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Remarks on Example Sheet 3

3.1 First,

$$\mathbb{P}(\min\{S,T\} > z) = \mathbb{P}(S > z)\mathbb{P}(T > z) = e^{-\alpha z}e^{-\beta z} = e^{-(\alpha+\beta)z}.$$

By integration, $P(S < T) = \alpha/(\alpha+\beta)$. Then

$$\mathbb{P}(S < T,\ \min\{S,T\} \geq t) = \mathbb{P}(t \leq S < T) = \frac{\alpha}{\alpha+\beta}e^{-(\alpha+\beta)t}$$

by integration. Students should be encouraged to think what these results mean, e.g., in terms of arrivals of rates α and β respectively. Also, more general results of this kind are obviously important when describing continuous time chains by specifying rates for individual transitions.

3.2 Compute $\mathbb{E}(e^{\theta T} \mid N = n)$ and then $\mathbb{E}(e^{\theta T})$.

3.3 Let F and W be the incoming Poisson processes, and let $N(t) = F(t) + W(t)$. Certainly $N(0) = 0$ and N is non-decreasing. Arrivals of flies during $[0, s]$ are independent of arrivals during $(s, t]$, if $s < t$; similarly for wasps. Therefore the aggregated arrival process during $[0, s]$ is independent of the aggregated process during $(s, t]$. Now

$$\mathbb{P}\big(N(t+h) = n+1 \mid N(t) = n\big) = \mathbb{P}(A \bigtriangleup B)$$

where

$$\begin{aligned} A &= \{\text{one fly arrives during } (t, t+h]\}, \\ B &= \{\text{one wasp arrives during } (t, t+h]\}. \end{aligned}$$

We have that

$$\begin{aligned} \mathbb{P}(A \bigtriangleup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \lambda h + \mu h - (\lambda h)(\mu h) + o(h) = (\lambda+\mu)h + o(h). \end{aligned}$$

Finally

$$\mathbb{P}\big(N(t+h) > n+1 \mid N(t) = n\big) \leq \mathbb{P}(A \cap B) + \mathbb{P}(C \cup D),$$

where $C = \{\text{two or more flies arrive in } (t, t+h]\}$ and $D = \{\text{two or more wasps arrive in } (t, t+h]\}$. This probability is no greater than $(\lambda h)(\mu h) + o(h) = o(h)$.

3.4 Let I be the incoming Poisson process, and let G be the process of arrivals of green insects. Matters of independence are dealt with as above. Finally

$$\begin{aligned} \mathbb{P}\big(G(t+h) = n+1 \mid G(t) = n\big) &= p\mathbb{P}\big(I(t+h) = n+1 \mid I(t) = n\big) + o(h) \\ &= p\lambda h + o(h), \end{aligned}$$

and

$$\mathbb{P}\big(G(t+h) > n+1 \mid G(t) = n\big) \leq \mathbb{P}\big(I(t+h) > n+1 \mid I(t) = n\big) = o(h).$$

3.5 Conditioning on T_1 and using the time-homogeneity of the process,

$$\mathbb{P}\big(E(t) > x \mid T_1 = u\big) = \begin{cases} \mathbb{P}\big(E(t-u) > x\big) & \text{if } u \le t \\ 0 & \text{if } t < u \le t + x \\ 1 & \text{if } u > t + x \end{cases}$$

(draw a diagram to help you see this). Therefore

$$\begin{aligned} \mathbb{P}\big(E(t) > x\big) &= \int_0^\infty \mathbb{P}\big(E(t) > x \mid T_1 = u\big)\lambda e^{-\lambda u}\, du \\ &= \int_0^t \mathbb{P}\big(E(t-u) > x\big)\lambda e^{-\lambda u}\, du + \int_{t+x}^\infty \lambda e^{-\lambda u}\, du. \end{aligned}$$

You may solve the integral equation using Laplace transforms. Alternately you may guess the answer and then check that it works. The answer is $\mathbb{P}\big(E(t) \le x\big) = 1 - e^{-\lambda x}$, the exponential distribution. Actually this answer is obvious since $E(t) > x$ if and only if there is no arrival in $[t, t+x]$, an event having probability $e^{-\lambda x}$.

3.6 The forward equations are

$$p_n'(t) = \lambda_{n-1}p_{n-1}(t) - \lambda_n p_n(t), \qquad n \ge 0,$$

where $\lambda_i = i\lambda + \nu$. The process is honest, and therefore $m(t) = \sum_n n p_n(t)$ satisfies

$$\begin{aligned} m'(t) &= \sum_{n=1}^\infty n\big[(n-1)\lambda + \nu\big]p_{n-1}(t) - \sum_{n=0}^\infty n(n\lambda + \nu)p_n(t) \\ &= \sum_{n=0}^\infty \big\{\lambda[(n+1)n - n^2] + \nu[(n+1) - n]\big\}p_n(t) \\ &= \sum_{n=0}^\infty (\lambda n + \nu)p_n(t) = \lambda m(t) + \nu. \end{aligned}$$

Solve subject to $m(0) = 0$ to obtain $m(t) = \nu(e^{\lambda t} - 1)/\lambda$.

