INFLUENCES IN PRODUCT SPACES: BKKKL RE-REVISITED

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Abstract. We prove a version of an assertion of Bourgain, Kahn, Kalai, Katznelson, Linial concerning influences in general product spaces, with an extension to the generalized influences of Keller.

1. Statement of result

The following extract is from the abstract of the paper [1], written by Bourgain, Kahn, Kalai, Katznelson, and Linial (BKKKL).

> Let X be a probability space and let $f : X^n \to \{0,1\}$ be a measurable map. Define the *influence of the k-th variable on f*, denoted by $I_f(k)$, as follows: For $u = (u_1, u_2, \dots, u_{n-1}) \in X^{n-1}$ consider the set $l_k(u) = \{(u_1, u_2, \dots, u_{k-1}, t, u_k, \dots, u_{n-1}) : t \in X\}$.
>
> $$I_f(k) = \Pr(u \in X^{n-1} : f \text{ is not constant on } l_k(u)).$$
>
> **Theorem 1.** There is an absolute constant c_1 so that for every function $f : X^n \to \{0,1\}$ with $\Pr(f^{-1}(1)) = p \le \frac{1}{2}$, there is a variable k so that
>
> $$I_f(k) \ge c_1 p \frac{\log n}{n}.$$

BKKKL gave a proof of the theorem for the special case in which X is the unit interval $[0,1]$ endowed with Lebesgue measure. They included no indication of the extension to general measure spaces, presumably in the (mistaken) belief that the relevant measure-theoretic arguments are widely known. We revisit BKKKL (following [2, 14]) in this short note, with a description of a proof of a stronger version of their Theorem 1.

Let $X = (\Omega, \mathcal{F}, P)$ be a probability space, and let E be a finite set with $|E| = n$. We write $X^E = (\Omega^E, \mathcal{F}^E, \mathbb{P} = P^E)$ for the product space.

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The *influence* of $e \in E$ on the event $A \in \mathcal{F}^E$ is defined as

$$I_A(e) = P^{E\setminus\{e\}}\big(\{\psi \in \Omega^{E\setminus\{e\}} : 0 < P(A \cap F_\psi) < 1\}\big), \tag{1.1}$$

where $F_\psi = \{\psi\} \times \Omega$, the 'fibre' of all $\omega \in X^E$ such that $\omega(f) = \psi(f)$ for $f \neq e$. For economy of notation, the space X is not listed explicitly in $I_A(e)$.

Remark 1.1. *Influence, as defined in (1.1), is never larger than that given by the BKKKL definition (presented above).*

Theorem 1.2. *Let $A \in \mathcal{F}^E$ satisfy $\mathbb{P}(A) \in (0,1)$. There exists an absolute constant $c \in (0,\infty)$ such that*

$$\sum_{e\in E} I_A(e) \geq c\mathbb{P}(A)(1-\mathbb{P}(A))\log[1/(2m)], \tag{1.2}$$

where $m = \max_e I_A(e)$.

It is immediate that (1.2) implies the existence of $e \in E$ with

$$I_A(e) \geq c'\mathbb{P}(A)(1-\mathbb{P}(A))\frac{\log n}{n}, \tag{1.3}$$

for some absolute contant $c' > 0$. By Remark 1.1, this is stronger than Theorem 1 of BKKKL.

It is now standard that the conclusion of Theorem 1.2 is valid in the special case when $X = \mathcal{L}$, where $\mathcal{L}$ denotes the Lebesgue space comprising the unit interval $[0,1]$ endowed with the Borel σ-field $\mathcal{B}[0,1]$ and Lebesgue measure λ. (The relevant history and literature is described in Section 2. By comment (a) there, the choice of Borel or Lebesgue σ-field is immaterial.) It suffices, therefore, to prove Theorem 1.3, following.

