Random-cluster representation of the Blume–Capel model

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Abstract

The so-called diluted-random-cluster model may be viewed as a random-cluster representation of the Blume–Capel model. It has three parameters, a vertex parameter a, an edge parameter p, and a cluster weighting factor q. Stochastic comparisons of measures are developed for the 'vertex marginal' when $q \in [1, 2]$, and the 'edge marginal' when $q \in [1, \infty)$. Taken in conjunction with arguments used earlier for the random-cluster model, these permit a rigorous study of part of the phase diagram of the Blume–Capel model.

Keywords Blume–Capel model, Ising model, Potts model, random-cluster model, first-order phase transition, tri-critical point.

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1 Introduction

The Ising model is one of the most studied models of statistical physics. It has configuration space $\{-1, +1\}^V$ where V is the vertex set of the (finite) graph G in question, and has Hamiltonian

$$\mathcal{H}(\sigma) = -J \sum_{\langle x,y \rangle} \sigma_x \sigma_y - h \sum_{x \in V} \sigma_x, \qquad \sigma \in \{-1, +1\}^V.$$

The first summation is over all (unordered) pairs of nearest neighbours, and $J \in [0, \infty)$, $h \in \mathbb{R}$. The Ising probability measure μ on $\{-1, +1\}^V$ is given by

$$\mu(\sigma) = \frac{1}{Z^{\mathrm{I}}} e^{-\beta \mathcal{H}(\sigma)}, \qquad \sigma \in \{-1, +1\}^V,$$

where Z^{I} is the appropriate normalizing constant. Here, $\beta = 1/(kT)$ where k is Boltzmann's constant and T is temperature.

It is standard that the Ising measure may be extended to a probability measure on the configuration space associated with an infinite graph. For physical and mathematical reasons, it is convenient that this graph have a good deal of symmetry, and it is usual to work with the d-dimensional hypercubic lattice $\mathbb{Z}^d$, where $d \geq 2$. In such a case, the model undergoes a phase transition, and this is the main phenomenon of interest in the theory. This transition is known to be of second-order (continuous) when $d = 2$ or $d \geq 4$, and is believed to be of second-order when $d = 3$ also. See [1, 4, 21].

The Ising model has two local states, namely ± 1. This may be generalized to any given number $q \in \{2, 3, \dots\}$ of local states by considering the so-called Potts model introduced in 1952, see [40]. The Potts phase transition is richer in structure than that of the Ising model, in that it is of first-order (discontinuous) if q is sufficiently large. See [28, 35, 36].

In 1966, Blume introduced a variant of the Ising model, see [9], with the physical motivation of studying magnetization in Uranium Oxide, UO_2, at a temperature of about 30°K. The Hamiltonian was given by

$$\mathcal{H}(\sigma) = -J \sum_{\langle x,y \rangle} \sigma_x \sigma_y + D \sum_{x \in V} \sigma_x^2 - h \sum_{x \in V} \sigma_x, \qquad \sigma \in \{-1, 0, +1\}^V, \tag{1.1}$$

where J, D, h are constants. The probability of a configuration σ was taken proportional to $e^{-\beta \mathcal{H}(\sigma)}$, $\beta = 1/(kT)$. Capel [13, 14, 15] used molecular field approximations to study the ferromagnetic case $J > 0$. A special case is the system with zero external-field, that is, $h = 0$. For a regular graph with vertex degree δ, Capel calculated that there is a first-order phase transition when $\frac{1}{3} J\delta \log 4 < D < \frac{1}{2} J\delta$, and a second-order phase transition when $D < \frac{1}{3} J\delta \log 4$. For $D > \frac{1}{2} J\delta$ he predicted that zero states would be dominant. These non-rigorous results have led to a certain amount of interest in the Blume–Capel model. According to the physics literature, there is a first-order transition even in the low-dimensional setting of $\mathbb{Z}^2$. Indeed, in the phase diagram with parameters (J, D), there is believed to be a so-called 'tri-critical point', at which a line of phase transitions turns from first- to second-order.

The so-called 'random-cluster representation' of Fortuin and Kasteleyn provides one of the basic methods for studying Ising and Potts models, see [25]–[28] and the references therein. Our target in the current paper is to demonstrate a random-cluster representation for the Blume–Capel model with $h = 0$. One of the principal advantages of this approach is that it allows the use of stochastic monotonicity for the corresponding random-cluster model. Thus, we shall explore monotonicity and domination methods for the ensuing measure, and shall deduce some of the structure of the Blume–Capel model on $\mathbb{Z}^d$. It may be possible to derive some, at least, of our results by other methods such as Pirogov–Sinai theory. One virtue of the current approach is relative simplicity.

There is some related literature. An apparently different proposal for a random-cluster representation of the Blume–Capel model is discussed in [10], where the target was to implement a Monte-Carlo method of Swendsen–Wang type, [43]. Of considerably more relevance is the Potts lattice gas of [5] and [17] to which we return near the start of Section 3. We note the early paper of Hu, [34], who considered a random-cluster representation for the Ising model with general ferromagnetic cell interaction on a square lattice.

For further results on the Blume–Capel model, see [7, 8, 11, 18, 23, 33, 39]. The usual random-cluster model is summarized in [27, 28].

The Blume–Capel model has three local states. There is an extension to a model with local state space $\{0, 1, 2, \ldots, q\}$ where $q \geq 1$. We introduce this new model in Section 3, where we dub it the Blume–Capel–Potts (BCP) model. We show there how to construct a random-cluster representation of the BCP model, and we call the corresponding model the 'diluted-random-cluster' (DRC) model. In the BCP model, vertices with state zero do not interact further with their neighbours, and the states of the other vertices have a Potts distribution. In the diluted-random-cluster model, the zero-state vertices of the BCP model are removed, and the remaining graph is subject to a conventional random-cluster model.

The diluted-random-cluster model is formulated on a finite graph in Section 3, and with boundary conditions on a (hyper)cubic lattice in Section 4. In Section 5, we establish stochastic orderings of measures, and we use these to study phase transitions. There are two types of stochastic ordering. In Section 5, we study the process of vertex-dilution, and we show that the set of remaining vertices has a law which is both monotonic and satisfies stochastic orderings with respect to different parameter values. In Section 6, we consider the set of open edges after dilution, and we prove stochastic orderings for the law of this set. The results so far are for finite graphs only.

The thermodynamic limit is taken in two steps, in Section 7. We prove first the existence of the infinite-volume limit of the vertex-measure, and the infinite-volume limits of the full measure and of the BCP measure follow for $1 \leq q \leq 2$. As in the case of the random-cluster model, a certain amount of uniqueness may be obtained using an argument of convexity of pressure. The comparison results for finite graphs carry through to infinite graphs, and enable a rigorous but incomplete study of part of the phase diagram of the Blume–Capel model. This is summarized in Sections 8 and 9, where it is shown that the rigorous theory of the $q = 1$ case gives support for the conjectured phase diagram of the Blume–Capel model.

2 Notation

A finite graph $G = (V, E)$ comprises a vertex-set V and a set E of edges $e = \langle x, y\rangle$ having endvertices x and y. We write $x \sim y$ if $\langle x, y\rangle \in E$, and we call x and y *neighbours* in this case. For simplicity, we shall assume generally that G has neither loops nor multiple edges. The degree $\deg_x$ of a vertex x is the number of edges incident to x.

Let $d \geq 2$. Let $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$, and let $\mathbb{Z}^d$ be the set of all d-vectors of integers. For $x \in \mathbb{Z}^d$, we write $x = (x_1, x_2, \dots, x_d)$, and we define

$$|x| = \sum_{i=1}^{d} |x_i|.$$

We write $x \sim y$ if $|x - y| = 1$, and we let $\mathbb{E}^d$ be the set of all unordered pairs $\langle x, y\rangle$ with $x \sim y$. The resulting graph $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ is called the d-dimensional *hypercubic lattice*.

Substantial use will be made later of the Kronecker delta,

$$\delta_{u,v} = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

3 The BCP and DRC measures

It is shown in this section how the Blume–Capel measure on a graph may be coupled with a certain 'diluted-random-cluster' measure. The Blume–Capel model has two non-zero local states, labelled ± 1. Just as in the Ising/Potts case, the corresponding random-cluster representation is valid for a general number, q say, of local states. Therefore, we first define a 'Potts extension' of the Blume–Capel model with zero external-field.

Let $G = (V, E)$ be a finite graph with neither loops nor multiple edges. Let $q \in \{1, 2, 3, \dots\}$, and let $\Sigma_q = \{0, 1, 2, \dots, q\}^V$. For $\sigma = (\sigma_x : x \in V) \in \Sigma_q$, we let E_σ be the subset of E comprising all edges $e = \langle x, y\rangle$ with $\sigma_x \neq 0$, $\sigma_y \neq 0$. After a change of notation, the Blume–Capel measure with zero external-field amounts to the probability measure on Σ_2 given by

$$\pi_2(\sigma) = \frac{1}{Z^{\mathrm{BC}}} \exp\left[-K|E_\sigma| + 2K \sum_{e\in E} \delta_e(\sigma) + \Delta \sum_{x \in V} \delta_{\sigma_x, 0}\right], \qquad \sigma \in \Sigma_2, \tag{3.1}$$

where

$$\delta_e(\sigma) = \delta_{\sigma_x, \sigma_y}(1 - \delta_{\sigma_x, 0}), \qquad e = \langle x, y\rangle \in E.$$

Note that

$$2\delta_e(\sigma) = \sigma_x\sigma_y + 1 \qquad \text{for } e = \langle x, y\rangle \text{ and } \sigma_x, \sigma_y \in \{-1, +1\},$$

and this accounts for the exponent in (3.1). The constants K and Δ are to be regarded as parameters of the model. We now define the 'Blume–Capel–Potts (BCP)' probability measure π_q on Σ_q by

$$\pi_q(\sigma) = \frac{1}{Z^{\mathrm{BCP}}} \exp\left[-K|E_\sigma| + 2K\sum_{e\in E}\delta_e(\sigma) + \Delta\sum_{x\in V}\delta_{\sigma_x,0}\right], \qquad \sigma \in \Sigma_q, \tag{3.2}$$

where $Z^{\mathrm{BCP}} = Z^{\mathrm{BCP}}_{K,\Delta,q}$ is the normalizing constant. We point out that the value $q = 1$ is permitted in the above definition of π_q.

Equation (3.2) may be compared with the corresponding definition of the Potts lattice gas, see equation (2.1) of [5] and equation (3.11) of [17]. The BCP measure is a special case of the Potts lattice gas in which the parameters J and κ of [5] are related by $J = -2\kappa > 0$. The analyses of [5, 16, 17] have something in common with the current work in that they include random-cluster representations of the Potts lattice gas. Paper [5] differs from the current work in the vital regard that the parameters of the Blume–Capel measure (3.2) do not satisfy the assumptions of [5]. The BCP model is a special case of the non-super-attractive lattice gas of [16, 17]. The graphical representation of the current paper is related to that of [17], and some of the stochastic inequalities obtained here may be extended to the models considered in [17]. We have chosen to work with the Blume–Capel formulation (3.2), since we wish to concentrate on the comparisons of Sections 8 and 9. The stochastic (FKG) orderings considered here have a superficial resemblance to those of [38] but the underlying ordering of the local state space is different.

We turn now to the random-cluster representation of the BCP measure. The support of the corresponding random-cluster-type measure is a subset of the product $\Psi \times \Omega$ where $\Psi = \{0, 1\}^V$, and $\Omega = \{0, 1\}^E$. For $\psi = (\psi_x : x \in V) \in \Psi$, we let

$$V_\psi = \{x \in V : \psi_x = 1\}, \quad E_\psi = \{\langle x, y\rangle \in E : x, y \in V_\psi\}.$$

Let $\omega = (\omega_e : e \in E) \in \Omega$. We say that ω and ψ are *compatible* if $\omega_e = 0$ whenever $e \notin E_\psi$, and we write Θ for the set of all compatible pairs $(\psi, \omega) \in \Psi \times \Omega$. Let $\theta = (\psi, \omega) \in \Theta$. A vertex $x \in V$ is called *open* (or ψ-open) if $\psi_x = 1$, and is called *closed* otherwise. An edge e is called *open* (or ω-open) if $\omega_e = 1$, and *closed* otherwise. We write $\eta(\omega)$ for the set of ω-open edges, and note that $(\psi, \omega) \in \Theta$ if and only if $\eta(\omega) \subseteq E_\psi$. For $\theta = (\psi, \omega) \in \Theta$ and $e \notin E_\psi$, we say that e has been *deleted*.

Let $\theta = (\psi, \omega) \in \Theta$. The connected components of the graph $(V_\psi, \eta(\omega))$ are called *open clusters*, and their cardinality is denoted by $k(\theta)$.

