INHOMOGENEOUS BOND PERCOLATION ON SQUARE, TRIANGULAR, AND HEXAGONAL LATTICES

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Abstract. The star–triangle transformation is used to obtain an equivalence extending over the set of all (in)homogeneous bond percolation models on the square, triangular, and hexagonal lattices. Amongst the consequences are box-crossing (RSW) inequalities for such models with parameter-values at which the transformation is valid. This is a step towards proving the universality and conformality of these processes. It implies criticality of such values, thereby providing a new proof of the critical point of inhomogeneous systems. The proofs extend to certain isoradial models to which previous methods do not apply.

1. Introduction and results

1.1. **Overview.** Two-dimensional percolation was studied intensively in the early 1980s, and again in the decade since 2000. The principal catalyst of the first period was the rigorous calculation by Kesten of a certain critical point (see [17]) and of the second the proof by Smirnov of Cardy's formula (see [27]). The techniques derived during the first period have proved well adapted to the needs of the second. For example, the RSW box-crossing lemmas of [25, 26] have a key role in the study of the conformality of critical percolation. Furthermore, work of Kesten [21] on scaling theory has provided the groundwork for exact calculations of critical exponents for two-dimensional percolation (see [24, 28], for example).

A great deal of rigorous mathematics exists for critical site percolation on the triangular lattice, but surprisingly little for other critical two-dimensional models. A major new idea is apparently needed if we are to develop a parallel theory for, say, critical bond percolation on the

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square lattice. Another question is how to extend methods for homogeneous models to inhomogeneous systems. The purpose of this article is to explain one part of how this may be done for (in)homogeneous bond models on square, triangular, and hexagonal lattices.

The Russo–Seymour–Welsh (RSW) theory of box-crossings (see [12, Sect. 11.7]) plays a significant part in the theory of critical site percolation on the triangular lattice — it is used, for example, in the proof of Cardy's formula, where it implies that certain crossing probabilities are uniformly Hölder (see [14, 30]). RSW theory applies also to homogeneous bond percolation on the square lattice. In contrast, it has been an open problem (see [4, 6, 19]) to derive an RSW theory for *inhomogeneous* bond percolation (with edge-probabilities depending on edge-orientation). The principal methodological advance of the current paper is a demonstration that box-crossing inequalities for one critical model may be translated into inequalities for all other critical models within the family of (in)homogeneous bond percolation processes on square, triangular, and hexagonal lattices. Since such inequalities are known for, say, homogeneous bond percolation on the square lattice, we establish them thus for all other models of this family.

This progress is achieved by judicious use of the star–triangle transformation, inspired by the work [3] of Baxter and Enting on the Ising model. The star–triangle transformation and its ramifications (known as the Yang–Baxter equation) have been at the heart of many advances in 'exact solutions' for systems such as the six-vertex model, the chiral Potts model, the dimer model, and so on (see, for example, [2, 7, 16, 23]). It turns out that the star–triangle map has a special affinity for certain physical models on so-called isoradial graphs — see, for example, the proof of conformality for the Ising model by Chelkak and Smirnov [10]. The relationship with isoradial graphs plays a role in the current work, particularly in the results for the 'highly inhomogeneous models' of Theorems 1.5–1.7.

Our basic approach is to use the star–triangle transformation to transport open paths from one lattice to another. Some complications arise during transportation, and these may be controlled using probabilistic estimates. In a second paper [15], we use these methods to prove the universality of certain critical exponents, including the arm exponents, within the above family of bond percolation models, under the assumption that they exist for any one such model. The current paper has therefore two principal targets: to prove a theory of box-crossings of critical inhomogeneous bond percolation models in two dimensions, and to develop techniques for the study of universality across families of such models.

For the percolation models studied in this paper, Theorem 1.3 verifies the assumption of [20] under which the incipient infinite cluster has been shown to exist. Box-crossing inequalities are useful also in proving scale-invariance for critical percolation in two-dimensions (see [8, 27]). Consider, for example, a domain $\mathcal{D}$ in the plane (with a superimposed lattice $\mathbb{L}$ with mesh-size δ) and four points, A, B, C and D distributed anti-clockwise along its boundary. Consider the limit (as $\delta \to 0$) of the probability that there exists an open path in $\mathcal{D}$ joining the boundary arcs AB and CD. Cardy [9] presented a formula for this limit, and this was proved by Smirnov [27] for the special case of critical site percolation on the triangular lattice. Corresponding statements are expected to hold for other lattices but no proofs are yet known. One may show that, if the underlying measure has the box-crossing property (see Definition 1.2), then such probabilities are bounded uniformly away from 0 and 1 as $\delta \to 0$.

The box-crossing property plays a significant role in the proof of Cardy's formula, in which one shows the uniform convergence of a certain triplet of discretely harmonic functions to a limiting triplet of harmonic functions. This is obtained in two steps; first, one proves tightness for the family of functions, then one identifies its subsequential limits. Tightness follows by an application of the Arzelà–Ascoli theorem, whose pre-compactness hypothesis is met by the fact that the discretely harmonic functions are uniformly Hölder. The proof of this last fact is via the box-crossing property. A full proof of Cardy's formula may be found in [14, Sect. 5.7] and [30, Sect. 2].

The structure of the paper is as follows. The necessary notation is introduced in Section 1.2, and our main results stated in Section 1.3. The star–triangle transformation is discussed in detail in Section 2, with particular attention to transformations of edge-configurations and open paths. Proofs for inhomogeneous bond percolation on the square, triangular and hexagonal lattices are found in Section 3, and for the highly inhomogeneous models in Section 4.

1.2. **Notation.** The lattices under study are the square, triangular, and hexagonal (or honeycomb) lattices illustrated in Figure 1.1. The hypotheses and conclusions of this paper may often be expressed in terms of their graph-theoretic properties. Nevertheless, we choose here to make use of certain planar embeddings of these lattices.

An embedding of a planar graph in $\mathbb{R}^2$ gives rise to a so-called *dual graph*. We shall make frequent use of duality, for a short account of which the reader is referred to [12, Sect. 11.2].

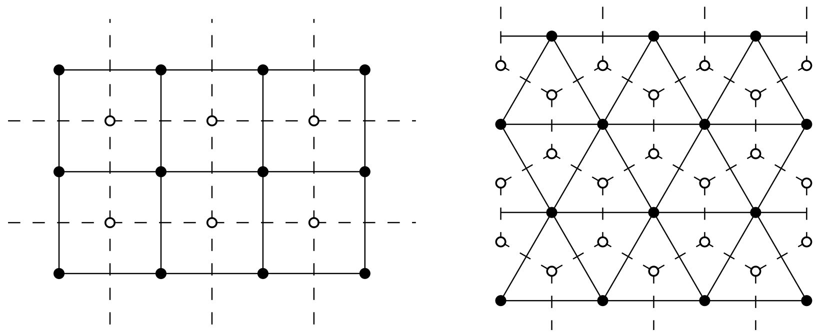

FIGURE 1.1. The square lattice and its dual square lattice. The triangular lattice and its dual hexagonal lattice.

With each of the lattices of Figure 1.1 we may associate a bond percolation model as follows. Let $G = (V, E)$ be a countable connected graph. A *configuration* on G is an element $\omega = (\omega_e : e \in E)$ of the set $\Omega = \{0, 1\}^E$. An edge with endpoints u, v is denoted uv. The edge e is called *open*, or ω-*open*, in $\omega \in \Omega$ (respectively, *closed*) if $\omega_e = 1$ (respectively, $\omega_e = 0$).

For $\omega \in \Omega$ and $A, B \in V$, we say A *is connected to* B (in ω), written $A \leftrightarrow B$ (or $A \xleftrightarrow{G,\omega} B$), if G contains a path of open edges from A to B. An *open cluster* of ω is a maximal set of pairwise-connected vertices. We write $A \leftrightarrow \infty$ if A is the endpoint of an infinite open self-avoiding path.

The simplest bond percolation model on G is that associated with the product measure $\mathbb{P}_p$ on Ω with given intensity $p \in [0, 1]$. Let 0 be a designated vertex of V called the *origin*, and define the *percolation probability*

$$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty).$$

The *critical probability* is given by

$$p_c(G) = \sup\{p : \theta(p) = 0\}.$$

It was proved in [17] that the square lattice has critical probability $\frac{1}{2}$, and the principle ingredient of the proof is the property of self-duality. The dual of the triangular lattice is the hexagonal lattice, and a further ingredient is required in order to compute the two corresponding critical probabilities, namely the so-called star–triangle transformation. This calculation was performed in [31]. General accounts of percolation may be found in [12, 14], and of aspects of two-dimensional percolation in [5].

We turn now to inhomogeneous percolation on the three lattices of Figure 1.1. The edges of the square lattice are naturally divided into two classes (horizontal and vertical) of parallel edges, while those of the triangular and hexagonal lattices may be split into three such classes. In *inhomogeneous* percolation, one allows the product measure to have different intensities on different edges, while requiring that any two parallel edges have the same intensity. Thus, inhomogeneous percolation on the square lattice has two parameters, p_{h} for horizontal edges and p_{v} for vertical edges. We denote the corresponding measure $\mathbb{P}_{\mathbf{p}}^{\square}$ where $\mathbf{p} = (p_{\mathrm{h}}, p_{\mathrm{v}})$. On the triangular and hexagonal lattices, the measure is defined by a triplet of parameters $\mathbf{p} = (p_0, p_1, p_2)$, and we denote these measures $\mathbb{P}_{\mathbf{p}}^{\triangle}$ and $\mathbb{P}_{\mathbf{p}}^{\hexagon}$, respectively.

These models have percolation probabilities and critical surfaces, and the latter were given explicitly in [12, 19]. Let

$$\kappa_{\square}(\mathbf{p}) = p_{\mathrm{h}} + p_{\mathrm{v}} - 1, \quad \mathbf{p} = (p_{\mathrm{h}}, p_{\mathrm{v}}), \tag{1.1}$$

$$\kappa_{\triangle}(\mathbf{p}) = p_0 + p_1 + p_2 - p_0 p_1 p_2 - 1, \quad \mathbf{p} = (p_0, p_1, p_2), \tag{1.2}$$

$$\kappa_{\hexagon}(\mathbf{p}) = -\kappa_{\triangle}(1 - p_0, 1 - p_1, 1 - p_2), \quad \mathbf{p} = (p_0, p_1, p_2). \tag{1.3}$$

Theorem 1.1. *The critical surfaces of inhomogeneous bond percolation are given as follows.*

(a) Square lattice: $\kappa_{\square}(\mathbf{p}) = 0$.
(b) Triangular lattice: $\kappa_{\triangle}(\mathbf{p}) = 0$.
(c) Hexagonal lattice: $\kappa_{\hexagon}(\mathbf{p}) = 0$.

Theorem 1.1 was predicted in [29], and discussed in [19, Sect. 3.4], where part (a) was proved and examples presented in support of parts (b) and (c). The complete proof of the theorem may be found in [12, Sect. 11.9]. For each of these three processes, the radii and volumes of open clusters have exponential tails when $\kappa(\mathbf{p}) < 0$, as in [12, Thms 5.4, 6.75].

We call a triplet $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1)^3$ *self-dual* if it satisfies $\kappa_{\triangle}(\mathbf{p}) = 0$. By Theorem 1.1, self-dual points are also critical points, but the neutral term 'self-duality' is chosen in order to emphasize that the methods of this paper do not make use of criticality *per se*.

We write $\alpha \pm \mathbf{p}$ for the triplet $(\alpha \pm p_0, \alpha \pm p_1, \alpha \pm p_2)$, and also $\mathbb{N} = \{1, 2, \dots\}$ for the natural numbers, $\mathbb{Z}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ for the integers.

1.3. **Main results.** We concentrate here on the probabilities of open crossings of boxes. Consider bond percolation on a connected graph G embedded in the plane $\mathbb{R}^2$. Let $h, l > 0$, and let $S_{h,l} = [0, h] \times [0, l]$, viewed as a subset of $\mathbb{R}^2$. A *box S of size $h \times l$* is a subset of $\mathbb{R}^2$ of the form $f(S_{h,l})$ for some map $f : \mathbb{R}^2 \to \mathbb{R}^2$ comprising a rotation and a translation. The box $S = f(S_{h,l})$ is said to *possess open crossings* if there exist two open paths of G (viewed as arcs in the plane) whose intersection with S are arcs having one endpoint in each of the sets $f(\{0\}\times[0,l])$ and $f(\{h\}\times[0,l])$ (respectively, each of the sets $f([0,h]\times\{0\})$ and $f([0,h]\times\{l\})$).

Definition 1.2. *Let $G = (V, E)$ be a countable connected graph embedded in the plane. We say that a measure $\mathbb{P}$ on $\Omega = \{0,1\}^E$ has the* box-crossing property *if, for $\alpha > 0$, there exists $\delta > 0$ such that: for all large $N \in \mathbb{N}$ and any box S of size $\alpha N \times N$, S possesses open crossings with probability at least δ.*

Note that the box-crossing property depends on the embedding. The lattices considered here will be embedded in $\mathbb{R}^2$ in very simple ways that will of themselves cause no difficulty in the current context. For the sake of definiteness at this point, the square lattice is embedded in $\mathbb{R}^2$ with vertex-set $\mathbb{Z}^2$, the triangular lattice has vertex-set $\{a\mathbf{i} + b\mathbf{j} : a, b \in \mathbb{Z}\}$ where $\mathbf{i} = (0, 2)$ and $\mathbf{j} = (1, \sqrt{3})$, and the hexagonal lattice is the dual of the triangular lattice. Note that the box-crossing property is invariant under affine maps of $\mathbb{R}^2$. It is standard that homogeneous bond percolation on the square lattice with parameter $p = \frac{1}{2}$ has the box-crossing property. (See, for example, [12, Sect. 11.7] and Proposition 3.1.)

Here is a simplified version of our main result; more general versions may be found at Theorems 1.4 and 1.5.

Theorem 1.3.

(a) *If $\mathbf{p} \in (0,1)^2$ satisfies $\kappa_\square(\mathbf{p}) = 0$, then $\mathbb{P}^\square_\mathbf{p}$ has the box-crossing property.*

(b) *If $\mathbf{p} \in [0,1)^3$ satisfies $\kappa_\triangle(\mathbf{p}) = 0$, then both $\mathbb{P}^\triangle_\mathbf{p}$ and $\mathbb{P}^{\hexagon}_{1-\mathbf{p}}$ have the box-crossing property.*

Square-lattice percolation may be obtained from triangular-lattice percolation by setting one of its three parameters to 0. Thus, part (b) includes part (a). We choose to distinguish the two cases, since this is in harmony with the method of proof.