3.7 By standard properties of the Poisson process the answers are: (a) $e^{-8}8^3/3!$, (b) $7^3/8^4$, (c) e^{-2}.

3.8 Let T be the time to cross and J the time at which the first car passes. Then

$$\begin{aligned} \mathbb{E}(T) &= a\mathbb{P}(J > a) + \int_0^a (s + \mathbb{E}(T))\lambda e^{-\lambda s} ds \\ &= (1 - e^{-\lambda a})\left(\mathbb{E}(T) + \frac{1}{\lambda}\right). \end{aligned}$$

Hence $\mathbb{E}(T)$. Check that it is correct as $\lambda \to 0$. Walking straight across replaces a by $2a$ and λ by 2λ, because the sum of two independent Poisson processes of rate λ is a Poisson process of rate 2λ. Using an island, it takes twice as long on average as a single lane. How much quicker is it to use an island?

3.9 For $0 \le s \le t$, the conditional distribution of X_t, given the first split J occurs at s, is the same as the distribution of $X_{t-s} + \widetilde{X}_{t-s}$, where $\widetilde{X}$ is an independent copy of X. So

$$\mathbb{E}(z^{X_t} \mid J = s) = \begin{cases} \mathbb{E}(z^{X_{t-s} + \widetilde{X}_{t-s}}) & \text{for } 0 \le s \le t, \\ z & \text{for } s \ge t, \end{cases}$$

and

$$\begin{aligned} \phi(t) &= \int_0^\infty \lambda e^{-\lambda s}\mathbb{E}(z^{X_t} \mid J = s) ds \\ &= z e^{-\lambda t} + \int_0^t \lambda e^{-\lambda s}\phi(t-s)^2 ds\,. \end{aligned}$$

Put $u = t - s$, multiply by $e^{\lambda t}$ and differentiate to see that $d\phi/dt = \lambda\phi(\phi - 1)$, and so $\phi = z(1-q)/(1-zq)$ as required.

3.10 (a) There are many ways of calculating G^n; let us use difference equations. Note first that $G^0 = I$, the identity matrix. Write

$$G^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \qquad n \geq 0,$$

and use the equation $G^{n+1} = G \cdot G^n$ to find that

$$a_{n+1} = -\mu a_n + \mu c_n, \quad c_{n+1} = \lambda a_n - \lambda c_n.$$

Hence $a_{n+1} = -(\mu/\lambda)c_{n+1}$ for $n \geq 0$, and the first difference equation becomes $a_{n+1} = -(\lambda+\mu)a_n$, $n \geq 1$, which, subject to $a_1 = -\mu$, has solution $a_n = (-1)^n \mu(\lambda+\mu)^{n-1}$, $n \geq 1$. Therefore $c_n = (-1)^{n+1}\lambda(\lambda+\mu)^{n-1}$ for $n \geq 1$, and one may see similarly that $b_n = -a_n$, $d_n = -c_n$ for $n \geq 1$. Using the facts that $a_0 = d_0 = 1$ and $b_0 = c_0 = 0$, we deduce that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} G^n = P_t = \frac{1}{\lambda+\mu} \begin{pmatrix} \lambda + \mu e^{-(\lambda+\mu)t} & \mu - \mu e^{-(\lambda+\mu)t} \\ \lambda - \lambda e^{-(\lambda+\mu)t} & \mu + \lambda e^{-(\lambda+\mu)t} \end{pmatrix}.$$

(b) With $\boldsymbol{\pi} = (\pi_1, \pi_2)$, we have that

$$-\mu\pi_1 + \lambda\pi_2 = 0, \quad \mu\pi_1 - \lambda\pi_2 = 0,$$

and therefore $\pi_1 = (\lambda/\mu)\pi_2$. In addition, $\pi_1 + \pi_2 = 1$ if $\pi_1 = \lambda/(\lambda+\mu) = 1 - \pi_2$.

3.11 Assume that $\lambda \geq \mu$, then

$$\mathbb{P}_0(\text{hit } i) = \begin{cases} 1 & \text{if } i \geq 0 \\ (\mu/\lambda)^{-i} & \text{if } i < 0. \end{cases}$$

Hence the return probability for any state is 2μ. On each visit, $(X_t)_{t\geq 0}$ spends an average time $1/q_i$ in state i. Hence

$$\mathbb{E}_0\Big(\text{total time in state } i\Big) = \begin{cases} \dfrac{1}{(\lambda-\mu)q_i} & \text{if } i \geq 0 \\ \dfrac{(\mu/\lambda)^{-i}}{(\lambda-\mu)q_i} & \text{if } i < 0. \end{cases}$$

If $\mu = 0$, then $(X_t)_{t\geq 0}$ is explosive if and only if $\sum_{i=1}^{\infty} 1/q_i < \infty$.

If $\mu \in [0, \frac{1}{2})$ then certainly the jump chain $Y_n \to \infty$ as $n \to \infty$ so $(X_t)_{t\geq 0}$ cannot explode to $-\infty$, but comparison with the case $\mu = 0$ shows that if $\sum_{i=1}^{\infty} 1/q_i = \infty$ then $(X_t)_{t\geq 0}$ cannot explode to ∞ either. To see that this condition is also necessary for non-explosion when $\mu \in [0, \frac{1}{2})$, observe that $(X_t)_{t\geq 0}$ makes only finitely many visits to states $i \leq 0$ and

$$\mathbb{E}_0\Big(\text{total time to the right of the origin}\Big) = \sum_{i=1}^{\infty} \frac{1}{(\lambda-\mu)q_i}.$$

Finally, in the case $\lambda = \mu$, $(Y_n)_{n\geq 0}$ is recurrent so $(X_t)_{t\geq 0}$ does not explode.