The parameters of the random-cluster measure in question are $a \in (0, 1]$, $p \in [0, 1)$, $q \in (0, \infty)$, and in addition we write $r = \sqrt{1-p}$. The *diluted-random-cluster measure* with parameters a, p, q is defined to be the probability measure on $\Psi \times \Omega$ given by

$$\phi(\theta) = \frac{1}{Z^{\mathrm{DRC}}} r^{|E_\psi|} q^{k(\theta)} \prod_{x \in V} \left(\frac{a}{1-a} \right)^{\psi_x} \prod_{e \in E_\psi} \left(\frac{p}{1-p} \right)^{\omega_e} \tag{3.3}$$

for $\theta = (\psi, \omega) \in \Theta$, and $\phi(\theta) = 0$ otherwise, where $Z^{\mathrm{DRC}} = Z^{\mathrm{DRC}}_{a,p,q}$ is the normalizing constant. The above formula may be interpreted when $a = 1$ as requiring that all vertices be open. We note for future use that the projection of ϕ onto the first component Ψ of the configuration space is the probability measure satisfying

$$\Phi(\psi) = \sum_{\omega \in \Omega} \phi(\psi, \omega) \propto r^{|E_\psi|} \left(\frac{a}{1-a} \right)^{|V_\psi|} Z^{\mathrm{RC}}_{p,q}(V_\psi, E_\psi), \qquad \psi \in \Psi, \tag{3.4}$$

where

$$Z^{\mathrm{RC}}_{p,q}(W, F) = \sum_{\omega \in \{0,1\}^F} q^{k(\omega)} \left(\frac{p}{1-p} \right)^{|\eta(\omega)|} \tag{3.5}$$

denotes the partition function of the random-cluster model on $G = (W, F)$ with parameters p, q. When $F = \varnothing$, we interpret $Z^{\mathrm{RC}}_{p,q}(W, F)$ as $q^{k(W,F)}$, where $k(W, F)$ is the number of components of the graph. We speak of Φ as the 'vertex-measure' of ϕ.

The diluted-random-cluster and BCP measures are related to one another in very much the same way as are the random-cluster and Potts measures, see [28]. This is not quite so obvious as it may first seem, owing to the factor $r^{|E_\psi|}$ in the definition of ϕ. We will not labour the required calculations since they follow standard routes, but we present the coupling theorem, and we will summarize some of the necessary facts concerning the conditional measures.

We turn therefore to a coupling between the diluted-random-cluster and BCP measures. Let $\Delta \in \mathbb{R}$, $K \in [0, \infty)$, $q \in \{1, 2, 3, \dots\}$, and let a and p satisfy

$$p = 1 - e^{-2K}, \quad \frac{a}{1-a} = e^{-\Delta}. \tag{3.6}$$

We will define a probability measure μ on the product space $\Sigma_q \times \Psi \times \Omega$. This measure μ will have as support the subset $\mathcal{S} \subseteq \Sigma_q \times \Psi \times \Omega$ comprising all triples (σ, ψ, ω) such that:

(i) $(\psi, \omega) \in \Theta$,

(ii) $\psi_x = 1 - \delta_{\sigma_x,0}$ for all $x \in V$, that is, $\psi_x = 0$ if and only if $\sigma_x = 0$, and

(iii) for all $e = \langle x, y\rangle \in E$, if $\sigma_x \neq \sigma_y$ then $\omega_e = 0$.

We define μ by

$$\mu(\sigma, \psi, \omega) = \begin{cases} \dfrac{1}{Z} r^{|E_\psi|} \displaystyle\prod_{x\in V} \left(\frac{a}{1-a}\right)^{\psi_x} \prod_{e\in E_\psi} \left(\frac{p}{1-p}\right)^{\omega_e} & \text{if } (\sigma, \psi, \omega) \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.7. *Let $q \in \{1, 2, 3, \dots\}$, let $\Delta \in \mathbb{R}$, $K \in [0, \infty)$ and let a, p satisfy* (3.6)*. The marginal measures of μ on Σ_q and on $\Psi \times \Omega$, respectively, are the BCP and diluted-random-cluster measures with respective parameters K, Δ, q and a, p, q.*

Proof. Let $\sigma \in \Sigma_q$. We fix ψ by $\psi_x = 1 - \delta_{\sigma_x,0}$ for all $x \in V$, so that $E_\psi = E_\sigma$. By (3.6),

$$\prod_{x\in V} \left(\frac{a}{1-a}\right)^{\psi_x} = \exp\left[-\Delta \sum_{x\in V} (1 - \delta_{\sigma_x,0})\right].$$

By summing over all ω such that $(\sigma, \psi, \omega) \in \mathcal{S}$,

$$\begin{aligned} \sum_\omega r^{|E_\psi|} \prod_{e\in E_\psi} \left(\frac{p}{1-p}\right)^{\omega_e} &= r^{|E_\psi|} \prod_{e\in E_\psi} \left[1 + \left(\frac{p}{1-p}\right)\delta_e(\sigma)\right] \\ &= \exp\left[-K|E_\psi| + 2K \sum_{e\in E} \delta_e(\sigma)\right]. \end{aligned}$$

By (3.2),

$$\sum_{(\psi,\omega)\in\Theta} \mu(\sigma, \psi, \omega) \propto \pi_q(\sigma), \qquad \sigma \in \Sigma_q.$$

Equality must hold here, since each side is a probability mass function. This proves that the marginal of μ on Σ_q is indeed the BCP measure π_q.

Turning to the second marginal, we fix $\theta = (\psi, \omega) \in \Theta$, and let $\mathcal{S}(\theta)$ be the set of all $\sigma \in \Sigma_q$ such that $(\sigma, \psi, \omega) \in \mathcal{S}$. We have that $\sigma_x = 0$ if and only if $\psi_x = 0$. The only further constraint on σ is that it is constant on each cluster of $(V_\psi, \eta(\omega))$. There are $k(\theta)$ such clusters, and therefore $|\mathcal{S}(\theta)| = q^{k(\theta)}$. It follows that

$$\sum_\sigma \mu(\sigma, \psi, \omega) = \phi(\theta), \qquad \theta = (\psi, \omega) \in \Theta,$$

as required. □

We make some observations based on Theorem 3.7 and the method of proof. First, subject to (3.6),

$$Z^{\mathrm{BCP}}_{K,\Delta,q} = Z^{\mathrm{DRC}}_{a,p,q} \cdot e^{|V|\Delta}. \tag{3.8}$$

Secondly, the conditional measure of μ, given the pair $(\psi, \omega) \in \Psi \times \Omega$, is that obtained as follows:

(a) for $x \in V$, $\sigma_x = 0$ if and only if $\psi_x = 0$,

(b) the spins are constant on every cluster of the graph $(V_\psi, \eta(\omega))$, and each such spin is uniformly distributed on the set $\{1, 2, \dots, q\}$,

(c) the spins on different clusters are independent random variables.

Thirdly, the conditional measure of μ, given the spin vector $\sigma \in \Sigma_q$, is that obtained as follows:

(i) for $x \in V$, $\psi_x = 0$ if and only if $\sigma_x = 0$,

(ii) $(\psi, \omega) \in \Theta$,

(iii) the random variables $(\omega_e : e \in E_\psi)$ are independent,

(iv) for $e = \langle x, y\rangle \in E_\psi$, $\omega_e = 0$ if $\sigma_x \neq \sigma_y$, and $\omega_e = 1$ with probability p if $\sigma_x = \sigma_y$.

In particular, conditional on the set $\{x \in V : \sigma_x = 0\}$, the joint distribution of σ and ω is the usual coupling of the Potts and random-cluster measures on the graph $G_\psi = (V_\psi, E_\psi)$.

As two-point correlation function in the BCP model, we may take the function

$$\tau_q(x, y) = \pi_q(\sigma_x = \sigma_y \neq 0) - \frac{1}{q}\pi_q(\sigma_x\sigma_y \neq 0), \qquad x, y \in V. \tag{3.9}$$

This is related as follows to the two-point connectivity function of the diluted-random-cluster model. For $x, y \in V$, we write $x \leftrightarrow y$ if there exists a path of ω-open edges joining x to y. Similarly, for $A, B \subseteq V$, we write $A \leftrightarrow B$ if there exist $a \in A$ and $b \in B$ such that $a \leftrightarrow b$.

Theorem 3.10. *Let $\Delta \in \mathbb{R}$, $K \in [0, \infty)$, $q \in \{1, 2, \dots\}$, and let a, p satisfy* (3.6). *The corresponding diluted-random-cluster measure ϕ and BCP measure π_q on the finite graph $G = (V, E)$ are such that*

$$\tau_q(x, y) = (1 - q^{-1})\phi(x \leftrightarrow y), \qquad x, y \in V.$$

The proof follows exactly that of the corresponding statement for the random-cluster model, see for example [28].

Two particular values of q are special, namely $q = 1, 2$. From the above, the diluted-random-cluster measure with $q = 2$ corresponds to the Blume–Capel measure. Theorem 3.7 is valid with $q = 1$ also. The BCP model with $q = 1$ has two local states labelled 0 and 1. By (3.2), the Hamiltonian may be written as

$$\begin{aligned}\mathcal{H}_q(\sigma) &= K|E_\sigma| - 2K\sum_{e\in E}\delta_e(\sigma) - \Delta\sum_{x\in V}\delta_{\sigma_x,0}\\ &= -\Delta|V| - K|E_\sigma| + \Delta\sum_{x\in V}\sigma_x\\ &= -\Delta|V| - K\sum_{e=\langle x,y\rangle\in E}\sigma_x\sigma_y + \Delta\sum_{x\in V}\sigma_x, \qquad \sigma\in\Sigma_1.\end{aligned}$$

We make the change of variables $\eta_x = 2\sigma_x - 1$, to find that

$$\mathcal{H}_q(\sigma) = -\tfrac{1}{2}\Delta|V| - \tfrac{1}{4}K|E| - J\sum_{e=\langle x,y\rangle\in E}\eta_x\eta_y - \sum_{x\in V}h_x\eta_x,$$

where $J = \frac{1}{4}K$ and $h_x = \frac{1}{4}(K\deg_x - 2\Delta)$. That is, we may work with the altered Hamiltonian

$$\mathcal{H}'_q(\sigma) = -J\sum_{e=\langle x,y\rangle\in E}\eta_x\eta_y - \sum_{x\in V}h_x\eta_x, \tag{3.11}$$

which is recognised as that of the Ising model with edge-interaction J and 'local' external field $(h_x : x \in V)$. If G is regular with (constant) vertex-degree δ, then $h_x = h = \frac{1}{4}(K\delta - 2\Delta)$ for all $x \in V$. That is, the BCP model with $q = 1$ is, after a re-labelling of the local states 0, 1, an Ising model with edge-interaction J and external field h. A great deal is known about this model, and we shall make use of this observation later.

4 The lattice DRC model

Until further notice, we shall study the diluted-random-cluster model rather than the BCP model, and thus we take q to be a positive real (number). The model has so far been defined on a finite graph only. In order to pass in Section 7 to the infinite-volume limit on $\mathbb{L}^d$, we shall next introduce the concept of boundary conditions.

Let V be a finite subset of $\mathbb{Z}^d$, and let E be the subset of $\mathbb{E}^d$ comprising all edges having at least one endvertex in V. We write $\Lambda = (V, E)$, noting that Λ is not a graph since it contains edges adjacent to vertices outside

V. Any such Λ is called a *region*. The corresponding graph $\Lambda^+ = (V^+, E)$ is defined as the subgraph of $\mathbb{L}^d$ induced by E. We write $\partial\Lambda = V^+ \setminus V$. The lattice $\mathbb{L}^d$ is regular with degree $\delta = 2d$.

Let $\Psi = \{0,1\}^{\mathbb{Z}^d}$ and $\Omega = \{0,1\}^{\mathbb{E}^d}$. Let Θ be the set of compatible vertex/edge configurations $(\psi,\omega) \in \Psi \times \Omega$ satisfying $\eta(\omega) \subseteq \mathbb{E}^d_\psi$. Each $\lambda = (\kappa,\rho) \in \Theta$ may be viewed as a boundary condition on the region Λ, as follows. Let Θ^λ_Λ be the subset of Θ containing configurations that agree with λ on $\mathbb{L}^d \setminus \Lambda$, in that Θ^λ_Λ contains all (ψ,ω) with $\psi_x = \kappa_x$ for $x \notin V$, $\omega_e = \rho_e$ for $e \notin E$. Let $\phi^\lambda_{\Lambda,a,p,q}$ denote the diluted-random-cluster measure on Λ with boundary condition λ, that is,

$$\phi^\lambda_{\Lambda,a,p,q}(\theta) = \frac{1}{Z^{\mathrm{DRC}}} r^{|E_\psi|} q^{k(\theta,\Lambda)} \prod_{x\in V} \left(\frac{a}{1-a}\right)^{\psi_x} \prod_{e\in E_\psi} \left(\frac{p}{1-p}\right)^{\omega_e}, \tag{4.1}$$

if $\theta = (\psi,\omega) \in \Theta^\lambda_\Lambda$, and $\phi^\lambda_{\Lambda,a,p,q}(\theta) = 0$ otherwise. Here, $E_\psi = \{\langle x,y\rangle \in E : \psi_x = \psi_y = 1\}$, $k(\theta,\Lambda)$ is the number of open clusters of $(\mathbb{Z}^d_\psi, \eta(\omega))$ that intersect V^+ , and $Z^{\mathrm{DRC}} = Z^{\mathrm{DRC}}_{\Lambda,\lambda,a,p,q}$ is a normalizing constant. See (3.3), and recall that $r = \sqrt{1-p}$.