Theorem 1.1 may be obtained from Theorem 1.3 by standard means. There are several possible choices for the details of the proof, but the core arguments comprise the combination of box-crossings, together with positive association and some type of sharp-threshold statement. The inhomogeneous models possess translation-invariance but not rotation-invariance, a gap that may be spanned by the box-crossing property. Full translation-invariance is in fact inessential, and our arguments may be applied to certain 'highly inhomogeneous models' that we describe next. The following is included as a demonstration of the use of the box-crossing property, and the connection between the current work and the geometry of isoradial graphs (see [16]).

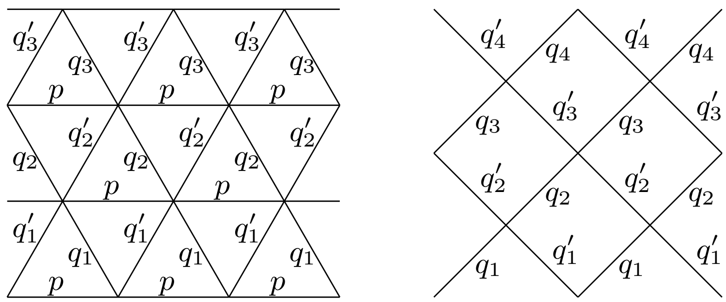

FIGURE 1.2. *Left*: The triangular lattice with the highly inhomogeneous product measure $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$. The probability for each edge to be open is described in the picture: all horizontal edges have probability p of being open, while the other edges have probability q_n (right edges of upwards pointing triangles) or q'_n (left edges of upwards pointing triangles) of being open, with n being their height. *Right*: The square lattice with a highly inhomogeneous product measure $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$, rotated by $\pi/4$. Edges inclined at angle $\pi/4$ have probability q_n of being open, while edges inclined at angle $3\pi/4$ have probability q'_n of being open, with n being their height.

Let $p \in (0,1)$, and let $\mathbf{q} = (q_n : n \in \mathbb{Z}) \in [0,1]^{\mathbb{Z}}$ and $\mathbf{q}' = (q'_n : n \in \mathbb{Z}) \in [0,1]^{\mathbb{Z}}$. These are the parameters of our highly inhomogeneous models, and one such parameter is allocated to any given edge of the square and triangular lattices. This is illustrated in Figure 1.2. See also Figure 1.3.

Consider first the triangular lattice, and write $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$ for the product measure on Ω under which: any horizontal edge is open with probability p; any right (respectively, left) edge of a upwards pointing triangle is open with probability q_n (respectively, q'_n). Here, $n \in \mathbb{Z}$ denotes the height of the edge as drawn in the figure. Let $\mathbb{P}^{\hexagon}_{1-p,1-\mathbf{q},1-\mathbf{q}'}$ be the measure on the hexagonal lattice that is dual to $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$.

Consider next the square lattice. The measure $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$ is defined similarly to the above, as in Figure 1.2. We refer to the three probability

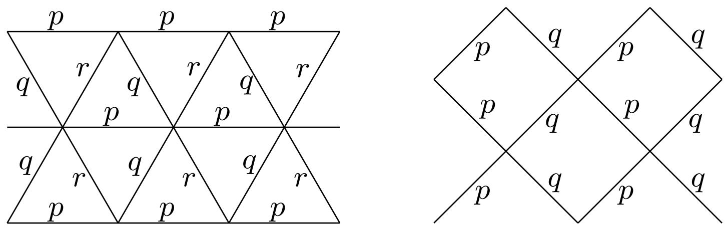

FIGURE 1.3. Each graph, when repeated periodically, constitutes a model included within the analysis of highly inhomogeneous models. *Left*: A triangular-lattice model with an axis of symmetry, as in [19, Sect. 2.1]. *Right*: A variant on the square-lattice model.

measures thus defined as *highly inhomogeneous*. Under suitable conditions on their parameters, each may be regarded as percolation on an isoradial graph with edge-parameters chosen in the canonical way according to the isoradial embedding (see [16, Sect. 5]).

We may show, under suitable conditions, that highly inhomogeneous models have the box-crossing property.

Theorem 1.4. *If* $p \in (0,1)$, $\mathbf{q}, \mathbf{q}' \in [0,1]^{\mathbb{Z}}$ *satisfy*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\triangle}(p, q_n, q'_n) = 0, \tag{1.4}$$

then both $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$ *and* $\mathbb{P}^{\hexagon}_{1-p,1-\mathbf{q},1-\mathbf{q}'}$ *have the box-crossing property.*

Theorem 1.5. *Let* $\mathbf{q}, \mathbf{q}' \in (0,1)^{\mathbb{Z}}$. *If there exists* $\epsilon > 0$ *such that*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\square}(q_n, q'_n) = 0 \quad \textit{and} \quad q_n, q'_n \ge \epsilon, \tag{1.5}$$

then $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$ *has the box-crossing property.*

The reader is reminded in Propositions 4.1 and 4.2 that the box-crossing property implies power-law behaviour of the radius of an open cluster.

An assumption along the lines of the second condition on $\mathbf{q}$, $\mathbf{q}'$ in (1.5) is necessary: if, for example, q_n is sufficiently small over an interval of values of n, then the chance of crossing a certain diagonally oriented rectangle is correspondingly small, and thus the box-crossing property could not hold. From the above theorems may be obtained a characterization of the critical surface of a highly inhomogeneous model. We call a percolation measure $\mathbb{P}$ *uniformly supercritical* if there exists $\theta > 0$ such that $\mathbb{P}(v \leftrightarrow \infty) \ge \theta$ for every vertex v. The open cluster at a vertex v, written C_v, is the set of vertices joined to v by open paths.

Theorem 1.6. *Let* $p \in (0,1)$ *and* $\mathbf{q}, \mathbf{q}' \in [0,1)^{\mathbb{Z}}$.

(a) *If*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\triangle}(p, q_n, q'_n) \le 0, \tag{1.6}$$

then there exists, $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$*-a.s., no infinite open cluster.*

(b) *If there exists* $\delta > 0$ *such that*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\triangle}(p, q_n, q'_n) \le -\delta, \tag{1.7}$$

then there exist $c, d > 0$ *such that, for every vertex* v,

$$\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}(|C_v| \ge k) \le ce^{-dk}, \quad k \ge 0.$$

(c) *If there exists* $\delta > 0$ *such that*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\triangle}(p, q_n, q'_n) \ge \delta, \tag{1.8}$$

then $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$ *is uniformly supercritical.*

The same holds for $\mathbb{P}^{\hexagon}_{p,\mathbf{q},\mathbf{q}'}$ *with* $\kappa_{\hexagon}$ *in place of* $\kappa_{\triangle}$.

Theorem 1.7. *Let* $\mathbf{q}, \mathbf{q}' \in (0,1)^{\mathbb{Z}}$.

(a) *If there exists* $\epsilon > 0$ *such that*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\square}(q_n, q'_n) \le 0 \quad \text{and} \quad q_n, q'_n \le 1 - \epsilon, \tag{1.9}$$

then there exists, $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$*-a.s., no infinite open cluster.*

(b) *If there exist* $\delta > 0$ *such that*

$$\forall n \in \mathbb{Z}, \qquad \kappa_{\square}(q_n, q'_n) \le -\delta,$$

then there exist $c, d > 0$ *such that, for every vertex* v,

$$\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}(|C_v| \ge k) \le ce^{-dk}, \quad k \ge 0.$$

(c) *If there exists* $\delta > 0$ *such that for all* n,

$$\kappa_{\square}(q_n, q'_n) \ge \delta,$$

then $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$ *is uniformly supercritical.*

Parts (c) of the above two theorems give conditions under which there exists an infinite open cluster. Such a cluster is necessarily (almost surely) unique since the box-crossing property implies the existence of open cycles in annuli.

We will prove Theorem 1.3 in Section 3, and Theorems 1.4–1.7 in Section 4.

2. Star–triangle transformation

2.1. **The basic transformation.** The star–triangle transformation was discovered first in the context of electrical networks, and adapted by Onsager and Kramers–Wannier to the Ising model. In its base form, it is a graph-theoretic transformation of the hexagonal lattice to the triangular lattice. Its importance stems from the fact that a variety of probabilistic models are conserved under this transformation, including the critical percolation, Potts, and random-cluster models. The methods of this paper extend to all such systems, but we concentrate here on percolation, for which we summarize its manner of operation as in [12, Sect. 11.9].

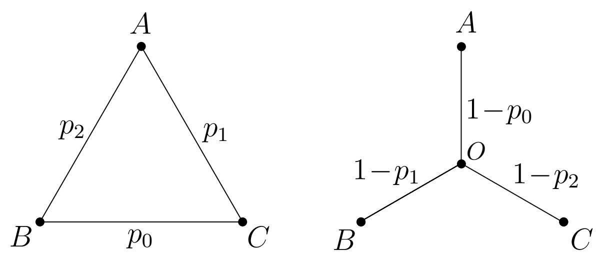

Figure 2.1. The star–triangle transformation

Consider the triangle $G = (V, E)$ and the star $G' = (V', E')$ drawn in Figure 2.1. Let $\mathbf{p} = (p_0, p_1, p_2)$. In order to eliminate trivialities, we shall assume throughout this paper that $\mathbf{p} \in [0, 1)^3$. Write $\Omega = \{0, 1\}^E$ with associated product probability measure $\mathbb{P}^{\triangle}_{\mathbf{p}}$, and $\Omega' = \{0, 1\}^{E'}$ with associated measure $\mathbb{P}^{\hexagon}_{1-\mathbf{p}}$. Let $\omega \in \Omega$ and $\omega' \in \Omega'$. For each graph we may consider open connections between its vertices, and we abuse notation by writing, for example, $x \xleftrightarrow{G,\omega} y$ for the *indicator function* of the event that x and y are connected by an open path of ω. Thus connections in G are described by the family $(x \xleftrightarrow{G,\omega} y : x, y \in V)$ of random variables, and similarly for G'.

Proposition 2.1 (Star–triangle transformation)**.** *Let $\mathbf{p} \in [0, 1)^3$ be self-dual in the sense that $\kappa_\triangle(\mathbf{p}) = 0$. The families*

$$\left(x \xleftrightarrow{G,\omega} y : x, y = A, B, C\right), \quad \left(x \xleftrightarrow{G',\omega'} y : x, y = A, B, C\right),$$

have the same law.

The proof may be found in [12, Sect. 11.9]. It will be helpful in the following to explore natural couplings of the two measures of the lemma. Let $\mathbf{p} \in [0, 1)^3$ be self-dual, and let Ω (respectively, Ω') have associated measure $\mathbb{P}^{\triangle}_{\mathbf{p}}$ (respectively, $\mathbb{P}^{\hexagon}_{1-\mathbf{p}}$) as above. There exist random mappings $T : \Omega \to \Omega'$ and $S : \Omega' \to \Omega$ such that $T(\omega)$ has law $\mathbb{P}^{\hexagon}_{1-\mathbf{p}}$, and $S(\omega')$ has law $\mathbb{P}^{\triangle}_{\mathbf{p}}$. Such mappings are given in Figure 2.2, and we shall not specify them more formally here. Note from the figure that $T(\omega)$ is deterministic for seven of the eight elements of Ω; only in the eighth case does $T(\omega)$ involve further randomness. Similarly, $S(\omega')$ is deterministic except for one special ω'. Each probability in the figure is well defined since $P := (1 - p_0)(1 - p_1)(1 - p_2) > 0$.

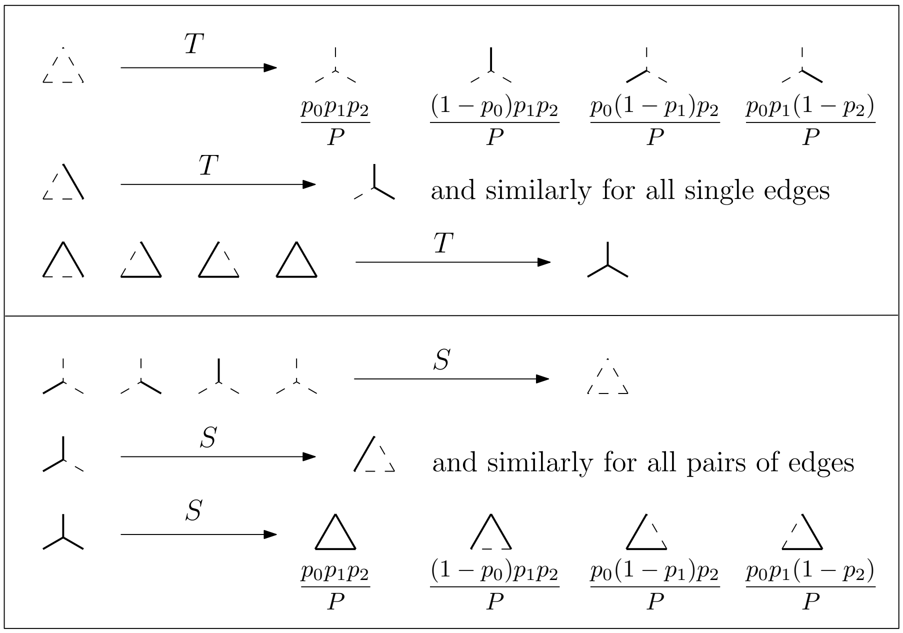

FIGURE 2.2. The 'kernels' T and S and their transition probabilities, with $P := (1 - p_0)(1 - p_1)(1 - p_2)$. Since $\kappa_{\triangle}(\mathbf{p}) = 0$, the probabilities in the first and last rows sum to 1.

Proposition 2.2 (Star–triangle coupling)**.** *Let* $\mathbf{p}$ *be self-dual and let* S *and* T *be given as in Figure 2.2. With* ω *and* ω' *sampled as above,*

(a) $T(\omega)$ *has the same law as* ω'*,*
(b) $S(\omega')$ *has the same law as* ω*,*
(c) *for* $x, y \in \{A, B, C\}$, $x \overset{G,\omega}{\longleftrightarrow} y$ *if and only if* $x \overset{G',T(\omega)}{\longleftrightarrow} y$*,*
(d) *for* $x, y \in \{A, B, C\}$, $x \overset{G',\omega'}{\longleftrightarrow} y$ *if and only if* $x \overset{G,S(\omega')}{\longleftrightarrow} y$*.*

2.2. **Transformations of lattices.** We show next how to use the star–triangle transformation to transform the triangular lattice into the square lattice. This transformation may be extended to transport self-dual measures on the first lattice to measures on the second lattice, via a coupling that preserves open connections. This permits the transportation of the box-crossing property from one lattice to the other. This general approach was introduced by Baxter and Enting [3] in a study of the Ising model, and has since been developed under the name Yang–Baxter equation, [23].

