(n, \lambda/n, q)$ for $q > 0$ (see Section 8).

Another random-graph model of interest is that with a fixed number of edges; the corresponding random-cluster model $G(n, m, q)$ is given by attaching weight $q^{c(E)}$ to each edge-set E of K_n having cardinality m, where m, n, and q are fixed in advance. This model is also obtained if we condition $G(n, p, q)$ (for any p) on having exactly m edges. For $G(n, m, q)$ one may obtain results similar to those described above for $G(n, \lambda/n, q)$, so long as m and n are related asymptotically in the manner of $m/n \to \mu$ for some positive constant μ. Some of these results depend on a certain hypothesis of which we have no complete proof (see Hypothesis 11.5). Note that the two models cannot behave (asymptotically) identically, for the following reason at least. The asymptotic number of edges per vertex in a.e. $G(n, \lfloor \mu n \rfloor, q)$ is exactly μ, where μ may take any value in $(0, \infty)$; on the other hand the corresponding 'edge-density' $\psi(\lambda, q)$ of $G(n, \lambda/n, q)$ has a jump discontinuity at $\lambda = \lambda_c(q)$ when $q > 2$, and therefore may not take all values in $(0, \infty)$.

The paper is laid out as follows. In Section 2 we state our main results for $G(n, p, q)$ under the assumption $q > 0$. This is followed in Section 3 by an account of the central technique of the paper. Proofs of main results when $q \geq 1$ appear in Sections 4–6. In Section 7, we summarise facts about the nature of the phase transition, and in Section 8 we describe an application of one of our calculations to the large-deviation behaviour of the number of components of a random graph. Similar conclusions are valid when $0 < q < 1$, as stated in Section 2; see Section 9. An external field is added in Section 10. Finally, Section 11 contains a brief adaptation of the arguments and conclusions to a 'fixed edge-number' random-cluster model $G(n, m, q)$.

9.2 Phase transition

We state our main results in this section. Until further notice we assume that $q > 0$, and we write $p = \lambda/n$ where λ is a positive constant. We shall sometimes suppress explicit reference to q. We define $\lambda_c(q)$ by

$$\lambda_c(q) = \begin{cases} q & \text{if } 0 < q \leq 2 \\ 2\left(\dfrac{q-1}{q-2}\right)\log(q-1) & \text{if } q > 2, \end{cases} \tag{2.1}$$

and $\theta(\lambda) = \theta(\lambda, q)$ by

$$\theta(\lambda) = \begin{cases} 0 & \text{if } \lambda < \lambda_c(q) \\ \theta_{\max} & \text{if } \lambda \geq \lambda_c(q) \end{cases} \tag{2.2}$$

where $\theta_{\max}$ is the largest root of the equation

$$e^{-\lambda\theta} = \frac{1-\theta}{1+(q-1)\theta}. \tag{2.3}$$

We shall need to study the properties of roots of (2.3) in some detail, but defer this until Lemma 2.5. For the moment we note only that $\theta(\lambda) = 0$ if and only if (a) $\lambda < \lambda_c(q)$, or (b) $\lambda = \lambda_c(q)$ and $q \leq 2$.

We now state three theorems, dealing respectively with the subcritical case $\lambda < \lambda_c(q)$, the supercritical case $\lambda > \lambda_c(q)$, and the critical case $\lambda = \lambda_c(q)$. In the matter of notation, for a sequence (X_n) of random variables, we write $X_n = \mathrm{O_p}(f(n))$ if $X_n/f(n)$ is bounded in probability, which is to say that

$$\mathbb{P}\big(|X_n| \leq f(n)\omega(n)\big) \to 1 \quad \text{as } n \to \infty$$

for any sequence $\omega(n)$ satisfying $\omega(n) \to \infty$ as $n \to \infty$. Similarly, we write $X_n = \mathrm{o_p}(f(n))$ if $X_n/f(n) \to 0$ in probability as $n \to \infty$, which is to say that

$$\mathbb{P}\big(|X_n| \leq f(n)/\omega(n)\big) \to 1 \quad \text{as } n \to \infty$$

for some sequence $\omega(n)$ satisfying $\omega(n) \to \infty$. Convergence in probability is denoted by '$\xrightarrow{\mathrm{P}}$'

Theorem 2.1. Subcritical case. *Assume that $q > 0$ and $\lambda < \lambda_c(q)$. Then a.e. $G(n, \lambda/n, q)$ comprises trees and unicyclic components only. There are $O_p(1)$ unicyclic components with a total number of $O_p(1)$ vertices. The largest component of a.e. $G(n, \lambda/n, q)$ is a tree having order $\alpha \log n + O_p(\log \log n)$ where $\alpha^{-1} = -\log(\lambda/q) + (\lambda/q) - 1 > 0$. The number of edges in $G(n, p, q)$ is $\lambda n/(2q) + o_p(n)$.*

Theorem 2.2. Supercritical case. *Assume that $q > 0$ and $\lambda > \lambda_c(q)$. Then a.e. $G(n, \lambda/n, q)$ consists of a giant component, trees, and unicyclic components. The giant component has $\theta(\lambda)n + o_p(n)$ vertices and $\lambda\theta(\lambda)\,\{q^{-1} + (\frac{1}{2} - q^{-1})\theta(\lambda)\}n + o_p(n)$ edges. The largest tree has order $\alpha \log n + o_p(\log n)$ where $\alpha^{-1} = -\log \beta + \beta - 1 > 0$ and $\beta = \lambda(1 - \theta(\lambda))/q$. There are $O_p(1)$ unicyclic components with a total number of $O_p(1)$ vertices. The number of edges in $G(n, \lambda/n, q)$ is $\lambda\{1 + (q-1)\theta(\lambda)^2\}n/(2q) + o_p(n)$.*

Theorem 2.3. Critical case. *Assume that $q \ge 1$ and $\lambda = \lambda_c(q)$.*

(a) *If $1 \le q \le 2$, then a.e. $G(n, \lambda/n, q)$ consists of trees, unicyclic components, and $O_p(1)$ components with more than one cycle. The largest component has order $o_p(n)$. The total number of vertices in unicyclic components is $O_p(n^{2/3})$. The largest tree has order $O_p(n^{2/3})$.*

(b) *If $q > 2$, then a.e. $G(n, \lambda/n, q)$ is as in either Theorem 2.1 or Theorem 2.2.*

We conjecture that Theorem 2.3(a) extends also to $0 < q < 1$, but we have not been able to prove this; see Section 9.

More detailed asymptotics are available for $G(n, \lambda/n, q)$ by looking deeper into the proofs. We remark that, in the case of Theorem 2.3(b) when $\lambda = \lambda_c(q)$ and $q > 2$, we have not checked whether both possibilities given above actually occur (with positive probabilities in the limit). If we allow $\lambda = \lambda_n$ to depend on n, where $\lambda_n \to \lambda_c(q)$ as $n \to \infty$, then one may see that, for $q > 2$ and $0 \le \pi \le 1$, there exist sequences (λ_n) such that a.e. $G(n, \lambda_n/n, q)$ is as in Theorem 2.1 with probability π and is as in Theorem 2.2 otherwise.

The giant component, when it exists, has order approximately $\theta(\lambda)n$, where $\theta(\lambda)$ is given in (2.2) and (2.3). We now study the roots of (2.3); note that $\theta = 0$ satisfies (2.3) for all λ and q, and that all positive roots satisfy $0 < \theta < 1$.

Let

$$f(\theta) = \frac{1}{\theta}\big\{\log(1 + (q-1)\theta) - \log(1-\theta)\big\}, \quad 0 < \theta < 1, \tag{2.4}$$

and note that θ ($\in (0, 1)$) satisfies (2.3) if and only if $f(\theta) = \lambda$. We start with two elementary lemmas concerning the function f.

Lemma 2.4. *The function f is strictly convex on $(0, 1)$ and satisfies $f(0+) = q$ and $f(1-) = \infty$.*

(a) *If $0 < q \le 2$, then f is strictly increasing.*

(b) *If* $q > 2$, *then there exists* $\theta_{\min} \in (0, 1)$ *such that* f *is strictly decreasing on* $(0, \theta_{\min})$ *and strictly increasing on* $(\theta_{\min}, 1)$.

Proof. If $t > -1$ then $(1+t\theta)^{-1}$ is a strictly convex function of θ for $0 < \theta < 1$; hence $f(\theta) = \int_{-1}^{q-1}(1+t\theta)^{-1}\,dt$ is also strictly convex. Furthermore,

$$\lim_{\theta\uparrow 1}\log\left\{\frac{1+(q-1)\theta}{1-\theta}\right\} = \lim_{\epsilon\downarrow 0}\log\left\{\frac{q-(q-1)\epsilon}{\epsilon}\right\} = \infty,$$

implying $f(1-) = \infty$. Applying Taylor's theorem about $\theta = 0$, we find that

$$f(\theta) = \frac{1}{\theta}\left\{(q-1)\theta - \tfrac{1}{2}(q-1)^2\theta^2 + \theta + \tfrac{1}{2}\theta^2 + \mathrm{O}(\theta^3)\right\} = q + \tfrac{1}{2}q(2-q)\theta + \mathrm{O}(\theta^2),$$

whence $f(0+) = q$ and $f'(0+) = \frac{1}{2}q(2-q)$. These facts imply assertions (a) and (b) of the lemma. □

In Figure 9.1, we plot f against θ in the three cases $1 \le q < 2$, $q = 2$, and $q > 2$. Since $0 < \theta < 1$ is a root of (2.3) if and only if $f(\theta) = \lambda$, Lemma 2.4 has the following immediate consequence.

Lemma 2.5. *The non-negative roots of equation (2.3) are given as follows, in addition to the root* $\theta = 0$.

(a) *Suppose* $0 < q \le 2$.

(i) *If* $0 < \lambda \le \lambda_c(q) = q$, *there is a unique root* $\theta = 0$.

(ii) *If* $q < \lambda$, *there is a unique positive root* $\theta_{\max}(\lambda)$, *which satisfies* $\theta_{\max}(q+) = 0$.

(b) *Suppose* $q > 2$, *and let* $\lambda_{\min} = f(\theta_{\min})$ *where* $\theta_{\min}$ *is given in Lemma 2.4.*

(i) *If* $0 < \lambda < \lambda_{\min}$, *there is a unique root* $\theta = 0$.

(ii) *If* $\lambda = \lambda_{\min}$, *then* $\theta_{\min}$ *is the unique positive root.*

(iii) *If* $\lambda_{\min} < \lambda < q$, *there are exactly two positive roots,* $\theta_1(\lambda)$ *and* $\theta_{\max}(\lambda)$.

(iv) *If* $\lambda \ge q$, *there is a unique positive root* $\theta_{\max}(\lambda)$.

We shall see later that $\lambda_{\min} < \lambda_c(q) < q$ when $q > 2$, and that the function $\theta(\lambda)$ of Theorems 2.1–2.3 satisfies

$$\theta(\lambda_c(q)) = \begin{cases} 0 & \text{if } q \le 2 \\ \dfrac{q-2}{q-1} & \text{if } q > 2. \end{cases} \tag{2.5}$$

Furthermore, we shall obtain the following asymptotic result for the 'free energy'.