The probability measure $\phi^\lambda_{\Lambda,a,p,q}$ is supported effectively on the product $\Psi_V \times \Omega_E$ where $\Psi_V = \{0,1\}^V$ and $\Omega_E = \{0,1\}^E$. We write $\Phi^\lambda_{\Lambda,a,p,q}$ for its marginal (or 'projected') measure on the first coordinate $\{0,1\}^V$ of this space, given as follows. Let $\lambda = (\kappa,\rho) \in \Theta$, let $\Lambda = (V,E)$ be a region, and let Ψ^λ_Λ be the set of all $\psi \in \Psi$ that agree with κ off V. For $\psi \in \Psi$, let $\Lambda(\psi)$ denote the subgraph of Λ^+ induced by the ψ-open vertices. Suppose $\psi \in \Psi^\lambda_\Lambda$. Let $Z^{\mathrm{RC}}_{\lambda,p,q}(\Lambda(\psi))$ denote the partition function of the random-cluster model on $\Lambda(\psi)$ with boundary condition λ, see (3.5). (This boundary condition is to be interpreted as: two vertices $u,v \in V^+$ are deemed to be connected off Λ^+ if there exists a path from u to v of ρ-open edges of $\mathbb{E}^d \setminus E$.) As in (3.4),

$$\begin{aligned}\Phi^\lambda_{\Lambda,a,p,q}(\psi) &= \sum_{\omega\in\Omega_E} \phi^\lambda_{\Lambda,a,p,q}(\psi,\omega)\\ &\propto r^{|E_\psi|} \left(\frac{a}{1-a}\right)^{|V_\psi|} Z^{\mathrm{RC}}_{\lambda,p,q}(\Lambda(\psi)),\end{aligned} \tag{4.2}$$

for $\psi \in \Psi_V$, where $V_\psi = \{v \in V : \psi_v = 1\}$. There is a slight abuse of notation here, in that ψ has been used as a member of both Ψ and Ψ_V.

Two especially interesting situations arise when $p = 0$ and/or $q = 1$.
(a) *Product measure.* If $p = 0$ then $\phi^\lambda_{\Lambda,a,p,q}$ is a product measure, and may therefore be extended to a product measure $\phi_{a,0,q}$ on $\mathbb{L}^d$ under which each vertex is open with probability $qa/(1-a+qa)$, and each edge is almost-surely closed. There exists, $\phi_{a,0,q}$-almost-surely, an infinite open vertex-cluster (respectively, infinite closed vertex-cluster) if $qa/(1-a+qa) > p_{\mathrm{c}}^{\mathrm{site}}$

(respectively, $(1-a)/(1-a+qa) > p_c^{\text{site}}$), where p_c^{site} denotes the critical probability of site percolation on $\mathbb{L}^d$.
(b) *Ising model with external field.* Let $q = 1$, and recall from the end of Section 3 that the BCP model is essentially an Ising model with edge-interaction $J = \frac{1}{4}K$ and local external field $h_x = \frac{1}{4}(K \deg_x - 2\Delta)$. For the sake of illustration, consider the box $B_n = [-n, n]^d$ of $\mathbb{L}^d$ with periodic boundary conditions, so that $\deg_x = \delta = 2d$ for all x. Then

$$J = -\tfrac{1}{8}\log(1-p), \quad h = \tfrac{1}{2}(Kd - \Delta) = \tfrac{1}{2}\log\left(\frac{a}{(1-a)(1-p)^{d/2}}\right). \tag{4.3}$$

On passing to the limit as $n \to \infty$, we obtain an infinite-volume Ising model with parameters J, h. If we restrict ourselves to pairs a, p such that $h = 0$, there is a critical value $K_c(d)$ of K given by $K_c(d) = -2\log(1-\pi_c)$ where $\pi_c = \pi_c(d)$ is the critical edge-parameter of the random-cluster model on $\mathbb{L}^d$ with cluster-weighting parameter 2. Rewritten in terms of a and p, the phase diagram possesses a special point $(\overline{a}, \overline{p})$, where

$$\overline{a} = \frac{(1-\pi_c)^{2d}}{1+(1-\pi_c)^{2d}}, \quad \overline{p} = 1-(1-\pi_c)^4. \tag{4.4}$$

By a consideration of the associated random-cluster measure or otherwise, we deduce that there is a line of first-order phase transitions along the arc

$$\frac{a}{1-a} = (1-p)^{d/2}, \qquad \overline{p} < p < 1. \tag{4.5}$$

To the left (respectively, right) of this arc in (a, p) space (see Figure 1 for the case $d = 2$), there is an infinite cluster of 0-state (respectively, 1-state) vertices. As the arc is crossed from left to right, there is a discontinuous increase in the density of the infinite 1-state cluster. Related issues concerning the percolation of $\pm$-state clusters in the zero-field Ising model are considered in [2].

We note when $d = 2$ that $\pi_c(2) = \sqrt{2}/(1+\sqrt{2})$, so that

$$\overline{a} = \frac{1}{1+(1+\sqrt{2})^4}, \quad \overline{p} = 1-(1+\sqrt{2})^{-4}. \tag{4.6}$$

5 Stochastic orderings of vertex-measures

Many of the results of this section have equivalents for general finite graphs, but we concentrate here on subgraphs of the lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$. While the route followed here is fairly standard, some of the calculations are novel. The vertex-measure $\Phi^\lambda_{\Lambda,a,p,q}$ plays an important part in the stochastic orderings relevant to the BCP model, and we turn next to its properties, beginning with a reminder about orderings.

Let I be a finite set, and let $\Sigma = \{0,1\}^I$ be viewed as a partially ordered set. For $J \subseteq I$ and $\sigma \in \Sigma$, we write σ^J for the configuration that equals 1 on J and agrees with σ off J. If $J = \{i\}$ or $J = \{i,j\}$ we may abuse notation by removing the braces. Let μ_1, μ_2 be probability measures on Σ. We write $\mu_1 \le_{\mathrm{st}} \mu_2$, and say that μ_1 is *stochastically dominated* by μ_2, if $\mu_1(f) \le \mu_2(f)$ for all increasing functions $f : \Sigma \to \mathbb{R}$. A probability measure μ on Σ is said to be strictly positive if $\mu(\sigma) > 0$ for all $\sigma \in \Sigma$. If μ_1, μ_2 are strictly positive, then $\mu_1 \le_{\mathrm{st}} \mu_2$ if the pair satisfies the so-called Holley condition,

$$\mu_2(\sigma_1 \vee \sigma_2)\mu_1(\sigma_1 \wedge \sigma_2) \ge \mu_1(\sigma_1)\mu_2(\sigma_2), \qquad \sigma_1, \sigma_2 \in \Sigma. \tag{5.1}$$

Here, $\vee$ denotes the coordinatewise maximum, and $\wedge$ the coordinatewise minimum. It is standard (see [28], Section 2.1) that it suffices to check (5.1) for pairs of the form $(\sigma_1, \sigma_2) = (\sigma^i, \sigma)$ and $(\sigma_1, \sigma_2) = (\sigma^i, \sigma^j)$, for $\sigma \in \Sigma$ and $i, j \in I$.

A probability measure μ on Σ is said to be *positively associated* if

$$\mu(A \cap B) \ge \mu(A)\mu(B)$$

for all increasing events $A, B \subseteq \Sigma$. For $\tau \in \Sigma$ and $J \subseteq I$, let Σ_J^τ be the subset of Σ containing all $\sigma \in \Sigma$ with $\sigma_i = \tau_i$ for $i \notin J$. The measure μ is said to be *strongly positively associated* if, for all pairs τ, J, the conditional measure, given Σ_J^τ, is positively associated when viewed as a measure on $\{0,1\}^J$. The measure μ is called *monotonic* if, for all $i \in I$, $\mu(\sigma_i = 1 \mid \Sigma_i^\tau)$ is a non-decreasing function of τ. It is standard (see [28], Section 2.2) that a strictly positive probability measure μ on Σ is strongly positively associated (respectively, monotonic) if and only if it satisfies the so-called FKG condition:

$$\mu(\sigma_1 \vee \sigma_2)\mu(\sigma_1 \wedge \sigma_2) \ge \mu(\sigma_1)\mu(\sigma_2), \qquad \sigma_1, \sigma_2 \in \Sigma. \tag{5.2}$$

Furthermore, it suffices to check (5.2) for pairs of the form $(\sigma_1, \sigma_2) = (\sigma^i, \sigma^j)$, for $\sigma \in \Sigma$ and $i, j \in I$. Further discussions of the FKG and Holley inequalities may be found in [22, 28, 32].

The proofs of the following theorems will be found later in this section.

Theorem 5.3. *Let $\Lambda = (V, E)$ be a region, let $\lambda \in \Theta$, and let $a \in (0,1), p \in [0,1)$. The probability measure $\Phi_{\Lambda,a,p,q}^\lambda$ is strongly positively associated, and hence monotonic, if $q \in [1,2]$.*

The condition $q \in [1,2]$ is important. If $q > 2$, then strong positive-association does not hold for all $p \in (0,1)$. The conclusion would be similarly false for the full diluted-random-cluster measure even for $q \in [1,2]$. For example, let G be the graph with exactly two vertices x, y joined by a single edge e, and consider the associated measure $\phi_{a,p,q}$ with $a, p \in (0,1)$ and $q \in (0,\infty)$. Then, with $r = \sqrt{1-p}$,

$$\begin{aligned}\phi(\psi_y = 1 \mid \psi_x = 0,\ \omega_e = 0) &= \frac{qa}{qa + 1 - a},\\ \phi(\psi_y = 1 \mid \psi_x = 1,\ \omega_e = 0) &= \frac{qar}{qar + 1 - a}.\end{aligned}$$

The first term exceeds the second strictly, and hence $\phi_{a,p,q}$ is not monotone on the product space $\{0,1\}^V \times \{0,1\}^E$.

We prove next that $\Phi^{\lambda}_{\Lambda,a,p,q}$ is increasing in λ, so long as $q \in [1,2]$.

Theorem 5.4. *Let $\Lambda = (V,E)$ be a region, and let $a \in (0,1), p \in [0,1)$ and $q \in [1,2]$. If $\lambda_1 \le \lambda_2$ then $\Phi^{\lambda_1}_{\Lambda,a,p,q} \le_{\mathrm{st}} \Phi^{\lambda_2}_{\Lambda,a,p,q}$.*

The two theorems above will be proved by checking certain inequalities related to (5.1) and (5.2). It is convenient to make use of a subsidiary proposition for this, and we state this next, beginning with some notation. For a region $\Lambda = (V,E)$, we abbreviate to Φ_i the marginal (or projected) measure on the space Ψ_V of the diluted-random-cluster measure $\phi^{\lambda_i}_{\Lambda,a_i,p_i,q_i}$. We abbreviate to $\mu^i_{\Lambda,\psi}$ the usual random-cluster measure on $\Lambda(\psi)$ with boundary condition λ_i and parameters (p_i,q_i). For $w \in \mathbb{Z}^d$, let $I_w \subseteq \Omega$ be the event that w has no incident ω-open edges.

Proposition 5.5. *Let $\lambda_1, \lambda_2 \in \Theta$, $a_i \in (0,1)$, $p_i \in [0,1)$ for $i = 1,2$, and $q_1 \in [1,\infty)$, $q_2 \in [1,2]$. Let $\psi \in \Psi$, let $\Lambda = (V,E)$ be a region, and let $x \in V$ be such that $\psi_x = 0$. Let $b = b(x,\psi)$ denote the number of edges of E of the form $\langle x,z\rangle$ with $\psi_z = 1$. If*

$$q_2\left(\frac{a_2}{1-a_2}\right)\frac{(1-p_2)^{b/2}}{\mu^2_{\Lambda,\psi^x}(I_x)} \ge q_1\left(\frac{a_1}{1-a_1}\right)\frac{(1-p_1)^{b/2}}{\mu^1_{\Lambda,\psi^x}(I_x)}, \tag{5.6}$$

then

$$\Phi_2(\psi^x)\Phi_1(\psi) \ge \Phi_1(\psi^x)\Phi_2(\psi), \tag{5.7}$$

$$\Phi_2(\psi^{x,y})\Phi_1(\psi) \ge \Phi_1(\psi^x)\Phi_2(\psi^y), \qquad y \in V \setminus V_\psi,\ y \ne x. \tag{5.8}$$

We examine next the monotonicity properties of $\Phi^{\lambda}_{\Lambda,a,p,q}$ as a, p, q vary. Recall that $\delta = 2d$.

Theorem 5.9. *Let $\Lambda = (V,E)$ be a region, and let $\lambda \in \Theta$. Let $a_i \in (0,1)$, $p_i \in [0,1)$, and $q_i \in [1,2]$ for $i = 1,2$, and let Φ_i be as above. Each of the following is a sufficient condition for the stochastic inequality $\Phi_1 \le_{\mathrm{st}} \Phi_2$:*

(i) *that $a_1 \le a_2$, $p_1 \le p_2$, and $q_1 = q_2$,*

(ii) *that*

$$q_2\left(\frac{a_2}{1-a_2}\right) \ge q_1\left(\frac{a_1}{1-a_1}\right)(1-p_1)^{-\delta/2},$$

(iii) *that* $p_1 \le p_2$, $q_1 \ge q_2$, *and*

$$q_2\left(\frac{a_2}{1-a_2}\right)(1-p_2)^{\delta/2} \ge q_1\left(\frac{a_1}{1-a_1}\right)(1-p_1)^{\delta/2}, \tag{5.10}$$

(iv) *that* $q_1 \le q_2$, (5.10) *holds, and*

$$\frac{p_2}{q_2(1-p_2)} \ge \frac{p_1}{q_1(1-p_1)}.$$

In the next section we shall pass to infinite-volume limits along increasing sequences of regions. In preparation for this, we note two further properties of stochastic monotonicity. The two extremal boundary conditions are the vectors $\mathbf{0} = (0,0) \in \Psi \times \Omega$ and $\mathbf{1} = (1,1) \in \Psi \times \Omega$.