Henceforth it is convenient to work with so-called *mixed lattices* that combine the square lattice with either the triangular or hexagonal lattice. We shall be precise about the manner in which a mixed lattice is embedded in $\mathbb{R}^2$. Let $i \in \mathbb{R}$, and let $I = \mathbb{R} \times \{i\}$ be the horizontal line of $\mathbb{R}^2$ with *height* i, called the *interface*; above I consider the triangular lattice and below I the square lattice. Our triangular lattice comprises equilateral triangles with side length $\sqrt{3}$, and our square lattice comprises rectangles whose horizontal (respectively, vertical) edges have length $\sqrt{3}$ (respectively, 1), as illustrated in the leftmost diagram of Figure 2.3. The embedding is specified up to horizontal translation and, in order to precise, we assume that the point $(0, i)$ is a vertex of the lattice. We call the ensuing graph the *mixed triangular lattice* $\mathbb{L}$ with interface $I = I_{\mathbb{L}}$.

The *mixed hexagonal lattice* $\mathbb{L}$ with interface $I = I_{\mathbb{L}}$ is similarly composed of a regular hexagonal lattice (of side length 1) above I and a square lattice below I (with edge-lengths as above), as drawn in the central diagram of Figure 2.3.

We define the *height* $h(A)$ of a subset $A \subseteq \mathbb{R}^2$ as the supremum of the y-coordinates of elements of A. A mixed lattice $\mathbb{L}$ may be identified with the subset of $\mathbb{R}^2$ belonging to its edge-set. Thus, for a mixed lattice $\mathbb{L}$, $h(I_{\mathbb{L}})$ is the height of its interface.

We next define two transformations, $T^{\vartriangle}$ and $T^{\triangledown}$ acting on a mixed triangular lattice $\mathbb{L}$.

(a) $T^{\vartriangle}$ transforms all upwards pointing triangles of $\mathbb{L}$ into stars, with centres at the circumcentres of the equilateral triangles.
(b) $T^{\triangledown}$ transforms all downwards pointing triangles into stars.

It is easily checked (and illustrated in Figure 2.3) that each transformation maps a mixed triangular lattice to a mixed hexagonal lattice.

We define similarly the transformations $S^{\curlywedge}$ and $S^{\curlyvee}$ on a mixed hexagonal lattice; these transform all upwards (respectively, downwards) pointing stars into triangles. They transform a mixed hexagonal lattice to a mixed triangular lattice.

The concatenated operators $S^{\curlywedge} \circ T^{\triangledown}$ and $S^{\curlyvee} \circ T^{\vartriangle}$ map the mixed triangular lattice $\mathbb{L}$ to another mixed triangular lattice, but with a

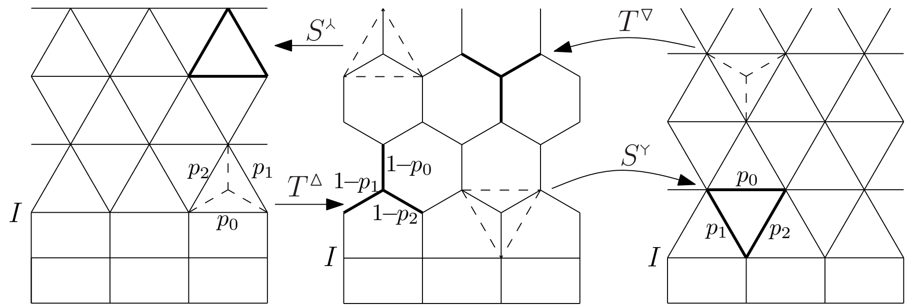

FIGURE 2.3. Transformations $S^{\curlywedge}$, $S^{\curlyvee}$, $T^{\vartriangle}$, and $T^{\triangledown}$ of mixed lattices. The transformations map the zones with dashes to the bold triangles/stars. The interface-height decreases by 1 from the leftmost to the rightmost graph.

different interface height:

$$h(I_{S^{\curlywedge}\circ T^{\triangledown}\mathbb{L}}) = h(I_{\mathbb{L}}) + 1,$$
$$h(I_{S^{\curlyvee}\circ T^{\vartriangle}\mathbb{L}}) = h(I_{\mathbb{L}}) - 1.$$

Loosely speaking, repeated application of $S^{\curlywedge} \circ T^{\triangledown}$ transforms $\mathbb{L}$ into the square lattice, while repeated application of $S^{\curlyvee} \circ T^{\vartriangle}$ transforms it into the triangular lattice.

We now extend the domains of the above maps to include configurations. Let $\mathbb{L} = (V, E)$ be a mixed triangular lattice with $\Omega_E = \{0,1\}^E$, and let $\omega \in \Omega_E$. The image of $\mathbb{L}$ under $T^{\vartriangle}$ is written $T^{\vartriangle}\mathbb{L} = (T^{\vartriangle}V, T^{\vartriangle}E)$ and we write $\Omega_{T^{\vartriangle}E} = \{0,1\}^{T^{\vartriangle}E}$. Let $\mathbf{p} \in [0,1)^3$. Let $T^{\vartriangle}(\omega)$ be chosen (randomly) from $\Omega_{T^{\vartriangle}E}$ by independent application of the kernel T within every upwards pointing triangle of $\mathbb{L}$. Note that the random map T depends on the choice of $\mathbf{p}$.

By Proposition 2.2, for any two vertices A, B on $\mathbb{L}$, we have:

$$\left(A \xleftrightarrow{\mathbb{L},\omega} B\right) \Leftrightarrow \left(A \xleftrightarrow{T^{\vartriangle}\mathbb{L},T^{\vartriangle}(\omega)} B\right). \tag{2.1}$$

The corresponding statements for $T^{\triangledown}$, $S^{\curlywedge}$, and $S^{\curlyvee}$ are valid also, with one point of note. In applying the transformations $S^{\curlywedge}$, $S^{\curlyvee}$ to a mixed hexagonal lattice, the points A and B in the corresponding versions of (2.1) must not be centres of transformed stars, since these points disappear during the transformations.

Let $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ be self-dual, and let $S^{\curlywedge}$, $S^{\curlyvee}$, $T^{\vartriangle}$, $T^{\triangledown}$ be given accordingly. We identify next the probability measures on the mixed lattices that are preserved by the operation of these transformations.

Let $\mathbb{L} = (V, E)$ be a mixed (triangular or hexagonal) lattice. The probability measure denoted $\mathbb{P}_{\mathbf{p}}$ on Ω_E is product measure whose intensity $p(e)$ at edge e is given as follows.

(a) $p(e) = p_0$ if e is horizontal,
(b) $p(e) = 1 - p_0$ if e is vertical,
(c) $p(e) = p_1$ if e is the right edge of an upwards pointing triangle,
(d) $p(e) = p_2$ if e is the left edge of an upwards pointing triangle,
(e) $p(e) = 1 - p_2$ if e is the right edge of an upwards pointing star,
(f) $p(e) = 1 - p_1$ if e is the left edge of an upwards pointing star.

When it becomes necessary to emphasize the lattice $\mathbb{L}$ in question, we shall write $\mathbb{P}^{\mathbb{L}}_{\mathbf{p}}$.

Proposition 2.3. *If $\mathbf{p} \in [0, 1)^3$ is self-dual in that $\kappa_{\triangle}(\mathbf{p}) = 0$, then $\mathbb{P}_{\mathbf{p}}$ is preserved by the transformations $S^{\curlywedge}$, $S^{\curlyvee}$, $T^{\vartriangle}$, and $T^{\triangledown}$. That is, if U is any of these four transformations acting on the mixed lattice $\mathbb{L} = (V, E)$ then*

$$\omega \in \Omega_E \textit{ has law } \mathbb{P}^{\mathbb{L}}_{\mathbf{p}} \quad \Leftrightarrow \quad U(\omega) \textit{ has law } \mathbb{P}^{U\mathbb{L}}_{\mathbf{p}}.$$

2.3. **Transformations of paths.** Since the star–triangle transformation preserves open connections (cf. (2.1)), there is a sense in which it maps open paths to open paths. Thus, if percolation on a mixed lattice $\mathbb{L}$ has the box-crossing property, one expects that its image also has the box-crossing property. Some difficulties occur in the proof of this, arising from the fact that the image of an open path tends to drift away from the original. We study this drift next. It is convenient to work with general paths in $\mathbb{R}^2$.

Recall that a *path* $\Gamma = (\Gamma_t)$ in $\mathbb{R}^2$ is a continuous function $\Gamma : [a, b] \to \mathbb{R}^2$ for some real interval $[a, b]$. Note that a path Γ may in general have self-intersections, and there may be sub-intervals of $[a, b]$ on which Γ is constant. Let $\phi : [c, d] \to [a, b]$ be continuous and strictly increasing with $\phi(c) = a$ and $\phi(d) = b$. We term the path $\Gamma_\phi = (\Gamma_{\phi(t)})$ a *reparametrization* of Γ over $[c, d]$.

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^2$. The space of paths may be metrized by

$$d(\Gamma, \Pi) = \inf \left\{ \sup_{t \in [0,1]} |\Gamma'_t - \Pi'_t| \right\},$$

where the infimum is over all reparametrizations Γ' (respectively, Π') of Γ (respectively, Π) over $[0, 1]$. Note that d is not a metric since $d(\Gamma, \Gamma') = 0$ if Γ' is a reparametrization of Γ, and thus the corresponding metric acts on a space of equivalence classes of paths (see [1, eqn (2.1)]).

We shall use the fact that, if two paths (parametrized over $[0, 1]$) satisfy $d(\Gamma, \Pi) < \delta$, then

$$\Gamma \subseteq \Pi^\delta, \quad |\Gamma_0 - \Pi_0| \le \delta, \quad |\Gamma_1 - \Pi_1| \le \delta,$$

where

$$A^\delta := \{x + y : x \in A,\, |y| \le \delta\}.$$

Henceforth all paths will be lattice-paths (we allow loops and repeated edges). Such a path is called *open* (in a given configuration) if it traverses only open edges.

Let ω be an edge-configuration on a mixed triangular lattice $\mathbb{L}$. Let Γ be an ω-open lattice-path of $\mathbb{L}$, and consider the action of the map $T^\vartriangle$ (illustrated in Figure 2.4). The image lattice $T^\vartriangle\mathbb{L}$ is endowed with the edge-configuration $T^\vartriangle(\omega)$, and we explain next the construction of a $T^\vartriangle(\omega)$-open path $T^\vartriangle(\Gamma)$ on $T^\vartriangle\mathbb{L}$. The path $T^\vartriangle(\Gamma)$ will remain close to Γ, and it will depend only locally on Γ and ω.

We summarize the argument for $T^\vartriangle(\Gamma)$ (the same argument is valid for $T^\triangledown(\Gamma)$). The path Γ passes through the sequence $\gamma_0, \gamma_1, \dots, \gamma_m$ of vertices of $\mathbb{L}$, in order. Since $\mathbb{L}$ is a mixed triangular lattice, each γ_i is present in $T^\vartriangle\mathbb{L}$ also. The edge $\gamma_i\gamma_{i+1}$ of $\mathbb{L}$ lies either in its square part (excluding the interface) or its triangular part (including the interface). If the former, it lies also in $T^\vartriangle\mathbb{L}$. If the latter, it lies in a unique upwards pointing triangle t of $\mathbb{L}$. Under $T^\vartriangle$, t is mapped to a star $T^\vartriangle(t)$, and the configuration on t is mapped to a configuration on $T^\vartriangle(t)$ in which γ_i is connected by an open path to γ_{i+1} via the centre O of the star. We replace the edge $\gamma_i\gamma_{i+1}$ of $\mathbb{L}$ by this open path. This is done for each edge of Γ, and the outcome, denoted $T^\vartriangle(\Gamma)$, is an open path of $T^\vartriangle\mathbb{L}$ with the same endpoints as Γ.

We turn now to a mixed hexagonal lattice $\mathbb{H}$ under the transformation $S^\curlyvee$ (the same argument holds for $S^\curlywedge$). Let ω be an edge-configuration on $\mathbb{H}$, and Γ an open path. Readers may be content with the illustration of Figure 2.4, but further details are given below.

We parametrize Γ as $(\Gamma_t : 0 \le t \le N)$ in such a way that:

(a) $(\Gamma_n : n = 0, 1, \dots, N)$ are the vertices visited by Γ in sequence (possibly with repetition),
(b) each $\Gamma_{[n,n+1]}$ is either an edge or a vertex of the lattice, and
(c) Γ is affine on the intervals $[n, n + 1]$.

It suffices to define the image under $S^\curlyvee$ of each edge of Γ. For simplicity, we assume that Γ has no stationary points; the argument is exactly similar otherwise. Let g_n be the edge $\Gamma_{[n,n+1]}$ of $\mathbb{H}$. If g_n lies in the square part of $\mathbb{H}$ (that is, in or below the interface) we set $S^\curlyvee(\Gamma)_n = \Gamma_n$ and $S^\curlyvee(\Gamma)_{n+1} = \Gamma_{n+1}$, with linear interpolation between.

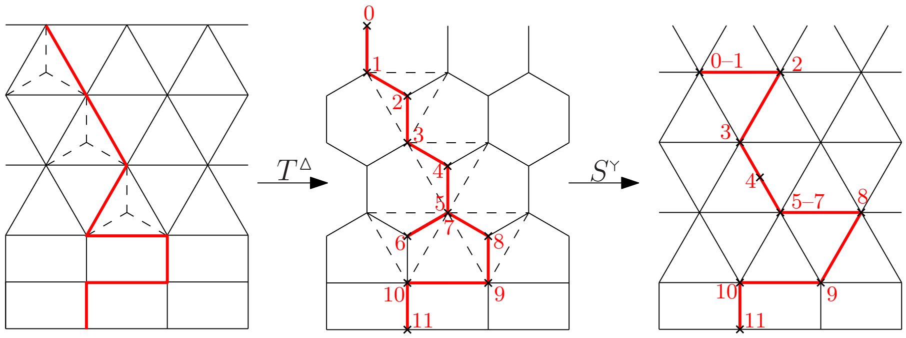

FIGURE 2.4. Transformations of lattice-paths. The transformation $T^{\triangle}$ acts deterministically on open paths, each edge of a triangle being transformed into two segments of an upwards pointing star. When applying $S^{\curlyvee}$, the segment labelled from 0 to 1 contracts to one point, as does that labelled from 5 to 7.