Following [11], we introduce a more general definition of influence. Let $\mathcal{M}$ be the set of measurable functions $h : [0,1] \to [0,1]$. For $h \in \mathcal{M}$, the *h-influence* of $e \in E$ on the event $A \in \mathcal{F}^E$ is defined as

$$I_A^h(e) = P^{E\setminus\{e\}}\big(h(P(A \cap F_\psi))\big), \tag{1.4}$$

where $\mu(f)$ denotes the expection of f under the probability measure μ. Thus $I_A^h(e) = I_A(e)$ when $h(x) = 1(0 < x < 1)$, with $1(B)$ the indicator function of B. The function $h(x) = x(1-x)$ has been considered in [7], and other functions h in [11].

One might define the influence $I_A(e)$ via a conditional expectation rather than the 'pointwise' definitions (1.1) and (1.4). With $\mathcal{F}_e^E$ the sub-σ-field of $\mathcal{F}^E$ generated by $\{\omega(f) : f \neq e\}$, (1.4) can be written

$$I_A^h(e) = P^{E\setminus\{e\}}\big(h(\mathbb{P}(A \mid \mathcal{F}_e^E))\big).$$

However, we retain the approach adopted in the prior literature.

Theorem 1.3. *Let $h \in \mathcal{M}$ and $A \in \mathcal{F}^E$. There exists a measurable event B in the Lebesgue product space $\mathcal{L}^E$ such that $\lambda^E(B) = \mathbb{P}(A)$, and $I_B^h(e) = I_A^h(e)$ for $e \in E$.*

This yields a positive answer to the question of Keller, [11, Footnote 2], asking whether h-influence inequalities may be extended from Lebesgue to general spaces.

2. History and literature

The first influence inequality of the type (1.2) was proved in the important paper [9] by Kahn, Kalai, and Linial. This was followed by a number of useful papers including BKKKL [1] and Talagrand [16]. The area has been surveyed by Kalai and Safra [10], and also in [5, Sect. 4.5]. Of more recent work, we mention [11, 12].

Rather than include here a full discussion of influence and sharp threshold, we draw the attention of the reader to five points.

(a) *Equivalence under null sets.* Let A, B be events in Ω such that $\mathbb{P}(A \triangle B) = 0$. By Fubini's theorem, $I_A(e) = I_B(e)$ for $e \in E$. Thus, when working with the definition of (1.1), one may use either the product σ-field $\mathcal{F}^E$ or its completion.

(b) *Definition of influence.* Influence as defined in [1] is generally unequal for two events that differ by a null set. This observation provoked the revised definition (1.1), introduced in [5]. Further notions of influence have been discussed in [7, 11, 12].

(c) *Form of inequality.* The influence inequality (1.2) is not quite in the same form as those proved in [1, 9]. The current form may be found in [3, Thm 3.4] (see also [4, 5]). It is useful when Russo's formula is to be deployed (see [5, Sect. 4.7]), and it makes no assumption of symmetry on the event under study.

(d) *General probability spaces.* The probability space of greatest practical value is the Lebesgue space $\mathcal{L}$, since this provides a coupling of many spaces of importance including the product spaces for Bernoulli variables. It was implied in [1] that the Lebesgue case implies the corresponding inequality for an arbitrary product space, and this is proved in the current note. The weaker assertion for *separable* spaces was discussed in [5, Sect. 4.5].

(e) *Separable versus non-separable.* Many probabilists consider non-separable spaces of limited interest. The current note was inspired by a desire to understand the assertion of [1], to tidy up a slightly dark corner of probability theory, and to give a correct proof for the separable case (cf. [5, Sect. 4.6]).

3. Proof of Theorem 1.3

For probability spaces $X_i = (\Omega_i, \mathcal{F}_i, P_i)$, a mapping $\phi : \Omega_1 \to \Omega_2$ is said to be *measure-preserving* if, for all $B_2 \in \mathcal{F}_2$, the inverse image $B_1 = \phi^{-1}(B_2)$ is measurable and satisfies $P_1(B_1) = P_2(B_2)$. Such a map ϕ is said to be measure-preserving from X_1 to X_2.