Theorem 5.11. *Let* $a \in (0,1), p \in [0,1)$, $q \in [1,2]$, *and let* Λ_1, Λ_2 *be regions with* $\Lambda_1 \subseteq \Lambda_2$. *Then*

$$\Phi^{\mathbf{0}}_{\Lambda_1,a,p,q} \le_{\mathrm{st}} \Phi^{\mathbf{0}}_{\Lambda_2,a,p,q}, \qquad \Phi^{\mathbf{1}}_{\Lambda_1,a,p,q} \ge_{\mathrm{st}} \Phi^{\mathbf{1}}_{\Lambda_2,a,p,q}.$$

It is noted that the boundary conditions $b = \mathbf{0}, \mathbf{1}$ contain information concerning both vertex and edge configuration off Λ. By (4.2), only the external edge configuration is in fact relevant. The above inequalities for the vertex-measures $\Phi_{a,p,q}$ imply a degree of monotonicity of the full diluted-random-cluster measure $\phi_{a,p,q}$. We shall not explore this in depth, but restrict ourselves to two facts for later use.

Theorem 5.12. *Let* $a \in (0,1)$, $p \in [0,1)$, $q \in [1,2]$, *and* $\lambda \in \Theta$. *For any region* Λ, *the diluted-random-cluster measure* $\phi^{\lambda}_{\Lambda,a,p,q}$ *is stochastically non-decreasing in* a, p, *and* λ.

A probability measure on a product space $\{0,1\}^I$ is said to have the finite-energy property if, for all $i \in I$, the law of the state of i, conditional on the states of all other indices, is (almost surely) strictly positive. See [28].

Theorem 5.13. *Let* $a \in (0,1)$, $p \in [0,1)$, $q \in [1,2]$, $\lambda \in \Theta$, *and let* Λ *be a region. The probability measure* $\Phi^{\lambda}_{\Lambda,a,p,q}$ *has the finite-energy property, and indeed,*

$$\frac{qa}{1-a+qa} \le \Phi^{\lambda}_{\Lambda,a,p,q}(J_x \mid \mathcal{T}_x) \le \frac{aq}{aq+(1-a)r^{\delta}}, \qquad \Phi^{\lambda}_{\Lambda,a,p,q}\text{-}a.s.,$$

where $J_x \subseteq \Psi$ *is the event that* x *is open, and* $\mathcal{T}_x$ *is the* σ*-field of* Ψ *generated by the states of vertices other than* x.

We turn now to the proofs, and begin with a lemma.

Lemma 5.14. *Under the conditions of Proposition 5.5, and with $x, y \in V \setminus V_\psi$,*

$$\mu^2_{\Lambda,\psi^{x,y}}(I_x) \le \mu^2_{\Lambda,\psi^x}(I_x) r_2^f,$$

where $r_2 = \sqrt{1-p_2}$ and $f \in \{0,1\}$ is the number of edges of $\mathbb{L}^d$ with endvertices x, y.

Proof. We note the elementary inequality

$$\frac{q(1-p)}{p+q(1-p)} \le \sqrt{1-p}, \qquad p \in [0,1],\ q \in [1,2]. \tag{5.15}$$

Let B (respectively, C) be the set of b (respectively, c) edges joining x (respectively, y) to ψ-open vertices of V^+, and let F be the set of edges with endvertices x, y. Let B_0 (respectively, C_0, F_0) be the (decreasing) event that all edges in B (respectively, C, F) are closed. Since a random-cluster measure with $q \ge 1$ is positively associated,

$$\mu^2_{\Lambda,\psi^{x,y}}(I_x) \le \mu^2_{\Lambda,\psi^{x,y}}(B_0 \cap F_0 \mid C_0).$$

By an elementary property of random-cluster measures, see [28],

$$\begin{aligned}\mu^2_{\Lambda,\psi^{x,y}}(B_0 \cap F_0 \mid C_0) &= \mu^2_{\Lambda\setminus C,\psi^{x,y}}(B_0 \cap F_0)\\ &= \mu^2_{\Lambda\setminus C,\psi^{x,y}}(B_0 \mid F_0)\mu^2_{\Lambda\setminus C,\psi^{x,y}}(F_0),\end{aligned}$$

where $\Lambda \setminus C$ is obtained from Λ by deleting all edges in C. In $\Lambda(\psi^{x,y}) \setminus C$, the only possible neighbour of y is x, whence, for $f = |F| = 0, 1$,

$$\mu^2_{\Lambda\setminus C,\psi^{x,y}}(F_0) = \frac{q_2(1-p_2)^f}{1+(q_2-1)(1-p_2)^f} \le (1-p_2)^{f/2} = r_2^f,$$

where we have used (5.15) and the fact that $q_2 \le 2$. Similarly,

$$\mu^2_{\Lambda\setminus C,\psi^{x,y}}(B_0 \mid F_0) = \mu^2_{\Lambda,\psi^x}(B_0) = \mu^2_{\Lambda,\psi^x}(I_x),$$

and the claim follows. ☐

Proof of Proposition 5.5. We prove (5.8) only, the proof of (5.7) is similar and simpler. Inequality (5.6) implies by Lemma 5.14 that

$$q_2\left(\frac{a_2}{1-a_2}\right)\frac{r_2^{b+f}}{\mu^2_{\Lambda,\psi^{x,y}}(I_x)} \ge q_1\left(\frac{a_1}{1-a_1}\right)\frac{r_1^b}{\mu^1_{\Lambda,\psi^x}(I_x)}, \tag{5.16}$$

where f is the number of edges of $\mathbb{L}^d$ joining x and y. Let $Z^{\mathrm{RC}}_{\lambda,p,q}(G)$ be the partition function of the random-cluster model on a graph G with parameters p, q and boundary condition λ, see (3.5). We have that

$$\mu^1_{\Lambda,\psi^x}(I_x) = q_1 \frac{Z^{\mathrm{RC}}_{\lambda_1,p_1,q_1}(\Lambda(\psi))}{Z^{\mathrm{RC}}_{\lambda_1,p_1,q_1}(\Lambda(\psi^x))}, \qquad \mu^2_{\Lambda,\psi^{x,y}}(I_x) = q_2 \frac{Z^{\mathrm{RC}}_{\lambda_2,p_2,q_2}(\Lambda(\psi^y))}{Z^{\mathrm{RC}}_{\lambda_2,p_2,q_2}(\Lambda(\psi^{x,y}))}.$$

We substitute these into (5.16) to find that

$$\begin{aligned}\left(\frac{a_2}{1-a_2}\right) Z^{\mathrm{RC}}_{\lambda_2,p_2,q_2}(\Lambda(\psi^{x,y})) Z^{\mathrm{RC}}_{\lambda_1,p_1,q_1}(\Lambda(\psi)) r_2^{b+f}& \\ \geq \left(\frac{a_1}{1-a_1}\right) Z^{\mathrm{RC}}_{\lambda_1,p_1,q_1}(\Lambda(\psi^x)) Z^{\mathrm{RC}}_{\lambda_2,p_2,q_2}(\Lambda(\psi^y)) r_1^b&.\end{aligned}$$

Now, $|V(\psi^x) \setminus V(\psi)| = 1$ and $|E(\psi^x) \setminus E(\psi)| = b$ where $V(\psi) = V \cap \mathbb{Z}^d_\psi$ and $E(\psi) = E \cap \mathbb{E}^d_\psi$, so that

$$\begin{aligned}&\left(\frac{a_2}{1-a_2}\right)^{|V(\psi^{x,y})|} Z^{\mathrm{RC}}_{\lambda_2,p_2,q_2}(\Lambda(\psi^{x,y})) r_2^{|E(\psi^{x,y})|} \\ &\qquad\qquad \times \left(\frac{a_1}{1-a_1}\right)^{|V(\psi)|} Z^{\mathrm{RC}}_{\lambda_1,p_1,q_1}(\Lambda(\psi)) r_1^{|E(\psi)|} \\ &\quad \geq \left(\frac{a_1}{1-a_1}\right)^{|V(\psi^x)|} Z^{\mathrm{RC}}_{\lambda_1,p_1,q_1}(\Lambda(\psi^x)) r_1^{|E(\psi^x)|} \\ &\qquad\qquad \times \left(\frac{a_2}{1-a_2}\right)^{|V(\psi^y)|} Z^{\mathrm{RC}}_{\lambda_2,p_2,q_2}(\Lambda(\psi^y)) r_2^{|E(\psi^y)|}.\end{aligned}$$

As in (4.2),

$$\Phi_i(\psi) = \sum_{\omega \in \Omega_E} \phi^{\lambda_i}_{\Lambda,a_i,p_i,q_i}(\psi,\omega) \propto r_i^{|E(\psi)|} \left(\frac{a_i}{1-a_i}\right)^{|V(\psi)|} Z^{\mathrm{RC}}_{\lambda_i,p_i,q_i}(\Lambda(\psi)),$$

for $\psi \in \Psi_V$, and (5.8) follows. □

Proof of Theorem 5.3. We apply Proposition 5.5 with $a_i = a$, $p_i = p$, $q_i = q$, and $\lambda_i = \lambda$. Inequality (5.6) is a triviality since $\mu^1_{\Lambda,\psi} = \mu^2_{\Lambda,\psi}$ for every ψ. By (5.8) and the comment after (5.2), $\Phi^\lambda_{\Lambda,a,p,q}$ satisfies the FKG condition (5.2), and the claim follows. □

Proof of Theorem 5.4. Since $\lambda_1 \leq \lambda_2$, $\mu^1_{\Lambda,\psi} \leq_{\mathrm{st}} \mu^2_{\Lambda,\psi}$ for every $\psi \in \Psi$. Now, I_x is a decreasing event, whence $\mu^1_{\Lambda,\psi}(I_x) \geq \mu^2_{\Lambda,\psi}(I_x)$. By Proposition 5.5 and the comment after (5.1), the $\Phi_i = \Phi^{\lambda_i}_{\Lambda,a,p,q}$ satisfy the Holley condition (5.1), and the claim follows. □

Proof of Theorem 5.9. In each case, we shall apply Proposition 5.5 and appeal to the Holley condition (5.1) and the comment thereafter. It suffices to check (5.6) for every relevant vertex x. We recall some basic facts about random-cluster measures to be found in, for example, [28]. Let $G = (W, F)$ be a graph and let $\mu_{p,q}$ be the random-cluster measure on $\{0,1\}^F$ with parameters $p \in [0,1]$, $q \in [1,\infty)$. By the comparison inequalities,

$$\frac{p}{p+q(1-p)} \le \mu_{p,q}(f \text{ is open}) \le p, \qquad f \in F, \tag{5.17}$$

and, if $x \in W$ has degree b,

$$(1-p)^b \le \mu_{p,q}(I_x) \le \left(1 - \frac{p}{p+q(1-p)}\right)^b. \tag{5.18}$$

We note from (5.15) that

$$\left(1 - \frac{p}{p+q(1-p)}\right)^b \le (1-p)^{b/2}, \qquad p \in [0,1],\ q \in [1,2]. \tag{5.19}$$

(i): We may adapt the exponential-steepness argument of [29], as in Section 2.5 of [28], to the decreasing event I_x to obtain, in the above notation,

$$\frac{d}{dp} \log \mu_{p,q}(I_x) \le -\frac{1}{p(1-p)} \sum_{f:\, f\sim x} \mu_{p,q}(f \text{ is open}), \tag{5.20}$$

where the sum is over the b edges f with endvertex x. Let $q \in [1,2]$. By (5.17),

$$\frac{d}{dp} \log \mu_{p,q}(I_x) \le -\frac{1}{p(1-p)} \sum_{f:\, f\sim x} \frac{p}{p+q(1-p)} \le -\frac{b}{2(1-p)}.$$

We integrate from p_1 to p_2 and apply to the measures $\mu_i^{\Lambda(\psi)}$ to obtain that

$$\frac{\mu^2_{\Lambda,\psi}(I_x)}{\mu^1_{\Lambda,\psi}(I_x)} \le \left(\frac{1-p_2}{1-p_1}\right)^{b/2}.$$

Inequality (5.6) follows as required.
(ii): Inequality (5.6) follows from (5.18)–(5.19) on noting that $b \le \delta$.
(iii), (iv): Under either set of conditions, $\mu^1_{\Lambda,\psi^x} \le_{\mathrm{st}} \mu^2_{\Lambda,\psi^x}$, implying that $\mu^1_{\Lambda,\psi^x}(I_x) \ge \mu^2_{\Lambda,\psi^x}(I_x)$. Inequality (5.6) follows on noting that $b \le \delta$. □

Proof of Theorem 5.11. These inequalities follow in the same way as for the random-cluster measure (see [28], Section 4.3) using the monotonicity of $\Phi^\lambda_{\Lambda,a,p,q}$ for $\lambda = \mathbf{0}, \mathbf{1}$. □

Proof of Theorem 5.12. Let $C \subseteq \Psi \times \Omega$ be an increasing cylinder event. By the coupling of Section 3,

$$\phi^{\lambda}_{\Lambda,a,p,q}(C) = \Phi^{\lambda}_{\Lambda,a,p,q}(\mu^{\lambda}_{\Lambda,\psi,p,q}(C_\psi)),$$

where $C_\psi = \{\omega \in \Omega : (\psi,\omega) \in C\}$ and $\mu^{\lambda}_{\Lambda,\psi,p,q}$ is the random-cluster measure on $V(\psi)$ with boundary condition λ. Now, C_ψ is an increasing event in Ω, and therefore $\mu^{\lambda}_{\Lambda,\psi,p,q}(C_\psi))$ is increasing in ψ, p, and λ. The claim follows by Theorem 5.9(i). □

Proof of Theorem 5.13. Since $q \in [1,2]$, $\Phi^{\lambda}_{\Lambda,a,p,q}$ is monotonic by Theorem 5.3. Since J_x is increasing, a lower bound for the conditional probability of J_x is obtained by considering the situation in which all other vertices are closed. In this case, x contributes $qa/(1-a)$ (respectively, 1) in (4.2) when open (respectively, closed), and the lower bound follows.