Suppose that g_n lies in the hexagonal part of $\mathbb{H}$, so that g_n is an edge of a downwards pointing star whose exterior vertices we denote as A, B, C, and whose central vertex as O. Thus g_n has O as one endvertex, and its other endvertex lies in $\{A, B, C\}$.

(a) If $n = 0$ and $\Gamma_0 = O$, set $S^{\curlyvee}(\Gamma)_{[0,1]} = \Gamma_1$.
(b) If $n = N-1$ and $\Gamma_N = O$, set $S^{\curlywedge}(\Gamma)_{[N-1,N]} = \Gamma_{N-1}$.
(c) Suppose $n \geq 1$ and $\Gamma_n = O$ (a similar argument holds if $n \leq N-2$ and $\Gamma_{n+1} = O$). Then $\Gamma_{n-1}, \Gamma_{n+1} \in \{A, B, C\}$.
  - If $\Gamma_{n-1} = \Gamma_{n+1}$, set $S^{\curlyvee}(\Gamma)_{[n-1,n+1]} = \Gamma_{n-1}$.
  - If $\Gamma_{n-1} \neq \Gamma_{n+1}$ and the edge $\Gamma_{n-1}\Gamma_{n+1}$ is open in $S^{\curlyvee}(\omega)$, set $S^{\curlyvee}(\Gamma)_{n-1} = \Gamma_{n-1}$ and $S^{\curlyvee}(\Gamma)_{n+1} = \Gamma_{n+1}$, with linear interpolation between.
  - Suppose $\Gamma_{n-1} \neq \Gamma_{n+1}$ and the edge $\Gamma_{n-1}\Gamma_{n+1}$ is closed in $S^{\curlyvee}(\omega)$, and let C denote the third exterior vertex of the star in question. The edges $\Gamma_n C$ and $\Gamma_{n+1} C$ are necessarily open in $S^{\curlyvee}(\omega)$. Set $S^{\curlyvee}(\Gamma)_{n-1} = \Gamma_{n-1}$, $S^{\curlyvee}(\Gamma)_n = C$, $S^{\curlyvee}(\Gamma)_{n+1} = \Gamma_{n+1}$, with linear interpolation between.

Proposition 2.4. *Let Γ be a path of a mixed lattice. We have that*

(a) $d(\Gamma, T^{\triangle}(\Gamma)) \leq \frac{1}{2}$ *and* $d(\Gamma, T^{\triangledown}(\Gamma)) \leq \frac{1}{2}$,
(b) $d(\Gamma, S^{\curlywedge}(\Gamma)) \leq 1$ *and* $d(\Gamma, S^{\curlyvee}(\Gamma)) \leq 1$,
(c) $d(\Gamma, (S^{\curlywedge} \circ T^{\triangledown})(\Gamma)) \leq 1$ *and* $d(\Gamma, (S^{\curlyvee} \circ T^{\triangle})(\Gamma)) \leq 1$,

whenever the transformations are matched to the mixed lattice.

Proof. This follows by examination of the cases above, and is illustrated in Figure 2.4. □

3. Proof of Theorem 1.3

3.1. **The box-crossing property.** Whereas the box-crossing property of Definition 1.2 involves crossing of boxes with arbitrary orientations, it is in fact necessary and sufficient that boxes with sides parallel to the axes possess horizontal and vertical open crossings with probabilities bounded away from 0.

Let $\mathbb{L} = (V, E)$ be a lattice embedded in the plane, and let $\omega \in \Omega_E = \{0,1\}^E$. Let $C_\mathrm{h}(m,n)$ (respectively, $C_\mathrm{v}(m,n)$) be the event that there is an open horizontal (respectively, vertical) crossing of the box $B_{m,n} = [-m,m] \times [0,n]$ of $\mathbb{R}^2$. Suppose now that $\mathbb{L}$ is invariant under translation by the non-zero real vectors $(a,0)$ and $(0,b)$ for some least positive a and b. A probability measure $\mathbb{P}$ on Ω_E is called *translation-invariant* if it is invariant under the actions of these translations. It is said to be *positively associated* if

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$$

for all increasing events A, B (see [13, Sect. 2.2]). By the Harris–FKG inequality (see [12, Sect. 2.2]), product measures are positively associated.

Proposition 3.1. *A translation–invariant, positively associated probability measure $\mathbb{P}$ on Ω_E has the box-crossing property if and only if the following hold:*

(a) *for $\alpha > 0$, there exists $\delta > 0$ such that, for all large $N \in \mathbb{N}$,*

$$\mathbb{P}\left[C_\mathrm{h}(\alpha N, N)\right] > \delta, \tag{3.1}$$

(b) *there exist $\beta, \delta > 0$ such that, for all large $N \in \mathbb{N}$,*

$$\mathbb{P}\left[C_\mathrm{v}(N, \beta N)\right] > \delta. \tag{3.2}$$

It is standard that $\mathbb{P}^{\square}_{\frac{1}{2},\frac{1}{2}}$ satisfies the above conditions and therefore has the box-crossing property. See [25, 26], and the accounts in [12, Sect. 11.7] and [14, Sect. 5.5].

Proof. This is sketched. It is trivial that the box-crossing property implies (3.1) and (3.2). Conversely, suppose (3.1) and (3.2) hold. The positive association permits the combination of box-crossings to obtain crossings of larger boxes. The claim is now obtained as illustrated in Figure 3.1. □

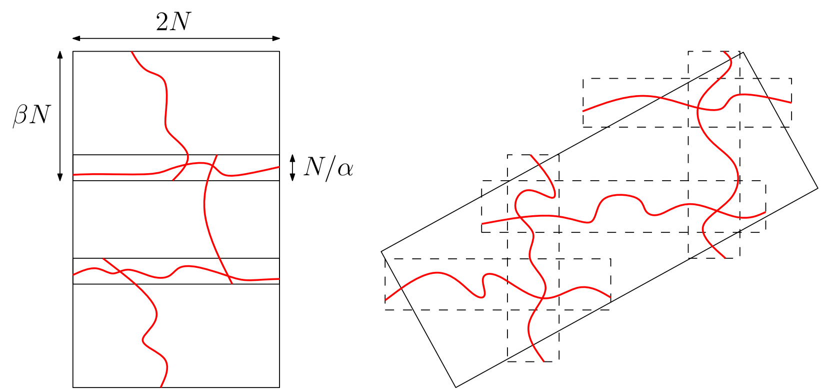

Figure 3.1. *Left*: Vertical crossings of copies of $B_{N,\beta N}$ and horizontal crossings of copies of $B_{N,N/\alpha}$, for large α, may be combined to obtain vertical crossings of boxes with arbitrary aspect ratio. *Right*: Crossings of the type $C_{\rm h}(\gamma n, n)$ and $C_{\rm v}(n, \gamma n)$ may be combined to obtain crossings of boxes with general inclination.

3.2. **Proof of Theorem 1.3.** Theorem 1.3 is an immediate consequence of the following theorem. Recall that a triplet $\mathbf{p} \in [0,1)^3$ is *self-dual* if $\kappa_\triangle(\mathbf{p}) = 0$, with $\kappa_\triangle$ given in (1.2).

Theorem 3.2. *Let $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ be self-dual.*

(a) *If $\mathbb{P}^\square_{(p_0,1-p_0)}$ has the box-crossing property, then so does $\mathbb{P}^\triangle_{\mathbf{p}}$.*
(b) *Let $p_0 > 0$. If $\mathbb{P}^\triangle_{\mathbf{p}}$ has the box-crossing property, then so does $\mathbb{P}^\square_{(p_0,1-p_0)}$.*
(c) *$\mathbb{P}^\triangle_{\mathbf{p}}$ has the box-crossing property if and only if $\mathbb{P}^{\hexagon}_{1-\mathbf{p}}$ has it.*

Since $\mathbb{P}^\square_{(\frac12,\frac12)}$ has the box-crossing property, we have by Theorem 3.2(a) that $\mathbb{P}^\triangle_{(\frac12,p_1,p_2)}$ has the box-crossing property for all self-dual triplets $(\frac12, p_1, p_2)$. As $(\frac12, p_1, p_2)$ ranges within the set of self-dual triplets, p_1 ranges over the interval $[0, \frac12]$. By Theorem 3.2(b), for all $p_1 \in (0, \frac12)$, $\mathbb{P}^\square_{(p_1,1-p_1)}$ has the box-crossing property. We then use Theorem 3.2(a) again to deduce that $\mathbb{P}^\triangle_{\mathbf{p}}$ has the box-crossing property for all self-dual triplets $\mathbf{p}$. Finally, the conclusion may be extended to the hexagonal lattice by Theorem 3.2(c).

Theorem 3.2(a, b) is proved in the remainder of this section. Part (c) is an immediate consequence of a single application of the star–triangle transformation, and no more will be said about this. We assume henceforth that all lattices are embedded in $\mathbb{R}^2$ in the style of Figure 2.3.

3.3. **Proof of Theorem 3.2(a).** It suffices to assume $p_0 > 0$, since the hypothesis does not hold when $p_0 = 0$. By Proposition 3.1, it suffices to prove the following two propositions.

Proposition 3.3. *Let* $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ *be self-dual with* $p_0 > 0$. *For* $\alpha > 1$ *and* $N \in \mathbb{N}$,

$$\mathbb{P}^{\triangle}_{\mathbf{p}}[C_{\mathrm{h}}((\alpha-1)N, 2N)] \ge \mathbb{P}^{\square}_{(p_0,1-p_0)}[C_{\mathrm{h}}(\alpha N, N)].$$

Proposition 3.4. *Let* $\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3$ *be self-dual with* $p_0 > 0$. *There exist* $\beta = \beta(p_0) > 0$, *and* $\rho_N = \rho_N(\beta) > 0$ *satisfying* $\rho_N \to 1$ *as* $N \to \infty$, *such that*

$$\mathbb{P}^{\triangle}_{\mathbf{p}}[C_{\mathrm{v}}(2N, \beta N)] \ge \rho_N \mathbb{P}^{\square}_{(p_0,1-p_0)}[C_{\mathrm{v}}(N, N)], \quad N \in \mathbb{N}.$$

The constant β is given by

$$\beta := \frac{1 - \sqrt{1 - p_0(1-p_0)}}{1 - p_0}, \tag{3.3}$$

and $\rho_N = \rho_N(\beta)$ may be calculated explicitly by the final argument of this subsection.

Proof of Proposition 3.3. Let $\mathbf{p} \in [0,1)^3$ be self-dual with $p_0 > 0$, and let $\alpha > 1$ and $N \in \mathbb{N}$. Let $\mathbb{L} = (V, E)$ be a mixed triangular lattice with interface-height $h(I_{\mathbb{L}}) = N$, and write $\mathbb{P}_{\mathbf{p}}$ for the associated product measure on $\mathbb{L}$. Since $B_{\alpha N, N} = [-\alpha N, \alpha N] \times [0, N]$ is beneath the interface,

$$\mathbb{P}^{\square}_{(p_0,1-p_0)}\left[C_{\mathrm{h}}(\alpha N, N)\right] = \mathbb{P}^{\mathbb{L}}_{\mathbf{p}}\left[C_{\mathrm{h}}(\alpha N, N)\right].$$

Let $\omega \in C_{\mathrm{h}}(\alpha N, N)$. We claim that there exists a horizontal open crossing of $B_{(\alpha-1)N,2N}$ in $(S^{\curlyvee} \circ T^{\triangle})^N(\omega)$, as illustrated in Figure 3.2.

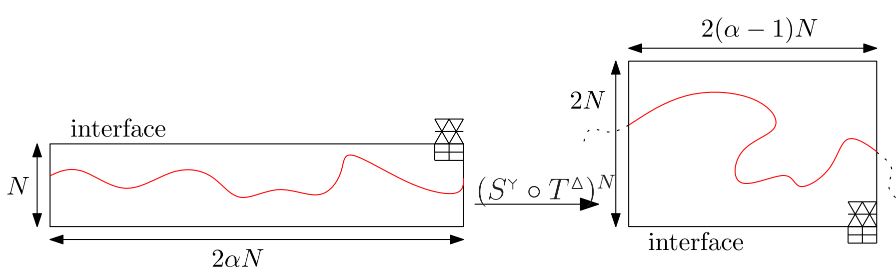

FIGURE 3.2. Transformation of a horizontal crossing of $B_{\alpha N,N}$ by $(S^{\curlyvee} \circ T^{\triangle})^N$. The interface moves down N steps. The path drifts by at most distance N and cannot go below the interface of the image lattice.

Let Γ be an open path of $\mathbb{L}$, parametrized by $[0,1]$, that crosses $B_{\alpha N,N}$ horizontally. By Proposition 2.4, $d(\Gamma, \Gamma(N)) \le N$ where $\Gamma(N) := (S^{\curlyvee} \circ T^{\vartriangle})^N(\Gamma)$, whence,

$$|\Gamma_0 - \Gamma(N)_0| \le N, \tag{3.4}$$

$$|\Gamma_1 - \Gamma(N)_1| \le N, \tag{3.5}$$

$$\Gamma(N) \subseteq \Gamma^N \subseteq B^N_{\alpha N,N}. \tag{3.6}$$

Since Γ contains no vertex with strictly negative y-coordinate, neither does $\Gamma(N)$. Hence,

$$\Gamma(N) \subseteq \Gamma^N \cap \big\{(x,y) \in \mathbb{R}^2 : y \ge 0\big\} \subseteq \mathbb{R} \times [0, 2N] .$$

Taken with (3.4)–(3.5), we deduce that $\Gamma(N)$ contains an open path Γ' that crosses $B_{(\alpha-1)N,2N}$ in the horizontal direction.

Since $B_{(\alpha-1)N,2N}$ lies entirely in the triangular part of $(S^{\curlyvee} \circ T^{\vartriangle})^N\mathbb{L}$, we have by Proposition 2.3 that

$$\begin{aligned}\mathbb{P}^{\mathbb{L}}_{\mathbf{p}}\left[C_{\mathrm{h}}(\alpha N, N)\right] &\le \mathbb{P}^{(S^{\curlyvee}\circ T^{\vartriangle})^N\mathbb{L}}_{\mathbf{p}}\left[C_{\mathrm{h}}((\alpha-1)N, 2N)\right] \\ &= \mathbb{P}^{\triangle}_{\mathbf{p}}\left[C_{\mathrm{h}}((\alpha-1)N, 2N)\right],\end{aligned}$$

and the proposition is proved. □

Proof of Proposition 3.4. Consider the box $B_{N,N}$ in the mixed triangular lattice $\mathbb{L}$ with interface-height $h(I_{\mathbb{L}}) = N$. We follow the strategy of the previous proof by considering the action of $S^{\curlyvee} \circ T^{\vartriangle}$ on a vertical open crossing Γ of the box. In N applications of $S^{\curlyvee} \circ T^{\vartriangle}$, the lattice within the box is transformed from square to triangular. By Proposition 2.4(c), the image of Γ may drift by distance 1 or less at each step. Drift of Γ in the horizontal direction can be accommodated within a box that is wider in that direction. Vertical drift is however more troublesome. Whereas the lower endpoint of Γ is unchanged by N applications of $S^{\curlyvee} \circ T^{\vartriangle}$, its upper endpoint may be reduced in height by 1 at each such application. If this were to occur at every application, both endpoints of the final path would be on the x-axis. This possibility will be controlled by proving that the downward velocity of the upper endpoint is strictly less than 1.