Lemma 3.1. *Let $X_i = (\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, 2$, be probability spaces, and let $\phi : \Omega_1 \to \Omega_2$ be measure-preserving. Let E be a finite set, and write Φ for the measure-preserving mapping ϕ^E from X_1^E to X_2^E. If $B_2 \in \mathcal{F}_2^E$ and $B_1 = \Phi^{-1}(B_2)$, then $I_{B_1}^h(e) = I_{B_2}^h(e)$ for all $e \in E$ and $h \in \mathcal{M}$.*

Proof. Let $e \in E$, $h \in \mathcal{M}$, $B_2 \in \mathcal{F}_2$, and $B_1 = \Phi^{-1}(B_2)$. For $\psi \in \Omega_i^{E\setminus\{e\}}$, let F_ψ be the fibre

$$F_\psi = \{\omega \in \Omega_i^E : \omega(f) = \psi(f) \text{ for } f \neq e\} \cong \{\psi\} \times \Omega_i.$$

Suppose $\nu \in \Omega_1^{E\setminus\{e\}}$ is such that $\phi^{E\setminus\{e\}}(\nu) = \psi$. Since ϕ is measure-preserving on each component,

$$P_1\Big(\{\nu\} \times \phi^{-1}(B_2 \cap F_\psi)\Big) = P_2(B_2 \cap F_\psi), \qquad \psi \in \Omega_2^{E\setminus\{e\}}. \tag{3.1}$$

Now $\{\nu\} \times \phi^{-1}(B_2 \cap F_\psi) = B_1 \cap F_\nu$, so that, for $u \in \mathbb{R}$,

$$P_1^{E\setminus\{e\}}\Big(h(P_1(B_1 \cap F_\nu)) > u\Big) = P_2^{E\setminus\{e\}}\Big(h(P_2(B_2 \cap F_\nu)) > u\Big).$$

We integrate over $u \geq 0$ to obtain the claim. □

Lemma 3.2. *Let X^E be as in Section 1, and let $A \in \mathcal{F}^E$. There exists a countably generated sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ such that $A \in \mathcal{G}^E$.*

Proof. Let $\{\mathcal{G}_i : i \in I\}$ be the collection of all countably generated sub-σ-fields of $\mathcal{F}$. Let $\mathcal{H}$ be the union of the $\mathcal{G}_i^E$ as i ranges over I. Then $\mathcal{H}$ is a σ-field. Furthermore, $\mathcal{H}$ is the smallest σ-field containing the *rectangles* $\prod_{e\in E} F_e$ as the F_e range over $\mathcal{F}$. Therefore, $\mathcal{H} = \mathcal{F}^E$. If $A \in \mathcal{F}^E$, we may pick $a \in I$ such that $A \in \mathcal{G}_a^E$. The claim is proved. □

The remainder of the proof conceals a version of the measure-space isomorphism theorem. In general terms the last states that, subject to certain assumptions, a measure space may be placed in correspondence with the Lebesgue space $\mathcal{L}$. The theorem comes in two forms: (i) there is an isomorphism between the associated measure rings (see, for example, [6, §40]), and (ii) there is a pointwise bijection between certain derived sample spaces (see, for example, [13, Thm 4.7]). Rather than appealing to a general theorem, we shall construct the required mappings in an explicit manner requiring no special consideration of the possible existence of atoms. This may be done either by repeated decimation of sub-intervals of $[0,1]$ (as in [15, Sect. 2.2]), or via a mapping to the Cantor set. We take the second route. See [8, App. A] for a discussion of measure-space isomorphisms.

For $T \subseteq \mathbb{R}^d$, the Borel σ-field of T is denoted $\mathcal{B}(T)$.