An upper bound is obtained by considering the situation in which $\lambda = \mathbf{1}$, and all vertices other than x are open and connected by open edges. This time, x contributes no more than

$$r^\delta q \left(\frac{a}{1-a}\right) \sum_{\omega \in \{0,1\}^\delta} \prod_{i=1}^{\delta} \left(\frac{p}{1-p}\right)^{\omega_i},$$

when open, and 1 when closed. □

6 Stochastic orderings of edge-measures

Let $G = (V,E)$ be a finite graph, and let $\phi_{a,p,q}$ be the diluted-random-cluster measure on the corresponding sample space $\Psi \times \Omega = \{0,1\}^V \times \{0,1\}^E$. Let $\Upsilon_{a,p,q}$ denote the marginal measure of $\phi_{a,p,q}$ on the second component Ω,

$$\Upsilon_{a,p,q}(\omega) = \sum_{\psi \in \Psi} \phi_{a,p,q}(\psi,\omega), \qquad \omega \in \Omega.$$

We first compare Υ_{1,p_1,q_1} with Υ_{a,p_2,q_2}.

Theorem 6.1. *Let* $0 < a_2 \le a_1 = 1$, $p_1, p_2 \in (0,1)$, $q_1, q_2 \in [1,\infty)$. *Let* $r_i = \sqrt{1-p_i}$, *and denote by* Υ_i *the probability measure* Υ_{a_i,p_i,q_i}.
(a) *If* $q_2 \le q_1$ *and*

$$\frac{1-p_2}{p_2}(1 + 2w_\delta + w_\delta w_{\delta-1}) \le \frac{1-p_1}{p_1}, \tag{6.2}$$

where δ *is the maximum vertex-degree of* G *and*

$$w_j = \frac{1}{q_2 r_2^j}\left(\frac{1-a_2}{a_2}\right), \qquad j = 0,1,2,\dots,\delta,$$

then $\Upsilon_1 \leq_{\text{st}} \Upsilon_2$.
(b) *If* $p_1 \geq p_2$ *and* $q_1 \leq q_2$, *then* $\Upsilon_1 \geq_{\text{st}} \Upsilon_2$.

Theorem 6.3. *Let* $0 < a_1 \leq a_2 < 1$, $0 < p_1 \leq p_2 < 1$, *and* $q \in [1, 2]$. *Then* $\Upsilon_{a_1,p_1,q} \leq_{\text{st}} \Upsilon_{a_2,p_2,q}$.

Proof of Theorem 6.1. (a) The quantity

$$w_j(a, p, q) = \frac{1}{qr^j}\left(\frac{1-a}{a}\right)$$

may be viewed as follows. Let $(\psi, \omega) \in \Theta$, and let $x \in V$ be such that $\psi_x = 0$. Then

$$\phi_{a,p,q}(\psi, \omega) = \phi_{a,p,q}(\psi^x, \omega) w_j, \tag{6.4}$$

where $j = j(x, \psi)$ is the number of neighbours u of x such that $\psi_u = 1$. Note that w_j is increasing in j.

Suppose (6.2) holds. We will show that the measures Υ_i satisfy (5.1). By the remark after (5.1), it suffices to show that, for $e, f \in E$ with $e \neq f$, and $\omega \in \Omega$ with $\omega_e = 0$,

$$\Upsilon_2(\omega^{e,f})\Upsilon_1(\omega) \geq \Upsilon_1(\omega^e)\Upsilon_2(\omega^f), \tag{6.5}$$

$$\Upsilon_2(\omega^e)\Upsilon_1(\omega) \geq \Upsilon_1(\omega^e)\Upsilon_2(\omega). \tag{6.6}$$

We will show (6.5) only, the proof of (6.6) is similar. We may assume that $\omega_f = 0$.

Since $a_1 = 1$, Υ_1 is the usual random-cluster measure on G with parameters p_1 and q. Therefore,

$$\Upsilon_1(\omega) = \Upsilon_1(\omega^e)\left(\frac{1-p_1}{p_1}\right) q_1^{k_1}, \tag{6.7}$$

where

$$k_1 = k(1, \omega) - k(1, \omega^e) = \begin{cases} 1 & \text{if } e \text{ is an isthmus of the graph } (V, \eta(\omega^e)), \\ 0 & \text{otherwise.} \end{cases}$$

For $\xi \in \Omega$, let $K(\xi) = \{\psi : (\psi, \xi) \in \Theta\}$ be the set of compatible $\psi \in \Psi$. Let $e = \langle x, y \rangle$, and write

$$B = \{\psi \in \Psi : \psi^{x,y} \in K(\omega^{e,f}),\ \psi_x = \psi_y = 0\}$$

Then $K(\omega^f)$ is the union of

(i) $\{\psi^{x,y} : \psi \in B\}$, and
(ii) $\{\psi^x : \psi \in B\}$ if y is isolated in ω^f, and

(iii) $\{\psi^y : \psi \in B\}$ if x is isolated in ω^f, and

(iv) B, if both x and y are isolated in ω^f.

Let $\psi \in B$. By (6.4), with $\phi_i = \phi_{a_i,p_i,q_i}$,

$$\begin{aligned}\phi_2(\psi^x,\omega^f) &\begin{cases} = 0 & \text{if } (\psi^x,\omega^f) \notin \Theta, \\ \le \phi_2(\psi^{x,y},\omega^f) w_\delta & \text{if } (\psi^x,\omega^f) \in \Theta. \end{cases} \\ &\le \phi_2(\psi^{x,y},\omega^f) w_\delta.\end{aligned}$$

Similarly,

$$\phi_2(\psi,\omega^f) \le \phi_2(\psi^x,\omega^f) w_\delta \le \phi_2(\psi^{x,y},\omega^f) w_\delta w_{\delta-1}.$$

Also,

$$\phi_2(\psi^{x,y},\omega^f) = \phi_2(\psi^{x,y},\omega^{e,f}) \left(\frac{1-p_2}{p_2}\right) q_2^{k_2}, \qquad \psi \in B,$$

where

$$k_2 = k(\psi^{x,y},\omega^f) - k(\psi^{x,y},\omega^{e,f}) \le k_1.$$

Therefore, for $\psi \in B$,

$$\begin{aligned}\phi_2(\psi^{x,y},\omega^f) + \phi_2(\psi^x,\omega^f) + \phi_2(\psi^y,\omega^f) + \phi_2(\psi,\omega^f)& \\ \le \phi_2(\psi^{x,y},\omega^{e,f}) \left(\frac{1-p_2}{p_2}\right) q_2^{k_2}(1 + 2w_\delta + w_\delta w_{\delta-1}).&\end{aligned}$$

We sum over $\psi \in B$ and use (6.2) and (6.7) to find as required that

$$\begin{aligned}\Upsilon_1(\omega^e)\Upsilon_2(\omega^f) &\le \Upsilon_1(\omega^e)\Upsilon_2(\omega^{e,f}) \left(\frac{1-p_2}{p_2}\right) q_2^{k_2}(1 + 2w^\delta + w_\delta w_{\delta-1}) \\ &\le \Upsilon_1(\omega^e)\Upsilon_2(\omega^{e,f}) \left(\frac{1-p_1}{p_1}\right) q_1^{k_1} \\ &= \Upsilon_1(\omega)\Upsilon_2(\omega^{e,f}).\end{aligned}$$

(b) The proof is similar but easier to that of (a), and is omitted. □

Proof of Theorem 6.3. Write $\phi_i = \phi_{a_i,p_i,q}$. For any increasing event $A \subseteq \Omega$,

$$\Upsilon_1(A) = \phi_1(\Psi \times A) = \phi_1(\phi_1(\Psi \times A \mid \psi)) = \Phi_1(\mu_{\psi,p_1,q}(A)),$$

where $\mu_{\psi,p,q}$ denotes the random-cluster measure on (V, E_ψ) with parameters p and q. Now, $\mu_{\psi,p_1,q} \le_{\mathrm{st}} \mu_{\psi,p_2,q}$, and $\mu_{\psi,p_2,q}(A)$ is non-decreasing in ψ. It follows by Theorem 5.9(i) that $\Psi_1(A) \le \Psi_2(A)$ as required. □

7 Infinite-volume measures

There are two ways of moving to infinite-volume measures on the lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, namely by passing to weak limits, and by the Dobrushin–Lanford–Ruelle (DLR) formalism. The associated theory is standard for the random-cluster model, and the same arguments are mostly valid for the diluted-random-cluster model. We shall not repeat them here, but refer the reader to [26, 28] for the details.

A subset of $\mathbb{Z}^d$ of the form $V_{a,b} = \prod_{i=1}^d [a_i, b_i]$ is called a *box*, and the associated region is denoted by $\Lambda_{a,b}$ and called a *box-region*. Write $\mathcal{B}$ for the set of all box-regions of $\mathbb{L}^d$. For a sequence Λ_n of box-regions, we write $\Lambda_n \uparrow \mathbb{L}^d$ if their vertex-sets increase to $\mathbb{Z}^d$. Let $\Psi = \{0,1\}^{\mathbb{Z}^d}$, $\Omega = \{0,1\}^{\mathbb{E}^d}$, and let Θ be the set of all compatible pairs $(\psi, \omega) \in \Psi \times \Omega$.

We begin with a consideration of vertex-measures. Let $a, p \in (0,1)$ and $q \in (0, \infty)$, and let $\mathcal{G}$ denote the σ-field generated by the cylinder events of $\Psi = \{0,1\}^{\mathbb{Z}^d}$. A probability measure Φ on $(\Psi, \mathcal{G})$ is called a *limit vertex-measure* with parameters a, p, q, if, for some $\lambda \in \Theta$, Φ is an accumulation point of the family $\{\Phi^\lambda_{\Lambda,a,p,q} : \Lambda \in \mathcal{B}\}$. Let $\mathcal{W}_{a,p,q}$ denote the set of all such measures, and $\overline{\mathrm{co}\,\mathcal{W}_{a,p,q}}$ its closed convex hull. It is standard by compactness that $\mathcal{W}_{a,p,q}$ is non-empty for all a, p, q.

We suppose henceforth that $q \in [1,2]$, so that we are within the domains of validity of the comparison and positive-correlation theorems of Sections 5 and 6. Arguing as for random-cluster measures, any $\Phi \in \mathcal{W}_{a,p,q}$ is positively associated, and any $\Phi \in \overline{\mathrm{co}\,\mathcal{W}_{a,p,q}}$ has the finite-energy property and satisfies the bounds of Theorem 5.13.

We may identify two special members of $\mathcal{W}_{a,p,q}$ as follows. Let $\mathbf{0} = (0,0) \in \Psi \times \Omega$ and $\mathbf{1} = (1,1)$. By positive-association in the usual way, the (monotonic) weak limits

$$\Phi^b_{a,p,q} = \lim_{\Lambda \uparrow \mathbb{L}^d} \Phi^b_{\Lambda,a,p,q}, \qquad b = \mathbf{0}, \mathbf{1},$$

exist. Furthermore, $\Phi^{\mathbf{0}}_{a,p,q}$ and $\Phi^{\mathbf{1}}_{a,p,q}$ are automorphism-invariant (that is, invariant with respect to automorphisms of $\mathbb{L}^d$), and are extremal in that

$$\Phi^{\mathbf{0}}_{a,p,q} \le_{\mathrm{st}} \Phi \le_{\mathrm{st}} \Phi^{\mathbf{1}}_{a,p,q}, \qquad \Phi \in \overline{\mathrm{co}\,\mathcal{W}_{a,p,q}}. \tag{7.1}$$

As in [6] (see also Section 4.3 of [28]), $\Phi^{\mathbf{0}}_{a,p,q}$ and $\Phi^{\mathbf{1}}_{a,p,q}$ are tail-trivial, and are ergodic with respect to the group $\mathbb{Z}^d$ of translations of $\mathbb{L}^d$. Since they have the finite-energy property, the number I of infinite open clusters satisfies either $I = 0$ or $I = 1$, $\Phi^b_{a,p,q}$-a.s. ($b = \mathbf{0}, \mathbf{1}$), see [12, 28]. As noted after Theorem 5.11, the boundary conditions $b = \mathbf{0}, \mathbf{1}$ contain information concerning both vertex and edge configuration off Λ, but only the external edge configuration is in fact relevant.

We shall perform comparisons in Sections 8 and 9 involving these two extremal measures, and towards that end we note that, by weak convergence, they satisfy the infinite-volume equivalents of Theorem 5.9.