Let $\mathbf{p} \in [0,1)^3$ be self-dual with $p_0 > 0$, and write $\mathbb{L}^k = (S^{\curlyvee} \circ T^{\vartriangle})^k\mathbb{L}$ for $0 \le k \le N$. The lattice $\mathbb{L}^k$ has edge-set E^k and configuration space $\Omega^k = \{0,1\}^{E^k}$. Let $\mathbb{P}^k_{\mathbf{p}}$ denote the probability measure on Ω^k given before Proposition 2.3. Recall from that proposition that $S^{\curlyvee} \circ T^{\vartriangle}$ acts as a *random* mapping from Ω^k to Ω^{k+1}, via the 'kernel' given in Figure 2.2. We shall assume that sequential applications of this kernel are independent of one another and of the choice of initial configuration. More specifically, let $(\omega^k : k \ge 0)$ satisfy:

(a) ω^k is a random configuration from Ω^k,
(b) the sequence $(\omega^k : k \geq 0)$ has the Markov property,
(c) given $\{\omega^0, \omega^1, \dots, \omega^k\}$, ω^{k+1} may be expressed as $\omega^{k+1} = S^{\curlyvee} \circ T^{\triangle}(\omega^k)$,
(d) the law of ω^0 is $\mathbb{P}^0_{\mathbf{p}}$.

Let $\mathbb{P}$ denote the joint law of the sequence $(\omega^0, \omega^1, \dots)$. By Proposition 2.3, the law of ω^k is $\mathbb{P}^k_{\mathbf{p}}$.

Let $D^k = B_{N+k,\infty} = [-N-k, N+k] \times [0, \infty]$ viewed as a subgraph of $\mathbb{L}^k$, and call the line $\mathbb{R} \times \{0\}$ the *base* of $\mathbb{R}^2$. We shall work with the sequence $(h^k : 1 \leq k \leq N)$ of random variables given by

$$h^k := \sup\bigl\{h : \exists x_1, x_2 \in \mathbb{R} \text{ with } (x_1, 0) \xleftrightarrow{D^k, \omega^k} (x_2, h)\bigr\}.$$

Note that h^k acts on Ω^k.

Since $\mathbb{L}^N$ is entirely triangular in the upper half-plane, it suffices to show the existence of $\rho_N = \rho_N(\beta) > 0$ such that $\rho_N \to 1$ and

$$\mathbb{P}(h^N \geq \beta N) \geq \rho_N \mathbb{P}(h^0 \geq N), \tag{3.7}$$

with β as in (3.3). The remainder of this subsection is devoted to proving this.

Lemma 3.5. *For $0 \leq k < N$, the following two statements hold:*

$$h^{k+1} \geq h^k - 1, \tag{3.8}$$

$$\mathbb{P}(h^{k+1} \geq h + \tfrac{1}{2} \mid h^k = h) \geq \beta, \quad h \geq 0. \tag{3.9}$$

Proof. We may assume that $h^k < \infty$ for $0 \leq k \leq N$, since the converse has zero probability. Let $k < N$, and let $\Gamma^k = \Gamma^k(\omega^k)$ be the leftmost path in D^k that reaches some point at height h^k. By Proposition 2.4(c), $\mathbb{L}^{k+1}$ possesses an open vertical crossing of B_{N+k+1,h^k-1}, so that $h^{k+1} \geq h^k - 1$. Inequality (3.8) is proved, and we turn to (3.9).

Let $0 \leq k < N$, and let $\mathcal{G}$ be the set of all paths γ of $\mathbb{L}^k$ such that there exists $h > 0$ with:

(a) all vertices of γ lie in $B_{N+k,h}$,
(b) γ has one endpoint (denoted γ_0) in $\mathbb{R} \times \{0\}$,
(c) its other endpoint (denoted γ_1) lies in $\mathbb{R} \times \{h\}$.

For $\gamma \in \mathcal{G}$, there is a unique such h, denoted $h(\gamma)$.

Let $\gamma \in \mathcal{G}$, and let $L(\gamma)$ be the closed sub-region of $[-N-k, N+k] \times [0, h(\gamma)] \subseteq \mathbb{R}^2$ lying 'to the left' of γ. Let $\mathcal{G}(\gamma)$ be the subset of $\mathcal{G}$ containing all paths γ' with $h(\gamma') = h(\gamma)$ and $\gamma' \subseteq L(\gamma)$. We write $\gamma' < \gamma$ if $\gamma' \subseteq L(\gamma)$ and $\gamma' \neq \gamma$.

Suppose that $p_1 \leq p_2$. The endpoint γ_1 is the lower *left* corner of some upwards pointing triangle denoted $ABC = ABC(\gamma)$, where

$A = \gamma(1)$ and O is its centre. If $p_2 > p_1$, we work instead with the similar triangle of which γ_1 is the lower *right* corner, and the ensuing argument is exactly similar. See Figure 3.3.

We claim that

$$\mathbb{P}(BC \text{ is } \omega^k\text{-closed} \mid \Gamma^k = \gamma) \geq 1 - p_1, \quad \gamma \in \mathcal{G}. \tag{3.10}$$

Since the marginal of $\mathbb{P}$ on Ω^k is $\mathbb{P}^k_{\mathbf{p}}$, it suffices to show that

$$\mathbb{P}^k_{\mathbf{p}}(BC \text{ closed} \mid \Gamma^k = \gamma) \geq 1 - p_1, \quad \gamma \in \mathcal{G}. \tag{3.11}$$

This is proved as follows. Let $\gamma \in \mathcal{G}$. Then $\{\Gamma^k = \gamma\} = F \cap G \cap \{\gamma \text{ open}\}$ where F is the event that there exists no $\gamma' < \gamma$ such that every edge of $\gamma' \setminus \gamma$ is open, and G is the event that there exists no $\gamma'' \in \mathcal{G}$ with $h(\gamma'') > h(\gamma)$ and every edge of $\gamma'' \setminus \gamma$ is open. Since $F \cap G$ is a decreasing event that is independent of the states of edges in γ, we have by the positive association of $\mathbb{P}^k_{\mathbf{p}}$ that

$$\begin{aligned}\mathbb{P}^k_{\mathbf{p}}(\Gamma^k = \gamma \mid BC \text{ closed}) &= \mathbb{P}^k_{\mathbf{p}}(\gamma \text{ open})\mathbb{P}^k_{\mathbf{p}}(F \cap G \mid BC \text{ closed})\\ &\geq \mathbb{P}^k_{\mathbf{p}}(\gamma \text{ open})\mathbb{P}^k_{\mathbf{p}}(F \cap G) = \mathbb{P}^k_{\mathbf{p}}(\Gamma^k = \gamma).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}^k_{\mathbf{p}}\left(BC \text{ closed} \mid \Gamma^k = \gamma\right) &= \mathbb{P}^k_{\mathbf{p}}(\Gamma^k = \gamma \mid BC \text{ closed})\frac{\mathbb{P}^k_{\mathbf{p}}(BC \text{ closed})}{\mathbb{P}^k_{\mathbf{p}}(\Gamma^k = \gamma)}\\ &\geq \mathbb{P}^k_{\mathbf{p}}(BC \text{ closed}) = 1 - p_1,\end{aligned}$$

and (3.10) is proved.

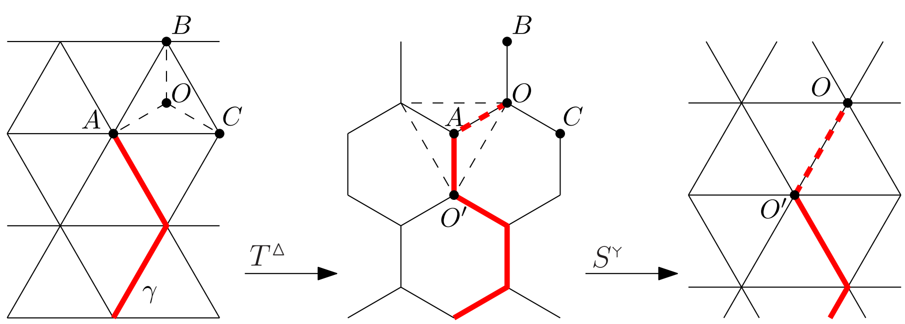

FIGURE 3.3. An illustration of the action of $S^{\curlyvee} \circ T^{\triangle}$ when $\Gamma^k = \gamma$. The top endpoint A of γ is preserved under $T^{\triangle}$. If $\omega^k(BC) = 0$, there is a strictly positive probability that AO is open in $T^{\triangle}(\omega^k)$, in which case $h^{k+1} \geq h^k + \frac{1}{2}$.

Consider the state of the edge AO in the configuration $T^{\triangle}(\omega^k)$. By Figure 2.2, for any $\omega \in \Omega^k$ with $\omega(BC) = 0$,

$$\mathbb{P}_{\mathbf{p}}^k\big(AO \text{ open in } T^{\triangle}(\omega) \mid \omega^k = \omega\big) \geq \frac{p_0 p_2}{(1-p_0)(1-p_2)}.$$

It follows that

$$\mathbb{P}(h^{k+1} \geq h^k + \tfrac{1}{2} \mid \omega^k = \omega) \geq \frac{p_0 p_2}{(1-p_0)(1-p_2)} 1_{\{\omega(BC)=0\}}, \quad \omega \in \Omega^k,$$

where 1_H is the indicator function of the event H. Recall that $BC = BC(\Gamma^k(\omega))$. Therefore, for $\gamma \in \mathcal{G}$,

$$\begin{aligned}\mathbb{P}(h^{k+1} \geq h^k + \tfrac{1}{2} \mid \Gamma^k = \gamma) &\geq \frac{p_0 p_2}{(1-p_0)(1-p_2)} \mathbb{P}(\omega^k(BC) = 0 \mid \Gamma^k = \gamma) \\ &\geq \frac{(1-p_1)p_0 p_2}{(1-p_0)(1-p_2)},\end{aligned}$$

by (3.10).

Now p_0 is fixed, $p_1 \leq p_2$, and $\kappa_{\triangle}(\mathbf{p}) = 0$. Hence, the last ratio is a minimum when $p_1 = p_2$, whence

$$\frac{(1-p_1)p_0 p_2}{(1-p_0)(1-p_2)} \geq \frac{1 - \sqrt{1 - p_0(1-p_0)}}{1-p_0} = \beta,$$

and the claim of the lemma follows. □

There are at least two ways to complete the proof of Proposition 3.4, of which one involves controlling the mean of $h^{k+1} - h^k$. We take a second route here, via a small standard lemma. For a real-valued discrete random variable X, we write $\mathcal{L}(X)$ for its law, and $\mathcal{S}(X) := \{x \in \mathbb{R} : P(X = x) > 0\}$ for its *support*. The inequality $\leq_{\text{st}}$ denotes stochastic domination.

Lemma 3.6. *Let (X_0, X_1) and (Y_0, Y_1) be pairs of real-valued discrete random variables such that:*

(a) $X_0 \leq_{\text{st}} Y_0$,
(b) *for $x \in \mathcal{S}(X_0)$, $y \in \mathcal{S}(Y_0)$ with $x \leq y$, the conditional laws of X_1 and Y_1 satisfy $\mathcal{L}(X_1 \mid X_0 = x) \leq_{\text{st}} \mathcal{L}(Y_1 \mid Y_0 = y)$.*

Then $X_1 \leq_{\text{st}} Y_1$.

Proof. We include a proof for completeness. By Strassen's Theorem (see [22, Sect. IV.1]), there exists a probability space and two random variables X_0', Y_0', distributed respectively as X_0 and Y_0, such that

$P(X_0' \le Y_0') = 1$. Now,

$$\begin{aligned} P(X_1 > u) &= \sum_{x \le y} P(X_1 > u \mid X_0 = x) P(X_0' = x, Y_0' = y) \\ &\le \sum_{x \le y} P(Y_1 > u \mid Y_0 = y) P(X_0' = x, Y_0' = y) \\ &= P(Y_1 > u), \end{aligned}$$

where the summations are restricted to $x \in \mathcal{S}(X_0)$ and $y \in \mathcal{S}(Y_0)$. □

Let $(H^k : k \ge 0)$ be a Markov process with $H^0 = h^0$ and transition probabilities

$$P(H^{k+1} = j \mid H^k = i) = \begin{cases} \beta & \text{if } j = i + \frac{1}{2}, \\ 1 - \beta & \text{if } j = i - 1, \end{cases} \tag{3.12}$$

with β as above. By Lemma 3.5 and an iterative application of Lemma 3.6,

$$\mathbb{P}(h^N \ge \beta N) \ge P(H^N \ge \beta N).$$

Since h^0 and H^0 have the same distribution,

$$\begin{aligned} \frac{\mathbb{P}(h^N \ge \beta N)}{\mathbb{P}(h^0 \ge N)} &\ge \frac{P(H^N \ge \beta N)}{P(H^0 \ge N)} \\ &\ge P(H^N \ge \beta N \mid H^0 \ge N) =: \rho_N(\beta). \end{aligned}$$

Now, (H_k) is a random walk with mean step-size $-1 + 3\beta/2$. By the law of large numbers, $\rho_N \to 1$ as $N \to \infty$. In addition, $\rho_N > 0$, and (3.7) follows. □

3.4. **Proof of Theorem 3.2(b).** By Proposition 3.1, it suffices to prove the following two propositions.