Lemma 3.3. *Let $A \in \mathcal{F}^E$ and let $\mathcal{G}$ be as in Lemma 3.2. There exists a probability space $Z = (C, \mathcal{B}(C), \mu)$ comprising the Cantor set C endowed with its Borel σ-field and a suitable probability measure, such that following hold.*

(a) *There exists a measure-preserving mapping ψ from X to Z.*
(b) *There exists $G \in \mathcal{B}(C^E)$ such that $A = \Psi^{-1}(G)$, where $\Psi = \psi^E$.*
(c) *There exists a measure-preserving mapping γ from $\mathcal{L}$ to Z.*

This lemma (and the forthcoming proof of Theorem 1.2) may be summarized in the diagrams

$$X \xrightarrow{\psi} Z \xleftarrow{\gamma} \mathcal{L}, \qquad A \xleftarrow{\Psi^{-1}} G \xrightarrow{\Gamma^{-1}} B. \tag{3.2}$$

Proof. (a) Let C be the Cantor set comprising all reals of the form $\sum_{k=1}^{\infty} 2 \cdot 3^{-k} a_k$ as $a = (a_k : k \in \mathbb{N})$ ranges over $\{0,1\}^{\mathbb{N}}$. Thus C is in one–one correspondence with $\{0,1\}^{\mathbb{N}}$. Let $\mathcal{G}$ be generated by the countable family $B = (B_k : k \in \mathbb{N})$ of subsets of Ω, and let $\psi : \Omega \to C$ be given by

$$\psi(x) = \sum_{k=1}^{\infty} 2 \cdot 3^{-k} 1(x \in B_k).$$

Write $\mathcal{G}' = \{\psi^{-1}(S) : S \in \mathcal{B}(C)\}$. We claim that $\mathcal{G} = \mathcal{G}'$. Since $B_k \in \mathcal{G}'$ for all k, we have $\mathcal{G} \subseteq \mathcal{G}'$. Conversely, since ψ is a sum of measurable functions, it is measurable, and hence $\mathcal{G}' \subseteq \mathcal{G}$.

Let μ be the probability measure on $(C, \mathcal{B}(C))$ induced by ψ, that is $\mu(S) = P(\psi^{-1}(S))$ for $S \in \mathcal{B}(C)$. By definition of μ, ψ is measure-preserving from X to $Z = (C, \mathcal{B}(C), \mu)$.

(b) Let $\mathcal{H}$ be the σ-field $\{\Psi^{-1}(S) : S \in \mathcal{B}(C^E)\}$ on Ω^E. By the above, $\mathcal{H} = \mathcal{G}^E$. Consequently, $A \in \mathcal{H}$, and hence $A = \Psi^{-1}(G)$ for some $G \in \mathcal{B}(C^E)$.

(c) Define $\kappa : C \to [0,1]$ by $\kappa(c) = \mu(C \cap [0,c])$. We may take as inverse the function

$$\gamma(y) = \inf\{c : \kappa(c) \geq y\}, \qquad y \in [0,1].$$

Since $\gamma(y) \le c$ if and only if $y \le \kappa(c)$, we have that

$$\gamma^{-1}(C \cap [0,c]) = [0,\kappa(c)], \qquad c \in C,$$

so that

$$\lambda\big(\gamma^{-1}(C \cap [0,c])\big) = \kappa(c) = \mu(C \cap [0,c]).$$

The set $\{C \cap [0,c] : c \in C\}$ is a π-system that generates $\mathcal{B}(C)$, and hence γ is measure-preserving from $\mathcal{L}$ to Z. $\square$

Proof of Theorem 1.2. See (3.2). By Lemmas 3.1 and 3.3(a,b), A and G have equal measure and influences. Write $\Gamma = \gamma^E$, and take $B = \Gamma^{-1}(G) \subseteq [0,1]^E$. Since Γ is measure-preserving, G and B have equal probability and influences. $\square$

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