Theorem 2.6. *If* $q > 0$ *and* $\lambda > 0$, *then*

$$\frac{1}{n}\log Z(n, \lambda/n, q) \to \phi(\lambda) \quad \textit{as } n \to \infty$$

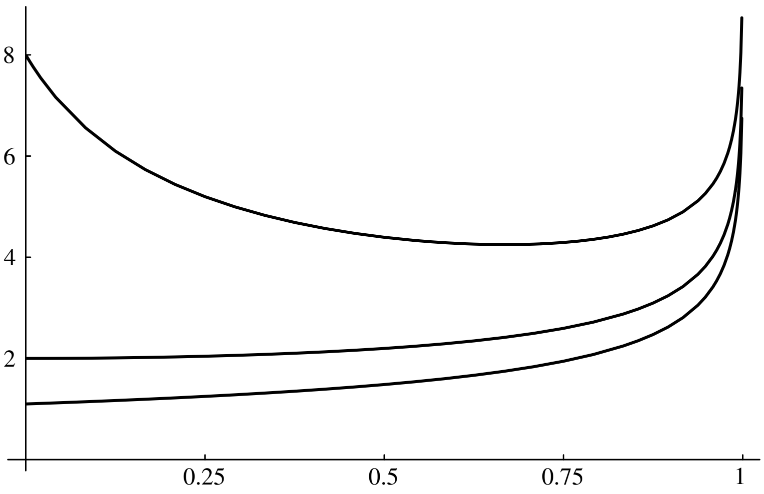


Figure 9.1. Sketches of the function $f(\theta)$ in the three cases $1 \le q < 2$, $q = 2$, and $q > 2$. The respective values of q can be read off from the y–axis, since $f(0+) = q$. Note that f is strictly increasing if and only if $q \le 2$, and $f'(0) = 0$ when $q = 2$. Recall that the positive roots of (2.3) are obtained by intersecting the graph of f by the horizontal line $f(\theta) = \lambda$.

9.fig1

where the 'free energy' $\phi(\lambda) = \phi(\lambda, q)$ is given by

$$\phi(\lambda) = \frac{g(\theta(\lambda))}{2q} - \frac{q-1}{2q}\lambda + \log q \qquad (2.6)$$

and $g(\theta)$ is defined in (6.17).

The proofs of the main theorems (2.1–2.3 and 2.6) are given in Sections 3–6 for $q > 1$, and Section 9 for $0 < q < 1$.

9.3 The fundamental lemma

The following technique is fundamental to the argument of this paper, and is used to reduce the study of $G(n, p, q)$ to the study of the usual random graph $G(n, p, 1)$.

Let $0 \le r \le 1$ be fixed. Given a random graph $G(n, p, q)$, colour each component either *red*, with probability r, or *green*, with probability $1-r$; different components are coloured independently of one other. Thus, if $C_1, \ldots, C_k$ are the components, then the probability that $C_{i_1}, \ldots, C_{i_l}$ are red and the other $k - l$ components are green is $r^l(1-r)^{k-l}$. The union of the red components is the *red* *subgraph* of $G(n, p, q)$, and the green components form the *green subgraph*. Let R be the set of *red* vertices, i.e., the vertex-set of the red subgraph.

Lemma 3.1. *Let $V_1 \subseteq V = \{1, 2, \ldots, n\}$, with $|V_1| = n_1$. Conditional on $R = V_1$, the red subgraph of $G(n, p, q)$ is distributed as $G(n_1, p, rq)$ and the green subgraph as $G(n - n_1, p, (1 - r)q)$; furthermore, the red subgraph is conditionally independent of the green subgraph.*

Proof. Set $V_2 = V \setminus V_1, n_2 = |V_2| = n - n_1$, and let $E_i \subseteq V_i^{(2)}$ for $i = 1, 2$. As in (1.1), let us write $c(U, F)$ for the number of components of the graph with vertex set U and edge set F. Then $c(V, E_1 \cup E_2) = c(V_1, E_1) + c(V_2, E_2)$, whence the probability that the red graph is (V_1, E_1) and the green graph is (V_2, E_2) satisfies

$$\left\{ \frac{p^{|E_1 \cup E_2|}(1 - p)^{\binom{n}{2} - |E_1 \cup E_2|} q^{c(V, E_1 \cup E_2)}}{Z(n, p, q)} \right\} r^{c(V_1, E_1)}(1 - r)^{c(V_2, E_2)}$$
$$= C(n, p, q, n_1)\mathbb{P}(E_1; V_1, p, rq)\mathbb{P}(E_2; V_2, p, (1 - r)q).$$

for some positive real $C(n, p, q, n_1)$ depending only on n, p, q, and n_1. Hence, conditional on $R = V_1$ and the green subgraph being (V_2, E_2), the probability that the red subgraph is (V_1, E_1) is precisely $P(E_1; V_1, p, rq)$. □

In what follows, we shall write N, rather than n_1, for the number of red vertices. Thus N is a random variable and $G(N, p, qr)$ is a random graph on a random number of vertices.

If $q \geq 1$ and $r = q^{-1}$, then the red subgraph is distributed as $G(N, p, 1)$. By studying the distribution of N and using known facts about $G(N, p, 1)$, one may deduce much about the structure of $G(n, p, q)$. Similarly, in order to study the random-cluster model with $q < 1$, one applies Lemma 3.1 to $G(n, p, 1)$ with $r = q$, obtaining that the red subgraph is distributed as $G(N, p, q)$; by using known facts about $G(n, p, 1)$, together with some distributional properties of N, we may obtain results for $G(m, p, q)$ when m is large.

As an example of the lemma in action, we present a corollary which will be of use later. The quantity $n^{3/4}$ is not optimal here, and may be replaced by $n^{2/3} \log n$.

Lemma 3.2. *Let $q \geq 1$. For any sequence $p = p(n)$, a.e. $G(n, p, q)$ has at most one component having order at least $n^{3/4}$.*

Proof. Let $L_{n,p,q}$ be the number of components of $G(n, p, q)$ having order at least $n^{3/4}$. Suppose $L_{n,p,q} \geq 2$, and pick two of these in some arbitrary way. With probability r^2 both of these are coloured red. Setting $r = q^{-1}$, we find that

$$r^2\mathbb{P}(L_{n,p,q} \geq 2) \leq \sum_{n^{3/4} \leq m \leq n} \mathbb{P}(L_{m,p,1} \geq 2)\mathbb{P}(|R| = m)$$
$$\leq \max_{n^{3/4} \leq m \leq n} \mathbb{P}(L_{m,p,1} \geq 2) \to 0 \quad \text{as } n \to \infty,$$

which follows from Bollobás (1985), Thm VI.9. We note the abuse of notation in writing $L_{m,p,1}$ without specifying the sample space. □

9.4 The order of the largest component

Until further notice we assume that $q \geq 1$. Let $\Theta_n n$ be the number of vertices in the largest component of $G(n, p, q)$ where $p = \lambda/n$ and $q \geq 1$; note that $0 < \Theta_n \leq 1$. If two or more 'largest components' exist, then we pick one of these at random. All other components are called 'small', and it is a consequence of Lemma 3.2 that all small components have orders less than $n^{3/4}$ (for a.e. $G(n, \lambda/n, q)$).

Consider the colouring scheme of Section 3, with $r = q^{-1}$, and suppose that $G(n, \lambda/n, q)$ has components of order $\Theta_n n, \nu_2, \nu_3, \ldots, \nu_c$ where c is the total number of components and we shall assume that $\nu_i \leq n^{3/4}$ for $i \geq 2$. The number of red vertices in the small components has conditional expectation

$$\sum_{i=2}^{c} \nu_i r = r(1 - \Theta_n)n$$

and variance given by

$$\sum_{i=2}^{c} \nu_i^2 r(1-r) \leq \sum_{i=2}^{c} \nu_i^2 \leq n \max_{i \geq 2} \nu_i \leq n^{7/4}.$$

Hence there are $r(1 - \Theta_n)n + \mathrm{O_p}(n^{7/8}) = r(1 - \Theta_n)n + \mathrm{o_p}(n)$ red vertices in the small components.

Since the largest component may or may not be coloured red, there are two possibilities for the red graph:

(i) with probability r, it has $\Theta_n n + r(1 - \Theta_n)n + \mathrm{o_p}(n) = \{r + (1-r)\Theta_n\}n + \mathrm{o_p}(n)$ vertices, of which $\Theta_n n$ belong to the largest component,

(ii) with probability $(1 - r)$, it has $r(1 - \Theta_n)n + \mathrm{o_p}(n)$ vertices and the largest component has order less than $n^{3/4}$.

In the first case, the red graph is distributed as a supercritical $G(n', \lambda'/n')$ graph, and in the second case as a subcritical $G(n'', \lambda''/n'')$ graph; here n' and n'' are random, and (with probability tending to 1) $\lambda' = n'p > 1 > \lambda'' = n''p$. This leads to the next lemma.

Lemma 4.1. *If $\lambda > q \geq 1$, then there exists θ_0 (> 0) such that $\Theta_n \geq \theta_0$ for a.e. $G(n, \lambda/n, q)$.*

Proof. For $q = 1$, the assertion is well known; see for example Bollobás (1985), Thm VI.11. Hence assume $q > 1$ and thus $r < 1$.

Let $\theta_0 = (\lambda - q)/(2\lambda)$, $\pi_n = \mathbb{P}(\Theta_n < \theta_0)$, and $\epsilon > 0$. By considering the event that the largest component is not coloured red, we see that, with probability at least $(1 - r)\pi_n + \mathrm{o}(1)$, the number N of red vertices satisfies $N \geq r(1 - \theta_0)n - \epsilon n$, and there are no red components of order at least $n^{3/4}$. When this happens,

$$Np \geq \lambda\{r(1 - \theta_0) - \epsilon\} = \frac{1}{2} + \frac{\lambda}{2q} - \epsilon\lambda > 1 \tag{4.1}$$

if ϵ is sufficiently small; pick ϵ accordingly. Conditional on N, a.e. $G(N, p)$ has a component of order at least δN ($\geq \delta n/\lambda$ by (4.1)) for some δ (> 0). Therefore $(1-r)\pi_n \to 0$ as $n \to \infty$. □

Lemma 4.2. *If $q \geq 1$, then for any sequence $\lambda = \lambda_n$, we have that*

$$e^{-\lambda_n \Theta_n} - \frac{1-\Theta_n}{1+(q-1)\Theta_n} \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty.$$

Proof. For $q = 1$ and constant $\lambda = \lambda_n$, this follows from the well known fact that $\Theta_n \xrightarrow{\mathrm{P}} \theta$ where $e^{-\lambda\theta} = 1 - \theta$ (see Bollobás (1985), Thm VI.11 and the remark after Thm V.7); the case of varying λ_n is not hard to deduce by standard arguments using the conclusion for constant λ (this is done by looking down subsequences along which λ_n converges, with limit lying in $[0, \infty]$). We may express this as

$$e^{-p_n \Theta_n n} + \frac{\Theta_n n}{n} - 1 \xrightarrow{\mathrm{P}} 0 \quad \text{when } q = 1,$$

for the model $G(n, p_n, 1)$ and any sequence (p_n). Applying this to the red subgraph, on the event that it contains the largest component of $G(n, \lambda/n, q)$, we obtain for general q (≥ 1) that

$$e^{-\lambda\Theta_n} + \frac{\Theta_n}{r+(1-r)\Theta_n} - 1 = e^{-p\Theta_n n} + \frac{\Theta_n n}{N} - 1 + \mathrm{o_p}(1) \xrightarrow{\mathrm{P}} 0$$

as $n \to \infty$, where N is the number of red vertices. The result follows. □

Combining these lemmas we arrive at the following theorem.

Theorem 4.3.

