MARKOV CHAINS

Remarks on Example Sheet 1

1.1 We have that

$$\begin{aligned}\mathbb{P}\big(Z_{k+1} = i_{k+1} \,\big|\, Z_r = i_r \text{ for } 0 \le r \le k\big) &= \mathbb{P}\big(X_{m+k+1} = i_{k+1} \,\big|\, X_{m+r} = i_r \text{ for } 0 \le r \le k,\ X_m = i\big)\\ &= \mathbb{P}\big(X_{m+k+1} = i_{k+1} \,\big|\, X_{m+k} = i_k\big)\end{aligned}$$

by the Markov property. The last probability does not depend on i_r for $r < k$. Small step now required.

1.2 The sequence $X_1, X_2, \dots$ of independent random variables satisfies

$$\mathbb{P}(X_{n+1} = j \mid X_1 = i_1, \dots, X_n = i_n) = \mathbb{P}(X_{n+1} = j),$$

whence the sequence is a Markov chain. The chain is homogeneous if the X_i are identically distributed.

1.3 With Y_n the outcome of the nth throw, $X_{n+1} = \max\{X_n, Y_{n+1}\}$, so that

$$p_{ij} = \begin{cases} 0 & \text{if } j < i \\ \frac{1}{6}i & \text{if } j = i \\ \frac{1}{6} & \text{if } j > i, \end{cases}$$

for $1 \le i, j \le 6$. Similarly,

$$p_{ij}(n) = \begin{cases} 0 & \text{if } j < i \\ \big(\frac{1}{6}i\big)^n & \text{if } j = i. \end{cases}$$

If $j > i$, then $p_{ij}(n) = \mathbb{P}(Z_n = j)$, where $Z_n = \max\{Y_1, Y_2, \dots, Y_n\}$, and an elementary calculation yields

$$p_{ij}(n) = \left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n, \qquad i < j \le 6.$$

1.4 (i) There is probably an easier way than the following. If $X_n = i$, then $X_{n+1} \in \{i-1, i+1\}$. Now, for $i \ge 1$,

$$(*)\qquad \begin{aligned}\mathbb{P}(X_{n+1} = i+1 \mid X_n = i, B) &= \mathbb{P}(X_{n+1} = i+1 \mid S_n = i, B)\mathbb{P}(S_n = i \mid X_n = i, B)\\ &\quad + \mathbb{P}(X_{n+1} = i+1 \mid S_n = -i, B)\mathbb{P}(S_n = -i \mid X_n = i, B)\end{aligned}$$

where $B = \{X_r = i_r \text{ for } 0 \le r < n\}$ and $i_0, i_1, \dots, i_{n-1}$ are integers. Clearly

$$\mathbb{P}(X_{n+1} = i+1 \mid S_n = i, B) = p, \quad \mathbb{P}(X_{n+1} = i+1 \mid S_n = -i, B) = q,$$

where p $(= 1 - q)$ is the chance of a rightward step. Let l be the time of the last visit to 0 prior to the time n, $l = \max\{r : i_r = 0\}$. During the time-interval $(l, n]$, the path lies entirely in either the positive integers or the negative integers. If the former, it is required to follow the route prescribed by the event $B \cap \{S_n = i\}$, and if the latter by the event $B \cap \{S_n = -i\}$. The absolute probabilities of these two routes are

$$\pi_1 = p^{\frac{1}{2}(n-l+i)} q^{\frac{1}{2}(n-l-i)}, \quad \pi_2 = p^{\frac{1}{2}(n-l-i)} q^{\frac{1}{2}(n-l+i)},$$

whence

$$\mathbb{P}(S_n = i \mid X_n = i, B) = \frac{\pi_1}{\pi_1 + \pi_2} = \frac{p^i}{p^i + q^i} = 1 - \mathbb{P}(S_n = -i \mid X_n = i, B).$$

Substitute into $(*)$ to obtain

$$\mathbb{P}(X_{n+1} = i+1 \mid X_n = i, B) = \frac{p^{i+1} + q^{i+1}}{p^i + q^i} = 1 - \mathbb{P}(X_{n+1} = i-1 \mid X_n = i, B).$$

Finally $\mathbb{P}(X_{n+1} = 1 \mid X_n = 0, B) = 1$. *Assuming* X is a Markov chain, its transition probabilities may be calculated more simply by conditioning on S_n.