The next two theorems concern the existence of the infinite-volume limits for the diluted-random-cluster measure and the BCP measure, when $1 \le q \le 2$. Here is a point of notation. Let $\Lambda = (V, E)$ be a box-region of $\mathbb{L}^d$, $q \in \{1, 2\}$, and $s \in \{0, 1, \dots, q\}$. We write $\pi^s_{\Lambda,K,\Delta,q}$ for the BCP measure on Λ with boundary condition s. The boundary condition $s = 0$ corresponds to the free boundary condition. For $\psi \in \Psi$, $\mu^b_{\Lambda,\psi,p,q}$ denotes the random-cluster measure on (V^+, E_ψ) with parameters p, q and boundary condition b. Similarly, $\mu^b_{\psi,p,q}$ denotes the corresponding random-cluster measure on the infinite graph $(\mathbb{Z}^d_\psi, \mathbb{E}^d_\psi)$. We write $\mathcal{H} = \sigma(\mathcal{G} \times \mathcal{F})$ for the product σ-field of $\Psi \times \Omega$. For $A \in \mathcal{H}$ and $\psi \in \Psi$, let A_ψ denote the section $\{\omega \in \Omega : (\psi, \omega) \in A\}$. For $B \subseteq \mathbb{Z}^d$, we write $B \leftrightarrow \infty$ if there exists $b \in B$ that is the endvertex of an infinite open path of the lattice.

Let $\mathcal{V}_{a,p,q}$ denote the set of all weak-limit diluted-random-cluster measures with parameters a, p, q, and let $\overline{\mathrm{co}\,\mathcal{V}_{a,p,q}}$ denote its closed convex hull. It is standard by compactness that $\mathcal{V}_{a,p,q} \ne \varnothing$ for $a, p \in (0, 1)$ and $q \in (0, \infty)$, and by taking a Cesàro average of measures that $\overline{\mathrm{co}\,\mathcal{V}_{a,p,q}}$ contains some translation-invariant measure. By part (a) of the next theorem, $\phi^b_{a,p,q} \in \mathcal{V}_{a,p,q}$ when $q \in [1, 2]$.

Theorem 7.2. *Let $a, p \in (0, 1)$, $q \in [1, 2]$, and $b \in \{\mathbf{0}, \mathbf{1}\}$.*
(a) *The limit diluted-random-cluster measure $\phi^b_{a,p,q} = \lim_{\Lambda\uparrow\mathbb{L}^d} \phi^b_{\Lambda,a,p,q}$ exists and satisfies*

$$\phi^{\mathbf{0}}_{a,p,q}(A) = \Phi^{\mathbf{0}}_{a,p,q}(\mu^0_{\psi,p,q}(A_\psi)), \qquad A \in \mathcal{H},$$

with a similar equation for the boundary condition $\mathbf{1}$.
(b) *The $\phi^b_{a,p,q}$ are stochastically increasing in a and p, and $\phi^{\mathbf{0}}_{a,p,q} \le_{\mathrm{st}} \phi \le_{\mathrm{st}} \phi^{\mathbf{1}}_{a,p,q}$ for $\phi \in \overline{\mathrm{co}\,\mathcal{V}_{a,p,q}}$.*
(c) *We have that*

$$\phi^{\mathbf{1}}_{\Lambda,a,p,q}(0 \leftrightarrow \infty) \to \phi^{\mathbf{1}}_{a,p,q}(0 \leftrightarrow \infty) \qquad \text{as } \Lambda \uparrow \mathbb{L}^d.$$

(d) *The number $L(\omega)$ of infinite open clusters of $\omega \in \Omega$ satisfies: either $\phi^b_{a,p,q}(L = 0) = 1$ or $\phi^b_{a,p,q}(L = 1) = 1$.*

Theorem 7.3. *Let $K \in [0, \infty)$, $\Delta \in \mathbb{R}$, and $q \in \{1, 2\}$. The limit BCP measure $\pi^s_{K,\Delta,q} = \lim_{\Lambda\uparrow\mathbb{L}^d} \pi^s_{\Lambda,K,\Delta,q}$ exists, for $s = 0, 1, \dots, q$.*

The proofs are deferred to the end of this section. We recall from Section 3 the 'usual' coupling of the diluted-random-cluster and BCP measures, and we shall see in the proof of the last theorem that the equivalent coupling is valid for the infinite-volume measures.

The limit measures $\phi^b_{a,p,q}$ are automorphism-invariant and have the finite-energy property, the proofs follow standard lines and are omitted. Similarly, the $\phi^b_{a,p,q}$ satisfy the comparison inequalities of Theorems 6.1 and 6.3.

We shall consider also the set of DLR measures. Let $\mathcal{T}_\Lambda$ be the sub-σ-field of $\mathcal{H}$ generated by the states of vertices and edges not belonging to the region Λ. A probability measure on $(\Psi \times \Omega, \mathcal{H})$ is called a *diluted-random-cluster measure* with parameters a, p, q if, for every $A \in \mathcal{H}$ and every region Λ,

$$\phi(A \mid \mathcal{T}_\Lambda)(\theta) = \phi^\theta_{\Lambda,a,p,q}(A) \qquad \text{for } \phi\text{-a.e. } \theta \in \Psi \times \Omega.$$

The set of such measures is denoted by $\mathcal{R}_{a,p,q}$. One way of showing that $\mathcal{R}_{a,p,q} \neq \varnothing$ is to prove that some measure in $\overline{\operatorname{co} \mathcal{V}_{a,p,q}}$ belongs to $\mathcal{R}_{a,p,q}$. The following theorem may be proved exactly as for random-cluster measures, see [26, 28].

Theorem 7.4. (i) *Let $a, p \in (0,1)$ and $q \in (0,\infty)$. If $\phi \in \overline{\operatorname{co} \mathcal{V}_{a,p,q}}$ and ϕ is such that $\phi(L \in \{0,1\}) = 1$, then $\phi \in \mathcal{R}_{a,p,q}$.*

(ii) *$\mathcal{R}_{a,p,q} \neq \varnothing$ for $a, p \in (0,1)$, $q \in (0,\infty)$.*

(iii) *Let $a, p \in (0,1)$ and $q \in [1,2]$. Then $\phi^b_{a,p,q} \in \mathcal{R}_{a,p,q}$ for $b = \mathbf{0}, \mathbf{1}$.*

Finally, we indicate how the convexity of the partition function may be used to show the uniqueness of certain infinite-volume measures. The proof follows [26], which in turn used the method of [37].

Theorem 7.5. *Let $q \in [1,2]$.*
(a) *For $p \in (0,1)$, the set of points $a \in (0,1)$ at which $|\mathcal{W}_{a,p,q}| \geq 2$ is countable.*
(b) *If $q \in \{1,2\}$, the set of pairs $(a,p) \in (0,1)^2$ at which $|\mathcal{V}_{a,p,q}| \geq 2$ may be covered by a countable family of rectifiable curves of $\mathbb{R}^2$.*

Proof of Theorem 7.2. (a) For simplicity in the following proofs, we shall suppress reference to the parameters. Consider first the boundary condition $\mathbf{0}$. Let $A \subseteq \Omega$ and $B \subseteq \Psi$ be increasing cylinder events, and let $U \subseteq \mathbb{Z}^d$ be a finite set such that A and B are defined in terms of the states of vertices in U and of edges joining members of U. By the discussion in Section 5,

$$\phi^{\mathbf{0}}_\Lambda(A \times B) = \Phi^{\mathbf{0}}_\Lambda(1_A(\psi)\mu^0_{\Lambda,\psi}(B)). \tag{7.6}$$

Since $\mathcal{H}$ is generated by the set of such events $A \times B$, it suffices to show that

$$\lim_{\Lambda \uparrow \mathbb{L}^d} \phi^{\mathbf{0}}_\Lambda(A \times B) = \Phi^{\mathbf{0}}(1_A(\psi)\mu^0_\psi(B)). \tag{7.7}$$

Let $\Lambda' = (V', E')$, Λ'' be box-regions such that $\Lambda' \subseteq \Lambda \subseteq \Lambda''$ and $U \subseteq V'$. By (7.6) and the monotonicity of $\Phi^{\mathbf{0}}_\Lambda$ in Λ, and of $\mu^0_{\Lambda,\psi}$ in Λ and ψ,

$$\Phi^{\mathbf{0}}_\Lambda(1_A(\psi)\mu^0_{\Lambda',\psi}(B)) \le \Phi^{\mathbf{0}}_\Lambda(A \times B) \le \Phi^{\mathbf{0}}_{\Lambda''}(1_A(\psi)\mu^0_{\Lambda,\psi}(B)).$$

Take the limits as $\Lambda'', \Lambda, \Lambda' \uparrow \mathbb{Z}^d$ in that order, and use the bounded convergence theorem to obtain (7.7). A similar argument holds with boundary condition **1**, and with the inequalities reversed.
(b) The necessary properties of monotonicity follow by Theorem 5.12.
(c) This follows the proof of the corresponding statement for random-cluster measures, see [3, 28], using part (a) and the representation (7.6) with boundary condition **0** replaced by **1**.
(d) The proof relies on the automorphism-invariance and the finite-energy property of the marginal measure of $\phi^b_{a,p,q}$ on Ω. This follows standard lines and is omitted. □

Proof of Theorem 7.3. Consider first the case $s = 0$. Let Λ^- be the graph obtained from the box-region $\Lambda = (V, E)$ by removing those edges that do not have both endvertices in V. Let μ be the coupled measure of Theorem 3.7 for Λ^-, having marginal measures $\pi^0_\Lambda = \pi^0_{\Lambda,K,\Delta,q}$ and $\phi^{\mathbf{0}}_\Lambda = \phi^{\mathbf{0}}_{\Lambda,a,p,q}$, where a, p satisfy (3.6).

Let $U \subset \mathbb{Z}^d$ be finite, $\tau \in \Sigma = \{0, 1, 2, \dots, q\}^{\mathbb{Z}^d}$, and let $\Sigma_{U,\tau}$ be the BCP cylinder event $\{\sigma \in \Sigma : \sigma_u = \tau_u \text{ for } u \in U\}$. Let $A = A_{U,\tau}$ be the set of $\theta = (\psi, \omega) \in \Theta$ that are compatible with $\Sigma_{U,\tau}$, that is, A is the set of θ such that:

(i) $\forall u \in U$, $\tau_u = 0$ if and only if $\psi_u = 0$, and

(ii) $\forall u, v \in U$, $\tau_u \ne \tau_v$ only if u and v are not ω-connected in $\mathbb{L}^d$.

For given $\theta \in A$, let $l(\theta)$ be the number of open clusters that intersect U. By the second observation after Theorem 3.7, subject to a slight abuse of notation, if $V \supseteq U$,

$$\pi^0_\Lambda(\Sigma_{U,\tau}) = \phi^{\mathbf{0}}_\Lambda\big(1_A(\theta)q^{-l(\theta)}\big). \tag{7.8}$$

Now, $\phi^{\mathbf{0}}_\Lambda \Rightarrow \phi^{\mathbf{0}}$ as $\Lambda \uparrow \mathbb{L}^d$ and, by Theorem 7.2(d), the random variable $1_A(\theta)q^{-l(\theta)}$ is $\phi^{\mathbf{0}}$-a.s. continuous. Therefore,

$$\lim_{\Lambda \uparrow \mathbb{L}^d} \pi^0_\Lambda(\Sigma_{U,\tau}) = \phi^{\mathbf{0}}\big(1_A(\theta)q^{-l(\theta)}\big).$$

Suppose now that $s \in \{1, 2, \dots, q\}$. Let μ be the coupled measure of Theorem 3.7 on the graph (V^+, E), and let μ^s denote the measure μ conditioned on the event that $\sigma_x = s$ for all $x \in \partial\Lambda$.

The marginal of μ^s on $\Sigma_V = \{1, 2, \dots, q\}^V$ is the measure $\pi^s_\Lambda = \pi^s_{\Lambda,K,\Delta,q}$, the marginal on $\Psi_V \times \Omega_E = \{0,1\}^V \times \{0,1\}^E$ is $\phi^{\mathbf{1}}_\Lambda = \phi^{\mathbf{1}}_{\Lambda,a,p,q}$. The conditional measure of μ^s on Σ_V, given the pair $(\psi, \omega) \in \Psi_V \times \Omega_E$, is that obtained as follows:

(a) $\forall v \in V$, the spin at v is 0 if and only if $\psi_v = 0$,

(b) the spins are constant on each given open cluster,

(c) the spins on any open cluster intersecting $\partial\Lambda$ are equal to s,

(d) the spins on the other open clusters are independent and uniformly distributed on the set $\{1, 2, \dots, q\}$.