(a) *If $1 \leq q \leq 2$ and $\lambda \leq q$, or if $q > 2$ and $\lambda < \lambda_{\min}$ where $\lambda_{\min}$ is given in Lemma 2.5, then $\Theta_n \xrightarrow{\mathrm{P}} 0$ as $n \to \infty$.*

(b) *If $q \geq 1$ and $\lambda > q$, then $\Theta_n \xrightarrow{\mathrm{P}} \theta(\lambda)$ where $\theta(\lambda)$ is the unique (strictly) positive solution of (2.3).*

This goes some way towards proving Theorems 2.1 and 2.2. Overlooking for the moment the more detailed asymptotical claims of those theorems, we note that the major remaining gap is when $q > 2$ and $\lambda_{\min} \leq \lambda \leq q$. In this case, it follows from Lemma 4.2 that Θ_n is approximately equal to one of the three roots of (2.3) (including the trivial root $\theta = 0$); only after the analysis of the next two sections shall we see which root is the correct one for a given value of λ.

Proof. The function

$$\phi(\theta) = e^{-\lambda\theta} - \frac{1-\theta}{1+(q-1)\theta}$$

is continuous on [0, 1]; the set Z of zeros of g is described in Lemma 2.5. Since $\phi(\Theta_n) \xrightarrow{\mathrm{P}} 0$, by Lemma 4.2, it follows that, for all $\epsilon > 0$,

$$\mathbb{P}\big(\Theta_n \in Z + (-\epsilon, \epsilon)\big) \to 1 \quad \text{as } n \to \infty.$$

Under the assumption of (a), Z contains the singleton 0, and the claim follows. Under (b), Z contains exactly one strictly positive number $\theta(\lambda)$, and the claim follows by Lemma 4.1. $\square$

9.5 The number of edges

Given $G(n, p, q)$, pick one of the largest components at random, and write $\Xi_n n$ for the number of edges in this component. Also, write $\Psi_n n$ for the number of edges of the entire graph $G(n, p, q)$. Assume $q > 1$. Arguing as in Sections 3 and 4 with $r = q^{-1}$, we see that a.e. $G(n, p, q)$ has at most $n^{3/4}$ edges in each small component (a 'small' component is any component except the largest, picked above). [For this we need the corresponding result for $q = 1$, which easily follows from the corresponding result for the number of vertices used above, and results on the components with more edges than vertices given in Bollobás (1984), Janson (1993), or Janson, Knuth, Łuczak, and Pittel (1993).] Furthermore the number of red edges in the small components is $r(\Psi_n - \Xi_n)n + \mathrm{o_p}(n)$. Hence the red subgraph has either

(i) (with probability r) $\{\Theta_n + r(1 - \Theta_n)\}n + \mathrm{o_p}(n)$ vertices and $\{\Xi_n + r(\Psi_n - \Xi_n)\}n + \mathrm{o_p}(n)$ edges, or

(ii) (otherwise) $r(1 - \Theta_n)n + \mathrm{o_p}(n)$ vertices and $r(\Psi_n - \Xi_n)n + \mathrm{o_p}(n)$ edges.

Assume that $p = \mathrm{O}(n^{-1})$. Since a.e. $G(N, p)$ graph has $\binom{N}{2}p + \mathrm{O_p}(Np^{1/2}) = \frac{1}{2}N^2 p + \mathrm{o_p}(N)$ edges, the following two equations follow from the two cases above,

$$\{\Xi_n + r(\Psi_n - \Xi_n)\}n = \tfrac{1}{2}\{\Theta_n + r(1 - \Theta_n)\}^2 n^2 p + \mathrm{o_p}(n), \tag{5.1}$$

$$r(\Psi_n - \Xi_n)n = \tfrac{1}{2}\{r(1 - \Theta_n)\}^2 n^2 p + \mathrm{o_p}(n), \tag{5.2}$$

yielding when $p = \lambda/n$ that

$$\Xi_n + r(\Psi_n - \Xi_n) = \tfrac{1}{2}\lambda\{\Theta_n + r(1 - \Theta_n)\}^2 + \mathrm{o_p}(1), \tag{5.3}$$

$$r(\Psi_n - \Xi_n) = \tfrac{1}{2}\lambda\{r(1 - \Theta_n)\}^2 + \mathrm{o_p}(1). \tag{5.4}$$

We solve for Ξ_n and Ψ_n, and pass to the limit as $n \to \infty$, to obtain the next theorem.

Theorem 5.1. *If $q > 1$ and $\lambda > 0$, then, as $n \to \infty$,*

$$\Psi_n - \frac{\lambda}{2q}\{1 + (q-1)\Theta_n^2\} \xrightarrow{\mathrm{P}} 0, \tag{5.5}$$

$$\Xi_n - \frac{\lambda}{q}\Theta_n\{1 + (\tfrac{1}{2}q - 1)\Theta_n\} \xrightarrow{\mathrm{P}} 0. \tag{5.6}$$

Whereas we proved this theorem under the assumption that $q > 1$, its conclusions are valid for $q = 1$ also (a consequence of Bollobás (1985), Thms VI.11 and VI.12). Note that the proof used information about the number of edges in a $G(n, p)$ graph but no information about the number of edges in its largest component.

9.6 Proofs of main results

In this section we combine the results derived so far with a new argument in order to prove Theorems 2.1–2.3 for $q \geq 1$.

We note first that the results of the theorems are well known when $q = 1$ (see Bollobás (1985), Chaps V and VI, and Łuczak, Pittel, and Wierman (1994)), and henceforth we assume that $q > 1$. We define the *acyclic* part of a graph to be the union of all components that are trees, and the *cyclic* part to be the union of the remaining components. We call a graph *cyclic* if its acyclic part is empty. We being by showing that the cyclic part of a.e. $G(n, \lambda/n, q)$ consists principally of the largest component only (when this component is cyclic).

Lemma 6.1. *The numbers of vertices and edges in the small cyclic components of $G(n, \lambda/n, q)$ are $\mathrm{o_p}(n)$.*

Proof. Let k be an integer satisfying $k \geq q$. In the colouring scheme of Section 3 with $r = q^{-1}$, we introduce the refinement that each component is coloured *dark red* with probability k^{-1} and *light red* with probability $r - k^{-1}$. Let M be the number of edges in the small cyclic components of $G(n, \lambda/n, q)$. By a symmetry argument, with probability at least k^{-1}, at least M/k of these edges are coloured dark red. [Let M_i be the number of such edges coloured χ_i, when each component is coloured by a random colour from the set $\{\chi_1, \chi_2, \ldots, \chi_k\}$, each having equal probability. If $\mathbb{P}(M_i \geq M/k) < k^{-1}$, then $\mathbb{P}(M_i \geq M/k$ for some $i) < 1$, which is a contradiction since $\sum_{i=1}^{k} M_i = M$.] Therefore, with probability at least r/k, the red subgraph contains the largest component together with small cyclic components having at least M/k edges. The result follows from the known case $q = 1$; see Bollobás (1984) and Bollobás (1985), Thm VI.11. □

Let $\widetilde{P}(m, j, k, l; n, p, q)$ be the sum of all $\widetilde{P}(E; n, p, q)$ over edge-sets E that define a graph with $|E| = m$ edges and a cyclic part with j components, k vertices, and l edges. Since such graphs have an acyclic part with $n-k$ vertices and $m-l$ edges, and therefore $n-k-m+l$ components, we obtain

(6.1)
$$\widetilde{P}(m,j,k,l;n,p,q) = \binom{n}{k} c(j,k,l) f(n-k,m-l) p^m (1-p)^{\binom{n}{2}-m} q^{n-k-m+l+j}$$

where $c(j,k,l)$ is the number of cyclic graphs with j components, k labelled vertices, and l edges, and similarly $f(n,m)$ is the number of forests with n labelled vertices and m edges.

Let us now assume that $n \to \infty$, that $\lambda = np > 0$ and $q \geq 1$ are fixed, and that

$$m/n \to \psi,\ k/n \to \theta,\ l/n \to \xi,\ \text{and } j/n \to 0, \tag{6.2}$$

where θ (≥ 0) satisfies (2.3), and

$$\xi = \frac{\lambda}{q}\theta\{1 + (\tfrac{1}{2}q - 1)\theta\}, \tag{6.3}$$

$$\psi = \xi + \frac{\lambda}{2q}(1-\theta)^2; \tag{6.4}$$

see (5.5) and (5.6). If $\lambda > q$, we also assume that $\theta > 0$, cf. Lemma 4.1 and Theorem 4.3(b).

Since $f(n,m) \leq \binom{\binom{n}{2}}{m}$, the total number of graphs with m edges on n vertices, we have that

(6.5)
$$\begin{aligned}
\widetilde{P}(m,j,k,l;n,p,q) &\leq \binom{n}{k} c(j,k,l) \binom{\binom{n-k}{2}}{m-l} p^m (1-p)^{\binom{n}{2}-m} q^{n-k-m+l+j} \\
&= \binom{n}{k} c(j,k,l) \binom{n-k}{2}^{m-l} \left(\frac{e}{m-l}\right)^{m-l} p^m q^{n-k-m+l} \exp(-\tfrac{1}{2}\lambda n + \mathrm{o}(n)) \\
&= c(j,k,l) \binom{n}{k} \left(\frac{(n-k)^2 e\lambda}{2(m-l)n}\right)^{m-l} p^l q^{n-k-m+l} \exp(-\tfrac{1}{2}\lambda n + \mathrm{o}(n)) \\
&= p^l c(j,k,l) \binom{n}{k} \left(\frac{(1-\theta)^2\lambda}{2(\psi-\xi)}\right)^{m-l} q^{n-k-m+l} \exp(m - l - \tfrac{1}{2}\lambda n + \mathrm{o}(n)) \\
&= p^l e^{-l} c(j,k,l) \binom{n}{k} q^{n-k} \exp(m - \tfrac{1}{2}\lambda n + \mathrm{o}(n)),
\end{aligned}$$

where we used (6.4) in the last step.

We shall be interested only in values of λ and roots θ of (2.3) satisfying

$$\text{either } \theta > 0, \text{ or } \theta = 0 \text{ and } \lambda \leq q. \tag{6.6}$$

We claim that, under these assumptions, (6.5) is an equality:

$$\widetilde{P}(m, j, k, l; n, p, q) = p^l e^{-l} c(j, k, l) \binom{n}{k} q^{n-k} \exp(m - \tfrac{1}{2}\lambda n + o(n)). \tag{6.7}$$

To see this when either $\theta > 0$, or $\theta = 0$ and $\lambda < q$, set $n_0 = n - k$ and $m_0 = m - l$, and observe that

$$\frac{m_0}{n_0} = \frac{m - l}{n - k} \to \frac{\psi - \xi}{1 - \theta} = \frac{\lambda}{2q}(1 - \theta) < \frac{1}{2},$$

where we have used the fact that, by (2.3),

$$\lambda\theta < e^{\lambda\theta} - 1 = \frac{q\theta}{1 - \theta} \quad \text{when } \theta > 0.$$

Hence in this case, the 'fixed edge-number' random graph $G(n_0, m_0)$ has average vertex degree not exceeding $1 - \epsilon$ for some positive constant ϵ independent of n, m, k, l. Therefore there exists a positive constant δ such that

$$\mathbb{P}\big(G(n_0, m_0) \text{ is a forest}\big) > \delta,$$

and hence

$$f(n_0, m_0) > \delta \binom{\binom{n_0}{2}}{m_0};$$

this implies (6.7), via (6.1) and (6.5). When $\theta = 0$ and $\lambda = q$, we have that $m_0/n_0 \to \psi = \frac{1}{2}$, and hence

$$f(n_0, m_0) = \binom{\binom{n_0}{2}}{m_0} \exp(o(n)), \tag{6.8}$$

implying (6.7). To see (6.8), note that with $0 < \epsilon < \frac{1}{4}$ and $s \asymp \epsilon n$ we have for large n, rather crudely by counting only forests where vertex 1 is an endpoint of an isolated path of length $s - 1$, that

$$\begin{aligned} f(n_0, m_0) &\geq (n_0 - 1)_{s-1} f(n_0 - s, m_0 - s + 1) \\ &\geq e^{-\epsilon n} n_0^s \binom{\binom{n_0 - s}{2}}{m_0 - s + 1} \geq e^{2\log(1-\epsilon)n} \binom{\binom{n_0}{2}}{m_0}. \end{aligned}$$

Alternatively, we may use the detailed asymptotics for $f(r, s)$ of Britikov (1988).