Proposition 3.7. *Let* $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1)^3$ *be self-dual with* $p_0 > 0$. *There exists* $\beta = \beta(p_0) \in \mathbb{N}$ *and* $N_0 = N_0(p_0) \in \mathbb{N}$ *such that, for* $\alpha \in \sqrt{3}\mathbb{N}$ *with* $\alpha > \beta$, *and* $N \ge N_0$,

$$\mathbb{P}^{\square}_{(p_0, 1-p_0)}[C_{\mathrm{h}}\big((\alpha - \beta)N, \beta N\big)] \ge (1 - \alpha e^{-N}) \mathbb{P}^{\triangle}_{\mathbf{p}}[C_{\mathrm{h}}(\alpha N, N)]. \tag{3.13}$$

Proposition 3.8. *Let* $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1)^3$ *be self-dual. For* $\alpha > 0$ *and* $N \in 2\mathbb{N}$,

$$\mathbb{P}^{\square}_{(p_0, 1-p_0)}\left[C_{\mathrm{v}}\big((\alpha + \tfrac{1}{2})N, \tfrac{1}{2}N\big)\right] \ge \mathbb{P}^{\triangle}_{\mathbf{p}}[C_{\mathrm{v}}(\alpha N, N)]. \tag{3.14}$$

Proof of Proposition 3.7. Let $\mathbf{p}$ satisfy the hypothesis of the proposition. The idea is to consider repeated applications of the transformation $S^{\curlywedge} \circ T^{\triangledown}$ to an open horizontal crossing of a box in the triangular part of a mixed lattice. The interface moves upwards, and the crossing may 'drift' upwards at each step. A new technique is required to control the rate of this drift. This will be achieved by bounding the vertical displacement of the path by a certain growth process.

We partition the plane into vertical columns

$$\mathcal{C}_n = \big(n\sqrt{3}, (n+1)\sqrt{3}\big) \times \mathbb{R}, \quad n \in \mathbb{Z},$$

of width $\sqrt{3}$. Let $\mathbb{L} = (V, E)$ be a mixed lattice, and $\omega \in \Omega_E$. The $\mathcal{C}_n$ correspond to the columns of the square sublattice of $\mathbb{L}$, as illustrated in Figure 3.4.

For any (parametrized) open path $\Lambda = (\Lambda_t : a \le t \le b)$ on $\mathbb{L}$, let

$$H_n(\Lambda) = \sup\big\{h(\Lambda_t) : t \text{ such that } \Lambda_t \in \mathcal{C}_n\big\}$$

be its *height* in $\mathcal{C}_n$. (The supremum of the empty set is taken to be $-\infty$.) Note that $h(\Lambda) = \sup_n H_n(\Lambda)$. The growth of the $H_n(\Lambda)$ may be bounded as follows under the action of the random map $S^{\curlywedge} \circ T^{\triangledown}$.

For future use, we define $\eta : (0,1) \to (0,1)$ by

$$\eta(x) = \Big(1 + x - \sqrt{1 - x + x^2}\Big)^2, \tag{3.15}$$

and note that η is increasing.

Lemma 3.9. *Let $\mathbb{L}$ be a mixed triangular lattice, and let ω, Λ be as above. There exists a family of independent Bernoulli random variables $(Y_n : n \in \mathbb{Z})$ with parameter $1 - \eta(p_0)$, such that, for all $n \in \mathbb{Z}$,*

$$H_n\big((S^{\curlywedge} \circ T^{\triangledown})(\Lambda)\big) \le \max\big\{H_{n-1}(\Lambda), H_n(\Lambda) + Y_n, H_{n+1}(\Lambda)\big\}.$$

We delay the proof of this lemma until later in this subsection.

Let $\mathbb{L}^0 = (V, E)$ be the mixed triangular lattice with interface-height $h(I_{\mathbb{L}^0}) = 0$, and let $\omega^0 \in \Omega_E$. Let $\alpha \in \sqrt{3}\mathbb{N}$, and let Γ^0 be an open path of $\mathbb{L}^0$ in the box $B_{\alpha n,n}$. We shall use the notation introduced at the start of the proof of Proposition 3.4, with the difference that the transformation $S^{\curlyvee} \circ T^{\triangle}$ there is replaced here by $S^{\curlywedge} \circ T^{\triangledown}$. Thus, $\mathbb{L}^k = (S^{\curlywedge} \circ T^{\triangledown})^k \mathbb{L}$, and ω^k is the edge-configuration on $\mathbb{L}^k$ given by $\omega^k = S^{\curlywedge} \circ T^{\triangledown}(\omega^{k-1})$ for $k \ge 1$. Recall that ω^k is a random function of ω^{k-1} generated via the kernel of Figure 2.2, and we assume as before that sequential applications of this kernel are independent. We shall study the heights of the image paths $\Gamma^k = (S^{\curlywedge} \circ T^{\triangledown})^k(\Gamma^0)$.

As before, if ω^0 is chosen according to $\mathbb{P}^0_{\mathbf{p}}$, then the law of ω^k is $\mathbb{P}^k_{\mathbf{p}}$. The law of the sequence $(\omega^k : k \ge 0)$ is written $\mathbb{P}$, although for the moment we take ω^0 to be fixed and write $\mathbb{P}(\cdot \mid \omega^0)$ for the corresponding conditional measure.

We shall show that the speed of growth of the maximal height of Γ^k is strictly less than 1. This will be proved by constructing a certain growth process that dominates (stochastically) the family $(H_n(\Gamma^k) : n \in \mathbb{Z},\, k \geq 0)$.

Let $\zeta \in (0,1)$. Let $(Y_n^k : n \in \mathbb{Z},\, k \geq 0)$ be a family of independent Bernoulli random variables with parameter $1-\zeta$. The Markov process $\mathbf{X}^k := (X_n^k : n \in \mathbb{Z})$ is given as follows.

(a) The initial value $\mathbf{X}^0$ is given by

$$X_n^0 = \begin{cases} N & \text{for } n \in \left[-\alpha N/\sqrt{3}, \alpha N/\sqrt{3}\right], \\ -\infty & \text{for } n \notin \left[-\alpha N/\sqrt{3}, \alpha N/\sqrt{3}\right]. \end{cases}$$

(b) For $k \geq 0$, conditional on $\mathbf{X}^k$, the vector $\mathbf{X}^{k+1}$ is given by

$$X_n^{k+1} = \max\{X_{n-1}^k, X_n^k + Y_n^k, X_{n+1}^k\}, \quad n \in \mathbb{Z}.$$

Lemma 3.10. *Let $\zeta \in (0,1)$. There exist $\beta, N_0 \in \mathbb{N}$ depending on ζ only (independent of α, N) such that, for $\alpha \in \sqrt{3}\mathbb{N}$ and $N \geq N_0$,*

$$P\left(\max_n X_n^{\beta N} \leq \beta N\right) \geq 1 - \alpha e^{-N}.$$

We postpone the proof of this lemma, first completing that of Proposition 3.7. Let $\zeta = \eta(p_0)$, and let β and N_0 be given as in Lemma 3.10. Since $H_n(\Gamma^0) \leq X_n^0$ for all n, we have by Lemma 3.9 that, given ω^0, $h(\Gamma^k)$ is dominated stochastically by $\max_n X_n^k$. By Lemma 3.10,

$$\mathbb{P}\big(h(\Gamma^{\beta N}) \leq \beta N \,\big|\, \omega^0\big) \geq 1 - \alpha e^{-N}, \quad N \geq N_0. \tag{3.16}$$

Since $h(I_{\mathbb{L}^0}) = 0$ and $h(I_{\mathbb{L}^N}) = N$,

$$\mathbb{P}_{\mathbf{p}}^{\triangle}[C_{\mathrm{h}}(\alpha N, N)] = \mathbb{P}\big(\omega_0 \in C_{\mathrm{h}}(\alpha N, N)\big),$$
$$\mathbb{P}_{(p_0,1-p_0)}^{\square}[C_{\mathrm{h}}((\alpha-\beta)N, \beta N)] = \mathbb{P}\big(\omega_{\beta N} \in C_{\mathrm{h}}((\alpha-\beta)N, \beta N)\big).$$

Hence,

$$\frac{\mathbb{P}_{(p_0,1-p_0)}^{\square}[C_{\mathrm{h}}((\alpha-\beta)N, \beta N)]}{\mathbb{P}_{p}^{\triangle}[C_{\mathrm{h}}(\alpha N, N)]} \geq \mathbb{P}\Big(\omega_{\beta N} \in C_{\mathrm{h}}((\alpha-\beta)N, \beta N) \,\Big|\, \omega_0 \in C_{\mathrm{h}}(\alpha N, N)\Big). \tag{3.17}$$

Let $\omega_0 \in C_{\mathrm{h}}(\alpha N, N)$ and let Γ^0 be an ω^0-open crossing of $B_{\alpha N,N}$. By Proposition 2.4, the leftmost point of $\Gamma^{\beta N}$ lies to the left of $B_{(\alpha-\beta)N,\beta N}$, and the rightmost point to the right of that box. Moreover $\Gamma^{\beta N}$ is contained in the upper half-plane, since the lower half-plane is in the square-lattice part of every $\mathbb{L}^k$. If, in addition, $h(\Gamma^{\beta N}) \le \beta N$, then $\Gamma^{\beta N}$ contains a $\omega^{\beta N}$-open horizontal crossing of $B_{(\alpha-\beta)N,\beta N}$. In conclusion,

$$\begin{aligned}\mathbb{P}\Big(\omega^{\beta N} \in C_{\rm h}((\alpha-\beta)N, \beta N) \,\Big|\, \omega_0 \in C_{\rm h}(\alpha N, N)\Big) \qquad\qquad \\ \ge \mathbb{P}\left(h(\Gamma^{\beta N}) \le \beta N \,\middle|\, \omega_0 \in C_{\rm h}(\alpha N, N)\right) \\ \ge 1 - \alpha e^{-N}, \quad N \ge N_0,\end{aligned}$$

by (3.16). The claim follows by (3.17). □

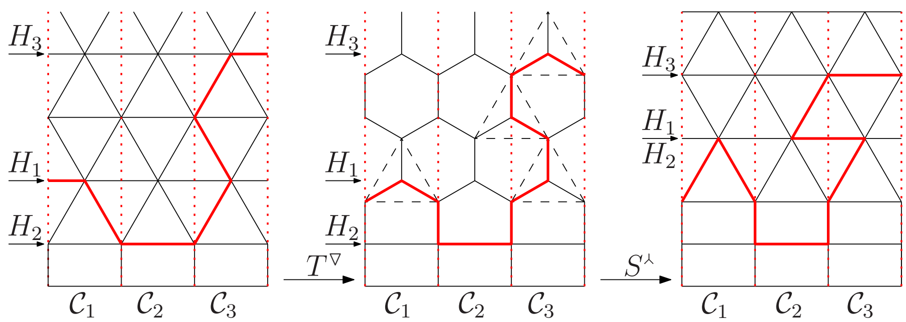

FIGURE 3.4. The evolution of the heights of a crossing within columns, when applying T^∇ and $S^\curlywedge$. The heights in each column are the same in the first and second lattice. In the third: H_1 increases by 1; H_2 increases by 2; H_3 does not change.

Proof of Lemma 3.9. We recall two properties of the transformations $S^\curlywedge$ and T^∇ when applied to an ω-open path Λ. In constructing $T^\nabla(\Lambda)$, we apply T^∇ to downwards pointing triangles of $\mathbb{L}$ containing either one or two edges of Λ. As illustrated in Figure 2.2, T^∇ acts deterministically on such triangles, and hence $T^\nabla(\Lambda)$ is specified by knowledge of Λ. By inspection of Figure 3.4 or otherwise,

$$H_n(T^\nabla(\Lambda)) = H_n(\Lambda), \quad n \in \mathbb{Z}. \tag{3.18}$$

The situation is less simple when applying $S^\curlywedge$ to $T^\nabla(\Lambda)$. Let $\mathcal{S}$ be the set of upwards pointing stars of $T^\nabla\mathbb{L}$, and let $(Z_{\rm l}^s, Z_{\rm r}^s : s \in \mathcal{S})$ be independent Bernoulli random variables with parameter

$$\nu := \sqrt{1-\nu_0} \quad \text{where} \quad \nu_0 := 1 - \frac{p_1 p_2}{(1-p_1)(1-p_2)}.$$

For $s \in \mathcal{S}$, let $\underline{Z}^s = \min\{Z_{\rm l}^s, Z_{\rm r}^s\}$, noting that

$$P(\underline{Z}^s = 1) = \nu^2 = 1 - \nu_0. \tag{3.19}$$

We call $s \in \mathcal{S}$ a *horizontal star* (for Λ) if $T^\triangledown(\Lambda)$ includes the two non-vertical edges of s.

By (3.18), any changes in the H_n occur only when applying $S^\curlywedge$. The height $H_n(\Lambda)$ may grow under the application of $S^\curlywedge \circ T^\triangledown$ for either of two reasons: (i) the highest part of Λ within $\mathcal{C}_n$ may move upwards, or (ii) part of Λ in a neighbouring column may drift into $\mathcal{C}_n$ (in which case, we say it 'invades' $\mathcal{C}_n$). These two possibilities will be considered separately.

Let $n \in \mathbb{Z}$. Assume first that

$$H_n(\Lambda) \le \max\big\{H_{n-1}(\Lambda), H_{n+1}(\Lambda)\big\} - 1. \tag{3.20}$$

By Proposition 2.4, the part of Λ within $\mathcal{C}_n$ cannot drift upwards by more than 1. By considering the ways in which parts of Λ may invade $\mathcal{C}_n$, we find that such invasions may occur only horizontally, and not diagonally upwards (see Figure 3.4). Combining these two observations, we deduce under (3.20) that

$$H_n(S^\curlywedge \circ T^\triangledown(\Lambda)) \le \max\big\{H_{n-1}(\Lambda), H_{n+1}(\Lambda)\big\}. \tag{3.21}$$

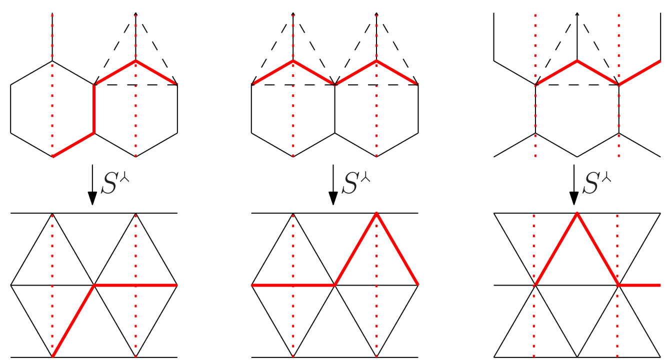

FIGURE 3.5. Three examples of growth of path-height within a column under the action of $S^\curlywedge$, under the assumption $H_n(\Gamma_k) \ge \max\{H_{n-1}(\Gamma_k), H_{n+1}(\Gamma_k)\}$. *Left*: The base of the marked triangle is present in the image, and the height does not increase. *Middle*: The base of the rightmost marked triangle is absent. The heights in the central and right columns increase. There is a strictly positive probability that both marked bases are present, and that the height in the central column does not increase. *Right*: The base of the marked triangle is absent, and the height increases by 1.