Equation (7.8) becomes

$$\pi^s_\Lambda(\Sigma_{U,\tau}) = \phi^{\mathbf{1}}_\Lambda\big(1_A(\theta)q^{-f(\theta)}\big), \tag{7.9}$$

where $f(\theta)$ is the number of finite open clusters that intersect U. [Recall that $\phi^{\mathbf{1}}_\Lambda$ has support $\Theta^{\mathbf{1}}_\Lambda$.] We may write $f(\theta) = l(\theta) - N(\omega)$ where $N = N(\omega)$ is the number of infinite open clusters of $\omega \in \Omega$ that intersect U. Clearly, $N \in \{0,1\}$ for $\theta = (\psi, \omega) \in \Theta^{\mathbf{1}}_\Lambda$, so that

$$\begin{aligned}\pi^s_\Lambda(\Sigma_{U,\tau}) &= \phi^{\mathbf{1}}_\Lambda(1_A q^{N-l}) \\ &= \phi^{\mathbf{1}}_\Lambda(1_A q^{-l}) + (q-1)\phi^1_\Lambda(1_A 1_{\{U \leftrightarrow \partial\Lambda\}} q^{-l}).\end{aligned} \tag{7.10}$$

Now, $1_A q^{-l}$ is $\phi^{\mathbf{1}}$-a.s. continuous by Theorem 7.2(d), so that

$$\phi^{\mathbf{1}}_\Lambda(1_A q^{-l}) \to \phi^{\mathbf{1}}(1_A q^{-l}) \qquad \text{as } \Lambda \uparrow \mathbb{L}^d. \tag{7.11}$$

It may be proved in a manner very similar to the proof of Theorem 7.2(c) that

$$\phi^{\mathbf{1}}_\Lambda(1_A 1_{\{U \leftrightarrow \partial\Lambda\}} q^{-l}) \to \phi^{\mathbf{1}}(1_A 1_{\{U \leftrightarrow \infty\}} q^{-l}) \qquad \text{as } \Lambda \uparrow \mathbb{L}^d. \tag{7.12}$$

By (7.10)–(7.12) and Theorem 7.2(d),

$$\pi^s_\Lambda(\Sigma_{U,\tau}) \to \phi^{\mathbf{1}}(1_A q^{-f}) \qquad \text{as } \Lambda \uparrow \mathbb{L}^d,$$

and the proof is complete. □

Proof of Theorem 7.5. (a) Let $\Lambda = (V, E)$ be a region in $\mathbb{L}^d$ with graph $\Lambda^+ = (V^+, E)$. Let $a, p \in (0,1)$ and $q \in [1, \infty)$. Consider the normalizing constant $Z^\lambda_\Lambda = Z^{\mathrm{DRC}}_{\Lambda,\lambda,a,p,q}$ of the diluted-random-cluster measure on Λ with boundary condition λ. Let the vectors (a, p) and (K, Δ) satisfy (3.6). By (4.1), we may write

$$Z^\lambda_\Lambda = \sum_{\theta=(\psi,\omega)\in\Theta^\lambda_\Lambda} r^{|E_\psi|} q^{k(\theta,\Lambda)} e^{-\Delta|V_\psi|} \left(\frac{p}{1-p}\right)^{|\eta(\omega)\cap E|}.$$

By a standard argument using subadditivity in Λ, see [26, 28], the limit

$$G(\Delta,p,q)=\lim_{\Lambda\uparrow\mathbb{L}^d}\left\{\frac{1}{|V|}\log Z_\Lambda^\lambda\right\}$$

exists and is independent of λ. The function G is termed *pressure*.

It is easily seen that

$$\frac{\partial}{\partial\Delta}\log Z_\Lambda^\lambda=-\phi_\Lambda^\lambda(|V_\psi|), \tag{7.13}$$

$$\frac{\partial^2}{\partial^2\Delta}\log Z_\Lambda^\lambda=\operatorname{var}(|V_\psi|), \tag{7.14}$$

where var denotes variance with respect to $\phi_{\Lambda,a,p,q}^\lambda$. Since variances are non-negative, $G(\Delta,p,q)$ is a convex function of Δ. Hence, for fixed p, q, the set of points Δ of non-differentiability of G is countable (that is, either finite or countably infinite). Wherever G is differentiable, its derivative is the limit as $\Lambda\uparrow\mathbb{L}^d$ of the derivative of $|V|^{-1}\log Z_\Lambda^\lambda$. This implies in turn that

$$\lim_{\Lambda\uparrow\mathbb{L}^d}\frac{1}{|V|}\phi_\Lambda^{\mathbf{0}}(|V_\psi|)=\lim_{\Lambda\uparrow\mathbb{L}^d}\frac{1}{|V|}\phi_\Lambda^{\mathbf{1}}(|V_\psi|),$$

so that $\Phi^{\mathbf{0}}(J_x)=\Phi^{\mathbf{1}}(J_x)$ for $x\in\mathbb{Z}^d$, where J_x is the event that x is open. The claim follows by (7.1) and a standard 'FKG' coupling (see, for example, Prop. 4.6 of [28]).

(b) When $q\in\{0,1\}$, we work with the constant $Z^{\mathrm{BCP}}=Z_{\Lambda,K,\Delta,q}^{\mathrm{BCP}}$ of (3.2). By the form of (3.2), $Z_{\Lambda,K,\Delta,q}^{\mathrm{BCP}}$ is a convex function of the pair (K,Δ). By (3.8) and the coupling of Chapter 3,

$$\begin{aligned}\frac{\partial}{\partial K}\log Z_\Lambda^b&=\pi_\Lambda^s\left(-|E_\sigma|+2\sum_{e\in E}\delta_e(\sigma)\right)\\&=\phi_\Lambda^b\left(-|E_\psi|+\frac{2}{p}\sum_{e\in E}\omega(e)\right),\end{aligned} \tag{7.15}$$

where $s=s(b)$ satisfies $s(\mathbf{0})=0$, $s(\mathbf{1})=1$. By Theorem 8.18 of [20] or Theorem 2.2.4 of [42], the set of points of $(0,1)^2$ at which G is not differentiable (when viewed as function of (a,p)) may be covered by a countable collection of rectifiable curves. Suppose G is differentiable at the point (a,p). By part (a), $\phi^{\mathbf{0}}(J_x)=\phi^{\mathbf{1}}(J_x)$ for $x\in\mathbb{Z}^d$ and, in particular, $|E_\psi|/|V|$ has the same (almost-sure and L^1) limit as $\Lambda\uparrow\mathbb{L}^d$ under either boundary condition. Therefore, by (7.15),

$$\lim_{\Lambda\uparrow\mathbb{L}^d}\frac{1}{|V|}\phi_\Lambda^{\mathbf{0}}(|\eta(\omega)\cap E|)=\lim_{\Lambda\uparrow\mathbb{L}^d}\frac{1}{|V|}\phi_\Lambda^{\mathbf{1}}(|\eta(\omega)\cap E|),$$

so that, by translation invariance, $\phi^{\mathbf{0}}(J_e) = \phi^{\mathbf{1}}(J_e)$ for $e \in \mathbb{E}^d$, where J_e is the event that e is open. The claim now follows by Theorem 7.2(b), as in part (a). □

8 Phase transitions

Let $d \geq 2$, $q \in [1, 2]$, and consider the 'wired' diluted-random-cluster measure $\phi^{\mathbf{1}}_{a,p,q}$ on $\mathbb{L}^d$. Several transitions occur as (a, p) increases from $(0, 0)$ to $(1, 1)$, and each gives rise to a 'critical surface' defined as follows.

Let Π be a monotonic property of pairs $(\psi, \omega) \in \Theta$ such that $\phi^{\mathbf{1}}_{a,p,q}(\Pi) \in \{0, 1\}$ for all a, p. Let

$$R(\Pi) = \{(a, p) \in (0, 1)^2 : \phi^{\mathbf{1}}_{a,p,q}(\Pi) = 1\}, \quad S(\Pi) = \overline{R(\Pi)} \cap \overline{R(\neg\Pi)},$$

where $\neg\Pi$ denotes the negation of Π. By Theorem 7.2(a), each $R(\Pi)$ is a monotonic subset of $(0, 1)^2$ with respect to the ordering $(a, p) \leq (a', p')$ if $a \leq a'$ and $p \leq p'$. The set $S(\Pi)$ $(= S(\neg\Pi))$ is called the 'critical surface' for Π.

Of principal interest here are the following three properties:

(i) Π_{icvc}, the property that there exists an infinite closed vertex-cluster,

(ii) Π_{iovc}, the property that there exists an infinite open vertex-cluster,

(iii) Π_{iec}, the property that there exists an infinite open edge-cluster.

It is easily checked that $\neg\Pi_{\text{icvc}}$, Π_{iovc}, Π_{iec} are increasing and satisfy the zero/one claim above. Furthermore, $\Pi_{\text{iec}} \Rightarrow \Pi_{\text{iovc}}$.

We do not know a great deal about the critical surfaces of the three properties above. Just as for percolation, it can occur that $R(\Pi_{\text{icvc}}) \cap R(\Pi_{\text{iovc}}) \neq \varnothing$ on any lattice whose critical site-percolation-probability $p_{\text{c}}^{\text{site}}$ satisfies $p_{\text{c}}^{\text{site}} < \frac{1}{2}$, see remark (a) following (4.2). When $d = 2$ however, $R(\Pi_{\text{icvc}}) \cap R(\Pi_{\text{iovc}}) = \varnothing$ by the main theorem of [24].

When $q = 2$, the critical surfaces of these three properties mark phase transitions for the Blume–Capel model. Consider the Blume–Capel measure $\pi^1_{K,\Delta,2}$ on $\mathbb{L}^d$, and let a, p satisfy (3.6). Then:

(i) $R(\Pi_{\text{icvc}})$ corresponds to the existence of an infinite vertex-cluster of spin 0,

(ii) $R(\Pi_{\text{iovc}})$ corresponds to the existence of an infinite vertex-cluster whose vertices have non-zero (and perhaps non-equal) spins,

(iii) $R(\Pi_{\text{iec}})$ corresponds to the existence of long-range order.

Statements (i)–(ii) are clear. Statement (iii) follows by Theorems 3.10 and 7.2(c), and the remark following Theorem 7.3, on noting by (3.9) that

$$\pi^1_{K,\Delta,2}(\sigma_0 = 1) - \tfrac{1}{2}\pi^1_{K,\Delta,2}(\sigma_0 \neq 0) = \tfrac{1}{2}\phi^{\mathbf{1}}_{a,p,2}(0 \leftrightarrow \infty). \tag{8.1}$$

Some numerical information may be obtained about the critical surfaces by use of the comparison inequalities proved earlier in this paper. This is illustrated in the next section, where we concentrate on the two-dimensional Blume–Capel model.

This section closes with some notes on the BCP model on $\mathbb{L}^2$ with $q = 1$, for use in Section 9. As remarked in Sections 3 and 4, this model may be transformed into the Ising model with edge-interaction $J = \frac{1}{4}K$ and external field $h = K - \frac{1}{2}\Delta$, see (4.3). The phase diagram is therefore well understood and is illustrated in Figure 1 with the parametrization (a, p) of (3.6).

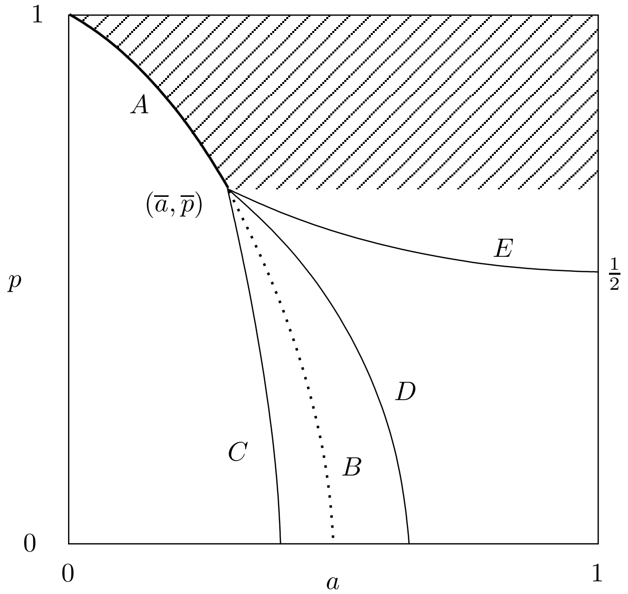

Figure 1: The phase diagram of the $q = 1$ BCP model on the square lattice. The model may be transformed into the Ising model with edge-interaction J and external field h, see (4.3). The arc $A \cup B$ with equation $a/(1-a) = 1 - p$ corresponds to $h = 0$, and the points to its right (respectively, left) correspond to $h > 0$ (respectively, $h < 0$). There is a 'tri-critical point' at $(\overline{a}, \overline{p})$, see (4.6), and the arc A joining this point to $(0, 1)$ marks a line of first-order phase transitions. The region to the left of $A \cup C$ is $R(\Pi_{\rm icvc})$, and that to the right of $A \cup D$ is $R(\Pi_{\rm iovc})$. The hatched region lies in $R(\Pi_{\rm iec})$, and the lower boundary of $R(\Pi_{\rm iec})$ is presumably as marked by E.

Some remarks concerning Figure 1 follow. The existence of the arcs

A, C, D follow by the established theory of the Ising model with edge-interaction J and external field h, see [19, 30, 31, 41] for the case $h = 0$. The arc A corresponds to $h = 0$, $J > J_c$, where J_c is the critical point of the zero-field model. Consider the corresponding random-cluster model RC_p with edge-parameter $\pi = 1 - (1-p)^{-4}$ and cluster-weighting factor 2. Then RC_p has (almost surely) an infinite open cluster I_p when $(a, p) \in A$. As one deviates rightwards from A with p held constant (that is, in the direction of positive h), the positive magnetic field attracts the vertices in I_p, together with at least one half of the finite clusters of RC_p. Write $\mathrm{P}_{p,h}$ for the resulting set of $+1$ spins. By the previous remark, and recalling the conditional law of the zero-field Ising model given the random-cluster configuration, we deduce that the bond percolation model on $\mathrm{P}_{p,h}$ with density π $(< p)$ possesses an infinite edge-cluster. It follows that the hatched region of Figure 1 lies in $R(\Pi_{\mathrm{iec}})$.