We estimate $c(j, k, l)$ next. Suppose first that $\theta = 0$. Then $c(j, k, l)$ is no greater than the total number of graphs with k vertices and l edges, i.e.,

$$\begin{aligned} p^l e^{-l} c(j, k, l) &\leq \left(\frac{p}{e}\right)^l \left(\frac{k^2}{2}\right)^l \frac{1}{l\,!} \leq \left(\frac{\lambda n}{l}\right)^l \\ &= \exp\left\{l\left(\log\lambda - \log\frac{l}{n}\right)\right\} = \exp(o(n)). \end{aligned}$$

Equality holds here for some suitable triple j, k, l: just set $j = k = l = 0$, for which $p^l e^{-l} c(j, k, l) = 1$. It is easily checked that $\binom{n}{k} = \exp(\mathrm{o}(n))$ when $\theta = 0$, and therefore

$$p^l e^{-l} c(j, k, l) \le \binom{n}{k}^{-1} \exp(\mathrm{o}(n)) \tag{6.9}$$

with equality for some suitable j, k, l.

Our estimate of $c(j, k, l)$ when $\theta > 0$ uses the fact that $\widetilde{P}(\cdot\,; n, p, q)$ is a probability measure when $q = 1$. Suppose $\theta > 0$, define $n_1 = n_1(\theta) = \lfloor \theta n + r(1-\theta)n \rfloor$ where $r = q^{-1}$ as usual, and set

$$\begin{aligned} m_1 &= l + r(m - l) + \mathrm{o}(n) = \{\xi + r(\psi - \xi)\}n + \mathrm{o}(n), \\ \lambda_1 &= \{\theta + r(1-\theta)\}\lambda. \end{aligned}$$

Then

$$\frac{m_1}{n_1} \to \psi_1 = \frac{\xi + r(\psi - \xi)}{\theta + r(1-\theta)} = \frac{1}{2}\lambda_1, \tag{6.10}$$

$$\frac{k}{n_1} \to \theta_1 = \frac{\theta}{\theta + r(1-\theta)}, \tag{6.11}$$

$$\frac{l}{n_1} \to \xi_1 = \frac{\xi}{\theta + r(1-\theta)}, \quad \frac{j}{n_1} \to 0; \tag{6.12}$$

it is easy to check the analogues of (2.3), (6.3), and (6.4), viz.

$$e^{-\lambda_1\theta_1} = 1 - \theta_1, \quad \xi_1 = \lambda_1\theta_1(1 - \tfrac{1}{2}\theta_1), \quad \psi_1 = \xi_1 + \tfrac{1}{2}\lambda_1(1-\theta_1)^2. \tag{6.13}$$

Now (6.7) is valid with $q = 1$, since $\theta > 0$. Hence

$$\begin{aligned} 1 &\ge \widetilde{P}(m_1, j, k, l; n_1, p_1, 1) \\ &= p_1^l e^{-l} c(j, k, l) \binom{n_1}{k} \exp(m_1 - \tfrac{1}{2}\lambda_1 n_1 + \mathrm{o}(n)) \\ &= p^l e^{-l} c(j, k, l) \binom{n_1}{k} \exp(\mathrm{o}(n)) \end{aligned} \tag{6.14}$$

by (6.10), where $p_1 = \lambda_1/n_1 = p(1 + \mathrm{O}(n^{-1}))$. Therefore

$$p^l e^{-l} c(j, k, l) \le \binom{n_1}{k}^{-1} \exp(\mathrm{o}(n)). \tag{6.15}$$

We claim that there exist suitable j, k, l such that equality holds in (6.15). To see this, note that $G(n_1, p_1)$ has $\binom{n_1}{2} p_1 + \mathrm{o_p}(n_1)$ edges, a giant component with $\theta_1 n_1 + \mathrm{o_p}(n_1)$ vertices and $\xi_1 n_1 + \mathrm{o_p}(n_1)$ edges, $\mathrm{O_p}(1)$ unicyclic components with a total of $\mathrm{O_p}(1)$ vertices and edges, and no other cyclic components (see Bollobás

(1985), Thm VI.11). By considering the number of possible combinations of values of m_1, j, k, l satisfying the above constraints, we find that there must exist m_1, j, k, l such that

$$\widetilde{P}(m_1, j, k, l; n_1, p_1, 1) \geq n^{-4}$$

for all large n. Combining this with (6.14), we deduce that equality holds in (6.15) for some suitable j, k, l.

In conclusion, we have that, whatever the root θ of (2.3) (subject to (6.6)), inequality (6.15) holds with equality for some suitable j, k, l, and where $n_1 = n_1(0)$ is interpreted as n (i.e., when $\theta = 0$). We substitute (6.15) into (6.5) to obtain

(6.16)

$$\begin{aligned}
\widetilde{P}(m, j, k, l; n, p, q) &\leq \binom{n_1}{k}^{-1}\binom{n}{k} q^{n-k}\exp(m - \tfrac{1}{2}\lambda n + \mathrm{o}(n)) \\
&= \frac{n^n}{(n-k)^{n-k}}\frac{(n_1-k)^{n_1-k}}{n_1^{n_1}} q^{n-k}\exp\left(\frac{\lambda}{2q}\left\{1+(q-1)\theta^2\right\}n - \tfrac{1}{2}\lambda n + \mathrm{o}(n)\right) \\
&= \Bigg[\frac{\{r(1-\theta)\}^{r(1-\theta)}}{(1-\theta)^{1-\theta}\{\theta + r(1-\theta)\}^{\theta+r(1-\theta)}} q^{1-\theta} \times \\
&\qquad\qquad \exp\left(\frac{\lambda}{2q}\left\{1+(q-1)\theta^2\right\} - \tfrac{1}{2}\lambda + \mathrm{o}(1)\right)\Bigg]^n \\
&= \exp\left(n\left[\frac{g(\theta)}{2q} - \frac{q-1}{2q}\lambda + \log q + \mathrm{o}(1)\right]\right), \qquad \blacksquare
\end{aligned}$$

where

(6.17) $g(\theta) = -(q-1)(2-\theta)\log(1-\theta) - \{2+(q-1)\theta\}\log\{1+(q-1)\theta\}.$

To obtain the second line of (6.16), we have used (6.3) and (6.4). To pass to the last line, we used the fact that θ is a root of (2.3), to enable the substitution

$$\exp\left(\frac{\lambda}{2q}\left\{1+(q-1)\theta^2\right\}\right) = e^{\lambda/(2q)}\left\{\frac{1+(q-1)\theta}{1-\theta}\right\}^{(q-1)\theta/(2q)}.$$

We have in addition that equality holds in (6.16) for at least one suitable choice of j, k, l. Let $\theta^* = \theta^*(\lambda)$ be the root of (2.3) which maximizes $g(\theta)$ and which satisfies (6.6) [we shall see that there is a unique such θ^*, except when $\lambda = \lambda_c(q)$ and $q > 2$]. By (6.16) and the equality observed above,

(6.18)

$$\begin{aligned}
Z(n, p, q) &= \sum_{m,j,k,l} \widetilde{P}(m, j, k, l; n, p, q) \\
&\geq \exp\left\{n\left[\frac{g(\theta^*)}{2q} - \frac{q-1}{2q}\lambda + \log q + \mathrm{o}(1)\right]\right\},
\end{aligned}$$

whence

$$\liminf_{n\to\infty}\left\{\frac{1}{n}\log Z(n,p,q)\right\} \geq \frac{g(\theta^*)}{2q} - \frac{q-1}{2q}\lambda + \log q. \tag{6.19}$$

On the other hand, by Lemmas 4.1 and 4.2, there exists a root θ of (2.3) satisfying (6.6), and a function $\omega(n)$ satisfying $\omega(n) \to \infty$, such that

$$\liminf_{n\to\infty} \mathbb{P}\big(|\Theta_n - \theta| < \omega(n)^{-1}\big) > 0. \tag{6.20}$$

For such θ there exist, by Lemma 6.1 and Theorem 5.1, sequences m, j, k, l satisfying (6.2)–(6.4) and such that

$$1 \geq \frac{\widetilde{P}(m,j,k,l;n,p,q)}{Z(n,p,q)} \geq n^{-4}$$

for all large n (this is shown by considering the number of possible combinations of m, j, k, l satisfying (6.2)–(6.4) and the above-mentioned results). By (6.16), this yields

$$\limsup_{n\to\infty}\left\{\frac{1}{n}\log Z(n,p,q)\right\} \leq \frac{g(\theta)}{2q} - \frac{q-1}{2q}\lambda + \log q, \tag{6.21}$$

which, by (6.19), implies $g(\theta) \geq g(\theta^*)$, and therefore $\theta = \theta^*$. Combining (6.19) and (6.21), we obtain Theorem 2.6. Furthermore, θ^* is the only root of (2.3) satisfying (6.6) such that (6.20) holds for some $\omega(n)$. Therefore

$$\Theta_n \xrightarrow{\mathrm{P}} \theta^* \quad \text{as } n \to \infty. \tag{6.22}$$

Next we calculate $\theta^*(\lambda)$. As in Theorem 4.3, when $1 \leq q \leq 2$, $\theta^*(\lambda)$ is the largest non-negative root of (2.3). Assume that $q > 2$. By straightforward computation,

$$g(0) = g\left(\frac{q-2}{q-1}\right) = 0, \quad g'(0) = 0,$$
$$g''(\theta) = -\frac{q(q-1)\{q-2-2(q-1)\theta\}\theta}{(1-\theta)^2\{1+(q-1)\theta\}^2}.$$

Therefore $g''(\theta)$ has a unique zero in (0, 1), at $\theta = \frac{1}{2}(q-2)/(q-1)$; at this point, $g'(\theta)$ has a negative minimum. It follows that $g(\theta) < 0$ on $(0, \theta_0)$, and $g(\theta) > 0$ on $(\theta_0, 1)$ where $\theta_0 = (q-2)/(q-1)$.

Substituting θ_0 into (2.3), we find that θ_0 satisfies (2.3) if

$$\lambda = \lambda_c(q) = 2\left(\frac{q-1}{q-2}\right)\log(q-1),$$

and, for this value of λ, the three roots of (2.3) are $0, \frac{1}{2}\theta_0, \theta_0$. It follows that $\lambda_{\min} < \lambda_c(q) < q$, and that $\theta^* = 0$ for $\lambda < \lambda_c(q)$, and $\theta^* = \theta_{\max}(\lambda)$ for $\lambda > \lambda_c(q)$. Finally, when $\lambda = \lambda_c(q)$, both $\theta = 0$ and $\theta = \theta_{\max}(\lambda)$ give the same value of $g(\theta)$ (namely 0), while the third root yields a smaller value. Hence we can in this case conclude only that

$$\mathbb{P}\Big(\Theta_n < \omega(n)^{-1} \text{ or } |\Theta_n - \theta_{\max}(\lambda_c(q))| < \omega(n)^{-1}\Big) \to 1$$

for some $\omega(n)$ satisfying $\omega(n) \to \infty$.