Suppose next that

$$H_n(\Lambda) \geq \max\big\{H_{n-1}(\Lambda), H_{n+1}(\Lambda)\big\}. \tag{3.22}$$

By Proposition 2.4, (3.18), and the above remark concerning invasion,

$$H_n(S^{\curlywedge} \circ T^{\triangledown}(\Lambda)) \leq H_n(\Lambda') + 1 = H_n(\Lambda) + 1,$$

where $\Lambda' = T^{\triangledown}(\Lambda)$. Assume that $H_n(S^{\curlywedge}(\Lambda')) = H_n(\Lambda')+1$. There must exist a star $s \in \mathcal{S}$ such that:

(a) s is a horizontal star for Λ,
(b) s intersects $\mathcal{C}_n$,
(c) $H_n(T^{\triangledown}(\Lambda)) = h(O)$ where O is the centre of s,
(d) the base of $S^{\curlywedge}(s)$ is closed in $S^{\curlywedge} \circ T^{\triangledown}(\omega)$.

(See the middle and rightmost cases of Figure 3.5 for illustrations.)

Let s satisfy (a), (b), and (c), and write A for the highest vertex of s, so that $T^{\triangledown}(\Lambda)$ includes the edges BO and CO. The edge BC is open in $S^{\curlywedge} \circ T^{\triangledown}(\omega)$ with (conditional) probability

$$\begin{cases} 1 & \text{if } AO \text{ is closed in } T^{\triangledown}(\omega), \\ \nu_0 & \text{if } AO \text{ is open in } T^{\triangledown}(\omega). \end{cases}$$

(See Figure 2.2.) This conditional probability is achieved by declaring BC to be open if and only if: either AO is closed in $T^{\triangledown}(\omega)$, or AO is open in $T^{\triangledown}(\omega)$ and $\underline{Z}^s = 0$. With this coupling,

if (d) above holds, then $\underline{Z}^s = 1$, and hence $Z_{\mathrm{l}}^s = Z_{\mathrm{r}}^s = 1$.

We return to (3.22). If the highest part of Λ in $\mathcal{C}_n$ comprises a single horizontal star s, as on the right of Figure 3.5,

$$H_n(S^{\curlywedge} \circ T^{\triangledown}(\Lambda)) - H_n(\Lambda) \leq \max\{Z_{\mathrm{l}}^s, Z_{\mathrm{r}}^s\} =: Y_n. \tag{3.23}$$

If, on the other hand, the highest part of Λ in $\mathcal{C}_n$ corresponds to two stars, s_1 and s_2, that also intersect $\mathcal{C}_{n-1}$ and $\mathcal{C}_{n+1}$ respectively (as in the first and second diagrams of the figure),

$$H_n(S^{\curlywedge} \circ T^{\triangledown}(\Lambda)) - H_n(\Lambda) \leq \max\{Z_{\mathrm{r}}^{s_1}, Z_{\mathrm{l}}^{s_2}\} =: Y_n. \tag{3.24}$$

Recalling the properties of the Z_{l}^s, Z_{r}^s, we have that the Y_n are independent Bernoulli variables with parameter $1 - \eta'$ where

$$\eta' := \big(1 - \sqrt{1-\nu_0}\big)^2 = \left(1 - \sqrt{\frac{p_1 p_2}{(1-p_1)(1-p_2)}}\right)^2. \tag{3.25}$$

The proof is completed by the elementary exercise of showing that $\eta' \geq \eta(p_0)$. □

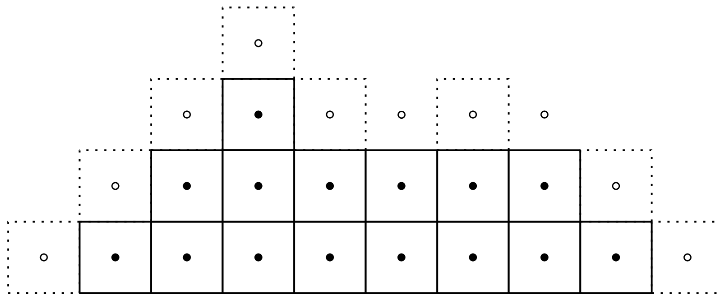

FIGURE 3.6. The solid squares represent the bricks at step k in the growth process. The dotted squares are the additions at time $k+1$. The lateral extensions occur with probability 1, and the vertical extensions with probability $1-\zeta$.

Proof of Lemma 3.10. The process $\mathbf{X} = (\mathbf{X}^k : k \geq 0)$ may be represented physically as follows. Above each integer is a pile of bricks, illustrated in Figure 3.6. At each epoch of time, each column gains a random number of bricks. If a column is as least as high as its two nearest neighbouring columns, a brick is added with probability $1-\zeta$. Otherwise, bricks are added to the column to match the height of its higher neighbour.

We study the process via the times at which bricks are placed at vertices. For each pair A, B of neighbours in the upper half-plane $\mathbb{Z}\times\mathbb{Z}_0$ of the square lattice with the usual embedding, we place a directed edge denoted AB from A to B, and similarly a directed edge BA from B to A. Let $\mathcal{E}$ be the set of all such directed edges. The random variables $(\tau_{AB} : AB \in \mathcal{E})$ are assumed independent with distributions as follows.

$$\tau_{AB} = \begin{cases} 1 & \text{if } AB \text{ is horizontal,} \\ 0 & \text{if } AB \text{ is directed downwards,} \end{cases}$$

and τ_{AB} has the geometric distribution with parameter $1-\zeta$ if AB is directed upwards, that is,

$$P(\tau_{AB} = r) = \zeta^{r-1}(1-\zeta), \quad r \geq 1.$$

Thinking about τ_{AB} as the time for the process to pass along the edge AB, we define the *passage-time* from C to D by

$$\tau(C,D) = \inf\left\{ \tau(\vec{\Lambda}) := \sum_{e\in\vec{\Lambda}} \tau_e : \vec{\Lambda} \in \mathcal{P}_{C,D} \right\},$$

where $\mathcal{P}_{C,D}$ is the set of all directed paths from C to D.

Let $\alpha \in \sqrt{3}\mathbb{N}$ and $L_i := [-\alpha N/\sqrt{3}, \alpha N/\sqrt{3}] \times \{i\}$. The initial state G_0 of this growth process is the set $\bigcup_{i=0}^{N} L_i$. It is easily seen that the state G_k at time k comprises exactly the set of all vertices D such that there exists $C \in L_N$ with $\tau(C, D) \le k$.

Let $\beta > 3$ be an integer, to be chosen later. By the above,

$$P(h(G_{\beta N}) \ge \beta N) \le \sum_{\substack{C,D:\\ C\in L_N,\, h(D)=\beta N}} P(\tau(C,D) \le \beta N). \tag{3.26}$$

Now, $\tau(C, D) \le \beta N$ if and only if there exists a directed path $\vec{\Lambda} \in \mathcal{P}_{C,D}$ with passage-time not exceeding βN, so that

$$P(h(G_{\beta N}) \ge \beta N) \le \sum_{\vec{\Lambda}\in\mathcal{P}_N} P(\tau(\vec{\Lambda}) \le \beta N), \tag{3.27}$$

where $\mathcal{P}_N$ is the set of directed paths whose endpoints C, D are as in (3.26). Consider such a path $\vec{\Lambda}$, and let u, d, h be the numbers of its upward, downward, and horizontal edges, respectively. Since upward and horizontal edges have passage-times at least 1, we must have $u + h \le \beta N$. By considering the heights of the first and last vertices, $u - d = (\beta - 1)N$. Therefore, $\vec{\Lambda}$ has no more than $(\beta + 1)N$ edges in total, of which at least $(\beta - 1)N$ are upward.

There are $|L_N| \le 2\alpha N$ possible choices for C, so that

$$|\mathcal{P}_N| \le 2\alpha N 4^{2N} \binom{(\beta+1)N}{2N}. \tag{3.28}$$

For $\vec{\Lambda} \in \mathcal{P}_N$, $\tau(\vec{\Lambda})$ is no smaller than the sum of the passage-times of its upward edges. Therefore,

$$P(\tau(\vec{\Lambda}) \le \beta N) \le P(S \le \beta N), \tag{3.29}$$

where S is the sum of $(\beta-1)N$ independent random variables with the $\mathrm{Geom}(1 - \zeta)$ distribution. It is elementary that

$$P(S \le \beta N) = P\big(T \ge (\beta - 1)N\big),$$

where T has the binomial distribution $\mathrm{bin}(\beta N, 1 - \zeta)$. By Markov's inequality (as in the proof of Cramér's Theorem),

$$\limsup_{N\to\infty} P\big(T \ge (\beta - 1)N\big)^{1/N} \le \beta \left(\frac{\beta(1-\zeta)}{\beta - 1}\right)^{\beta}, \tag{3.30}$$

when $\beta(1 - \zeta) < \beta - 1$, that is, $\beta > 1/\zeta$.

By (3.27)–(3.30), there exists $N_0 = N_0(\beta, \zeta)$ such that, for $N \geq N_0$,

$$P(h(G_{\beta N}) \geq \beta N) \leq 2\alpha N 4^{2N} \binom{(\beta+1)N}{2N} \left\{ 2\beta \left(\frac{\beta(1-\zeta)}{\beta - 1} \right)^{\beta} \right\}^{N}.$$

By Stirling's formula, there exists $c = c(\zeta)$ and $N_1 = N_1(\beta, \zeta)$ such that, for $N \geq N_1$,

$$P(h(G_{\beta N}) \geq \beta N) \leq \alpha \left\{ c\beta^3 \left(\frac{\beta(1-\zeta)}{\beta - 1} \right)^{\beta} \right\}^{N}. \tag{3.31}$$

Choose $\beta = \beta(\zeta)$ sufficiently large that the last term is smaller than αe^{-N}, and the proof is complete. □

This concludes the proof of Lemma 3.10 and thus of Proposition 3.7.

Proof of Proposition 3.8. Let $N \in 2\mathbb{N}$. Let $\mathbb{L} = (V, E)$ be the mixed triangular lattice with interface-height 0, so that

$$\mathbb{P}^{\triangle}_{\mathbf{p}}\big[C_{\rm v}(\alpha N, N)\big] = \mathbb{P}^{\mathbb{L}}_{\mathbf{p}}\big[C_{\rm v}(\alpha N, N)\big].$$

Let $\omega \in \Omega_E$, and let Γ be an ω-open vertical crossing of $B_{\alpha N, N}$. In $\frac{1}{2}N$ applications of $S^{\curlywedge} \circ T^{\triangledown}$, the images of the lower endpoint of Γ remain in the square part of the lattice, and thus are immobile. By Proposition 2.4, $(S^{\curlywedge} \circ T^{\triangledown})^{N/2}(\Gamma)$ contains a vertical crossing of $B_{(\alpha+\frac{1}{2})N, N/2}$ that is open in $(S^{\curlywedge} \circ T^{\triangledown})^{N/2}(\omega)$. Since $B_{(\alpha+\frac{1}{2})N, N/2}$ lies entirely within the square part of $(S^{\curlywedge} \circ T^{\triangledown})^{N/2}\mathbb{L}$, we deduce that

$$\begin{aligned} \mathbb{P}^{\square}_{(p_0, 1-p_0)}\big[C_{\rm v}((\alpha + \tfrac{1}{2})N, \tfrac{1}{2}N)\big] &= \mathbb{P}^{(S^{\curlywedge} \circ T^{\triangledown})^{N/2}\mathbb{L}}_{\mathbf{p}}\big[C_{\rm v}((\alpha + \tfrac{1}{2})N, N)\big] \\ &\geq \mathbb{P}^{\triangle}_{\mathbf{p}}\big[C_{\rm v}(\alpha N, N)\big], \end{aligned}$$

and the claim is proved. □

4. Remaining proofs

4.1. **Using the box-crossing property.** Our target in this subsection is to summarize how certain properties of inhomogeneous percolation models may be deduced from the box-crossing property. These properties will be used later in this section, and are of independent interest.

We shall consider bond percolation on the square, triangular, and hexagonal lattices, and any reference to a lattice shall mean one of these three, duly embedded in $\mathbb{R}^2$ as described after Definition 1.2. Our arguments may be applied to many other graphs, but for the sake of simplicity this is not discussed here. Fix a lattice $\mathbb{L} = (V, E)$ with *origin* 0, and let $\mathbf{p} = (p_e : e \in E) \in [0, 1]^E$. Denote by $\mathbb{P}_{\mathbf{p}}$ the product measure on $\Omega = \{0,1\}^E$ under which edge $e \in E$ is open with probability p_e. The lattice $\mathbb{L}$ has a dual lattice $\mathbb{L}^* = (V^*, E^*)$, and we write $\mathbb{P}^*_{1-\mathbf{p}}$ for the product measure on $\Omega^* = \{0,1\}^{E^*}$ under which an edge $e^* \in E^*$, dual to $e \in E$, is open with probability $1 - p_e$.

For $\nu > 0$, let $\overline{\mathbb{P}}_{\mathbf{p}+\nu}$ be the product measure on E under which e is open with probability $1_{\{p_e>0\}} \min\{p_e+\nu, 1\}$, and $\underline{\mathbb{P}}_{\mathbf{p}-\nu}$ that under which e is open with probability $1_{\{p_e=1\}} + 1_{\{p_e<1\}} \max\{p_e - \nu, 0\}$. Under these measures, any edge with $\mathbb{P}_{\mathbf{p}}$-parameter equal to 0 or 1 retains this property. Let $|\cdot|$ denote the Euclidean norm as before and, for $x \geq 0$, define the box $S_x = \{z \in \mathbb{R}^2 : |z| < x\}$. The open cluster at vertex v is denoted C_v, and its *radius* $\mathrm{rad}(C_v)$ is the supremum of all x such that C_v intersects $\mathbb{R}^2 \setminus (v + S_x)$.