Similarly, as one deviates leftwards from A with p held constant, the resulting negative magnetic field attracts the vertices in I_p, and an infinite closed vertex-cluster forms.

9 The Blume–Capel phase diagram

Throughout this final section, we consider the Blume–Capel model on the square lattice $\mathbb{L}^2$, and the associated diluted-random-cluster measure. [Related but partial conclusions are valid similarly on $\mathbb{L}^d$ with $d \geq 3$.] The respective parameters are $K \in [0, \infty)$, $\Delta \in \mathbb{R}$, and the values a, p given at (3.6). The three putative phases of the models are illustrated in Figure 2. We recall from the last section the fact that, since $d = 2$, $R(\Pi_{\mathrm{iovc}}) \cap R(\Pi_{\mathrm{icvc}}) = \varnothing$.

The three regions of Figure 2 are characterized as follows.

(a) The top region is $R(\Pi_{\mathrm{iec}})$, in which the diluted-random-cluster measure possesses (almost surely) an infinite open edge-cluster, and the Blume–Capel model has long-range order.

(b) The left region is $R(\Pi_{\mathrm{icvc}})$, in which the measures possess an infinite vertex-cluster of zero states.

(c) The central region is $R(\neg\Pi_{\mathrm{icvc}}) \cap R(\neg\Pi_{\mathrm{iec}})$, in which either all closed and open vertex-clusters are finite, or there exists an infinite open vertex-cluster which is too small to support an infinite open edge-cluster. There is no long-range order.

In the more normal parametrization (1.1) of the Blume–Capel model, there is a parameter β denoting inverse-temperature, and one takes $K = \beta J$, $\Delta = \beta D$. If we hold the ratio D/J fixed and let β vary, the arc of corresponding

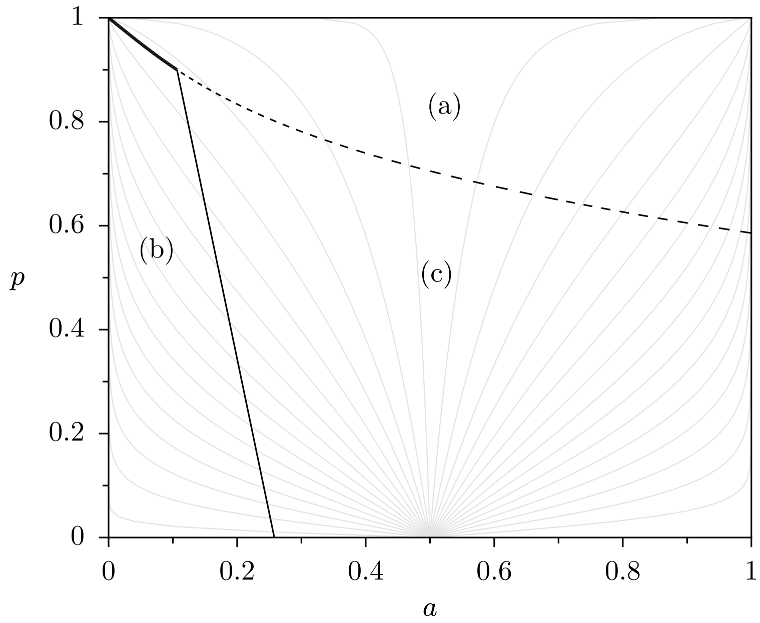

Figure 2: The Blume–Capel phase diagram in two dimensions as proposed by Capel. Note the three phases labelled (a), (b), (c) as in the text. The boundary between (a) and (b) is thought to be a line of first-order phase transitions, whereas that between (a) and (c) is expected be a line of second-order transitions. The point at which the the phases are expected to meet is termed the *tri-critical point*. Moral support for such a phase diagram is provided by the rigorously known $q = 1$ diagram of Figure 1.

pairs (a, p) satisfies

$$\frac{a}{1-a} = (1-p)^{D/2J}.$$

As the ratio D/J varies, such arcs are plotted in the gray lines of Figure 2.

The region labelled (c) may be split into two sub-regions depending on whether or not there exists an infinite open vertex-cluster. We shall not pursue this distinction here.

A key prediction of Capel for this model is the existence of a so-called tri-critical point where the three phases meet. The common boundary between the regions $R(\Pi_{\text{icvc}})$ and $R(\Pi_{\text{iovc}})$ is thought to be a line of first-order phase transitions. Based on a mean-field analysis, Capel has made the numerical proposals that the tri-critical point lies on the line $a/(1-a) = (1-p)^{\frac{2}{3}\log 4}$, and that the line of first-order transitions arrives at the corner $(0,1)$ with the same gradient as the line $a/(1-a) = 1-p$. The remaining boundary of $R(\Pi_{\text{iec}})$ is thought to mark a line of second-order phase transitions, and to meet the line $a = 1$ at the point $p = \sqrt{2}/(1+\sqrt{2})$.

The $q = 2$ random-cluster (Ising) measure on $\mathbb{L}^2$ has critical point $p = \sqrt{2}/(1+\sqrt{2})$, which for numerical clarity we shall approximate by 0.586. Site percolation on $\mathbb{L}^2$ has critical probability p_c^{site}, to which we shall approximate with the value 0.593. Figure 3 indicates certain regions of the phase diagram about which we may make precise observations.

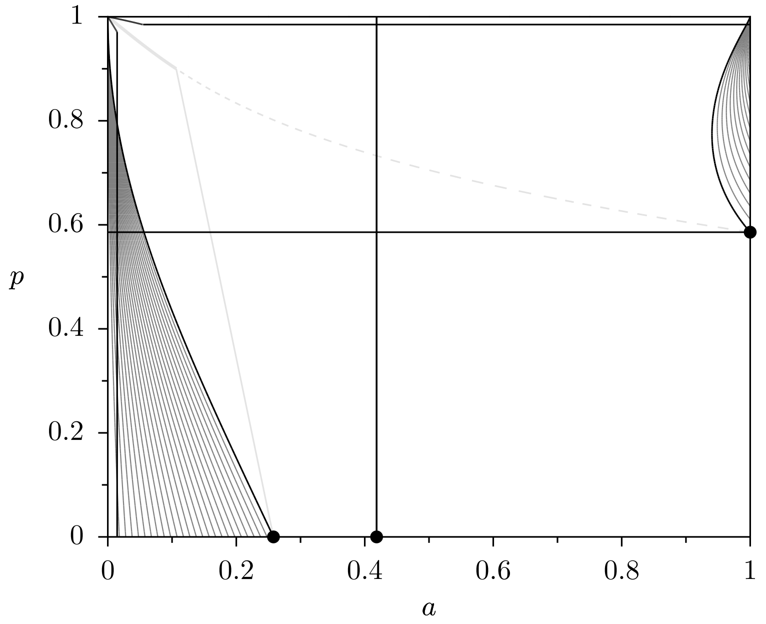

Figure 3: Regions of the phase diagram of the Blume–Capel model on $\mathbb{L}^2$ about which one may make rigorous statements on the basis of comparisons with other models. The three points referred to in (i)–(iii) are marked. The narrow vertical strip along the p-axis is a subset of $R(\Pi_{\text{icvc}})$, and the horizontal strip along the line $p = 1$ is a subset of $R(\Pi_{\text{iec}})$; see the comments around (9.1) and (9.2).

For three special vectors (a, p, q), the corresponding diluted-random-cluster measure $\phi^{\mathbf{1}}_{a,p,q}$ provides information concerning the phase diagram. These vectors are given as follows. For simplicity, we shall refer to the comparison theorems for measures on finite graphs; the corresponding inequalities for infinite-volume measures are easily seen to hold, see Section 7.

(i) *The triple* $a = 1$, $p = \sqrt{2}/(1+\sqrt{2}) \approx 0.586$, $q = 2$. The corresponding $\phi^{\mathbf{1}}_{1,p,2}$ is a critical random-cluster measure. By Theorem 6.1(a), the shaded region to the right of the given curve joining $(1, 0.586)$ to $(1, 1)$ lies within Π_{iec}. The corresponding Blume–Capel models have long-range order. By Theorem 6.1(b), no point below the horizontal line $p = 0.586$ lies in Π_{iec}, and the corresponding Blume–Capel models do not have long-range order.

(ii) *The triple* $a = (1 - p_{\mathrm{c}}^{\mathrm{site}})/(1 + p_{\mathrm{c}}^{\mathrm{site}}) \approx 0.26$, $p = 0$, $q = 2$. To the left of this point on the horizontal line $p = 0$, the vertex-measure $\Phi^{\mathbf{1}}_{a',0,2}$ is a product measure with density $2a'/(1 + a')$ and possessing (almost surely) an infinite cluster of closed vertices. By Theorem 5.9(ii), $\Phi^{\mathbf{1}}_{a',0,2}$ dominates the vertex-measures on the given arc joining $(a', 0)$ to $(0, 1)$. The interior of the shaded area is thus a subset of $R(\Pi_{\mathrm{icvc}})$, and the corresponding BCP measures possess (almost surely) an infinite cluster of 0-spin vertices.

(iii) *The vector* $a = p_{\mathrm{c}}^{\mathrm{site}}/(2 - p_{\mathrm{c}}^{\mathrm{site}}) \approx 0.42$, $p = 0$, $q = 2$. To the right of this point on the horizontal line $p = 0$, the vertex-measure $\Phi^{\mathbf{1}}_{a',0,2}$ is a supercritical product measure with an infinite open vertex-cluster. It follows by Theorem 5.9(i) that the interior of the region to the right of the vertical line $a = p_{\mathrm{c}}^{\mathrm{site}}/(2 - p_{\mathrm{c}}^{\mathrm{site}})$ lies in $R(\Pi_{\mathrm{iovc}})$.

Finally, we shall make comparisons involving the diluted-random-cluster model with parameters $(a, p, 2)$ and the $q = 1$ models lying on the arc A of Figure 1. Let $\overline{a} \approx 0.029$ and $\overline{p} \approx 0.971$ be given by (4.6), and consider the BCP model with parameters $(a_2, p_2, 1)$ where $a_2/(1 - a_2) = 1 - p_2$. Take $q_2 = 1$ and $q_1 = 2$ in Theorem 5.9(iii) to find that: if $(a, p) \in (0, 1)^2$ satisfies

$$2\left(\frac{a}{1-a}\right)(1-p)^2 < (1-p_2)^3$$

for some $p_2 \geq \max\{p, \overline{p}\}$, then $(a, p) \in R(\Pi_{\mathrm{icvc}})$. This holds in particular if

$$\frac{2a}{1-a} < 1 - p \quad \text{and} \quad p > \overline{p}. \tag{9.1}$$

Taken in conjunction with Theorem 5.9(i), this implies that the narrow vertical strip marked along the p-axis of Figure 3 is a subset of $R(\Pi_{\mathrm{icvc}})$.

Secondly, take $q_1 = 1$, $q_2 = 2$ in Theorem 5.9(iv) to find similarly that: if $(a, p) \in (0, 1)^2$ satisfies

$$2\left(\frac{a}{1-a}\right)(1-p)^2 > \left(1 - \frac{p}{2-p}\right)^3 \quad \text{and} \quad \frac{p}{2-p} > \overline{p},$$

then $(a, p) \in R(\Pi_{\mathrm{iovc}})$. This occurs if

$$\frac{2a}{1-a} > \frac{8(1-p)}{(2-p)^3} \quad \text{and} \quad p > \frac{2\overline{p}}{1+\overline{p}}. \tag{9.2}$$

We indicate next that $(a, p) \in R(\Pi_{\text{iec}})$ whenever (9.2) holds. Assume (9.2). By Theorem 5.9(iv), $\Phi^{\mathbf{1}}_{a,p,2} \geq_{\text{st}} \Phi^{\mathbf{1}}_{a_1,p_1,1}$ where $a_1/(1-a_1) = 1 - p_1$ and $p_1 = p/(2-p) > \overline{p}$. Since the inequalities of (9.2) are strict, we may replace a_1 by $a_1 + \epsilon$ for some small $\epsilon > 0$, and we deduce that $\Phi^{\mathbf{1}}_{a,p,2}$ dominates (stochastically) the law, μ^+_J say, of the set S of $+$-spins of the infinite-volume Ising model with zero external field, edge-interaction $J = -\frac{1}{8}\log(1-p) > J_{\text{c}}$, and $+$ boundary condition. Recalling the coupling between the Ising model and the random-cluster model, the critical probability $p_{\text{c}}^{\text{bond}}(S)$ of bond percolation on S satisfies $p_{\text{c}}^{\text{bond}}(S) < \pi$, μ^+_J-a.s., where π is the 'effective' edge-parameter of the random-cluster model RC_{p_1} given by

$$(1-\pi)^4 = 1 - p_1 = 1 - \frac{p}{2-p}.$$

The random-cluster measure with parameters p, 2 on the graph induced by the open vertex-set of $\mathbb{L}^2$ dominates (stochastically) the product measure with intensity $p_1 = p/(2-p)$. Since $p_1 \geq \pi$, there exists an infinite open edge-cluster, $\phi^{\mathbf{1}}_{a,p,2}$-a.s. That is, $(a, p) \in R(\Pi_{\text{iec}})$ if (9.2) holds. This implies as above that the narrow horizontal strip marked along the line $p = 1$ in Figure 3 is a subset of $R(\Pi_{\text{iec}})$.

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