This completes the proof of the assertions concerning the order of the largest component. The claims concerning the number of edges in $G(n, p, q)$ and in the giant component follow by Theorem 5.1. We omit explicit proofs of the remaining assertions about the structure of $G(n, p, q)$; as in Lemma 6.1, these follow easily by the colouring argument from known facts for $G(n, p)$; see Bollobás (1985) and Łuczak, Pittel, and Wierman (1994).

9.7 The nature of the singularity

Let q (≥ 1) be fixed, and consider the functions $\theta(\lambda)$ and

$$\psi(\lambda) = \frac{\lambda}{2q}\{1 + (q-1)\theta(\lambda)^2\}, \quad \xi(\lambda) = \frac{\lambda}{q}\{\theta(\lambda) + (\tfrac{1}{2}q - 1)\theta(\lambda)^2\},$$

describing the order of the giant component, and the numbers of edges in the graph and in its giant component, respectively. All three functions are nondecreasing on $(0, \infty)$. Actually ψ is strictly increasing, while $\theta(\lambda)$ and $\xi(\lambda)$ equal 0 for $\lambda < \lambda_c$ and are strictly increasing on $[\lambda_c, \infty)$.

A fourth function of interest is the free energy $\phi(\lambda)$ given in Theorem 2.6. All these four functions are real analytic on $(0, \infty)$ except at λ_c. At the singularity λ_c, the following is valid, as may be verified with ease.

(a) Assume $q < 2$. Then θ, ψ, ξ, and ϕ are continuous at $\lambda_c(q) = q$. The functions θ and ξ have discontinuous first derivatives at λ_c; in fact

$$\theta'(\lambda_c-) = \xi'(\lambda_c-) = 0, \quad \theta'(\lambda_c+) = \xi'(\lambda_c+) = \frac{2}{q(2-q)}.$$

In particular

$$\theta(\lambda) \sim \frac{2(\lambda - \lambda_c)}{q(2-q)} \quad \text{as } \lambda \downarrow \lambda_c.$$

Similarly, ψ' and ϕ'' are continuous, but ψ'' and ϕ''' have discontinuities at λ_c, except when $q = 1$.

(b) Assume $q = 2$. Still θ, ψ, ξ, and ϕ are continuous at λ_c. In this case

$$\theta(\lambda) \sim \xi(\lambda) \sim \{\tfrac{3}{2}(\lambda - \lambda_c)\}^{1/2} \quad \text{as } \lambda \downarrow \lambda_c;$$

thus $\theta'(\lambda_c+) = \xi'(\lambda_c+) = \infty$. Also, ψ' has a jump at λ_c: $\psi'(\lambda_c-) = \frac{1}{4}$, $\psi'(\lambda_c+) = 1$. And ϕ' is continuous, but ϕ'' has a jump at λ_c: $\phi''(\lambda_c-) = 0$, $\phi''(\lambda_c+) = \frac{3}{8}$. The functions ψ and ϕ are real analytic on $(0, \lambda_c]$ and on $[\lambda_c, \infty)$.

(c) Assume $q > 2$. Then θ, ψ, and ξ have jumps at λ_c, and it may be verified that $\psi(\lambda_c-) = \lambda_c/(2q) < \frac{1}{2} < \psi(\lambda_c+)$. The free energy ϕ is continuous at λ_c, but its derivative ϕ' has a jump at λ_c:

$$\phi'(\lambda_c-) = -\frac{q-1}{2q}, \quad \phi'(\lambda_c+) = -\frac{2q-3}{2q(q-1)}.$$

9.8 Large deviations

Let C_n be the number of components of the graph $G(n, \lambda/n, q)$. We show next how the exact calculation of the free energy in Theorem 2.6 may be used to estimate probabilities of the form $\mathbf{P}(C_n \le \alpha n)$ and $\mathbf{P}(C_n \ge \beta n)$ for given constants α, β. In the case $q = 1$, this gives new information about the probabilities of large deviations of C_n. (We will use Theorem 2.6 also for $q < 1$, even though this case is not proven until the next section.)

As in the language of large-deviation theory (see Dembo and Zeitouni (1993)), we set

$$\Lambda_{n,\lambda,q}(\nu) = \log \mathbb{E}\big(\exp\{\nu C_n/n\}\big), \quad \nu \in \mathbb{R},$$

and note that

$$\Lambda_{n,\lambda,q}(\nu) = \log\left\{\frac{Z(n, \lambda/n, qe^{\nu/n})}{Z(n, \lambda/n, q)}\right\},$$

whence

$$\frac{1}{n}\,\Lambda_{n,\lambda,q}(n\nu) \to \Lambda_{\lambda,q}(\nu) = \phi(\lambda, qe^{\nu}) - \phi(\lambda, q) \tag{8.1}$$

as $n \to \infty$ (here, $\phi(\lambda, q)$ denotes the free-energy function of Theorem 2.6). We define the Legendre transform of $\Lambda_{\lambda,q}$ by

$$\Lambda^*_{\lambda,q}(x) = \sup_{\nu\in\mathbb{R}}\{\nu x - \Lambda_{\lambda,q}(\nu)\}, \quad x \in \mathbb{R}. \tag{8.2}$$

It may be proved directly (or see Dembo and Zeitouni (1993), Lemma 2.3.9) that $\Lambda_{\lambda,q}$ and $\Lambda^*_{\lambda,q}$ are convex functions, and that

$$\Lambda^*_{\lambda,q}(x) = \delta x - \Lambda_{\lambda,q}(\delta) \quad \text{if} \quad \Lambda'_{\lambda,q}(\delta) = x. \tag{8.3}$$

Since we have an exact formula for $\Lambda_{\lambda,q}$ (see (8.1) and (2.6)), we may compute its derivative whenever it exists (see Theorem 8.2). As a consequence of such computations, we find that

$$\Lambda^*_{\lambda,q}(x) \begin{cases} < \infty & \text{if } 0 \le x \le 1 \\ = \infty & \text{otherwise.} \end{cases}$$

(The cases $x = 0, 1$ are not used in the sequel, and therefore we omit some extra details required for these cases.)

A large-deviation principle (LDP) may be established for $n^{-1}C_n$, in terms of the 'rate function' $\Lambda^*_{\lambda,q}$. The details of the LDP depend on the set of points x at which $\Lambda^*_{\lambda,q}$ is strictly convex, and we investigate this next. There is a slight complication in this, arising from the discontinuity of the phase transition when $q > 2$. The proof of the following theorem is deferred to the end of this section.

Theorem 8.2. *Let* $\lambda, q > 0$.

(a) *The derivative*

$$\kappa(\lambda, q) = q\frac{\partial\phi(\lambda, q)}{\partial q} \tag{8.4}$$

exists except when $\lambda = \lambda_c(q)$ *and* $q > 2$, *and satisfies*

$$\kappa(\lambda, q) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(C_n).$$

Furthermore

$$\kappa(\lambda, q) = 1 - \theta(\lambda) - \big(1 - \theta(\lambda)\big)^2 \frac{\lambda}{2q}. \tag{8.5}$$

When $\lambda = \lambda_c(q)$ *and* $q > 2$, *the limits*

$$\kappa^{\pm}(\lambda, q) = q\frac{\partial\phi(\lambda, q\pm)}{\partial q}$$

exist with $\kappa^-(\lambda, q) < \kappa^+(\lambda, q)$. *Also,* $\kappa^-(\lambda, q)$ *is given by the formula in (8.5), and*

$$\kappa^+(\lambda, q) = 1 - \frac{\lambda}{2q}.$$

(b) *Suppose* $\lambda \le 2$. *Then* $\Lambda_{\lambda,q}$ *is differentiable on* $\mathbb{R}$, *and* $\Lambda^*_{\lambda,q}$ *is strictly convex on* $(0, 1)$.

(c) *Suppose* $\lambda > 2$, *and let* Q *be such that* $\lambda = \lambda_c(Q)$. *Then* $\Lambda_{\lambda,q}$ *is differentiable except at the point* ν *satisfying* $qe^\nu = Q$, *at which point it has left* *(resp. right) derivative $\kappa^-(\lambda, Q)$ (resp. $\kappa^+(\lambda, Q)$). Also, $\Lambda^*_{\lambda,q}$ is strictly convex on $(0,1)\backslash\big(\kappa^-(\lambda, Q), \kappa^+(\lambda, Q)\big)$ and is linear on the deleted subinterval. Furthermore, $\Lambda^*_{\lambda,q}$ is differentiable on $(0,1)$.*

(d) *Suppose that either $\lambda \le 2$, or $\lambda > 2$ and $\lambda \ne \lambda_c(q)$. Then the derivative $\Lambda'_{\lambda,q}(0) = \kappa(\lambda, q)$ exists, and $\Lambda^*_{\lambda,q}$ is strictly decreasing (resp. increasing) and strictly positive on $\big(0, \kappa(\lambda, q)\big)$ (resp. $\big(\kappa(\lambda, q), 1\big)$), while $\Lambda^*_{\lambda,q}(\kappa(\lambda, q)) = 0$.*

(e) *Suppose $\lambda > 2$ and $\lambda = \lambda_c(q)$. Then $\Lambda^*_{\lambda,q}$ is strictly decreasing (resp. increasing) and strictly positive on $\big(0, \kappa^-(\lambda, q)\big)$ (resp. $\big(\kappa^+(\lambda, q), 1\big)$), while $\Lambda^*_{\lambda,q}(x) = 0$ on $[\kappa^-(\lambda, q), \kappa^+(\lambda, q)]$.*

We write $\mathcal{F}_{\lambda,q}$ for the set of 'exposed points' of $\Lambda^*_{\lambda,q}$, i.e., following (b) and (c) above,

$$\mathcal{F}_{\lambda,q} = \begin{cases} (0,1) & \text{if } \lambda \le 2 \\ (0,1)\backslash\big[\kappa^-(\lambda, Q), \kappa^+(\lambda, Q)\big] & \text{if } \lambda > 2 \end{cases} \tag{8.6}$$

where Q satisfies $\lambda = \lambda_c(Q)$. The following LDP is a consequence of Thm 2.3.6 of Dembo and Zeitouni (1993).

Theorem 8.3. *Assume $\lambda, q > 0$.*

(a) *For any closed subset F of $\mathbb{R}$,*

$$\limsup_{n\to\infty} \left\{ \frac{1}{n} \log \mathbf{P}(n^{-1}C_n \in F) \right\} \le -\inf_{x\in F} \Lambda^*_{\lambda,q}(x).$$

(b) *For any open subset G of $\mathbb{R}$,*

$$\liminf_{n\to\infty} \left\{ \frac{1}{n} \log \mathbf{P}(n^{-1}C_n \in G) \right\} \ge -\inf_{x\in G\cap\mathcal{F}_{\lambda,q}} \Lambda^*_{\lambda,q}(x).$$

Of especial interest are the cases when F takes the form $[0,\alpha]$ or $[\beta, 1]$, analysed as follows using Theorem 8.3.

(a) Suppose $q \le 2$. Then, as $n \to \infty$,

$$\frac{1}{n} \log \mathbf{P}(C_n \le \alpha n) \to -\Lambda^*_{\lambda,q}(\alpha), \tag{8.7}$$

$$\frac{1}{n} \log \mathbf{P}(C_n \ge \beta n) \to -\Lambda^*_{\lambda,q}(\beta), \tag{8.8}$$

whenever $0 < \alpha \le \kappa(\lambda, q) \le \beta < 1$.