Proposition 4.1. *Suppose $\mathbb{P}^*_{1-\mathbf{p}}$ has the box-crossing property.*

(a) *There exist $a, b > 0$ such that, for every vertex v,*

$$\mathbb{P}_{\mathbf{p}}(\mathrm{rad}(C_v) \geq k) \leq ak^{-b}, \quad k \geq 0.$$

(b) *There exists, $\mathbb{P}_{\mathbf{p}}$-a.s., no infinite open cluster.*

(c) *For $\nu > 0$, there exist $c, d > 0$ such that, for every vertex v,*

$$\underline{\mathbb{P}}_{\mathbf{p}-\nu}(|C_v| \geq k) \leq ce^{-dk}, \quad k \geq 0.$$

Proposition 4.2. *Suppose $\mathbb{P}_{\mathbf{p}}$ has the box-crossing property.*

(a) *There exist $a, b > 0$ and $M \in \mathbb{N}$ such that: for every $v \in V$, there exists $w = w(v) \in V$ with $|v - w| \leq M$ and*

$$\mathbb{P}_{\mathbf{p}}(\mathrm{rad}(C_w) \geq k) \geq ak^{-b}, \quad k \geq 0.$$

(b) *Let $\nu > 0$. There exist $\alpha > 0$ and $M \in \mathbb{N}$ such that: for every $v \in V$, there exists $w = w(v) \in V$ with $|v - w| \leq M$ and $\overline{\mathbb{P}}_{\mathbf{p}+\nu}(w \leftrightarrow \infty) > \alpha$. There exists, $\overline{\mathbb{P}}_{\mathbf{p}+\nu}$-a.s., a unique infinite open cluster.*

The unusually complicated formulation of this proposition arises from the possible existence of edges e with $p_e = 0$.

Sketch proof of Proposition 4.1. Further details of the arguments used here may be found in [14, Chap. 5]. For definiteness, we consider only the square lattice, and analogous proofs are valid for the triangular and hexagonal lattices. Assume $\mathbb{P}^*_{1-\mathbf{p}}$ has the box-crossing property.
(a) If $\mathrm{rad}(C_v) \geq k$, there exist order $\log k$ disjoint annuli around v none of which contains a dual open cycle surrounding v. By the box-crossing property, this event has probability less than $a(1-\gamma)^{\alpha \log k}$ for some $a, \gamma, \alpha > 0$, and the claim follows. Part (b) is a trivial consequence.

(c) Let $N \geq 1$. Let B_N be a box of size $3N \times N$, and let H_N be the event that B_N has an open dual crossing in the long direction, in the dual lattice $\mathbb{L}^*$. By the box-crossing property, there exists $\tau = \tau(\mathbf{p}) > 0$, independent of B_N and N, such that $\mathbb{P}^*_{1-\mathbf{p}}(H_N) \geq \tau$ for all large N. By part (a) above, Russo's formula, and the theory of influence (as in [11, 14], for example),

$$\frac{d}{d\eta}\overline{\mathbb{P}}^*_{1-\mathbf{p}+\eta}(H_N) \geq c_1 \overline{\mathbb{P}}^*_{1-\mathbf{p}+\eta}(H_N)\big(1 - \overline{\mathbb{P}}^*_{1-\mathbf{p}+\eta}(H_N)\big)\log(c_2 N^a), \tag{4.1}$$

for $\eta > 0$ and some absolute constants c_1, c_2. This inequality, when integrated over $(0, \nu)$, yields $\overline{\mathbb{P}}^*_{1-\mathbf{p}+\nu}(H_N) \to 1$ as $N \to \infty$, uniformly in the choice of B_N.

For $\zeta > 0$, we may choose N sufficiently large that $\overline{\mathbb{P}}^*_{1-\mathbf{p}+\nu}(H_N) \geq 1 - \zeta$ for all B_N. By passing to the dual and using the method of proof of [18, Thm 1] with ζ small (see also [13, Thm 5.86]), one obtains the exponential decay of cluster-volume. □

Sketch proof of Proposition 4.2. We consider only the square lattice.
(a) Let $v \in V$. For odd $i \geq 1$, let $A_i(v)$ be the event that $v + [0, 2^i] \times [0, 2^{i-1}]$ has a horizontal open crossing; for even $i \geq 1$, let $A_i(v)$ be the event that $v + [0, 2^{i-1}] \times [0, 2^i]$ has a vertical open crossing. By the box-crossing property, there exist $\tau > 0$ and $I \in \mathbb{N}$ such that $\mathbb{P}_{\mathbf{p}}(A_i(v)) > \tau$ for $i \geq I$ and $v \in V$. Let $M = 2^{I-1}$ and $J > I + \log_2 k$. There exists w with $|v - w| \leq M$ such that: on the event $\bigcap_{i=I}^J A_i(v)$, we have $\mathrm{rad}(C_w) \geq k$. The claim follows by positive association and the box-crossing property.
(b) The proof follows that of Proposition 4.1(c) and [14, Thm 5.64]. Let $N \geq 1$, let $B_N = [0, 8N] \times [0, 2N]$, and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ comprise a rotation and a translation. Let H_N be the event that $f(B_N)$ has an open 'horizontal' crossing, and in addition the boxes $f([0, 2N] \times [0, 2N])$ and $f([6N, 8N] \times [0, 2N])$ have open 'vertical' crossings ('horizontal' and 'vertical' refer to the orientation of B_N). By the box-crossing property and positive association, there exists $\tau > 0$ such that $\mathbb{P}_{\mathbf{p}}(H_N) \geq \tau$ for all large N, uniformly in f. By the argument leading to (4.1), $\overline{\mathbb{P}}_{\mathbf{p}+\nu}(H_N) \to 1$ as $N \to \infty$, uniformly in f. We pick N sufficiently large, and adapt the block argument of [14, Sect. 5.8] to deduce the claim. The second assertion is an elementary consequence of the box-crossing property and the existence of open paths in annuli. □

4.2. **Proofs of Theorems 1.4 and 1.6.** The proof of Theorem 1.4 follows exactly that of Section 3.3 on noting that: each triangle of the mixed triangular lattice of Figure 4.1 has three edges with parameters forming a self-dual triplet, and the constants of Propositions 3.3 to 3.8

depend only (in the current setting) on the value of p and not otherwise on $\mathbf{q}$ and $\mathbf{q}'$. The hexagonal-lattice case follows by a single application of the star–triangle transformation.

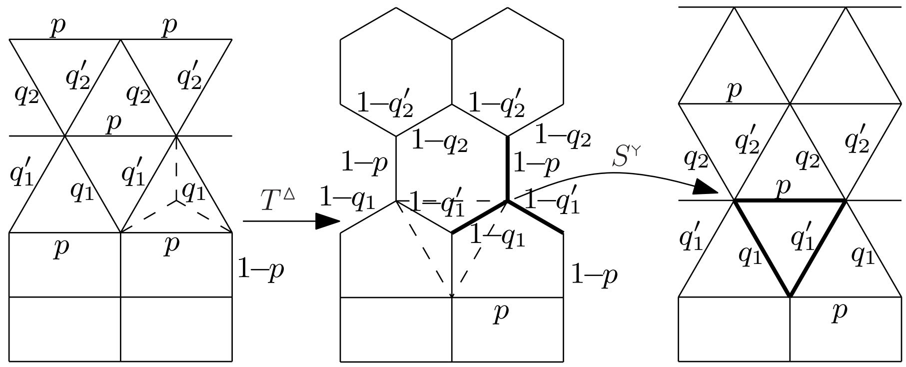

FIGURE 4.1. A mixed triangular lattice (left) with the highly inhomogeneous measure above the interface. The transformation $S^\curlyvee \circ T^\triangle$ moves the interface down by one unit. Every triangle is parametrized by a self-dual triplet.

Since $\mathbb{P}^\triangle_{p,\mathbf{q},\mathbf{q}'}$ is increasing in $\mathbf{q}$ and $\mathbf{q}'$, and since the non-existence of an infinite component is a decreasing event, Theorem 1.6(a) follows from Proposition 4.1(b).

Turning to part (b) of Theorem 1.6, assume (1.7) holds with $\delta > 0$. Let $\epsilon = \frac{1}{4}\delta$ and note from (1.7) that $p, q_n, q_n' < 1 - \epsilon$ for $n \in \mathbb{Z}$. Therefore, $p + \epsilon, q_n + \epsilon, q_n' + \epsilon < 1$ for all n, and

$$\kappa_\triangle(p + \epsilon, q_n + \epsilon, q_n' + \epsilon) \le 0, \quad n \in \mathbb{Z}.$$

By Theorem 1.4 and the monotonicity of measures, the dual measure, $\mathbb{P}^{\hexagon}_{1-p-\epsilon,1-\mathbf{q}-\epsilon,1-\mathbf{q}'-\epsilon}$ has the box-crossing property. The claim follows by Proposition 4.1(c) with $\nu = \epsilon$.

Assume finally that (1.8) holds with $\delta > 0$. Let $\epsilon = \frac{1}{3}\min\{\delta, p\}$ and write

$$x^+ = \max\{x, 0\}, \quad \widehat{x} = x1_{\{x \ge \epsilon\}}.$$

Then

$$\kappa_\triangle\big((p - \epsilon)^+, (q_n - \epsilon)^+, (q_n' - \epsilon)^+\big) \ge 0, \quad n \in \mathbb{Z}.$$

By Theorem 1.4 and the monotonicity of measures, the associated product measure on the triangular lattice has the box-crossing property. By Proposition 4.2(b) with $\nu = \epsilon$, and the fact that $\mathbb{P}_{\widehat{p},\widehat{\mathbf{q}},\widehat{\mathbf{q}}'}((v + S_M) \subseteq C_v)$ is bounded from 0 uniformly in $v \in V$, we have that $\mathbb{P}_{\widehat{p},\widehat{\mathbf{q}},\widehat{\mathbf{q}}'}$ is uniformly supercritical. By monotonicity of measures, $\mathbb{P}_{p,\mathbf{q},\mathbf{q}'}$ is uniformly supercritical as claimed.

The same arguments are valid for the hexagonal lattice.

4.3. **Proofs of Theorems 1.5 and 1.7.** Let $\mathbf{q} = 1 - \mathbf{q}'$ satisfy (1.5) with $\epsilon > 0$, and let $p = 1 - p' = \frac{1}{2}\epsilon$. We may pick $r_n \in (0,1)$ such that $\kappa_\triangle(p, q_n, r_n) = 0$ for all n, and we write $r_n' = 1 - r_n$. By Theorem 1.4, the measure $\mathbb{P}^\triangle_{p,\mathbf{q},\mathbf{r}}$ has the box-crossing property, and we propose to transport this property to the square-lattice measure $\mathbb{P}_{\mathbf{q},\mathbf{q}'}$ via the star-triangle transformation.

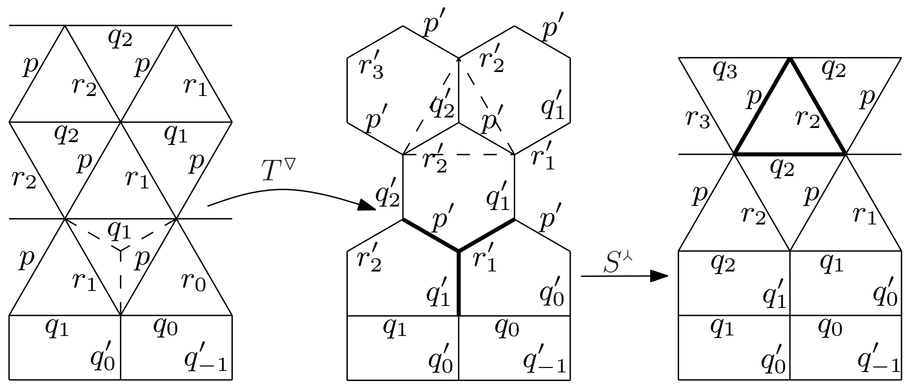

FIGURE 4.2. *Left*: The measure $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$ on $\mathbb{L}$. In the triangular part the measure is $\mathbb{P}^\triangle_{p,\mathbf{q},\mathbf{r}}$ on a rotated lattice, and in the square part it is $\mathbb{P}^\square_{\mathbf{q},\mathbf{q}'}$. *Middle, right*: Application of $S^{\curlywedge} \circ T^{\triangledown}$ transforms $\mathbb{L}$ to a copy of itself shifted upwards and sideways.

Let $\mathbb{L} = (V, E)$ be the mixed triangular lattice on the left of Figure 4.2, and denote by $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$ the product measure given there. Under $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$, all triangles in $\mathbb{L}$ have self-dual triplets. Thus, $T^{\triangledown}$ acts on Ω_E endowed with $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$ in the manner of Section 2 (with parameters varying between triangles), and the ensuing measure is given in the middle figure. Then $S^{\curlywedge}$ acts on edge-configurations of $T^{\triangledown}\mathbb{L}$ (with parameters varying between stars). The ensuing lattice $(S^{\curlywedge} \circ T^{\triangledown})\mathbb{L}$ is illustrated on the right, and it may be noted that the corresponding measure is precisely that of $\mathbb{L}$ shifted upwards and rightwards.

In the triangular part of $\mathbb{L}$, $\mathbb{P}_{\mathbf{q},\mathbf{r},p}$ corresponds to the measure $\mathbb{P}^\triangle_{p,\mathbf{q},\mathbf{r}}$, while in the square part it corresponds to $\mathbb{P}^\square_{\mathbf{q},\mathbf{q}'}$. By Theorem 1.4, $\mathbb{P}^\triangle_{p,\mathbf{q},\mathbf{r}}$ has the box-crossing property, and thus it remains to adapt the proofs of Propositions 3.7 and 3.8.

Proposition 3.8 holds because of its non-probabilistic bound for the drift of a path under $S^{\curlywedge} \circ T^{\triangledown}$. Its proof is easily adapted to give, as there, that, for $\alpha > 0$ and $N \in 2\mathbb{N}$,

$$\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}\left[C_{\mathrm{v}}\left(\left(\alpha+\tfrac{1}{2}\right)N, N/2\right)\right] \geq \mathbb{P}^{\triangle}_{\mathbf{q},\mathbf{r},p}[C_{\mathrm{v}}(\alpha N, N)].$$

The proof of Proposition 3.7 requires the probabilistic estimate of Lemma 3.9. This hinges on the application of $S^{\curlywedge}$ to configurations on upwards pointing stars. The key fact is that $\eta(p_0) > 0$, with η as in (3.15) and p_0 the parameter associated with a horizontal edge in the triangular lattice. In the present situation, such edges have parameters q_n. Since $q_n \geq \epsilon$, we have that $\eta(q_n) \geq \eta(\epsilon) > 0$. This results in an altered version of Lemma 3.9 with $\eta(p_0)$ replaced by $\eta(\epsilon)$. The proof continues as before, and a version of (3.13) results. Theorem 1.5 is proved.

Finally, consider Theorem 1.7, and assume (1.9). Let $\nu_n = (1 - q_n - q'_n)/2$, and apply Theorem 1.5 to the self-dual measure $\mathbb{P}^{\square}_{\mathbf{q}+\boldsymbol{\nu},\mathbf{q}'+\boldsymbol{\nu}}$. Part (a) then follows by Proposition 4.1(b). The proofs of (b, c) hold as for the triangular lattice.

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