(ii) Note that $Y_n \ge 0$. If $Y_n > 0$, then $Y_n - Y_{n+1}$ equals the $(n+1)$th step, a random variable which is independent of the past history of the process. If $Y_n = 0$ then $S_n = M_n$, so that Y_{n+1} takes the values 0 and 1 with respective probabilities p and q, independently of the past history. [Appealing to an appropriate equation:] We deduce that Y is a Markov chain with transition probabilities

$$\text{for } i > 0, \quad p_{ij} = \begin{cases} p & \text{if } j = i-1 \\ q & \text{if } j = i+1, \end{cases} \qquad p_{0j} = \begin{cases} p & \text{if } j = 0 \\ q & \text{if } j = 1. \end{cases}$$

The sequence Y is a random walk with a retaining barrier at 0.

1.5 For any sequence $i_0, i_1, \dots$ of states,

$$\begin{aligned}\mathbb{P}(Y_{k+1} = i_{k+1} \mid Y_r = i_r \text{ for } 0 \le r \le k) &= \frac{\mathbb{P}(X_{n_s} = i_s \text{ for } 0 \le s \le k+1)}{\mathbb{P}(X_{n_s} = i_s \text{ for } 0 \le s \le k)} \\ &= \frac{\prod_{s=0}^{k} p_{i_s, i_{s+1}}(n_{s+1} - n_s)}{\prod_{s=0}^{k-1} p_{i_s, i_{s+1}}(n_{s+1} - n_s)} \\ &= p_{i_k, i_{k+1}}(n_{k+1} - n_k) = \mathbb{P}(Y_{k+1} = i_{k+1} \mid Y_k = i_k),\end{aligned}$$

where $p_{ij}(n)$ denotes the appropriate n-step transition probability of X.

With the usual notation, the transition matrix of Y is

$$\pi_{ij} = \begin{cases} p^2 & \text{if } j = i+2 \\ 2pq & \text{if } j = i \\ q^2 & \text{if } j = i-2. \end{cases}$$

1.6 Not necessarily! Take as example the chains S and Y of Exercise 1.4. The sum is $S_n + Y_n = M_n$, which is not a Markov chain.

1.7 A simple way to deal with the first flea is to reduce to a two-state chain on {initial vertex, another} with transition matrix $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ so that

$$p_{11}(n) = \tfrac{1}{3} + \tfrac{2}{3}(-\tfrac{1}{2})^n.$$

The second flea has transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}.$$

The eigenvalues of P are 1, $-\frac{1}{\sqrt{3}} e^{\pm i\pi/6}$ so P is diagonalizable and

$$p_{11}(n) = \alpha + \left(-\frac{1}{\sqrt{3}}\right)^n \left(\beta \cos \frac{n\pi}{6} + \gamma \sin \frac{n\pi}{6}\right)$$

for some constants α, β, γ. Then $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$, $\gamma = 0$ by working out $p_{11}(n)$ for $n = 0, 1, 2$. This method is discussed in the lectures. A very desirable alternative to using $p_{11}(2)$ is to spot that $p_{11}(n) \to \frac{1}{3}$ as $n \to \infty$ by convergence to equilibrium (not yet proved in lectures), which identifies α. An alternative method altogether is to exploit symmetry to write

$$\begin{aligned} p_{11}(n+1) &= \tfrac{2}{3} p_{12}(n) + \tfrac{1}{3}(1 - p_{11}(n) - p_{12}(n)) \\ p_{12}(n+1) &= \tfrac{1}{3} p_{11}(n) + \tfrac{2}{3}(1 - p_{11}(n) - p_{12}(n)) \end{aligned}$$

and then solve for $p_{11}(n)$. (Take the first equation. Substituting for $p_{12}(n)$ using the second equation and then for $p_{12}(n-1)$ using the first, you obtain $3p_{11}(n+1) + 3p_{11}(n) + p_{11}(n-1) = \frac{7}{3}$ and so obtain the constant $\alpha = \frac{1}{3}$ without using initial conditions or convergence to equilibrium.)

1.8 In the first case the die is like the first flea of the last question, and $p = \frac{1}{6} + \frac{5}{6}(-\frac{1}{5})^{n-1}$. By symmetry, the other probabilities are $\frac{1}{5}(1-p)$. The second die may be simulated by throwing the first die but taking the nth score to be $i_n + n - 1$ mod 6, where i_n is the nth score on the first die. Hence

$$p = \begin{cases} \frac{1}{6} + \frac{5}{6}(-\frac{1}{5})^{n-1} & \text{if } n = 1 \pmod 6 \\ \frac{1}{6} - \frac{1}{6}(-\frac{1}{5})^{n-1} & \text{otherwise.} \end{cases}$$