(b) Suppose $q > 2$ and $\lambda = \lambda_c(q)$. Then (8.7) and (8.8) hold for $0 < \alpha \le \kappa^-(\lambda, q) < \kappa^+(\lambda, q) \le \beta < 1$.

(c) Suppose $q > 2$ and $\lambda \neq \lambda_c(q)$. Let Q be such that $\lambda = \lambda_c(Q)$. We have that (8.7) and (8.8) hold for any α, β satisfying $0 < \alpha \leq \kappa(\lambda, q) \leq \beta < 1$ except possibly when

$$(8.9) \qquad \kappa^-(\lambda, Q) < \alpha \leq \kappa^+(\lambda, Q) \quad \text{or} \quad \kappa^-(\lambda, Q) \leq \beta < \kappa^+(\lambda, Q).$$

(Note that $\kappa^+(\lambda, Q) < \kappa(\lambda, q)$ if $Q < q$, and $\kappa^-(\lambda, Q) > \kappa(\lambda, q)$ if $Q > q$, so that only one of the cases in (8.9) can occur for any given value of q.) We do not know whether the appropriate inferior and superior limits of (8.7) and (8.8) coincide when (8.9) holds. Suppose that $Q < q$ and α satisfies the inequality of (8.9). In this case, we have from the LDP that (only)

$$\begin{aligned} -\Lambda^*_{\lambda,q}\big(\kappa^-(\lambda, Q)\big) &\leq \liminf_{n\to\infty} \left\{ \frac{1}{n} \log \mathbf{P}(C_n \leq \alpha n) \right\} \\ &\leq \limsup_{n\to\infty} \left\{ \frac{1}{n} \log \mathbf{P}(C_n \leq \alpha n) \right\} \leq -\Lambda^*_{\lambda,q}(\alpha). \end{aligned}$$

Similar inequalities hold for $\mathbf{P}(C_n \geq \beta n)$ when $Q > q$ and β satisfies the appropriate part of (8.9). A complete account of the limits in (8.7) and (8.8), under (8.9), would be provided by a more detailed analysis of first-order phase transitions than that presented in this paper.

We summarise the above facts before proving Theorem 8.2. Except in the special case when $\lambda = \lambda_c(q)$ and $q > 2$, the limit $\kappa = \lim_{n\to\infty}\{n^{-1}\mathbb{E}(C_n)\}$ exists, and $\mathbf{P}(C_n \leq \alpha n)$, $\mathbf{P}(C_n \geq \beta n)$ decay at least exponentially when $\alpha < \kappa < \beta$. The exact (exponential) rate of decay can be determined except when the levels αn and βn lie within the discontinuity of a first-order phase transition. In the exceptional case ($\lambda = \lambda_c(q)$ and $q > 2$), a similar conclusion holds when $\alpha < \kappa^-$ and $\beta > \kappa^+$, for suitable κ^-, κ^+. (We do not know whether $\lim_{n\to\infty}\{n^{-1}\mathbb{E}(C_n)\}$ exists in the exceptional case.)

Since first-order transitions occur only when $q > 2$, and since the critical λ-values of such q fill the interval $(2, \infty)$, there is a weak sense in which the value $\lambda = 2$ marks a singularity of the asymptotics of the random graph $G(n, \lambda/n, q)$. This holds for any value of q, including $q = 1$. Therefore the usual Erdős–Rényi random graph senses the existence of a first-order phase transition in the random-cluster model, but only through its large deviations.

Proof of Theorem 8.2. (a) The function

$$\zeta_n(\nu) = \frac{1}{n} \log Z(n, \lambda/n, e^\nu), \qquad \nu \in \mathbb{R},$$

is convex (as may be seen directly by differentiating twice and using Schwarz's inequality) and has derivative

$$\zeta_n'(\nu) = \frac{1}{n}\,\mathbb{E}(C_{n,e^\nu}).$$

It follows that

$$\zeta(\nu) = \lim_{n\to\infty} \zeta_n(\nu) = \phi(\lambda, e^\nu),$$

is convex and satisfies

$$\zeta_n'(\nu) \to \zeta'(\nu) \quad \text{if } \zeta'(\nu) \text{ exists}.$$

It may be verified from the formula for ϕ in Theorem 2.6 that ζ is differentiable on $(-\infty, \infty)$ except for pairs (λ, ν) such that $\lambda = \lambda_c(e^\nu)$ and $e^\nu > 2$, cf. the similar results for the λ-derivative $\partial\phi/\partial\lambda$ in Section 7. Moreover, with $q = e^\nu$, it is found that $\zeta' = q\,\partial\phi/\partial q = \kappa(\lambda, q)$.

The explicit formulae for $\kappa(\lambda, q)$ may be derived in a number of ways. One is to calculate, using the value of ϕ in Theorem 2.6. Another is to use the colouring method, together with the fact that $\kappa(\lambda, q)$ is the limit, as $n \to \infty$, of the mean of $|C_1|^{-1}$, where C_1 is the order of the component of $G(n, \lambda/n, q)$ containing the vertex 1 (cf. Grimmett (1989), Thm 4.2). We note the implication that $\mathbf{E}(Z^{-1}) = 1 - \frac{1}{2}\lambda$, where Z is the total size of a branching process with Poisson-distributed family-sizes having parameter λ (< 1). Cf. Jagers (1975), Thm 2.11.2.

The existence and values of the left and right derivatives $\kappa^\pm(\lambda, q)$ follow from the convexity of ζ.

(b,c) The differentiability of $\Lambda_{\lambda,q}$ follows from (8.1) and the above remarks. The strict convexity of $\Lambda^*_{\lambda,q}$ on $\mathcal{F}_{\lambda,q}$ (defined in (8.6)) follows from Lemma 2.3.9 of Dembo and Zeitouni (1993); the linearity of $\Lambda^*_{\lambda,q}$ on the deleted sub-interval (in (c)) follows by a computation involving (8.2). Using (8.3), we find that the derivative of $\Lambda^*_{\lambda,q}$ equals $\big(\Lambda'_{\lambda,q}\big)^{-1}$, whenever the latter quantity is uniquely defined. It follows from (a) and the strict convexity of $\Lambda_{\lambda,q}$ that $\Lambda^*_{\lambda,q}$ is differentiable off the interval $\big[\kappa^-(\lambda, Q), \kappa^+(\lambda, Q)\big]$. Clearly it is differentiable in the interior of this interval, since it is linear there. Finally, by the above remarks, it has left and right derivatives equal to ν at both $x = \kappa^-, \kappa^+$.

(d) The existence and value of $\Lambda'_{\lambda,q}(0)$ follow by (8.1) and part (a). The monotonicity of $\Lambda^*_{\lambda,q}$ will follow from the facts that

$$\Lambda^*_{\lambda,q}(x) = \sup_{\nu>0}\{\nu x - \Lambda_{\lambda,q}(\nu)\}, \quad x > \kappa(\lambda, q), \tag{8.10}$$

$$\Lambda^*_{\lambda,q}(x) = \sup_{\nu<0}\{\nu x - \Lambda_{\lambda,q}(\nu)\}, \quad x < \kappa(\lambda, q). \tag{8.11}$$

We check (8.10) and (8.11) next. Clearly $\Lambda^*_{\lambda,q}\big(\kappa(\lambda, q)\big) = -\Lambda_{\lambda,q}(0) = 0$, by (8.3). Now, if $x > \kappa(\lambda, q)$ and $\nu < 0$, or if $x < \kappa(\lambda, q)$ and $\nu > 0$, then

$$\nu x - \Lambda_{\lambda,q}(\nu) \le \nu\kappa(\lambda, q) - \Lambda_{\lambda,q}(\nu) \le \Lambda^*_{\lambda,q}\big(\kappa(\lambda, q)\big) = 0,$$

and (8.10) and (8.11) follow. The strict monotonicity is a consequence of strict convexity together with the fact that $\Lambda^*_{\lambda,q}\big(\kappa(\lambda, q)\big) = 0$.

(e) The proof is similar to that of (d). □

9.9 The case $q < 1$

In our applications of the colouring method so far, we have assumed that $q > 1$. Assume now that $q < 1$, let $n_1 \geq 1$, and colour each component of $G(n_1, p) = G(n_1, p, 1)$ *red* with probability $r = q$, independently of the other components. By Lemma 3.1, the red graph is distributed as $G(N, p, q)$ on a random number N of vertices. This observation will enable us to draw conclusions about $G(n, p, q)$ when $q < 1$ (just as we did earlier when $q > 1$) from the case $q = 1$; this time we shall use the colouring in the 'opposite direction', going from 1 to q, rather than from q to 1. One major difference, when $q < 1$, is that we must first show that $\mathbf{P}(N = n)$ may be made reasonably large by a suitable choice of n_1.

Given $p > 0$, define $\gamma = \gamma(n_1)$ to be the largest root of $e^{-n_1 p \gamma} = 1 - \gamma$, and note that $\gamma = 0$ if and only if $n_1 p \leq 1$. Next, given n and $p = \lambda/n$, choose $n_1 \geq n$ such that

$$\left|\{\gamma(n_1) + (1 - \gamma(n_1))q\}n_1 - n\right| \leq 1. \tag{9.1}$$

This is possible since changing n_1 to $n_1 + 1$ alters $\{\gamma + (1 - \gamma)q\}n_1$ by at least q and at most 2, as may easily be verified by calculating $d(\gamma n_1)/dn_1$ (temporarily regarding n_1 as a real variable).

In the sequel we fix $\lambda > 0$, and let $c_1, c_2, \ldots$ denote suitable positive constants that may depend on λ (and on q, which is fixed in this section). Note that $n_1 p = \lambda_1 = \lambda/(\gamma + (1 - \gamma)q) + \mathrm{O}(n^{-1})$. It is easily verified that (for large n at least) $\lambda_1 > 1$ (resp. $\lambda < 1$) if $\lambda > q$ (resp. $\lambda < q$).

Lemma 9.1. *Suppose that $p = \lambda/n$, where $\lambda \neq q$ is fixed, and let n_1 satisfy (9.1). Consider the standard random graph $G(n_1, p) = G(n_1, p, 1)$, colour each component red with probability q, independently of the other components, and let N be the number of vertices in the red components. Then, for some positive constant $c_1 = c_1(\lambda)$,*

$$\mathbb{P}(N = n) \geq c_1 n^{-1/2}. \tag{9.2}$$

Proof. We consider the case $\lambda > q$, so that $\lambda_1 > 1$ for large n. The case $\lambda < q$ is similar but somewhat simpler, since it requires no special treatment for the largest component.

