MARKOV CHAINS

Remarks on Example Sheet 4

4.1 The position evolves as a Markov chain with generator

$$G = \begin{pmatrix} -\lambda-\mu & \lambda & \mu \\ \mu & -\lambda-\mu & \lambda \\ \lambda & \mu & -\lambda-\mu \end{pmatrix}$$

and the problem is to compute $p_{1,1}(t)$. Find the eigenvalues, and follow the method of Problem 3.10. Convergence to equilibrium gives $p_{1,1}(\infty) = \frac{1}{3}$, which is useful in fixing the constants.

4.2 Condition on the time of the first event as in Problem 3.9 and use $\mathbb{P}(X_t = 0 \mid X_s = 2) = h(t-s)^2$.

4.3 The transition matrix $\mathbf{Q} = (q_{i,j})$ of Y is given by

$$q_{i,j} = \begin{cases} \dfrac{i\mu}{\lambda + i\mu} & \text{if } j = i-1 \\ \dfrac{\lambda}{\lambda + i\mu} & \text{if } j = i+1. \end{cases}$$

To find the stationary distribution of Y, either solve the equation $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{Q}$, or look for a solution of the detailed balance equations $\pi_i q_{i,i+1} = \pi_{i+1} q_{i+1,i}$. Following the latter route, we have that

$$\pi_i = \pi_0 \frac{q_{01}q_{12}\cdots q_{i-1,i}}{q_{i,i-1}\cdots q_{21}q_{10}}, \qquad i \geq 1,$$

whence $\pi_i = \pi_0 \rho^i (1 + i/\rho)/i!$ for $i \geq 1$. Choosing π_0 accordingly, we obtain the result.

It is a standard calculation that X has stationary distribution $\boldsymbol{\nu}$ given by $\nu_i = \rho^i e^{-\rho}/i!$ for $i \geq 0$. The difference between $\boldsymbol{\pi}$ and $\boldsymbol{\nu}$ arises from the fact that the holding-times of X have distributions which depend on the current state.

4.4 Compute the integral by

$$\begin{aligned} \int_0^\infty p_{i,i}(t)dt &= \int_0^\infty \mathbb{E}_i(1_{X_t=i})dt = \mathbb{E}_i \int_0^\infty 1_{X_t=i}dt \\ &= \mathbb{E}_i \sum_{n=0}^\infty S_{n+1} 1_{Y_n=i} = \sum_{n=0}^\infty \mathbb{E}_i(S_{n+1} \mid Y_n = i)\mathbb{P}_i(Y_n = i) = \sum_{n=0}^\infty \frac{1}{q_i}\pi_{ii}(n), \end{aligned}$$

where S_n is the nth holding time and $(Y_n)_{n\geq 0}$ the jump chain.

4.5 The generator is given by $g_{A_i,A_{i+1}} = g_{B_i,B_{i+1}} = \lambda$, $g_{A_i,A_{i-1}} = \mu$, $g_{A_i,B_i} = \alpha$, $g_{B_i,A_i} = \beta$. In order to reach A_0 from A_k you must pass through $A_{k-1}, \ldots, A_1$ and the formula results from the strong Markov property applied at the times you first hit these states. By considering the jump chain, $\theta = \mathbb{P}_{A_1}(\text{hit } A_0)$ and $\mu = \mathbb{P}_{B_1}(\text{hit } A_1)$ satisfy

$$(\alpha + \mu + \lambda)\theta = \mu + \lambda\theta^2 + \alpha\eta\theta, \quad (\lambda + \beta)\eta = \beta + \lambda\eta\theta.$$

Hence $(\theta - 1)f(\theta) = 0$. By general results θ is the smallest non-negative root of $(\theta-1)f(\theta)$, so you have to work out whether $f(\theta)$ has a root in $(0, 1)$.

4.6 Recurrence is decided by finding $h_i = \mathbb{P}_i(\text{hit } 0)$ as the minimal non-negative solution to

$$(\lambda_i + \mu_i)h_i = \mu_i h_{i-1} + \lambda_i h_{i+1}, \quad h_0 = 1.$$

Divergence of $\sum i^{-\alpha}$ is necessary and sufficient.