1.9 The characteristic polynomial of P is $\frac{1}{3}(x-1)(3x^2+x+p)$. The roots of $3x^2+x+p=0$ are $-\frac{1}{4}, -\frac{1}{12}$ in case (a), $-\frac{1}{6} \pm \frac{1}{6}i$ in case (b), and $-\frac{1}{6}$ (twice) in case (c). In fact if $p = \frac{1}{12}$, P is not diagonalizable and the usual trick fails; instead P is similar to

$$\begin{pmatrix} -\frac{1}{6} & 1 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so $p_{11}(n) = \alpha + (-\frac{1}{6})^n(\beta + \gamma n)$ for some constants α, β, γ which can be identified as usual. Another approach is to solve the system of recurrences

$$\begin{aligned} p_{11}(n+1) &= p.\, p_{13}(n) \\ p_{12}(n+1) &= p_{11}(n) + \tfrac{2}{3}p_{12}(n) + (1-p)p_{13}(n) \\ p_{13}(n+1) &= \tfrac{1}{3}p_{12}(n) \end{aligned}$$

coming from the Chapman–Kolmogorov equations. Using $p_{11}(n) + p_{12}(n) + p_{13}(n) = 1$ and proceeding as in Exercise 1.7 you obtain $3p_{11}(n+1) + p_{11}(n) + p.p_{11}(n-1) = p$ which can be solved in the usual way. Using this method you obtain the constant term $p/(p+4)$ in the general case without using initial conditions.

1.10 Classes $\{1,5\}$ and $\{3\}$ are closed.

1.11 If the communicating class of i is not closed, there exists j with $i \to j$, $j \nrightarrow i$. Suppose there is no closed class. Then we get a sequence $i_1, i_2, \dots$ of states with $i_n \to i_{n+1}$, $i_n \nrightarrow i_{n+1}$. There can be no repeats, so the state-space is infinite.

1.12 Set $h_i = \mathbb{P}_i(\text{hit } 10)$ then

$$h_2 = \tfrac{1}{2}h_4,\ h_4 = \tfrac{1}{2}h_8,\ h_8 = \tfrac{1}{2}h_6 + \tfrac{1}{2},\ h_6 = \tfrac{1}{2}h_2 + \tfrac{1}{2}.$$

Hence $h_2 = \frac{1}{5}$. Set $k_i = \mathbb{E}_i(\text{time to hit 0 or 10})$, then $k_2 = k_8$ and $k_4 = k_6$, by symmetry. Also $k_2 = 1 + \frac{1}{2}k_4$ etc, so $k_2 = 2$. Students should be encouraged to spot and use the symmetry in this question.

1.13 The calculation of hitting probabilities for a general birth and death chain will be discussed in lectures. Set $h_i = \mathbb{P}_i(\text{hit } 0)$, then $h_0 = 1$ and

$$h_i = p_{i,i+1}h_{i+1} + p_{i,i-1}h_{i-1}$$

so $u_i = h_{i-1} - h_i$ satisfies $u_i \geq 0$,

$$u_{i+1} = \frac{p_{i,i-1}}{p_{i,i+1}}u_i = \left(\frac{i}{i+1}\right)^2 u_i = \cdots = \left(\frac{1}{i+1}\right)^2 u_1,$$

and

$$h_i = h_0 - (h_0 - h_i) = 1 - u_1 \sum_{j=1}^{i} \frac{1}{j^2}\ .$$

Since $(h_i : i \geq 0)$ is the minimal non-negative solution, we have that

$$u_1 = \left(\sum_{j=1}^{\infty} \frac{1}{j^2}\right)^{-1} = \frac{6}{\pi^2}$$

and

$$\mathbb{P}_0(X_n \geq 1 \text{ for all } n \geq 1) = 1 - h_1 = \frac{6}{\pi^2}.$$

1.14 By conditioning on X_1 and using shift invariance and the strong Markov property, we obtain that

$$\phi = \tfrac{1}{2}\mathbb{E}(s^{H_0} \mid X_1 = 2) + \tfrac{1}{2}\mathbb{E}(s^{H_0} \mid X_1 = 0) = \tfrac{1}{2}s\phi^2 + \tfrac{1}{2}s.$$

whence $\phi(s) = \{1 \pm \sqrt{1-s^2}\}/s$. Now $\phi(0) = 0$, and $\phi(s)$ is continuous on $[0,1)$; hence $\phi(s) = \{1 - \sqrt{1-s^2}\}/s$. For the second part,

$$\phi = \tfrac{1}{2}\mathbb{E}(s^{H_0} \mid X_1 = 3) + \tfrac{1}{2}\mathbb{E}(s^{H_0} \mid X_1 = 0) = \tfrac{1}{2}s\phi^3 + \tfrac{1}{2}s.$$