Let T_k, $k = 1, 2, \ldots, n_1$, denote the number of components of $G(n_1, p)$ that are trees of order k (excluding the largest component, in the unlikely event that this is itself a tree), let $M = \sum_2^{n_1} kT_k$ be the number of vertices in the tree components having at least two vertices, let L denote the order of the largest component, and let U denote the number of vertices in the remaining components. Thus

$$n_1 = \sum_{k=1}^{n_1} kT_k + L + U = T_1 + M + L + U.$$

Moreover, let T'_k be the number of trees of order k that are coloured red, let $M' = \sum_2^{n_1} kT'_k$ be the number of vertices in the red tree-components, except isolated vertices, and let U' be the number of vertices in the red cyclic components except the giant. Hence

$$N = \begin{cases} T'_1 + M' + U' + L & \text{if the giant is coloured red} \\ T'_1 + M' + U' & \text{otherwise.} \end{cases} \tag{9.3}$$

We claim that (at least for large n) with probability at least $\frac{1}{2}$, all the following four inequalities hold:

$$T_1 \geq c_2 n, \tag{9.4}$$

$$\sum_{k=2}^{n_1} k^2 T_k \leq c_3 n, \tag{9.5}$$

$$|L - \gamma n_1| \leq c_4 n^{1/2}, \tag{9.6}$$

$$U \leq c_5 n^{1/2}. \tag{9.7}$$

In fact, T_1, the number of isolated vertices in $G(n_1, p)$, is asymptotically normal with mean $\sim c_6 n_1 \sim c_7 n$ and variance $\sim c_8 n$; see Barbour (1982). Hence (9.4) holds with probability $1 - \mathrm{o}(1)$ for any $c_2 < c_7$. Similarly, $\sum_{k=2}^{n_1} k^2 T_k$ is asymptotically normal with mean $\sim c_9 n$ and variance $\sim c_{10} n$, by Pittel (1990), and (9.5) follows. Moreover, again by Pittel (1990), $(L - \gamma n_1)/n_1^{1/2}$ converges in distribution to a normal distribution, and thus (9.6) holds with probability at least $\frac{2}{3}$, say, if c_4 is sufficiently large. Finally, $U = \mathrm{O}_p(1)$, by Bollobás (1985), Thm VI.11, and thus (9.7) holds with probability $1 - \mathrm{o}(1)$ for any $c_5 > 0$.

Now condition on a fixed realization of the random graph $G(n_1, p)$ satisfying (9.4)–(9.7), and consider the colouring process. Note that, by (9.3), we have that $N = n$ if the giant component is coloured red and $T'_1 = n - M' - U' - L$; moreover, by (9.6) and (9.7),

$$\begin{aligned} &|n - M' - U' - L - qT_1| \\ &\quad = \big|n - M' - U' - L - q(n_1 - L - M - U)\big| \\ &\quad \leq |n - \{\gamma + q - \gamma q\}n_1| + (1 - q)|L - \gamma n_1| + |M' - qM| + |U' - qU| \\ &\quad \leq 1 + c_4\sqrt{n} + c_5\sqrt{n} + |M' - qM|. \end{aligned}$$

By colouring the isolated vertices after all other components, we see, using independence and the fact that T'_1 has the binomial distribution $\mathrm{Bin}(T_1, q)$ where $c_2 n \leq T_1 \leq n$ by (9.4), that

$$\begin{aligned} \mathbb{P}(N = n) &\geq \mathbb{P}(\text{the giant is red})\mathbb{P}\big(|M' - qM| \leq c_{11} n^{1/2}\big)\times \\ &\qquad \inf\Big\{\mathbb{P}(T'_1 = j) : |j - qT_1| \leq c_{12} n^{1/2}\Big\} \\ &\geq q\mathbb{P}\Big(|M' - qM| \leq c_{11} n^{1/2}\Big) c_{13} n^{-1/2}. \end{aligned} \tag{9.8}$$

Moreover, the (conditional) expectation of M' equals qM, and the (conditional) variance is, using (9.5),

$$\operatorname{var}(M') = \sum_{k=2}^{n_1} k^2 T_k q(1-q) \le c_3 n.$$

Thus, by Chebyshev's inequality,

$$\mathbb{P}\big(|M' - qM| \le c_{11} n^{1/2}\big) \ge 1 - \frac{c_3}{c_{11}^2} \ge \frac{1}{2}, \tag{9.9}$$

for a suitable c_{11}.

Consequently, the probability that $N = n$, conditional on (9.4)–(9.7), is at least $c_{14} n^{-1/2}$. This implies that $\mathbb{P}(N = n) \ge \frac{1}{2} c_{14} n^{-1/2}$. □

Note that Lemma 9.1 is false for the critical case $\lambda = q$. We conjecture that in this case $\mathbb{P}(N = n) \ge cn^{-2/3}$, which would enable us to extend also Theorem 2.3 to the case $q < 1$.

Lemma 9.1 may be used as follows. Suppose that Π is a graph property such that the probability that the red part of the standard random graph $G(n_1, p)$ has property Π is sufficiently small, namely $o(n^{-1/2})$. Then, by Lemmas 3.1 and 9.1,

$$\begin{aligned}
&\mathbb{P}\big(G(n, p, q) \text{ has property } \Pi\big)\\
&\qquad = \mathbb{P}\big(\text{the red part of } G(n_1, p) \text{ has property } \Pi \mid N = n\big)\\
&\qquad \le \mathbb{P}\big(\text{the red part of } G(n_1, p) \text{ has property } \Pi\big)/\mathbb{P}(N = n)\\
&\qquad = o(1).
\end{aligned}$$

This argument can be used to verify many claims about the structure of the graph $G(n, p, q)$, but we prefer to use the following sharper result.

Lemma 9.2. *Suppose that Π is a graph property that is not affected by addition or deletion of isolated vertices. Let n_1 satisfy (9.1), and colour the components of $G(n_1, p)$ as above. Then*

$$\begin{aligned}
&\mathbb{P}\big(G(n, p, q) \text{ has property } \Pi\big)\\
&\qquad \le c_{15} n^{-1/2} + c_{16} \mathbb{P}\big(\text{the red part of } G(n_1, p) \text{ has property } \Pi\big).
\end{aligned} \tag{9.10}$$

Proof. Let again T_1 be the number of isolated vertices and let T_1' be the number of these that are coloured red. We observe first that, by a straightforward application of Chebyshev's inequality, (9.4) holds with probability $1 - O(n^{-1})$, and therefore

$$\mathbb{P}(T_1' < c_{17} n) \le c_{18} n^{-1}. \tag{9.11}$$

We do the colouring of $G(n_1, p)$ in two stages. First colour all components with at least two vertices and let G' denote the resulting coloured graph; then colour also the isolated vertices. Let G_r be the red part of the final graph and G'_r the red part of G'. By assumption, G_r has property Π if and only if G'_r has. Now, condition on G' and the colours of components thereof; note that this fixes G'_r and T_1. If G' is such that $T_1 \geq c_1 n$, and m is the number of red vertices in G', then

$$\mathbb{P}\big(N = n \mid G'\big) = \mathbb{P}\big(T'_1 = n - m \mid G'\big) \leq c_{19} T_1^{-1/2} \leq c_{20} n^{-1/2},$$

because T'_1 has the binomial distribution $\mathrm{Bin}(T_1, q)$. Consequently,

$$\begin{aligned}
&\mathbb{P}(G_r \text{ has property } \Pi,\ N = n) \\
&\qquad \leq \mathbb{P}(T_1 < c_1 n) + \mathbb{P}\big(G'_r \text{ has property } \Pi,\ T_1 \geq c_1 n,\ N = n\big) \\
&\qquad \leq \mathbb{P}(T_1 < c_1 n) + c_{20} n^{-1/2} \mathbb{P}\big(G'_r \text{ has property } \Pi,\ T_1 \geq c_1 n\big) \\
&\qquad \leq c_{18} n^{-1} + c_{20} n^{-1/2} \mathbb{P}(G_r \text{ has property } \Pi).
\end{aligned}$$

The estimate (9.10) now follows by Lemma 9.1 and the fact that

$$\begin{aligned}
\mathbb{P}\big(G(n, p, q) \text{ has property } \Pi\big) &= \mathbb{P}(G_r \text{ has property } \Pi \mid N = n) \\
&= \frac{\mathbb{P}(G_r \text{ has property } \Pi,\ N = n)}{\mathbf{P}(N = n)}. \qquad \square
\end{aligned}$$

The claims in Theorems 2.1 and 2.2 for $q < 1$ now follow almost immediately from the corresponding results for $G(n_1, p, 1)$. In obtaining these theorems, we fix n and λ, we set $p = \lambda/n$ and choose n_1 to satisfy (9.1), and we then let $n \to \infty$. For the supercritical case we also need the easy fact that

$$\mathbb{P}\big(N = n, \text{ and the giant component is not coloured red}\big) = \mathrm{o}(n^{-1/2}),$$

when n_1 is chosen to satisfy (9.1). For the assertion on the largest tree, we use the fact that for every fixed m, a.e. $G(n_1, p, 1)$ contains m trees of the given size, see Bollobás (1985), Thms V.10 and VI.11. Thus the probability that the red part contains no such tree is at most $(1 - q)^m + \mathrm{o}(1)$, which can be made arbitrarily small.

The same method implies the conclusion of Theorem 5.1 when $q < 1$ and $\lambda \neq q$. The calculations in Section 6 hold without change, which implies the asymptotic formula for the partition function in Theorem 2.6 when $\lambda \neq q < 1$. By certain properties of monotonicity of the partition function as a function of either q or λ, we have that Theorem 2.6 holds also for $\lambda = q$.

Appendix

Historical Observations

The basic theory of the random-cluster process was enunciated in the series of papers of Kees Fortuin and Piet Kasteleyn around 1970, and in the 1971 doctoral thesis of Fortuin. Contemporaneously, and in collaboration with Ginibre, these authors established the FKG inequality for functions and measures on finite distributive lattices. This early work on the random-cluster model contains several of the main ingredients of Chapter 2 of the current work. The impact of the approach was attenuated by the combinatorial style of the papers.

In response to an enquiry concerning the discovery of random-cluster measures, Piet Kasteleyn kindly contributed the following historical material, extracted from a letter to the present author dated 11 November 1992.

You asked me about the origin of Kees Fortuin's and my ideas on the random-cluster model. I have excavated my recollections and here, and in the subsequent pages, is what came up.

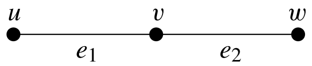
When in the late '60s Fortuin came from the Technical University of Delft to Leiden for a PhD study, I had for some time been intrigued by a similarity between a number of very elementary facts concerning three different models defined on finite graphs. I was at that time actively interested in graph theory and I had begun to toy with a few ideas in order to find out if there was more behind this similarity than sheer accidents or trivialities.

I told Fortuin about the data that had struck me and proposed him to look closer at them and at related problems. So we began to cooperate and first attacked the case of finite graphs. When this led to success, we turned to infinite graphs. Since Kees was very good, he mastered in a short time the necessary mathematics and began to work more and more independently. It became a good piece of PhD research, of which the results were set down in his thesis and (identically) in our and his papers on the percolation model and the random-cluster model. The details you find on the following pages. As you will see, the first few steps were all extremely simple. Therefore you may find my account unnecessarily detailed. I found it fun, however, to go through this history once again.

Let $G = (V, E)$ be a finite connected graph with vertex set V and edge set E (multiple edges allowed).

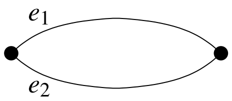
A. Consider a function $R : E \to [0, \infty)$. Then (G, R) may be considered as representing an *electric network* consisting of 'branches' (resistors) and 'nodes' (for brevity identify edges with branches and nodes with vertices), where $R(e)$ is the resistance of the branch e.

(i) Suppose E contains two edges e_1 and e_2, placed 'in series' (i.e. having end points (u, v) and (v, w), respectively, with $u \neq w$):



Write $R(e_1) = R_1$, $R(e_2) = R_2$. We can replace e_1 and e_2 by a *single* new edge e between u and w without affecting the electric currents and potentials that arise in the rest of the network when a potential difference is imposed on an arbitrary pair of nodes ($\neq v$), *provided* we attribute to e a resistance $R(e) = R = R_1 + R_2$.