Positive recurrence is then decided by seeing whether invariant measures have finite mass, which they do provided $\alpha - \beta < -1$. Students should be asked what they think is happening when $\alpha - \beta < -1$ and $\alpha > 1$.

4.7 The theorem on convergence to equilibrium given in lectures applies to irreducible, positive recurrent chains, which (b), (c) and (d) are not. In (a) the invariant distribution is $(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3})$ and the theorem applies to give $\frac{1}{6}$. In (b) state 2 is transient, so the answer is 0. In (c) any distribution of the form $(\frac{1}{2}p, \frac{1}{2}p, \frac{1}{2}(1-p), \frac{1}{2}(1-p))$ is invariant, but the answer is $\frac{1}{2}$. In (d) the chain enters the closed class $\{2, 3\}$ with probability $\frac{1}{2}$; the invariant distribution on $\{2, 3\}$ is $(\frac{1}{3}, \frac{2}{3})$, so the answer is $\frac{1}{6}$.

4.8 Let n be the number of service times needed to clear the queue in the absence of any arrivals. $g_{i,i+2} = \lambda$; $g_{i,i-1} = \mu$. An invariant measure satisfies $\pi G = 0$, so

$$\lambda\pi_0 = \mu\pi_1; \qquad (\lambda + \mu)\pi_1 = \mu\pi_2; \qquad \mu\pi_{i+1} - (\lambda + \mu)\pi_i + \lambda\pi_{i-2} = 0 \qquad i \geq 2.$$

Put $x = \lambda/\mu$ and take $\pi_0 = 1$. Then $\pi_1 = x$, $\pi_2 = x(x + 1)$, and the roots of the equation coming from the recurrence are 1 and $\alpha, \beta = \frac{1}{2}[x \pm \sqrt{x^2 + 4x}]$. For the case $x \neq \frac{1}{2}$ write $\pi_i = A\alpha^i + B\beta^i + C$. The initial values can be used to show that $C = 0$, and $|\alpha|, |\beta| < 1$ for $x < \frac{1}{2}$. And so on.

4.9 The generator is given by $g_i = \lambda + i\mu$, $g_{i,i+1} = \lambda$ and $g_{i,i+1} = i\mu$, and has invariant distribution

$$\pi_i = \frac{e^{-\nu}\nu^i}{i!}, \quad \nu = \lambda/\mu.$$

To apply the standard result that $p_{i,j}(t) \to \pi_j$ as $t \to \infty$, you must also check irreducibility and recurrence (or non-explosion). Here the jump chain is obviously recurrent, also the process is dominated by a Poisson process of rate λ, so cannot explode. The expected return time to state i is given by $m_i = \{q_i\pi_i\}^{-1}$ so the mean length of the busy periods is $m_0 - q_0^{-1} = \{\lambda(e^{\lambda/\mu} - 1)\}^{-1}$. By considering one call separately we see

$$\mathbb{E}(X_t \mid X_0 = i) = e^{-\mu t} + \mathbb{E}(X_t \mid X_0 = i - 1) = ie^{-\mu t} + \mathbb{E}(X_t \mid X_0 = 0).$$

Also, in equilibrium,

$$\frac{\lambda}{\mu} = \mathbb{E}(X_t) = \sum_{i=0}^{\infty} \pi_i \mathbb{E}(X_t \mid X_0 = i) = \mathbb{E}(X_t \mid X_0 = 0) + \sum_{i=0}^{\infty} \frac{e^{-\nu}\nu^i ie^{-\mu t}}{i!}$$

hence $\mathbb{E}(X_t \mid X_0 = i)$.

The chain is reversible, so in equilibrium N_t has the same distribution as the number of calls starting in $[0, t]$. Indeed the same argument shows that the joint distribution of $N_{t_1}, \ldots, N_{t_n}$ for $0 \leq t_1 \leq \cdots \leq t_n$ coincides with that of a Poisson process, so $(N_t)_{t\geq 0}$ is a Poisson process. This may be surprising if one is thinking of $(N_t)_{t\geq 0}$ as governed by $(X_t)_{t\geq 0}$, but remember that $(X_t)_{t\geq 0}$ is in equilibrium.