(ii) Suppose next that E contains two edges e_1 and e_2 which are 'parallel' (i.e. which have the same pair of end points):



We can now replace e_1 and e_2 by a single edge e without affecting the rest of the network, as before, provided we choose $R(e) = R$ according to the following 'substitution rule':

$$R = \frac{R_1 R_2}{R_1 + R_2} \,.$$

B. Consider a function $J : E \to [0, \infty)$. Then (G, J) may be considered as representing an *Ising model* consisting of spins taking values ± 1, having ferromagnetic interactions described by the Hamiltonian

$$H(\sigma) = -\sum_{e \in E} J(e)\sigma(e) \,,$$

where $\sigma : V \to \{-1, 1\}$ and $\sigma(e) = \sigma(u)\sigma(v)$ where u and v are the end points of e. The probability of the spin state σ is

$$\pi(\sigma) = Z^{-1} e^{-\beta H(\sigma)}$$

with Z the normalizing factor (partition function).

(i) Let e_1 and e_2 be two edges in series with $J_{1,2} = J(e_{1,2})$, as before. We can replace e_1 and e_2 by a single edge e without affecting the probabilities of the spin states on $V \setminus \{v\}$ *provided* we choose

$$J(e) = J = (2\beta)^{-1} \log \frac{1 + e^{2\beta(J_1+J_2)}}{e^{2\beta J_1} + e^{2\beta J_2}}\,.$$

This can be seen as the result of 'summing out' the variable $\sigma(v)$.

(ii) Let e_1 and e_2 be parallel edges. We can replace them by a single edge e without affecting the rest of the system provided we choose $J(e) = J = J_1 + J_2$. Trivial: the two similar terms in H are combined to one.

C. Consider $p : E \to [0, 1]$. Then (G, p) may be considered as representing a *percolation model* with the usual interpretation of $p(e)$.

(i) As before, with suitable translations of concepts. The substitution rule is now $p = p_1 p_2$.

(ii) As before. The substitution rule is $p = p_1 + p_2 - p_1 p_2$.

So far for the facts. *Question*: do they reflect some relation between the three systems? To answer this question we began with the following elementary steps (in which order, I do not remember exactly; the one I give here will not be far from the actual one).

1. To bring case B somewhat more in line with A and C, go over from $J(e)$ to $w(e) = \exp(-2\beta J(e))$. Then the substitution rules are

$$\text{(i)} \quad w = \frac{w_1 + w_2}{1 + w_1 w_2}; \qquad \text{(ii)} \quad w = w_1 w_2.$$

2. Introduce new variables, viz.
in case A: $R^* = R^{-1}$ (conductivity);
in case B: $w^* = (1 - w)/(1 + w) = \tanh(\beta J)$;
in case C: $p^* = 1 - p$. This reduces the substitution rules to:

	(i)	(ii)
A	$R = R_1 + R_2$	$R^* = R_1^* + R_2^*$
B	$w = \dfrac{w_1 + w_2}{1 + w_1 w_2}$	$w^* = \dfrac{w_1^* + w_2^*}{1 + w_1^* w_2^*}$
C	$p = p_1 p_2$	$p^* = p_1^* p_2^*$

3. (Sideline) Note that if G is planar and G^* is its dual, then the situation in G corresponds to  in G^* and conversely. So the starred variables can be considered as 'dual' to the original ones (note that $R^{**} = R$ etc).

4. Consider now a q-state Potts model with $\sigma : V \to \{1, 2, \ldots, q\}$ and

$$H = 2 \sum_{e \in E} J_e (1 - \delta_\sigma(e))$$

with $J(e) \geq 0$ for all e; the factor 2 is inserted for the sake of comparison and a constant term is omitted. Define $w(e)$ as for the Ising model. For case (i), a simple calculation (summation over $\sigma(v)$) shows that the substitution rule is now

$$w = \frac{w_1 + w_2 + (q-2)w_1 w_2}{1 + (q-1)w_1 w_2}.$$

That for case (ii) is, as in the Ising model, $w = w_1 w_2$. It takes the same form as for case (i), with w^* instead of w, if we define

$$w^* = \frac{1 - w}{1 + (q-1)w}.$$

5. Now note that by substituting $q = 1$ in the last few formulae we get $w = w_1 + w_2 - w_1 w_2$, or $1 - w = (1 - w_1)(1 - w_2)$ and $w^* = 1 - w$. Hence, if we write $1 - w = p$, we recover the rules for C. So in this very special sense, the percolation model behaves, just like the Ising model, as a special case of the Potts model.

6. At first sight the electric network does not seem to fit into the Potts model. It does so, however, if we take an appropriate formal limit. Define for the Potts model $S = q^{-\frac{1}{2}}(1 - w)$. In the limit $q \to 0$ (which at this stage is still meaningless, since q is an integer), the substitution rules reduce to

$$\text{(i)} \quad S = \frac{S_1 S_2}{S_1 + S_2}; \qquad \text{(ii)} \quad S = S_1 + S_2$$

and the duality rule is $S^* = 1/S$. In other words, we recover the rules for the network, with $S = R^{-1} (= R^*)$.

7. So far we had got only a first indication about a relationship between the systems A–C and the Potts model. We then wanted to turn to more general situations. We had observed that (for arbitrary G) certain characteristic quantities in A–C, such as

(A) the total current flowing through an electric network when a unit potential difference is imposed on two arbitrary vertices x and y,

(B) the two-spin correlation $E[\sigma(x)\sigma(y)]$ of the Ising model,

(C) the pair connectivity $E[I(x \leftrightarrow y)]$ of the percolation model

can all be written in the form $P(X)/Q(X)$ (where X stands for $S = R^{-1}$, $p = 1 - w$, and p, respectively), with P and Q polynomials in the edge variables $X(e)$ that are linear in each variable separately. For the electric network, P and Q are homogeneous in all variables $S(e)$ together (of degree $|V| - 2$ and $|V| - 1$, respectively); Q is the generating function of spanning trees of G, and P is the generating function of spanning forests which consist of two trees, with x and y in different trees. For the Ising model, Q is the partition function (which has also a graph-theoretical interpretation, viz. in terms of cycles). For the percolation model, $Q = 1$ identically.

8. From the linearity in the $X(e)$ it follows that $P(X) = P(X, G)$ satisfies, for each edge e, the recursion relation

$$P(X, G) = P_1 + X(e)P_2 ,$$

with P_1 and P_2 polynomials in all $X(f)$ with $f \neq e$. This relation holds also for the Potts model with arbitrary q, again with $p = 1 - w$. A similar relation holds for Q.

9. Now if in the Potts model we have $p(e) = 0$, i.e. $w(e) = 1$, for some edge e, this means that $J(e) = 0$; this is equivalent to having an interaction graph G with the edge e *deleted*. Similarly, $p(e) = 1$, $w(e) = 0$, means that the interaction is infinitely strong; this is equivalent to having an interaction graph with the edge e *contracted* (i.e. deleted and its end points identified). If D_eG and C_eG, respectively, are the graphs thus obtained from G, then obviously,

$$\begin{aligned} P(p, D_eG) &= P_1, \\ P(p, C_eG) &= P_1 + P_2. \end{aligned}$$

Hence, we can write

$$\begin{aligned} P(p, G) &= P(p, D_eG) + p(e)\{P(p, C_eG) - P(p, D_eG)\} \\ &= [1 - p(e)]P(p, D_eG) + p(e)P(p, C_eG). \end{aligned}$$

10. Iteration of the last step leads directly to the expansion of P and Q in the variables $p(e)$ and $1 - p(e)$. Since $0 \le p(e) \le 1$, we could interpret the $p(e)$ as probabilities and the entire system as an example (the first one we knew) of 'weighted' (we would now say 'dependent') percolation. The generalization to arbitrary positive q (and even to complex q!) was now obvious. Then the limit $q \to 0$, as described above, could be taken correctly, and what came out was the electric network with all its properties.

11. It is the system obtained in this way which — for lack of a more instructive name — we called the random-cluster model. Most people have just called it the Potts model, and of course, it is closely related to the generalized spin model bearing this name. (We referred to the latter model as the Ashkin–Teller–Potts model, because Ashkin and Teller were really the first to consider generalizations of the Ising to more than two spin states; one of these was the 4-state Potts model.) Fortuin and I preferred, however, to distinguish between the two systems, because they are different in principle. It is only in the paper by Edwards and Sokal that the relation between the two was fully established for integral values of q. It is now obvious that to every function $f(\sigma)$ of the spin state in the Potts model there corresponds a function $F(\omega)$ of the edge state in the random-cluster model such that the expectation of F with respect to the random-cluster measure equals the expectation of f with respect to the Potts measure, and conversely. F and f are transformed into each other via kernels which are nothing but the conditional probabilities of Edwards and Sokal[14]. The relations which Fortuin and I found between spin correlations in the Ising model and certain connectivity probabilities in the random-cluster model with $q = 2$ were special cases.

12. After thus having introduced the random-cluster model for finite graphs, we were prepared to tackle infinite graphs. Fortuin wanted absolutely to treat these in as general a setting as possible and not restrict himself from the outset to regular lattices, as I had suggested to him. This admittedly makes his papers less accessible, but, in my opinion, also richer than they would have been if he had followed my suggestion. But this is a matter of taste.

So far about history. Looking at the subsequent developments, I am somewhat surprised by the fact that (to my knowledge) no one has given any attention to the domain of q between 0 and 1, not even to the limit in which one recovers the electric network, where life becomes much simpler. Of course, the FKG inequality does not hold in this domain, but does that imply that nothing of interest can be done? I admit that Fortuin was the first to restrict himself to the region where FKG holds, but that was because the time for his PhD research was limited! In fact, if I remember correctly, some mathematician once published a paper in which a graph-theoretical interpretation was given to the random-cluster model (probably under the name of dichromatic polynomial) for $q = -1$ (or $q = -2$, I am not sure).

Then there is the ‘antiferromagnetic’ Potts model, where $J(e)$ is allowed to be negative. If it is, $p(e)$ is also negative, so that a standard probability interpretation is impossible. This case has not been investigated either, as far as I know. Still, it is of interest, if only because, for integral positive q, the limit where all $J(e)$ become infinitely negative leads one into the theory of vertex colourings with q colours! In this connection I may point out that for two-dimensional regular lattices the value $q = 4$ plays a very special role in the random-cluster model: for $q < 4$ the phase transition is ‘of second order’ (i.e. the percolation probability is continuous), for

[14] Kasteleyn returned to this point in a further letter dated 17 November 1992. He pointed out that, in general, there exist functions $F(\omega)$ with no corresponding function $f(\sigma)$ independent of p. The random-cluster process is thereby *doubly* richer than the Potts model: it *contains* the Potts model (for integer q), and in addition there are questions which may be asked about it for which there are no corresponding questions about the Potts model.

$q > 4$ it is 'of first order'. So it may be that there is more to be said about the four-colour problem than we know at present!

Postscript by Cees Fortuin, 11 September 2003.

I remember especially the first time he [Piet] told me about his ideas (end of 1966 when I still was doing my military service and he already had invited me to work with him): we were sitting next each other at the table in front of the window, which he used for working sessions, and he explained his ideas (the ABC of the letter) while I was listening. My first work was then 'putting the electrical current of a network into the scheme'. The actual formulating of the model took some time: I guess that it was end 1968/begin 1969 before on my blackboard the formula with 2^n appeared (the reformulation of the Ising model); I then went to his office and said something like: "I have found what we sought" (but half and half expecting he would say that he already knew!). We walked back to my office where he overlooked the blackboard and remarked that this was a special moment(!).

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